CS 2LC3
Logical Reasoning for Computer Science

Tutorial 4

Mahdee Jodayree

Habib Ben Abdallah

McMaster University

October 4th, 2022

Outline

- Announcements / Reminders
- Format for writing proofs
- Does superman exist?
- Several Problems from Chapter 5, similar to Q31 and Q32 of Assignment 1.

Announcements

- Mid-term date has changed to **October 25**, Tuesday, during the class, before it was scheduled to ~~October 20~~).
- The deadline for assignment 1 has extended to Oct 17.
- Question about the list of axioms.
- Assignment 1, Q32 has changed, you only need to answer part a and b. but you will get bonus if you have answered the entire question.
- Final exam sheet sheet.

Format for writing proofs

Writing format for the proofs are in chapter 1.5.

Use $\langle\ \rangle$ Angle brackets

The number any previously proven axiom should be inside the parenthesis.

Proof of theorem (3.78), $(p \Rightarrow r) \land (q \Rightarrow r) \equiv (p \lor q \Rightarrow r)$.

$$
\begin{array}{ll}
 & (p \Rightarrow r) \land (q \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
 & (\neg p \lor r) \land (\neg q \lor r) \\
= & \quad \langle \text{Distributivity of } \lor \text{ over } \land \text{ (3.45)} \rangle \\
 & (\neg p \land \neg q) \lor r \\
= & \quad \langle \text{De Morgan (3.47b)} \rangle \\
 & \neg(p \lor q) \lor r \\
= & \quad \langle \text{Implication (3.59)} \rangle
\end{array}
$$

Proof of theorem (3.78), $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.

$$
\begin{array}{ll}
 & (p \Rightarrow r) \wedge (q \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
 & (\neg p \vee r) \wedge (\neg q \vee r) \\
= & \quad \langle \text{Distributivity of } \vee \text{ over } \wedge \text{ (3.45)} \rangle \\
 & (\neg p \wedge \neg q) \vee r \\
= & \quad \langle \text{De Morgan (3.47b)} \rangle \\
 & \neg(p \vee q) \vee r \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & p \vee q \Rightarrow r
\end{array}
$$

Proof of theorem (3.79), $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$.

$$
\begin{array}{ll}
 & (p \Rightarrow r) \wedge (\neg p \Rightarrow r) \\
= & \quad \langle \text{Case analysis (3.78), with } q := \neg p \rangle \\
 & p \vee \neg p \Rightarrow r \\
= & \quad \langle \text{Excluded middle (3.28)} \rangle \\
 & \mathit{true} \Rightarrow r \\
= & \quad \langle \text{Left identity of } \Rightarrow \text{ (3.73)} \rangle \\
 & r
\end{array}
$$

Proof of Mutual implication (3.80), $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv p \equiv q$.

$$
\begin{array}{ll}
& (p \Rightarrow q) \wedge (q \Rightarrow p) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
& (\neg p \vee q) \wedge (\neg q \vee p) \\
= & \quad \langle \text{Golden rule (3.35)} \rangle \\
& \neg p \vee q \vee \neg q \vee p \equiv \neg p \vee q \equiv \neg q \vee p \\
= & \quad \langle \text{Excluded middle (3.28); Zero of } \vee \text{ (3.29)} \rangle \\
& \mathit{true} \equiv \neg p \vee q \equiv \neg q \vee p \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
& \neg p \vee q \equiv \neg q \vee p \\
= & \quad \langle \text{(3.32), } p \vee q \equiv p \vee \neg q \equiv p \text{, twice} \rangle \\
& p \vee q \equiv q \equiv q \vee p \equiv p \\
= & \quad \langle \text{Symmetry of } \equiv \text{ (3.2)} \rangle \\
& p \equiv q
\end{array}
$$

Does Superman Exist?'

A Problem from pages 88-90

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

Similar to Question Q31 and Q32 of assignment 1.

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

We want to use the propositional calculus to determine whether this argument is sound —whether the conclusion "Superman does not exist" follows from the previous sentences. As on page 37, we associate variables with the primitive propositions:

a : Superman is able to prevent evil.
w : Superman is willing to prevent evil.
i : Superman is impotent.
m : Superman is malevolent.
p : Superman prevents evil.
e : Superman exists.

We want to use the propositional calculus to determine whether this argument is sound —whether the conclusion "Superman does not exist" follows from the previous sentences. As on page 37, we associate variables with the primitive propositions:

a : Superman is able to prevent evil.
w : Superman is willing to prevent evil.
i : Superman is impotent.
m : Superman is malevolent.
p : Superman prevents evil.
e : Superman exists.

