Graphs: Shortest Paths

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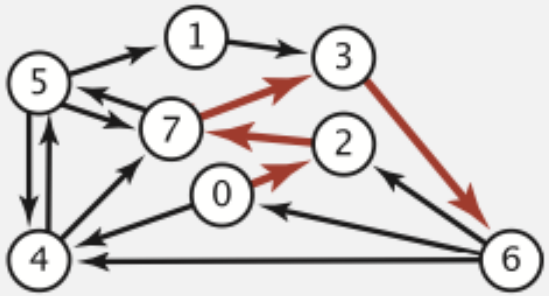
Shortest paths in an edge-weighted digraph

- In this lecture, we study the shortest path problem, given an edge-weighted digraph.

edge-weighted digraph

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



shortest path from 0 to 6

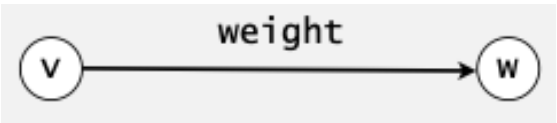
```
0->2  0.26
2->7  0.34
7->3  0.39
3->6  0.52
```

Shortest paths in an edge-weighted digraph problems

1. Given an edge-weighted digraph, find the shortest path from s to t.
2. Given an edge-weighted digraph, find the shortest paths from s to all other vertices (single source shortest paths).
3. It turns out that both time and space complexities of (1) and (2) are the same!
4. We cannot efficiently solve (1) without solving (2) in general case.

Weighted directed edge API

public class DirectedEdge		
	DirectedEdge(int v, int w, double weight)	*weighted edge v→w*
int	from()	*vertex v*
int	to()	*vertex w*
double	weight()	*weight of this edge*
String	toString()	*string representation*



Idiom for processing an edge e: int $v = e.from()$, $w = e.to()$;

Weighted directed edge: implementation in Java

Similar to Edge for undirected graphs, but a bit simpler.

```
public class DirectedEdge
{
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int from()
   {  return v;  }

   public int to()
   {  return w;  }

   public int weight()
   {  return weight; }
}
```

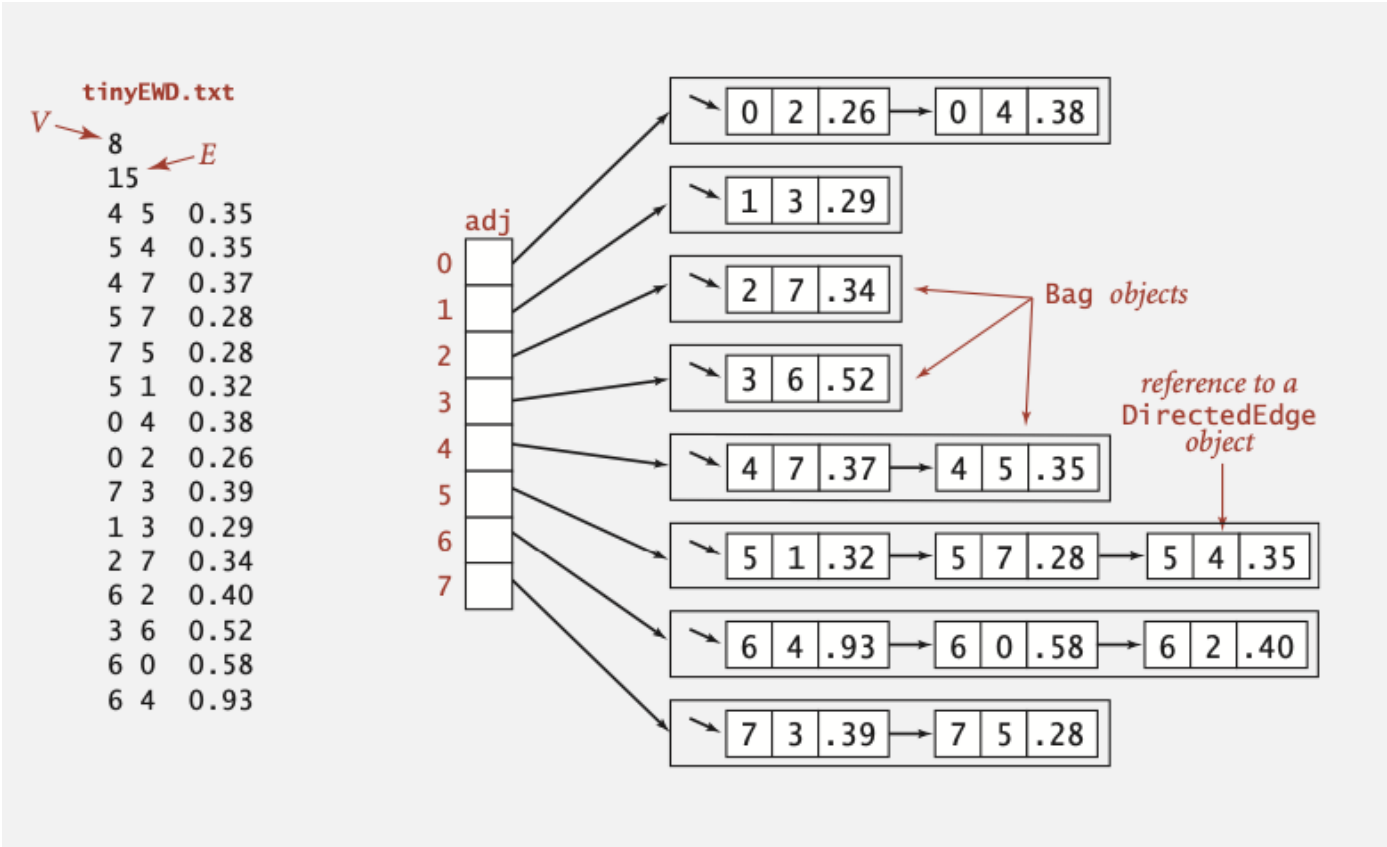
from() and to() replace either() and other()

Edge-weighted digraph API

public class EdgeWeightedDigraph		
	EdgeWeightedDigraph(int V)	*edge-weighted digraph with V vertices*
	EdgeWeightedDigraph(In in)	*edge-weighted digraph from input stream*
void	addEdge(DirectedEdge e)	*add weighted directed edge e*
Iterable<DirectedEdge>	adj(int v)	*edges pointing from v*
int	V()	*number of vertices*
int	E()	*number of edges*
Iterable<DirectedEdge>	edges()	*all edges*
String	toString()	*string representation*

Conventions: Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation



Summary of properties and assumptions

Properties. Several important properties and assumptions made for this topic:

- Paths are directed. A shortest path must respect the direction of its edges.
- Weights are not necessarily distances. The edge weights might represent time or cost.
- Not all vertices need be reachable. If t is not reachable from s, there is no path at all, and therefore there is no shortest path from s to t.
- Shortest paths are not necessarily unique. There may be multiple paths of the lowest weight from one vertex to another; we are content to find any one of them.

Data structures for single-source shortest paths

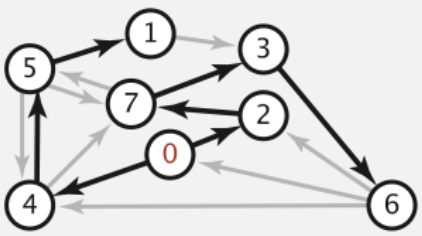
Given an edge-weighted digraph and a designated vertex s, a **shortest-paths tree (SPT)** for a source s is a subgraph containing s and all the vertices reachable from s that forms a directed tree rooted at s such that every tree path is a shortest path in the digraph.

A shortest-paths tree (SPT) solution exists. Why?

Data structures for single-source shortest paths

We can represent the SPT with two vertex-indexed arrays:

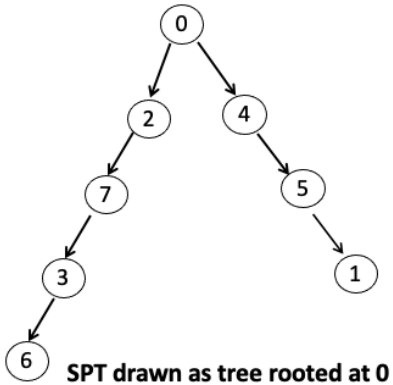
- distTo[v] is length of shortest path from s to v (in the figures this weight is associated to the vertex v).
- edgeTo[v] is last edge on shortest path from s to v.



shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

parent-link representation



SPT drawn as tree rooted at 0

Goal of all the algorithms is to compute SPT, that is, compute the $edgeTo[]$ and $distTo[]$ arrays.

Edge and Vertex relaxation

To compute SPT two techniques are useful:

1. Edge relaxation
2. Vertex relaxation

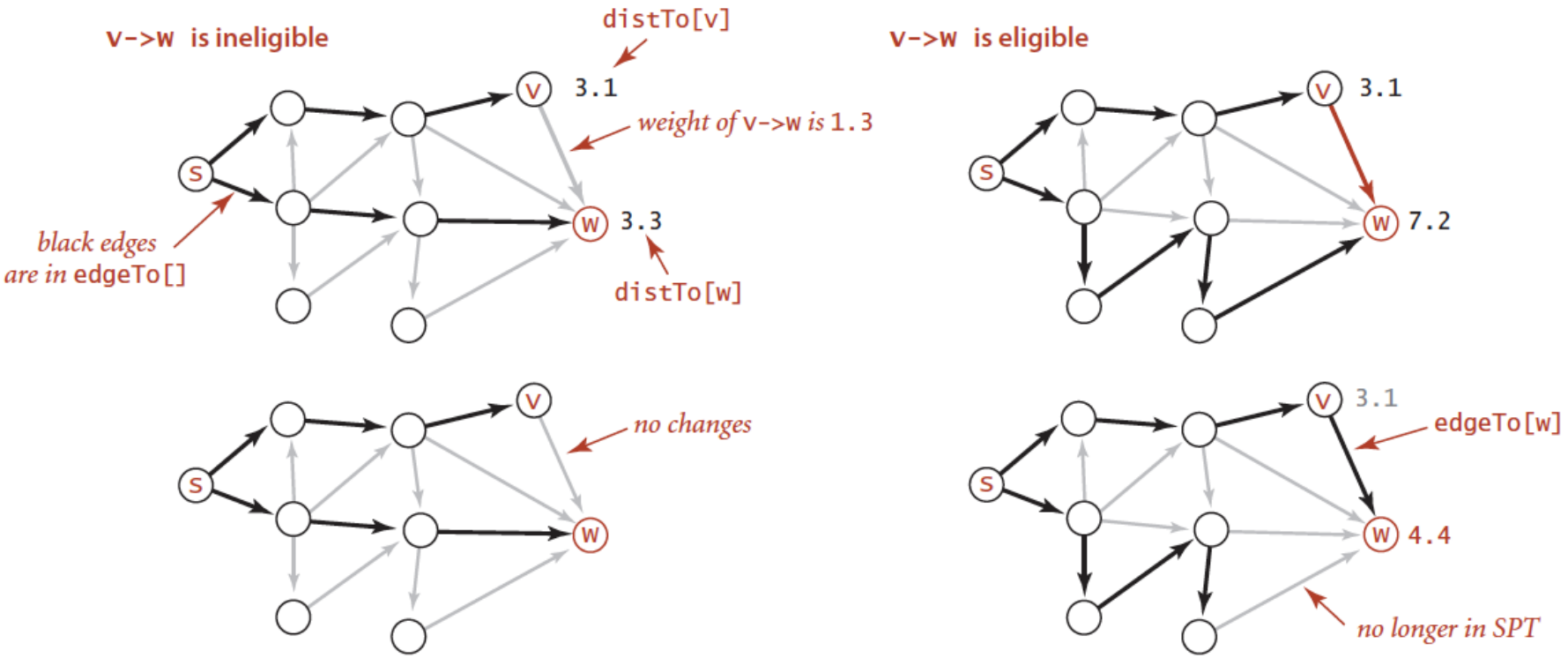
Edge relaxation - I

Edge relaxation idea:

- $distTo[v]$ is length of shortest path known from s to v.
- $distTo[w]$ is length of shortest known path from s to w.
- $edgeTo[w]$ is last edge on shortest known path from s to w.
- If $e = v \rightarrow w$ gives shorter path to w through v, update both $distTo[w]$ and $edgeTo[w]$ - in which case v$\rightarrow$w is called eligible edge; otherwise, it is called an ineligible edge.

Edge relaxation - II

Example.



Edge relaxation (two cases)

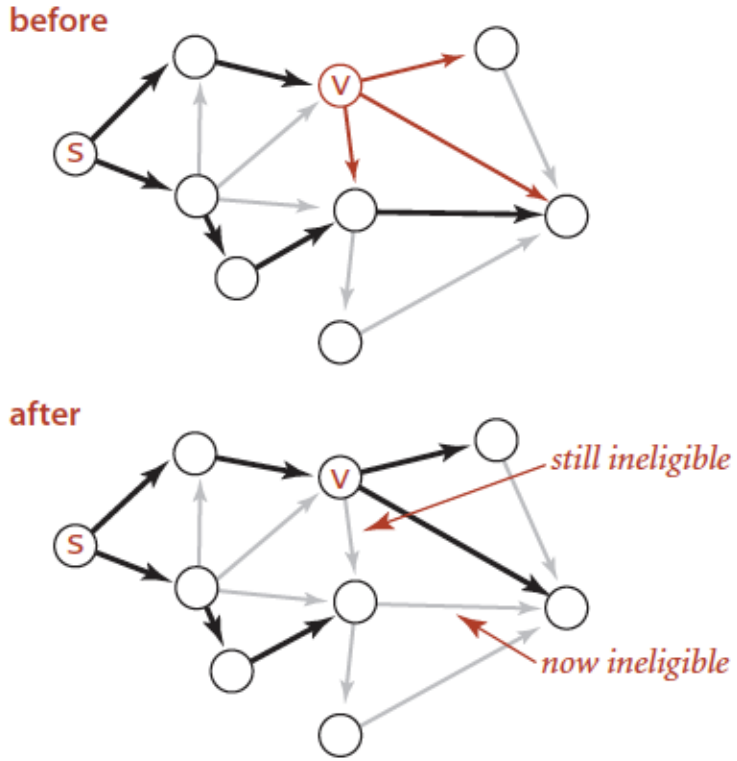
Edge relaxation code

```
private void relax(DirectedEdge e)
{
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight())
   {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
   }
}
```

Edge relaxation

Vertex relaxation

- All the edges pointing **from** a given vertex are relaxed.
- A edge from a vertex whose distTo[v] entry is finite to a vertex whose distTo[] entry is infinite is eligible and will be added to edgeTo[] if relaxed.



Vertex relaxation

Vertex relaxation code

```
private void relax(EdgeWeightedDigraph G, int v)
{
   for (DirectedEdge e : G.adj(v))
   {
      int w = e.to();
      if (distTo[w] > distTo[v] + e.weight())
      {
         distTo[w] = distTo[v] + e.weight();
         edgeTo[w] = e;
      }
   }
}
```

Vertex relaxation

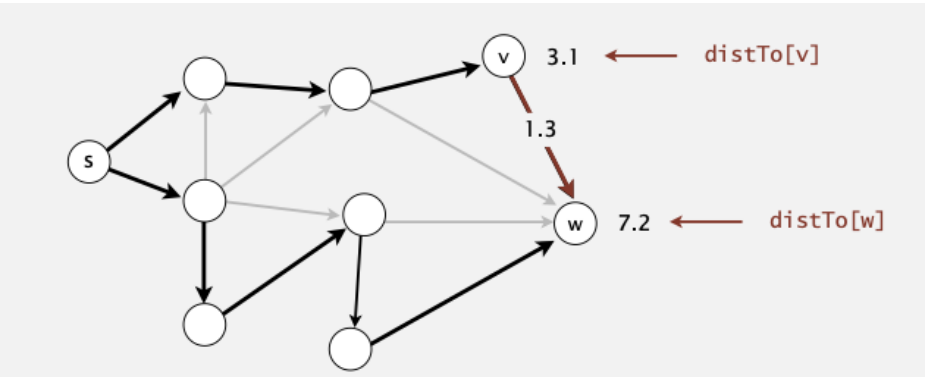
Shortest-paths optimality conditions I

Proposition. Let G be an edge-weighted digraph.
Then distTo[] are the shortest path distances from s (source) iff:

1. distTo[s] = 0.
2. For each vertex v, distTo[v] is the length of some path from s to v.
3. For each edge e = v → w, distTo[w] ≤ distTo[v] + e.weight().

Proof. Pf. (⇒)

- Suppose that distTo[w] > distTo[v] + e.weight() for some edge e = v→w.
- Then, e gives a path from s to w (via v) of length less than distTo[w] – a contradiction.



Shortest-paths optimality conditions II

Proposition. Let G be an edge-weighted digraph.
Then distTo[] are the shortest path distances from s iff:

1. distTo[s] = 0.
2. For each vertex v, distTo[v] is the length of some path from s to v.
3. For each edge e = v → w, distTo[w] ≤ distTo[v] + e.weight().

Proof. Pf. (⇐)

- Suppose that $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k = w$ is a shortest path from s to w.
- Then,

```
distTo[v1]   ≤  distTo[v0]   + e1.weight()
distTo[v2]   ≤  distTo[v1]   + e2.weight()
...
distTo[vk]   ≤  distTo[vk-1] + ek.weight()
```

  ↔ e_i = i^{th} edge on shortest path from s to w

- Add inequalities; simplify; and substitute distTo[v0] = distTo[s] = 0:

```
distTo[w] = distTo[vk] ≤ e1.weight() + e2.weight() + … + ek.weight()
```

  (e1.weight() + e2.weight() + … + ek.weight(): weight of shortest path from s to w)

- Thus, distTo[w] is the weight of shortest path to w. ∎

  (distTo[w]: weight of some path from s to w)

Generic shortest-paths algorithm I

The previous proposition shows an equivalence between the global condition that the distances in $distTo[]$ are shortest-paths distances, and the local condition that we test to relax an edge.

Hence, the optimality conditions lead immediately to a generic algorithm that encompasses all of the shortest-paths algorithms that we consider. For now we consider nonnegative weights.

Generic algorithm (to compute SPT from s)

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat until optimality conditions are satisfied:

- **Relax any edge.**

Generic shortest-paths algorithm II

Proposition. Generic algorithm computes SPT from s.

Pf sketch.

- The entry distTo[v] is always the length of a simple path[1] from s to v.
- Each successful relaxation decreases distTo[v] for some v.
- The entry distTo[v] can decrease at most a finite number of times.

[1] A directed path is simple if it has no repeated vertices.

Generic shortest-paths algorithm III

The generic algorithm does not specify in which order the edges are to be relaxed.

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort shortest path (SP) algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

Dijkstra's algorithm (see demo)

Dijkstra's algorithm outline.

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest $distTo[]$ value).
- Add the vertex closest to s (min.weight) to the SPT, and relax it; that is, relax all edges pointing from that it.

Dijkstra's algorithm: Implementation outline

The implementation maintains a priority queue (pq) containing vertices. It

1. Initializes the elements of the array $distTo[]$ to positive infinity.
2. Adds the source $s = 0$ to the priority queue pq and set $distTo[0] = 0$.
3. While pq is not empty, it deletes the min. priority/weighted vertex from pq and relaxes it; that is, relaxes all the edges from it.
    1. while doing so it adds the the end points (say w) of these edges to the pq, if it is not already in pq;
    2. otherwise, it updates $distTo[w]$ and the weight of w in pq - only if the resulting weight is less.