The first four sentences can be formalized as

$F0: \ a \wedge w \Rightarrow p$ If Superman were able and willing to prevent evil, he would do so.

$F1: \ (\neg a \Rightarrow i) \wedge (\neg w \Rightarrow m)$

$F2: \ \neg p$ Superman does not prevent evil.

$F3: \ e \Rightarrow \neg i \wedge \neg m$ If Superman exists, he is neither impotent not malevolent.

Therefore, Superman does not exist.

and the Superman argument is equivalent to the boolean expression

(5.4) $\quad F0 \wedge F1 \wedge F2 \wedge F3 \Rightarrow \neg e$. **Therefore, Superman does not exist.**

One way to prove (5.4) is to assume the four conjuncts of the antecedent and prove the consequent. That is, we begin by manipulating the consequent $\neg e$. Beginning with $\neg e$, we see only *one* way to proceed. The only assumption in which e appears is $F3$. If we translate $F3$ into its contrapositive $\neg(\neg i \wedge \neg m) \Rightarrow \neg e$, $\neg e$ emerges. (See (3.61) for the contrapositive of an implication).

$$\boxed{F3: \; e \Rightarrow \neg i \wedge \neg m}$$

Assume $F0$, $F1$, $F2$, $F3$

$$\begin{array}{ll}
 & \neg e \\
\Leftarrow & \langle \text{Contrapositive } \neg(\neg i \wedge \neg m) \Rightarrow \neg e \text{ of } F3 \\
 & \quad \text{—the only other place } e \text{ appears} \rangle \\
 & \neg(\neg i \wedge \neg m) \\
= & \langle \text{De Morgan (3.47a); Double negation (3.12), twice} \rangle \\
 & i \vee m \\
\Leftarrow & \langle \text{First conjunct of } F1 \text{ and Monotonicity (4.2)} \rangle \\
 & \neg a \vee m
\end{array}$$

$$
\begin{array}{ll}
\Leftarrow & \langle \text{Second conjunct of } F1 \text{ and Monotonicity (4.2)} \rangle \\
& \neg a \vee \neg w \\
= & \langle \text{De Morgan (3.47a)} \rangle \\
& \neg(a \wedge w) \\
\Leftarrow & \langle \text{Contrapositive } \neg p \Rightarrow \neg(a \wedge w) \text{ of } F0 \rangle \\
& \neg p \quad \text{—this is } F2
\end{array}
$$

We conclude that (5.4) is a theorem, so the argument of the Superman paragraph is sound.

This calculation illustrates an important point. We started with the consequent $\neg e$ and worked "backward" toward the assumptions. In this case, working backwards was a real help, for at each step there was essentially no choice about what to do next! The only choice was in the order in which to use the conjuncts of $F1$, and this choice was immaterial to the proof development. Proofs in which there is no choice at each step are particularly nice, because the reader can see that each step is directed by a formula's structure and is not a rabbit pulled out of a hat.

Does Superman Exist?

Assume $F0$, $F1$, $F2$, $F3$

$$
\begin{array}{ll}
 & \neg e \\
\Leftarrow & \langle \text{Contrapositive } \neg(\neg i \wedge \neg m) \Rightarrow \neg e \text{ of } F3 \\
 & \quad \text{—the only other place } e \text{ appears} \rangle \\
 & \neg(\neg i \wedge \neg m) \\
= & \langle \text{De Morgan (3.47a); Double negation (3.12), twice} \rangle \\
 & i \vee m \\
\Leftarrow & \langle \text{First conjunct of } F1 \text{ and Monotonicity (4.2)} \rangle \\
 & \neg a \vee m \\
\Leftarrow & \langle \text{Second conjunct of } F1 \text{ and Monotonicity (4.2)} \rangle \\
 & \neg a \vee \neg w \\
= & \langle \text{De Morgan (3.47a)} \rangle \\
 & \neg(a \wedge w) \\
\Leftarrow & \langle \text{Contrapositive } \neg p \Rightarrow \neg(a \wedge w) \text{ of } F0 \rangle \\
 & \neg p \quad \text{—this is } F2
\end{array}
$$

Sound Theorem.

Soundness

In logic, more precisely in deductive reasoning, an argument is **sound** if it is both **valid in form** and its **premises are true**.

Soundness also has a related meaning in mathematical logic, wherein logical systems are sound if and only if every formula that can be proved in the system is logically valid with respect to the semantics of the system.

Chapter 7 – page 128.

Solving word problems

5.1 Formalize the following arguments and either prove that they are valid or find a counterexample.

(a) Either the program does not terminate or n eventually becomes 0. If n becomes 0, m will eventually be 0. The program terminates. Therefore, m will eventually be 0.