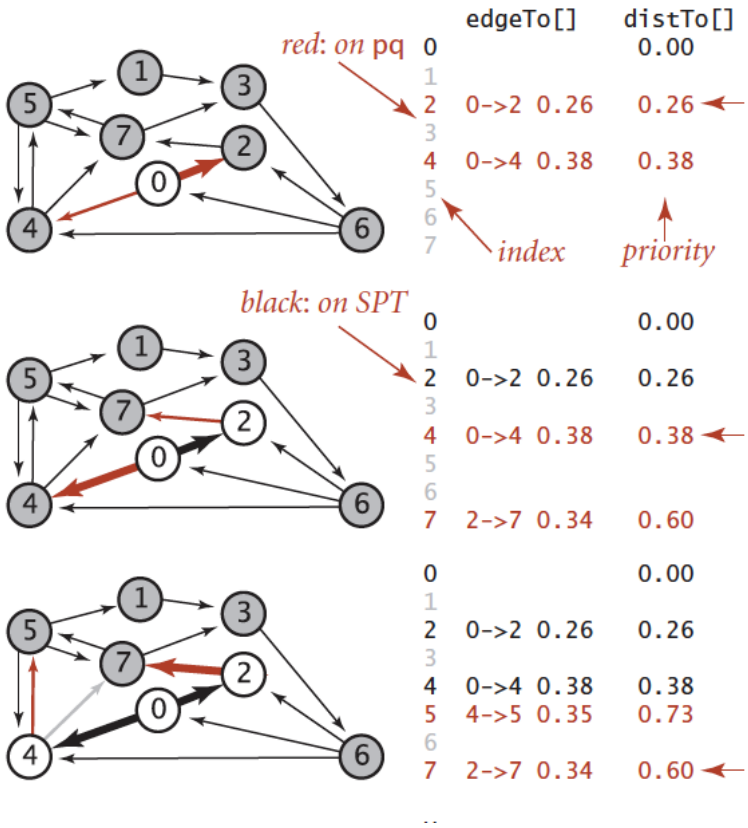
Dijkstra's algorithm implementation: Observations

Observations:

- For each vertex w on the priority queue, $distTo[w]$ = weight of a shortest path from s to w, that uses only intermediate vertices in the SPT and ends in the crossing edge $edgeTo[w]$.
- The $distTo[]$ entry for the vertex with the smallest priority is a shortest path weight, and therefore is the next to be relaxed. As a result, reachable vertices are relaxed in order of the weight of their shortest path from s.

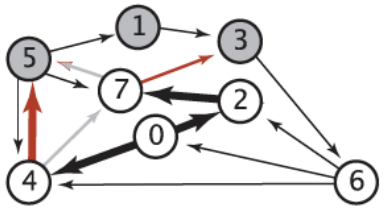
Dijkstra's algorithm - Trace I

```
public DijkstraSP(EdgeWeightedDigraph G, int s)
{
   edgeTo = new DirectedEdge[G.V()];
   distTo = new double[G.V()];
   pq = new IndexMinPQ<Double>(G.V());

   for (int v = 0; v < G.V(); v++)
      distTo[v] = Double.POSITIVE_INFINITY;
   distTo[s] = 0.0;

   pq.insert(s, 0.0);
   while (!pq.isEmpty())
      relax(G, pq.delMin())
}
```

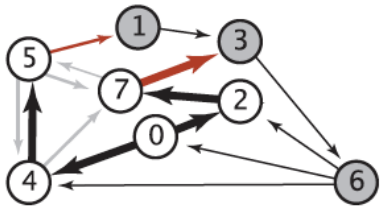


Dijkstra's algorithm - Trace II

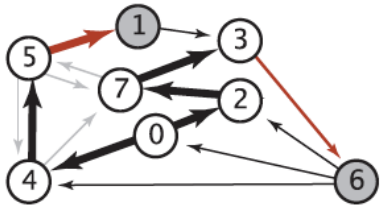
```
private void relax(EdgeWeightedDigraph G, int v)
{
   for(DirectedEdge e : G.adj(v))
   {
      int w = e.to();
      if (distTo[w] > distTo[v] + e.weight())
      {
         distTo[w] = distTo[v] + e.weight();
         edgeTo[w] = e;
         if (pq.contains(w)) pq.change(w, distTo[w]);
         else                pq.insert(w, distTo[w]);
      }
   }
}
```



0		0.00
1		
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73 ←
6		
7	2->7 0.34	0.60

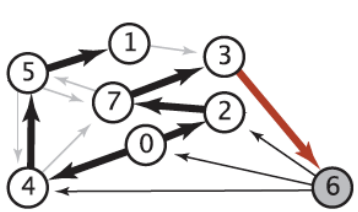


0		0.00
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97 ←
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6		
7	2->7 0.34	0.60

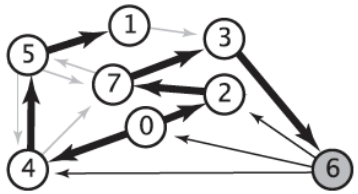


0		0.00
1	5->1 0.32	1.05 ←
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

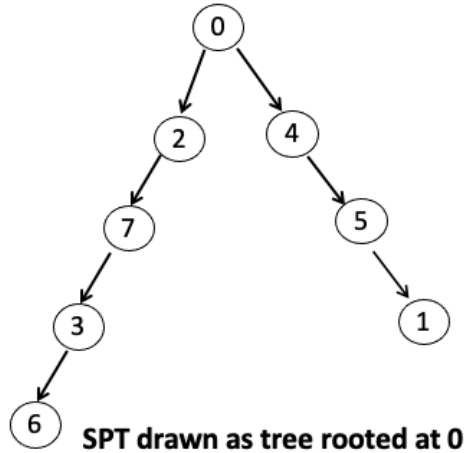
Dijkstra's algorithm - Trace III



0			0.00
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.37	0.97
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.49 ←
7	2->7	0.34	0.60



0			0.00
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.37	0.97
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.49
7	2->7	0.34	0.60

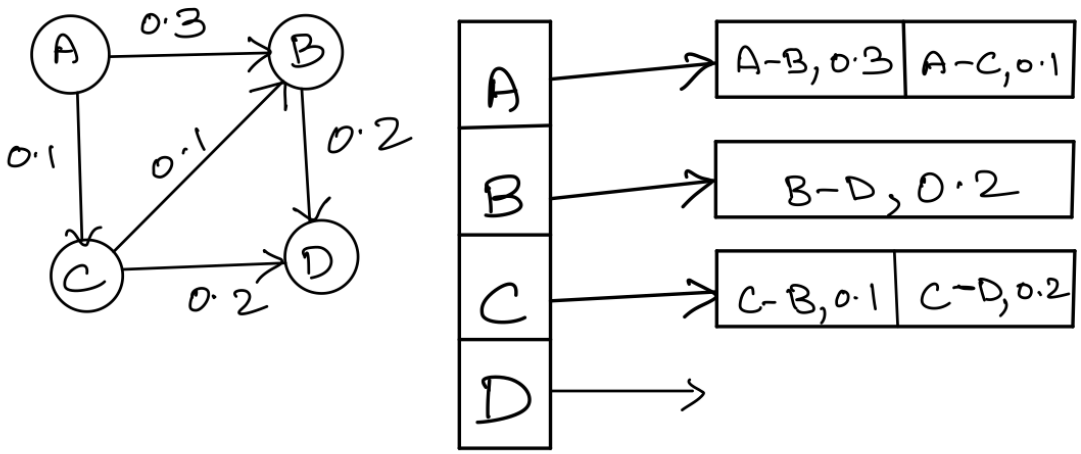


SPT drawn as tree rooted at 0

This implementation of Dijkstra's algorithm grows the SPT by adding an edge at a time, always choosing the edge from a tree vertex to a non-tree vertex whose destination w is closest to s.

Moreover, vertices are added to the SPT in increasing order of their distance from the source, as indicated by the red arrows at the right edge of the diagram.

Shortest Path - Another example



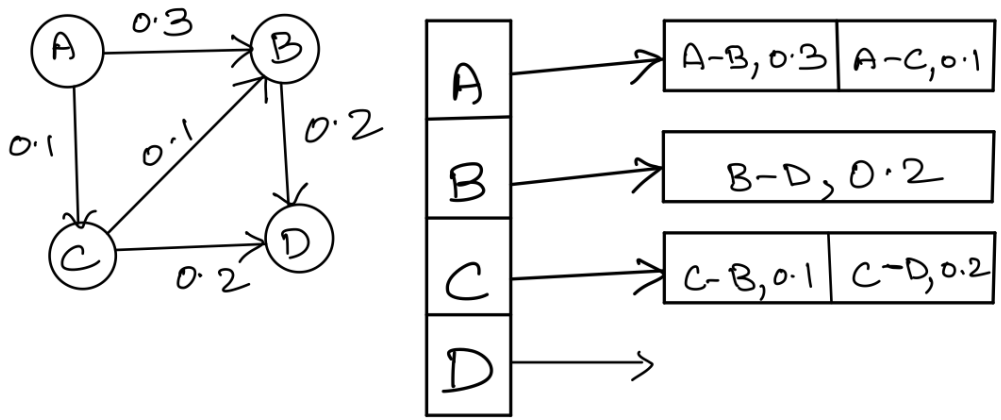
In the above graph with source $= A$:

Shortest path from A to B is: $A \to C \to B$ with path weight $= 0.1 + 0.1 = 0.2$

Shortest path from A to C is: $A \to C$ with path weight $= 0.1$

Shortest path from A to D is: $A \to C \to D$ with path weight $= 0.1 + 0.2 = 0.3$