(b) If the initialization is correct and if the loop terminates, then P is *true* in the final state. P is *true* in the final state. Therefore, if the initialization is correct, the loop terminates.

(c) If there is a man on the moon, the moon is made of cheese, and if the moon is made of cheese then I am a monkey. Either no man is on the moon or the moon is not made of cheese. Therefore either the moon is not made of cheese or I am a monkey.

(d) If Joe loves Mary, then either mom is mad or father is sad. Father is sad. Therefore, if mom is mad then Joe doesn't love Mary.

Solving word problems

(a) Either the program does not terminate or n eventually becomes 0. If n becomes 0, m will eventually be 0. The program terminates. Therefore, m will eventually be 0.

(a) We associate identifiers with the primitive propositions:

T : The program terminates,
N : n becomes 0,
M : m becomes 0.

The boolean expression is then

$$(\neg T \lor N) \land (N \Rightarrow M) \land T \Rightarrow M \quad .$$

We calculate to show that it is a theorem, proving that the English argument is sound.

$$\begin{array}{ll}
& (\neg T \lor N) \land (N \Rightarrow M) \land T \\
= & \quad \langle \text{Absoportion (3.44a)} \rangle \\
& N \land (N \Rightarrow M) \land T \\
= & \quad \langle (3.66),\ P \land (P \Rightarrow Q) \equiv P \land Q \rangle \\
& N \land M \land T \\
\Rightarrow & \quad \langle \text{Strengthening (3.76b)} \rangle \\
& M
\end{array}$$

Solving word problems

(b) If the initialization is correct and if the loop terminates, then P is *true* in the final state. P is *true* in the final state. Therefore, if the initialization is correct, the loop terminates.

(b) We associate identifiers with the primitive propositions:

C : The initialization is correct,
T : The loop terminates,
S : P is true in the final state.

The boolean expression is then

$$(C \wedge T \Rightarrow S) \wedge S \Rightarrow (C \Rightarrow T) \quad .$$

We calculate:

$$\begin{array}{ll} & (C \wedge T \Rightarrow S) \wedge S \\ = & \quad \langle (3.59),\ P \Rightarrow Q \equiv \neg P \vee Q \rangle \\ & (\neg(C \wedge T) \vee S) \wedge S \\ = & \quad \langle \text{Absorption (3.43a)} \rangle \\ & S \end{array}$$

At this point, we don't believe that S implies $C \Rightarrow T$. The expression $S \Rightarrow (C \Rightarrow T)$ gives us a hint on what assignment of values for C, T, and S to choose so that the original expression is *false*: $S, T, C := \mathit{true}, \mathit{false}, \mathit{true}$. Hence, the argument is invalid.

Solving word problems

(c) If there is a man on the moon, the moon is made of cheese, and if the moon is made of cheese then I am a monkey. Either no man is on the moon or the moon is not made of cheese. Therefore either the moon is not made of cheese or I am a monkey.

(c) We associate identifiers with the primitive propositions:

M : There is a man on the moon,
C : The moon is made of cheese,
I : I am a monkey.

The boolean expression is then

$$(M \Rightarrow C) \land (C \Rightarrow I) \land (\neg M \lor \neg C) \Rightarrow \neg C \lor I \quad .$$

Note that the consequent $\neg C \lor I$ is equivalent to $C \Rightarrow I$, which is one of the conjuncts in the antecedent. Therefore, we believe that the expression is a theorem. Taking this cue, we begin with the antecedent and calculate as follows.

$$\begin{array}{ll}
 & (M \Rightarrow C) \land (C \Rightarrow I) \land (\neg M \lor \neg C) \\
\Rightarrow & \langle \text{Strengthening (3.76b)} \rangle \\
 & C \Rightarrow I \\
= & \langle (3.59),\ P \Rightarrow Q \equiv \neg P \lor Q \rangle \\
 & \neg C \lor I
\end{array}$$

Solving word problems

(d) If Joe loves Mary, then either mom is mad or father is sad. Father is sad. Therefore, if mom is mad then Joe doesn't love Mary.

(d) We associate identifiers with the primitive propositions:

J : Joe loves Mary,
M : Mom is mad,
F : Father is sad.

The boolean expression is then

$$(J \Rightarrow M \vee F) \wedge F \Rightarrow (M \Rightarrow \neg J) \quad .$$

At first glance, we guessed that this was not a theorem and looked for a counterexample. Choosing the assignment $M, J := true, true$, makes the consequent $false$. It also makes the first conjunct $true$. Choosing $F := true$ makes the second conjunct of the antecedent $true$ as well. Hence, this assignment makes the expression $false$, so the expression is not a theorem and the original argument is not valid.

Solving word problems

5.3 Suppose we have the following facts. Prove that if the maid told the truth, the butler lied.

> The maid said she saw the butler in the living room. The living room adjoins the kitchen. The shot was fired in the kitchen and could be heard in all adjoining rooms. The butler, who had good hearing, said he did not hear the shot.

Solving word problems

5.3 Suppose we have the following facts. Prove that if the maid told the truth, the butler lied.

> The maid said she saw the butler in the living room. The living room adjoins the kitchen. The shot was fired in the kitchen and could be heard in all adjoining rooms. The butler, who had good hearing, said he did not hear the shot.

MT : The maid told the truth,
BT : The butler told the truth,
BL : The butler was in the living room,
LK : The living room adjoins the kitchen,
BH : The butler heard the shot.

$$MT \Rightarrow BL \quad .$$

$$LK \quad .$$

$$(LK \wedge BL) \Rightarrow BH \quad .$$

$$BT \Rightarrow \neg BH \quad .$$

$$MT \Rightarrow \neg BT?$$

Solving word problems

$$
\begin{array}{ll}
 & MT \\
\Rightarrow & \langle \text{Assumption } MT \Rightarrow BL \text{ ; Modus ponens (3.77)} \rangle \\
 & BL \\
\Rightarrow & \langle \text{Assumptions } LK \text{ and } (LK \wedge BL) \Rightarrow BH \text{ ; Modus ponens (3.77)} \rangle \\
 & BH
\end{array}
$$

$$
\begin{array}{ll}
= & \langle \text{Assumption } BT \Rightarrow \neg BH \text{ ; Contrapositive (3.61); Double negation (3.12)} \rangle \\
 & BH \wedge (BH \Rightarrow \neg BT) \\
\Rightarrow & \langle \text{Modus ponens (3.77)} \rangle \\
 & \neg BT
\end{array}
$$

5.4 Suppose Portia puts her picture into one of three caskets and places the following inscriptions on them:

Gold casket:	The portrait is in here.
Silver casket:	The portrait is in here.
Lead casket:	At least two of the caskets have a false inscription.

Which casket should the suitor choose? Formalize and calculate an answer.

5.4 Suppose Portia puts her picture into one of three caskets and places the following inscriptions on them:

Gold casket:	The portrait is in here.
Silver casket:	The portrait is in here.
Lead casket:	At least two of the caskets have a false inscription.

Which casket should the suitor choose? Formalize and calculate an answer.

Solution:

5.4 The suitor should immediately note that the situation is symmetric in gold and silver —the inscriptions on the gold and silver caskets are the same. Hence, if the portrait were in the silver or gold casket, there would be no way to determine this from the information given. Hence, the suitor should choose the lead casket!

Armed with this clue, let us see whether we can prove that the portrait is in the lead casket. Introduce the following identifiers:

G : The portrait is in the gold casket,
S : The portrait is in the silver casket,
L : The portrait is in the lead casket.

We formalize the fact that the portrait is in exactly one casket. Using the symmetry clue, we formalize only the part of this fact that is symmetric in G and S.

Fact 1: $L \equiv \neg G \wedge \neg S$
Fact 2: $G \wedge S \equiv false$

For the inscriptions ig and is on the gold and silver caskets, we have $ig \equiv G$ and $is \equiv S$. Therefore, we just use G and S in place of ig and is. The inscription il on the lead casket is more complicated. For our purposes, it is best to write down what it means as two implications. If the inscription is true, then the other two inscriptions are false; if the inscription is false, then at most one of the inscriptions is false, which means that the other two are true. Hence, for the third inscription we have

$$\begin{array}{l} \text{Fact 3: } il \Rightarrow \neg G \wedge \neg S \quad , \\ \text{Fact 4: } \neg il \Rightarrow G \wedge S \quad . \end{array}$$

We hope that Facts 1, 2, 3, and 4 imply L. To prove this, we assume the facts and prove L. Because of the different facts concerning il and $\neg il$, we proceed as follows.

$$\begin{array}{ll} & il \vee \neg il \quad \text{—Excluded middle (3.28)} \\ \Rightarrow & \quad \langle \text{Fact 3; Fact 4} \rangle \\ & (\neg G \wedge \neg S) \ \vee \ (G \wedge S) \\ = & \quad \langle \text{Fact 1; Fact 2} \rangle \\ & L \ \vee \ \mathit{false} \\ = & \quad \langle \text{Identity of } \vee \ (3.30) \rangle \\ & L \end{array}$$

Any Questions?