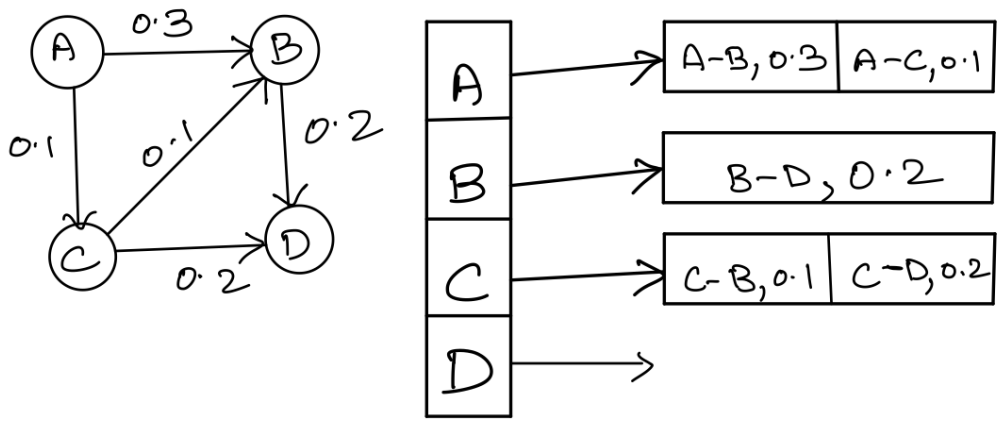
Shortest Path Dijkstra's algorithm - Another example - I



Dijkstra's algorithm maintain the below data structures to compute shortest paths from source to its reachable vertices:

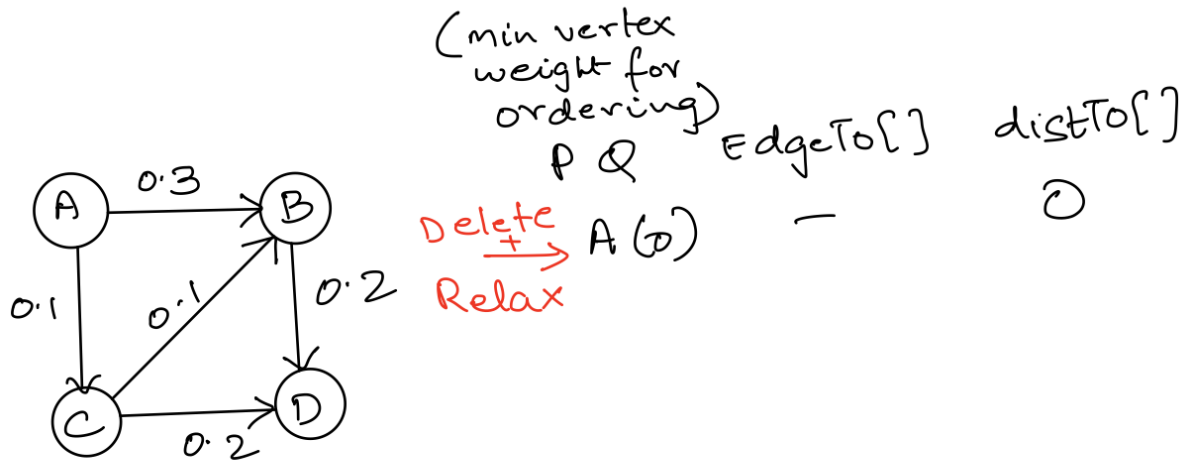
- pq to store vertices with priority = weight of the current shortest path from source to itself.
- $distTo[]$ = vertex index array to store the weight of the shortest path from source to the respective vertex.
- $EdgeTo[]$ = vertex index array to store the last edge connecting source to the vertex in the SPT.

Shortest Path Dijkstra's algorithm - Another example - II



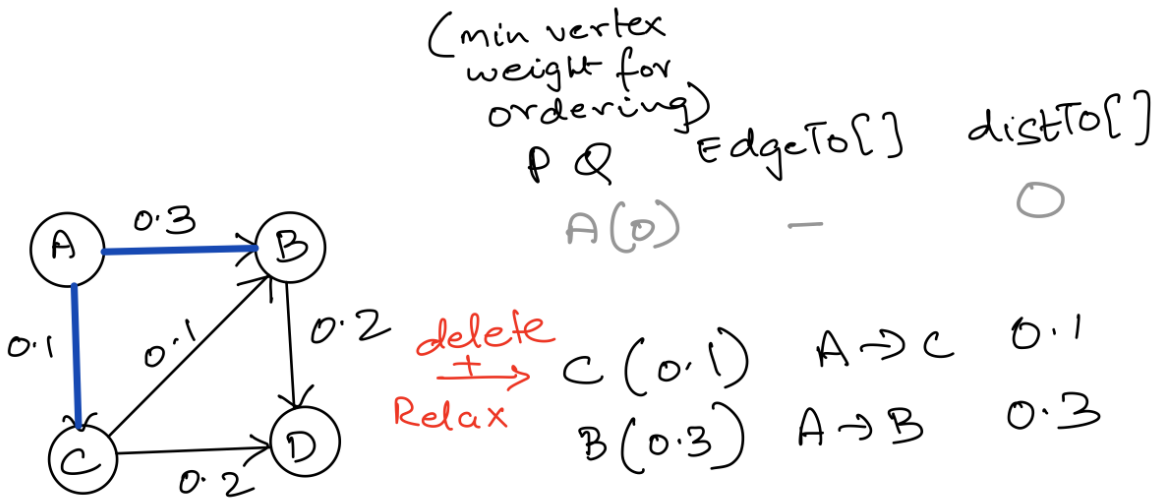
1. Dijkstra's algorithm, initially marks the elements of $distTo[]$ to positive infinity.
2. Then adds the source $A(0)$ to the priority queue pq and sets $distTo[0] = 0$.

Shortest Path Dijkstra's algorithm - Another example - III



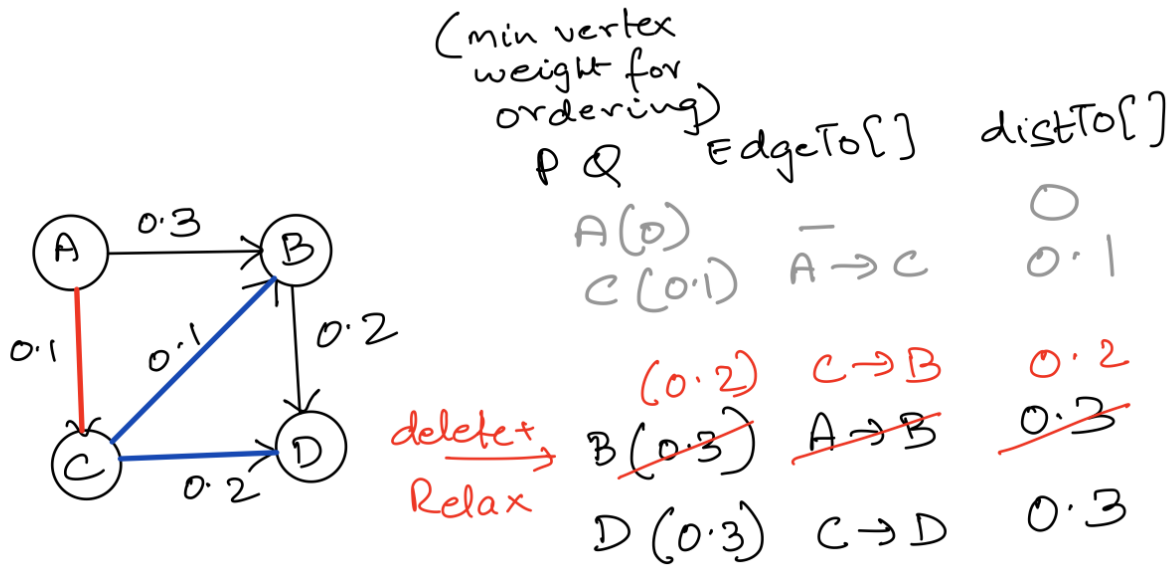
1. Delete the vertex in pq having min.priority [$pq.delMin()$] - in this example, it gives A.
2. Relax A: that is,
   - examine A's adjacency list.
   - update distances (if $distTo[w] > distTo[A] + e_{A \to w}.weight$) of vertices connected to A and already in pq; otherwise, add them to pq, and updates $edgeTo[]$ and $distTo[]$ data structures accordingly.

Shortest Path Dijkstra's algorithm - Another example IV



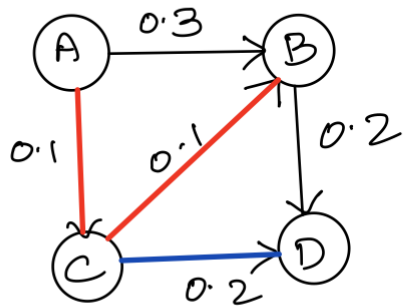
1. Deleting A from pq and relaxing it results in adding B, C to pq.
2. Blue arrows indicate edges under consideration.
3. Red arrows indicate edges in the SPT
4. Grey elements in $pq, edgeTo[], distTo[]$ indicate finalized/vertices added to SPT.

Shortest Path Dijkstra's algorithm - Another example V



1. Delete the vertex in pq having min.priority $[pq.delMin()] = C$ and relax it [results in updating B's values, adding D to pq, and updating their priority, $distTo[]$ and $edgeTo[]$ values].

Shortest Path Dijkstra's algorithm - Another example VI

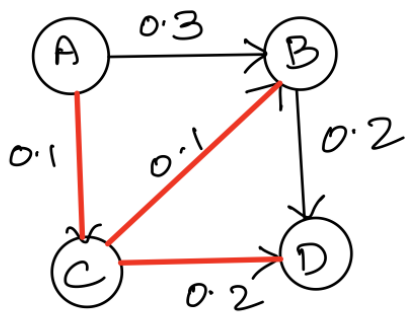


(min vertex weight for ordering)

PQ	EdgeTo[]	distTo[]
A(0)	–	0
C(0.1)	A→C	0.1
B(0.2)	C→B	0.2
Delete → D(0.3) / Relax	C→D	0.3

1. Delete the vertex in pq having min.priority $[pq.delMin()] = B$ and relax it [results in no change].

Shortest Path Dijkstra's algorithm - Another example VII



(min vertex weight for ordering)

PQ	EdgeTo[]	distTo[]
A(0)	–	0
C(0.1)	A→C	0.1
B(0.2)	C→B	0.2
D(0.3)	C→D	0.3

PQ is Empty ⇒ EXIT !

1. Delete the vertex in pq having min.priority $[pq.delMin()] = D$ and relax it [results in no change].

Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Vertices are added to SPT (in increasing order of weights), after which they are relaxed; that is, all edges from it are relaxed.
- Also, when v is added to SPT, each edge $e = v \rightarrow w$ is relaxed exactly once, leaving $distTo[w] \leq distTo[v] + e.weight()$. This inequality holds until algorithm terminates because:
  - $distTo[w]$ cannot increase [$distTo[]$ values are monotone decreasing]
  - $distTo[v]$ will not change [we choose lowest $distTo[]$ value at each step (and edge weights are nonnegative)]
- Thus, upon termination, shortest-paths optimality conditions hold.

Dijkstra's algorithm: analysis

Dijkstra's algorithm (using a min.priority queue implemented as a binary heap) uses extra space proportional to V and time proportional to E log V (in the worst case) to compute the SPT rooted at a given source in an edge-weighted digraph with E edges and V vertices.

Proof. The PQ implementation: V insert, V delete-min, E decrease-key[2].

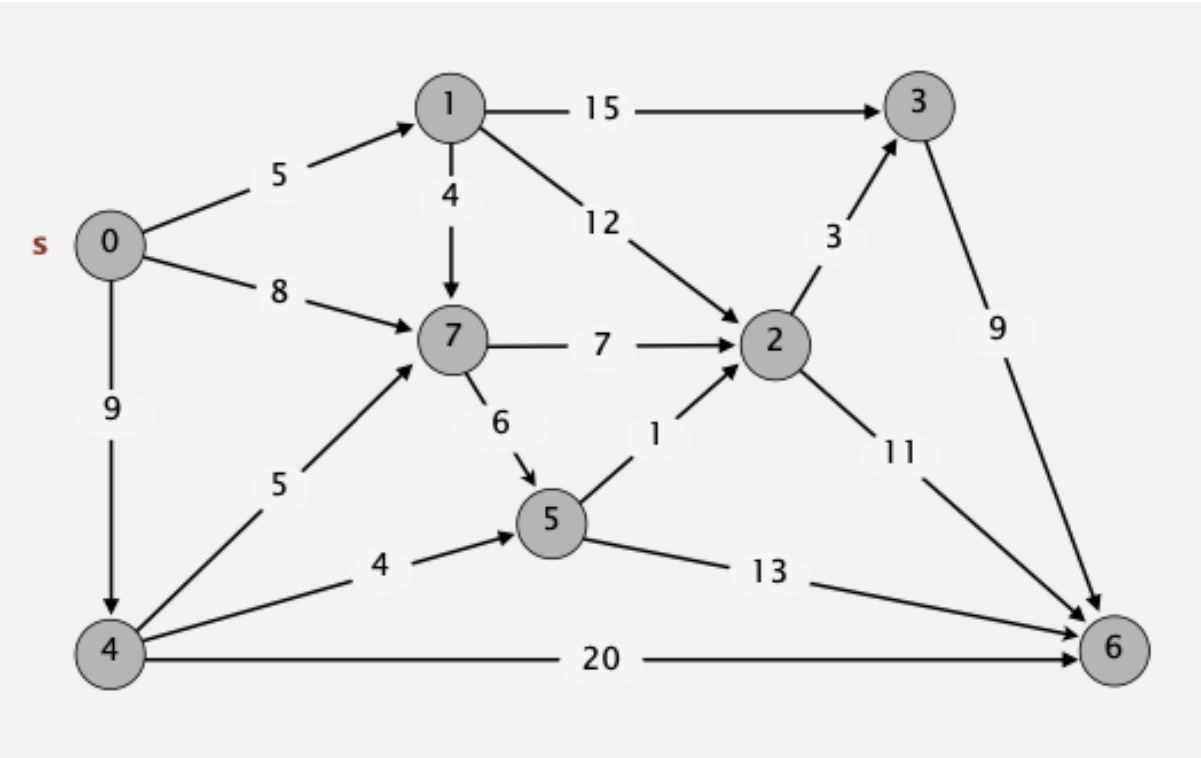
PQ implementation	insert	delete-min	decrease-key	total
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$

[2]Uses change operation on priority queue, which uses $\log V$ time

Acyclic edge-weighted digraphs

An edge-weighted DAG refers to an acyclic edge-weighted digraph.

Q. Suppose that we are given an edge-weighted DAG. Is it easier and faster to find shortest paths than in a general digraph?



A. Yes!

Topological sort shortest path algorithm - I

The topological sort shortest path (SP) algorithm :

- Solves the single-source problem in linear time $O(E + V)$
- Handles negative edge weights
- Solves related problems, such as finding longest paths.

In the course we will use Acyclic edge-weighted digraphs shortest path and Topological sort shortest path interchangeably.

Topological sort shortest path algorithm - II

Topological sort SP algorithm: combines vertex relaxation with topological sorting, and is outlined below:

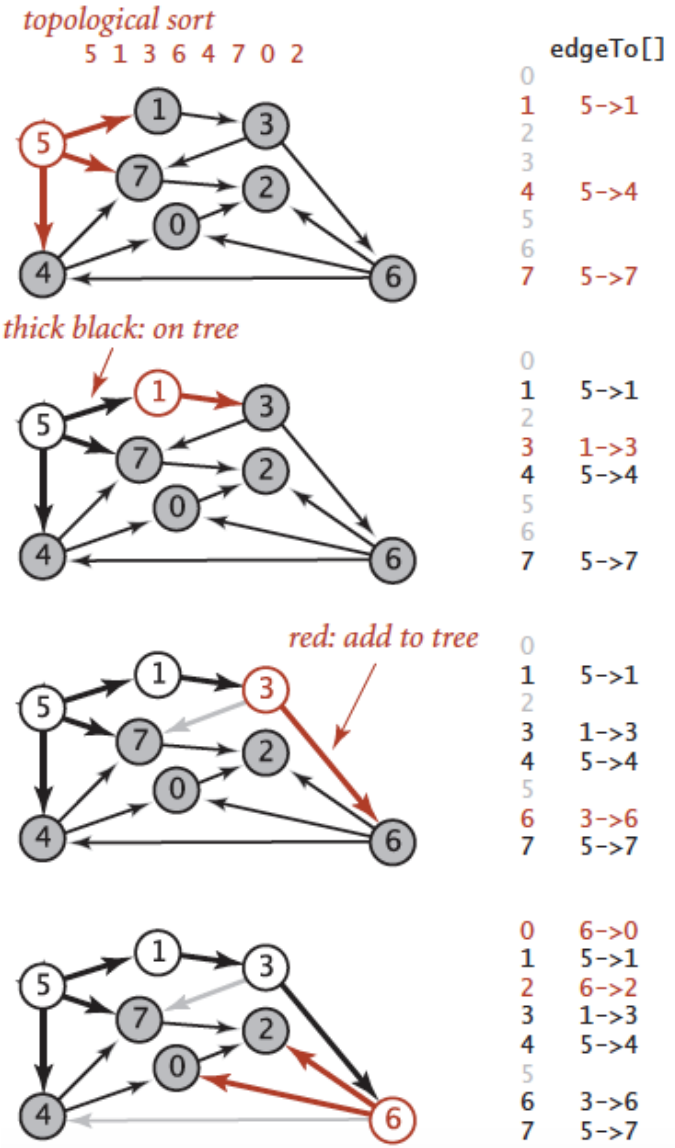
- The algorithm, first initializes distTo[s] to 0 and all other distTo[] values to infinity,
- Then it relaxes the vertices, one by one, taking the vertices in topological sort order.

Note that, since we are processing vertices in an acyclic digraph in topological order, we never re-encounter a vertex that we have already relaxed.

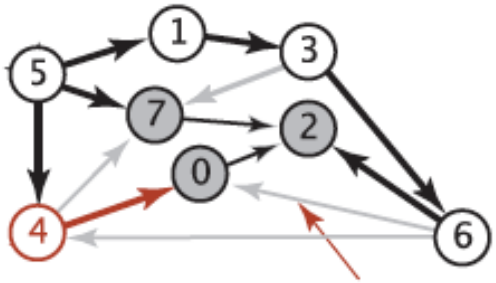
Acyclic edge-weighted digraphs shortest path - Trace I

Topological sort SP algorithm

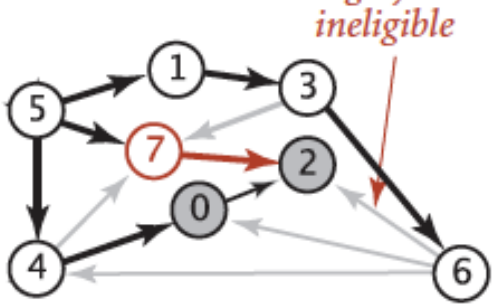
```
public AcyclicSP(EdgeWeightedDigraph G, int s)
{
   edgeTo = new DirectedEdge[G.V()];
   distTo = new double[G.V()];

   for (int v = 0; v < G.V(); v++)
      distTo[v] = Double.POSITIVE_INFINITY;
   distTo[s] = 0.0;

   Topological top = new Topological(G);

   for (int v : top.order())
      relax(G, v);
}
```



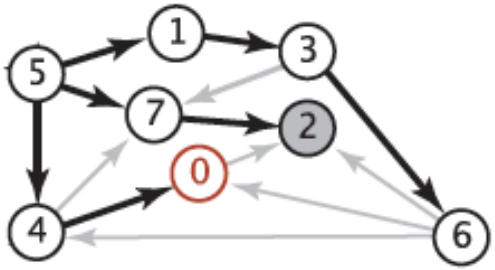
Acyclic edge-weighted digraphs shortest path - Trace II



0	4->0
1	5->1
2	6->2
3	1->3
4	5->4
5	
6	3->6
7	5->7



0	4->0
1	5->1
2	7->2
3	1->3
4	5->4
5	
6	3->6
7	5->7



0	4->0
1	5->1
2	7->2
3	1->3
4	5->4
5	
6	3->6
7	5->7

Acyclic edge-weighted digraphs shortest path - Another Example - I

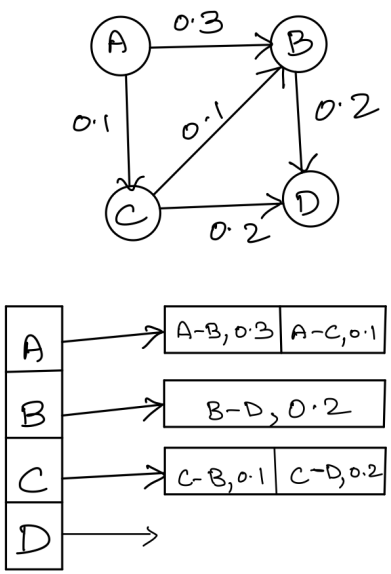


Figure 1: Sample graph G

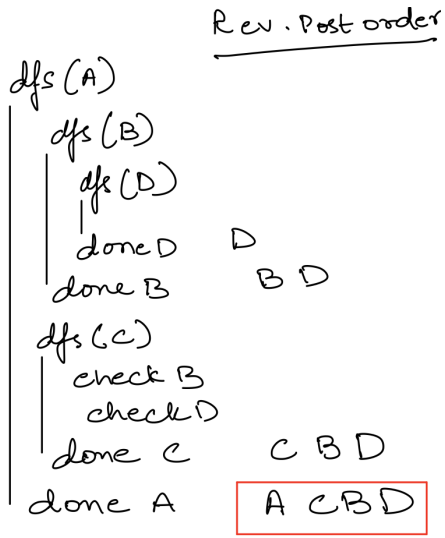
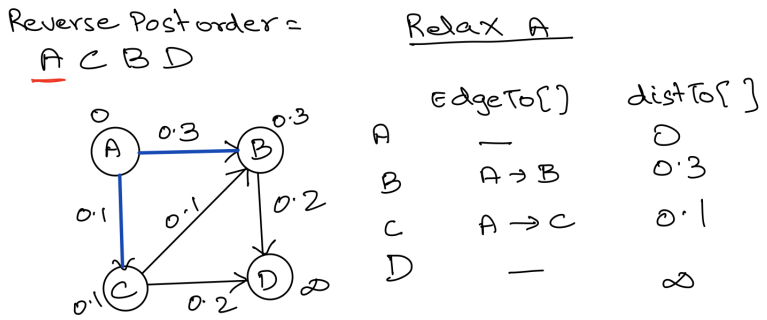
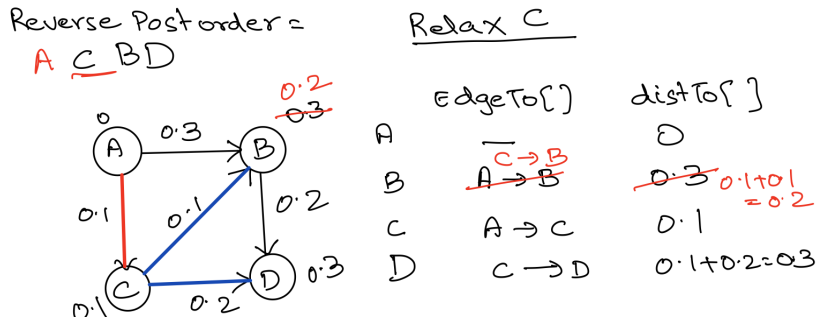


Figure 2: Reverse Postorder of $G = ACBD$

Acyclic edge-weighted digraphs shortest path - Another Example - II

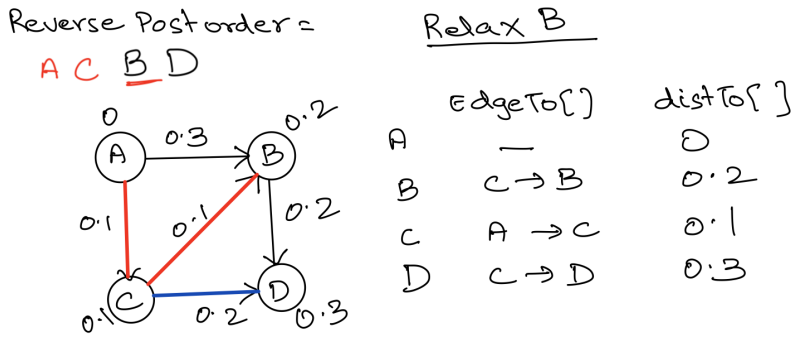


- We relax A; that is, update the $EdgeTo[]$ and $distTo[]$ values of B, C, and A to SPT.

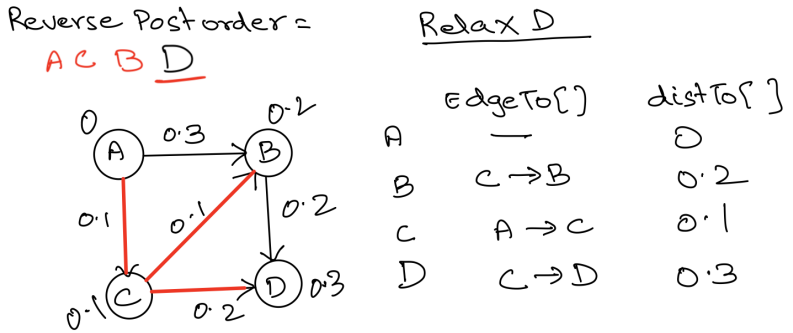


- The next vertex to be relaxed in reverse post order is C, therefore we add it to SPT along with the edgeTo[C] edge. Then we relax it.

Acyclic edge-weighted digraphs shortest path - Another Example - III



- The next vertex to be relaxed in reverse post order is B, therefore we add it to SPT along with the edgeTo[B] edge. Then we relax it [does not change anything].



- The last vertex to be relaxed in reverse post order is D, therefore we add it to SPT along with the edgeTo[D] edge. Then we relax it [does change anything].

Shortest paths in edge-weighted DAGs: complexity

Topological sort SP algorithm computes SPT in any edge- weighted DAG in time proportional to E + V – [edge weights can be negative.]

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights can be negative!

Pf.

- Each edge `e = v→w` is relaxed exactly once (when vertex `v` is relaxed), leaving `distTo[w] ≤ distTo[v] + e.weight()`.
- Inequality holds until algorithm terminates because:
  - `distTo[w]` cannot increase ← `distTo[]` values are monotone decreasing
  - `distTo[v]` will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ▪

Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

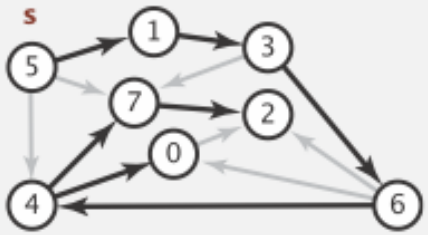
- Negate all weights.
- Find shortest paths.
- Negate weights in result.

longest paths input

```
5->4  0.35
4->7  0.37
5->7  0.28
5->1  0.32
4->0  0.38
0->2  0.26
3->7  0.39
1->3  0.29
7->2  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

shortest paths input

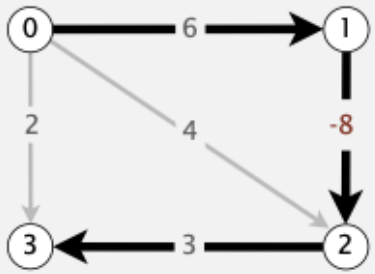
```
5->4 -0.35
4->7 -0.37
5->7 -0.28
5->1 -0.32
4->0 -0.38
0->2 -0.26
3->7 -0.39
1->3 -0.29
7->2 -0.34
6->2 -0.40
3->6 -0.52
6->0 -0.58
6->4 -0.93
```



Key point. Topological sort algorithm works even with negative weights.

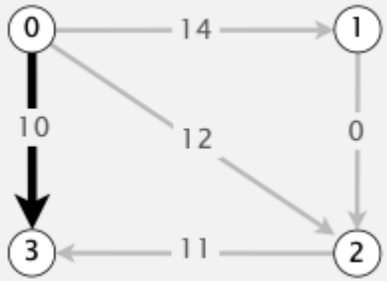
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.

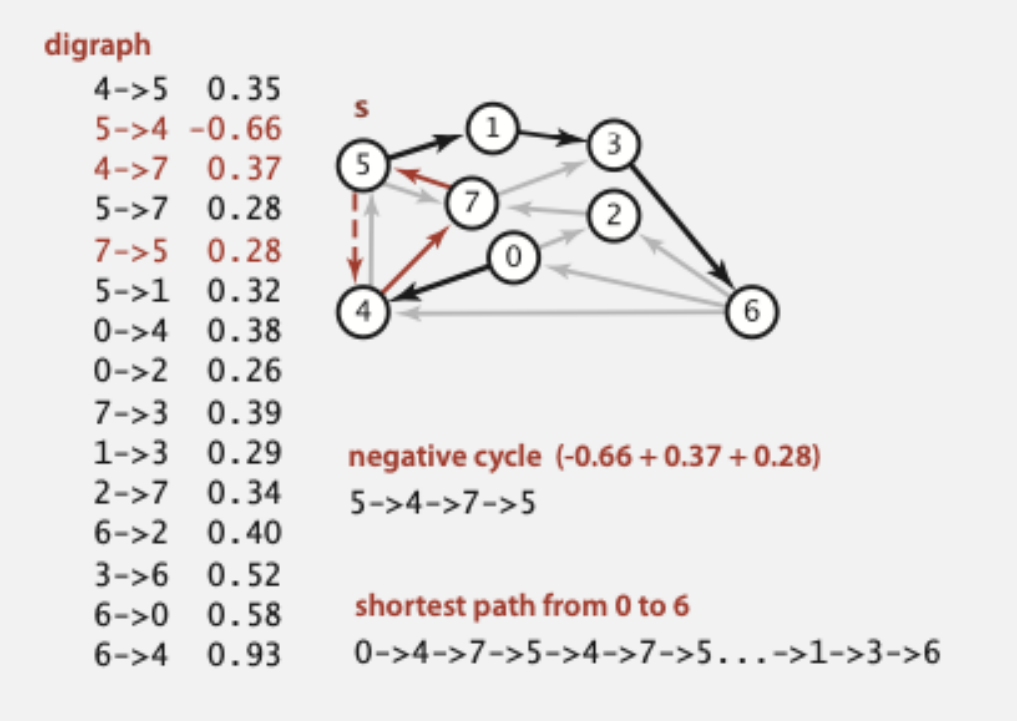


Adding 8 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

Conclusion. Need a different algorithm.

Shortest paths with negative cycle

A negative cycle is a directed cycle whose sum of edge weights is negative.



The shortest path problem is meaningless with negative cycles (every time a cycle is added to the path, its weight decreases), and hence:
Proposition. An SPT exists iff no negative cycles.

Bellman-Ford algorithm - General idea

Bellman–Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:

- **Relax each edge.**

Bellman-Ford algorithm - Queue Based

Queue-based Bellman-Ford. [for improved running time]

- We can easily determine a priori that numerous edges are not going to lead to a successful relaxation in any given pass: the only edges that could lead to a change in distTo[] are those leaving a vertex whose distTo[] value changed in the previous pass.
- To keep track of such vertices, the algorithm uses a FIFO queue. It contains the vertices to be relaxed.
- To avoid duplicates in the queue, a vertex-indexed boolean array onQ[] that indicates which vertices are on the queue is also maintained.

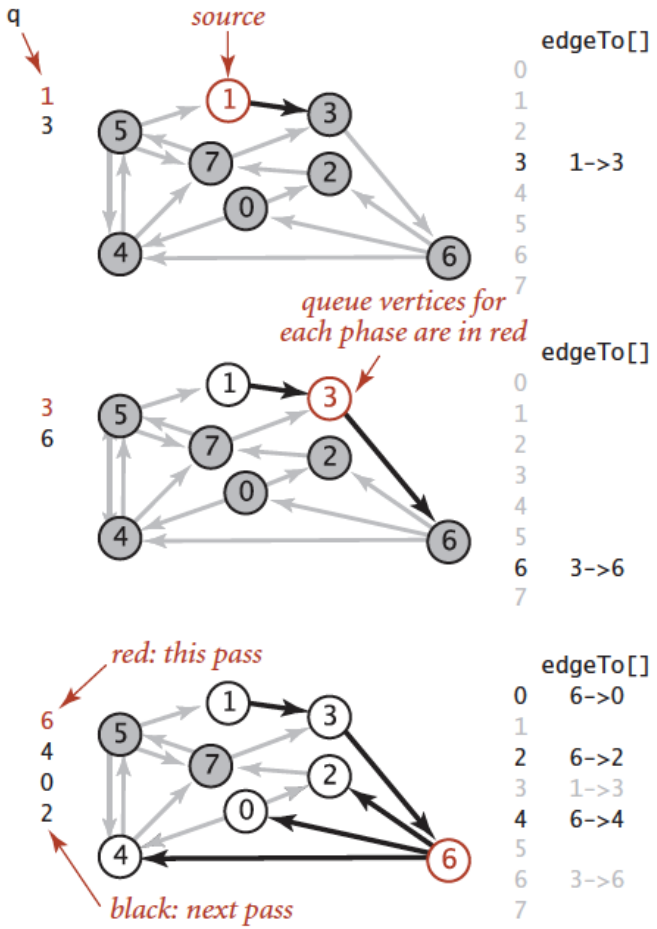
Queue-based Bellman-Ford: Algorithm Implementation

```
public BellmanFordSP(EdgeWeightedDigraph G, int s)
{
   distTo = new double[G.V()];
   edgeTo = new DirectedEdge[G.V()];
   onQ = new boolean[G.V()];
   queue = new Queue<Integer>();
   for (int v = 0; v < G.V(); v++)
      distTo[v] = Double.POSITIVE_INFINITY;
   distTo[s] = 0.0;
   queue.enqueue(s);
   onQ[s] = true;
   while (!queue.isEmpty() && !this.hasNegativeCycle())
   {
      int v = queue.dequeue();
      onQ[v] = false;
      relax(v);
   }
}
```

Queue-based Bellman-Ford: Trace I

tinyEWDn.txt

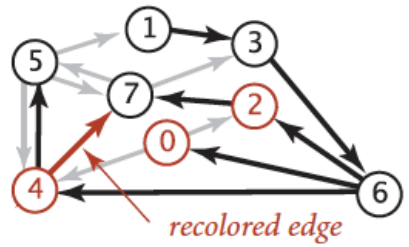
```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2 -1.20
3->6  0.52
6->0 -1.40
6->4 -1.25
```



Queue-based Bellman-Ford: Trace II

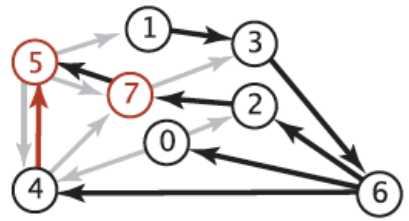
```
private void relax(EdgeWeightedDigraph G, int v)
{
   for (DirectedEdge e : G.adj(v)
   {
      int w = e.to();
      if (distTo[w] > distTo[v] + e.weight())
      {
         distTo[w] = distTo[v] + e.weight();
         edgeTo[w] = e;
         if (!onQ[w])
         {
            q.enqueue(w);
            onQ[w] = true;
         }
      }
      if (cost++ % G.V() == 0)
         findNegativeCycle();
   }
}
```

4
0
2
7
5

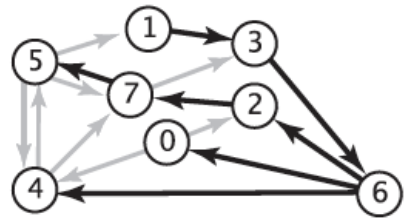


edgeTo[]	
0	6->0
1	
2	6->2
3	1->3
4	6->4
5	4->5
6	3->6
7	2->7

7
5

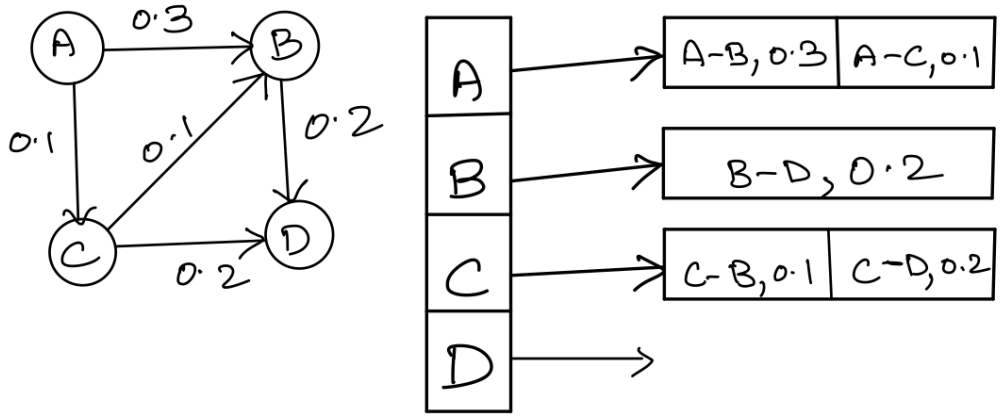


edgeTo[]	
0	6->0
1	
2	6->2
3	1->3
4	6->4
5	7->5
6	3->6
7	2->7



edgeTo[]	
0	6->0
1	
2	6->2
3	1->3
4	6->4
5	7->5
6	3->6
7	2->7

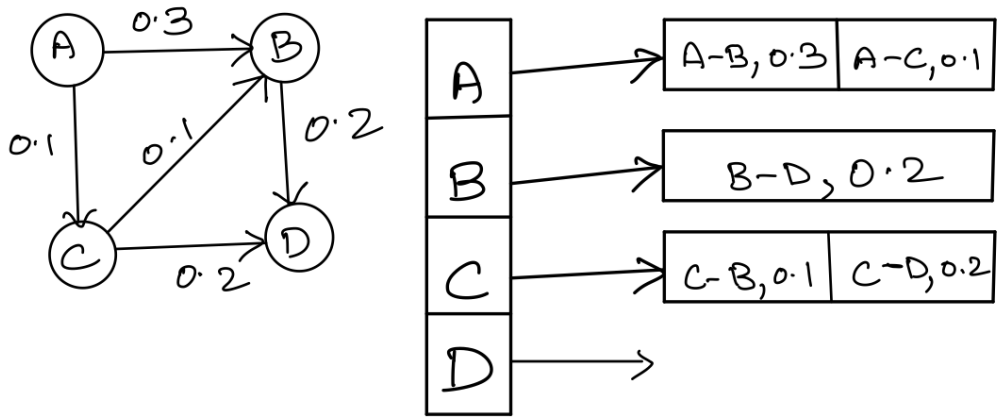
Queue-based Bellman-Ford: Another example I



Queue-based Bellman-Ford algorithm maintains the below data structures to compute shortest paths from source to its reachable vertices:

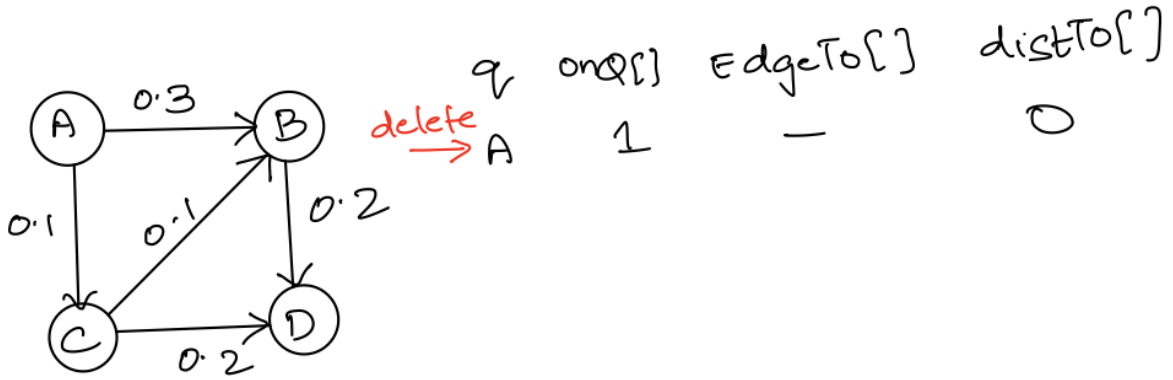
- q to store vertices who weight is reduced during edge relaxation.
- $distTo[]$ = vertex indexed array to store the weight of the shortest path from source to the respective vertex.
- $EdgeTo[]$ = vertex indexed array to store the last edge connecting source to the vertex in the SPT.
- $onQ[]$ = vertex indexed boolean array to store if the vertex is on q. 0 means not on queue q, and 1 means the vertex is on queue q.

Queue-based Bellman-Ford: Another example - II



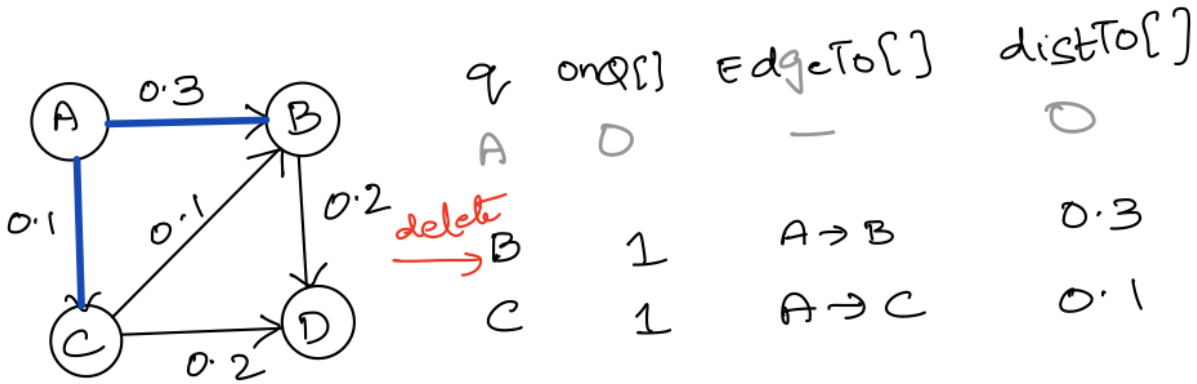
1. Queue-based Bellman-Ford algorithm, initially marks the elements of $distTo[]$ to positive infinity.
2. Then adds the source A to the FIFO queue q and sets $distTo[0] = 0$ and $onQ[0] = 1$.

Queue-based Bellman-Ford: Another example - III



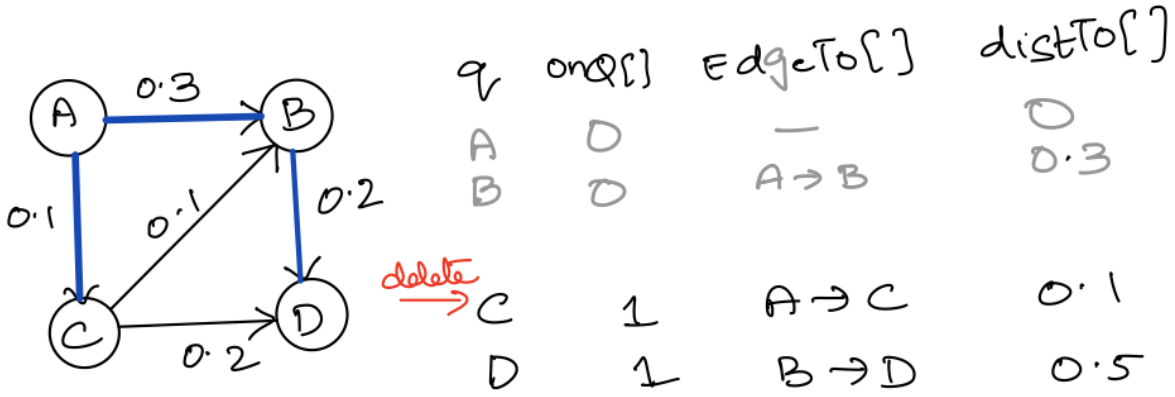
1. Delete the least recently added vertex from q [$q.dequeue()$] - in this example, it gives A.
2. Set $onQ[A] = 0$ and relax A; that is,
   - examine A's adjacency list.
   - update distances (if $distTo[w] > distTo[A] + e_{A \to w}.weight$) of vertices with an edge from A, and add only those vertices to q if they are not on queue q and whose weights have been decreased by relaxing A. Therefore, we add B, C to q.

Queue-based Bellman-Ford; Another example - IV



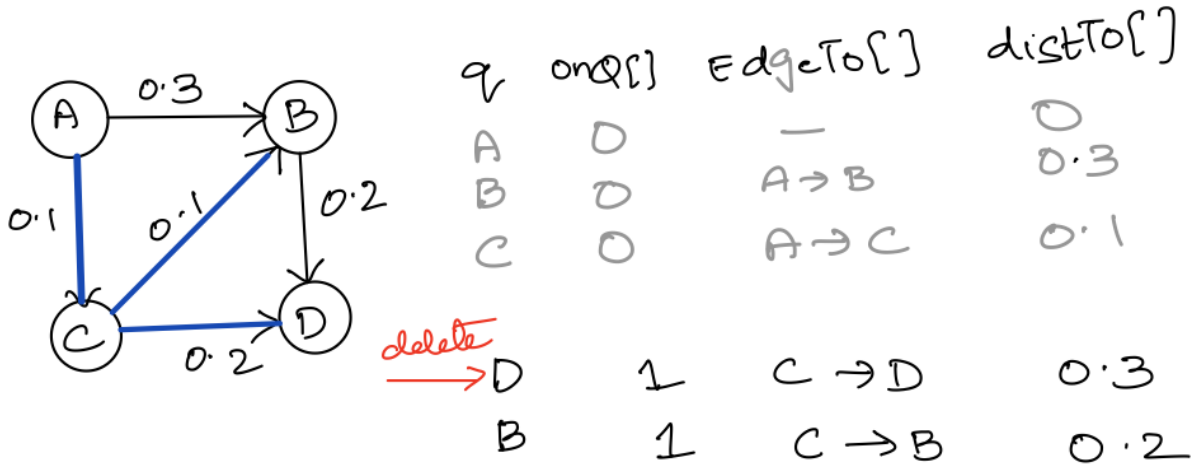
1. Blue arrows indicate edges under consideration.
2. Greyed elements indicate vertices NOT in queue.
3. B will be the next vertex dequeued from q.

Queue-based Bellman-Ford: Another example - V



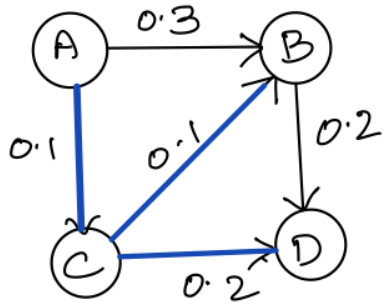
1. Delete B and relax it [results in adding D to q].
2. C will be the next vertex dequeued from q.

Queue-based Bellman-Ford: Another example - VI



1. Delete C and relax it [results in adding B back to q!].
2. Notice how some blue edges go back to being black ($A \to B, B \to D$) and new blue edges are added ($C \to B, C \to D$).
3. D will be the next vertex dequeued from q.

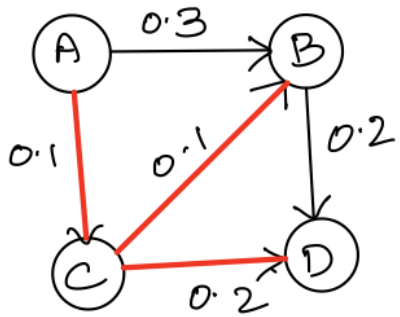
Queue-based Bellman-Ford: Another example - VII



q	onQ[]	EdgeTo[]	distTo[]
A	D	—	0
B	D	A→B	0.3
C	D	A→C	0.1
D	D	C→D	0.3
B	1	C→B	0.2

1. Delete D and relax it [results in no changes in $edgeTo[]$ and $distTo[]$ data structures].
2. B will be the next vertex dequeued from q.

Queue-based Bellman-Ford: Another example - VIII



q	onQ[]	edgeTo[]	distTo[]
A	D	—	0
B	0	A→B	0·3
C	0	A→C	0·1
D	0	C→D	0·3
B	1	C→B	0·2

queue empty ⇒ EXIT!

1. Delete B and relax it [results in no changes in $edgeTo[] and distTo[]$ data structures].
2. q is empty. Therefore, exit.
3. Red arrows indicate the final SPT.

Bellman-Ford algorithm: analysis

Proposition. Queue-based Bellman-Ford computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$ in the worst case.

However, it is much faster than that in practice [takes (E + V) time].

The queue-based Bellman-Ford algorithm is an effective and efficient method for solving the shortest-paths problem that is widely used in practice, even for the case when edge weights are positive.