# Mergesort and Quicksort

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# Mergesort!

Over the next few lectures, we will study two **Divide and Conquer** algorithms: Mergesort and Quicksort.

- The goal with divide and conquer is to divide a problem into two or more **sub-problems**, which are similar to or the same as the original problem.
- This sub-problem division continues until problems are easy to solve directly.
- Finally, all of the subproblems are combined in such a manner as to produce the desired result.

# Mergesort!

The Mergesort divide and conquer algorithm proceeds as follows.

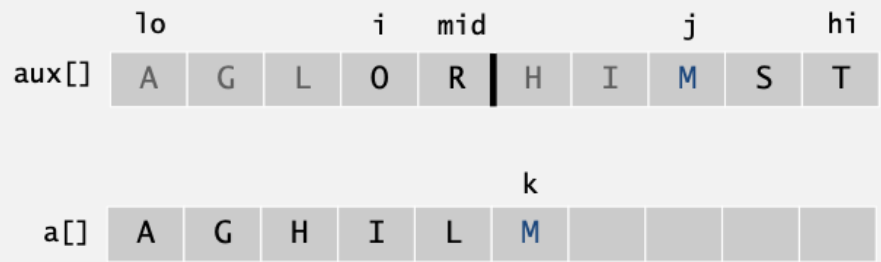
- Given an array to sort, we decide if it is small enough to solve directly (i.e, only one element!).
- If not, we divide the array in half and recursively call Mergesort on both halves in sequence.
- Then, we combine the two halfs and pass the sorted array to the calling function.


Mergesort overview

# Mergesort in Java I

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];                                                    copy

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)                 a[k] = aux[j++];                merge
      else if (j > hi)                  a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                              a[k] = aux[i++];
   }
}
```

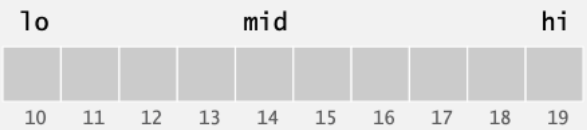


# Mergesort in Java II

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



# Merging on the Highway

Up to now we've only talked about runtime complexity, but we can also examine algorithms in terms of **memory complexity**.

- In general, we are concerned with the amount of memory an algorithm uses *aside from it's inputs*.
- Sorts which don't require separate arrays to be instantiated are called "in-place" sorts.
- Because of the use of the `aux[]` array above, mergesort is not an in-place algorithm.
- In-place algorithms require $N + O(\lg N)$ memory.

# Mergesort Algorithm: Trace

# Polymerization!

Some things of note about the foregoing algorithm:

- Note that, while it's theoretically possible to merge any two subarrays, the two subarrays merged by mergesort are **always adjacent.**
- Arguments to `merge`:
  - `a[]` $\Rightarrow$ the array we are sorting
  - `aux[]` $\Rightarrow$ a pre-declared region of memory for swap space.
  - `lo` $\Rightarrow$ Beginning of the first half
  - `mid` $\Rightarrow$ End of the first half
  - `hi` $\Rightarrow$ End of the second half

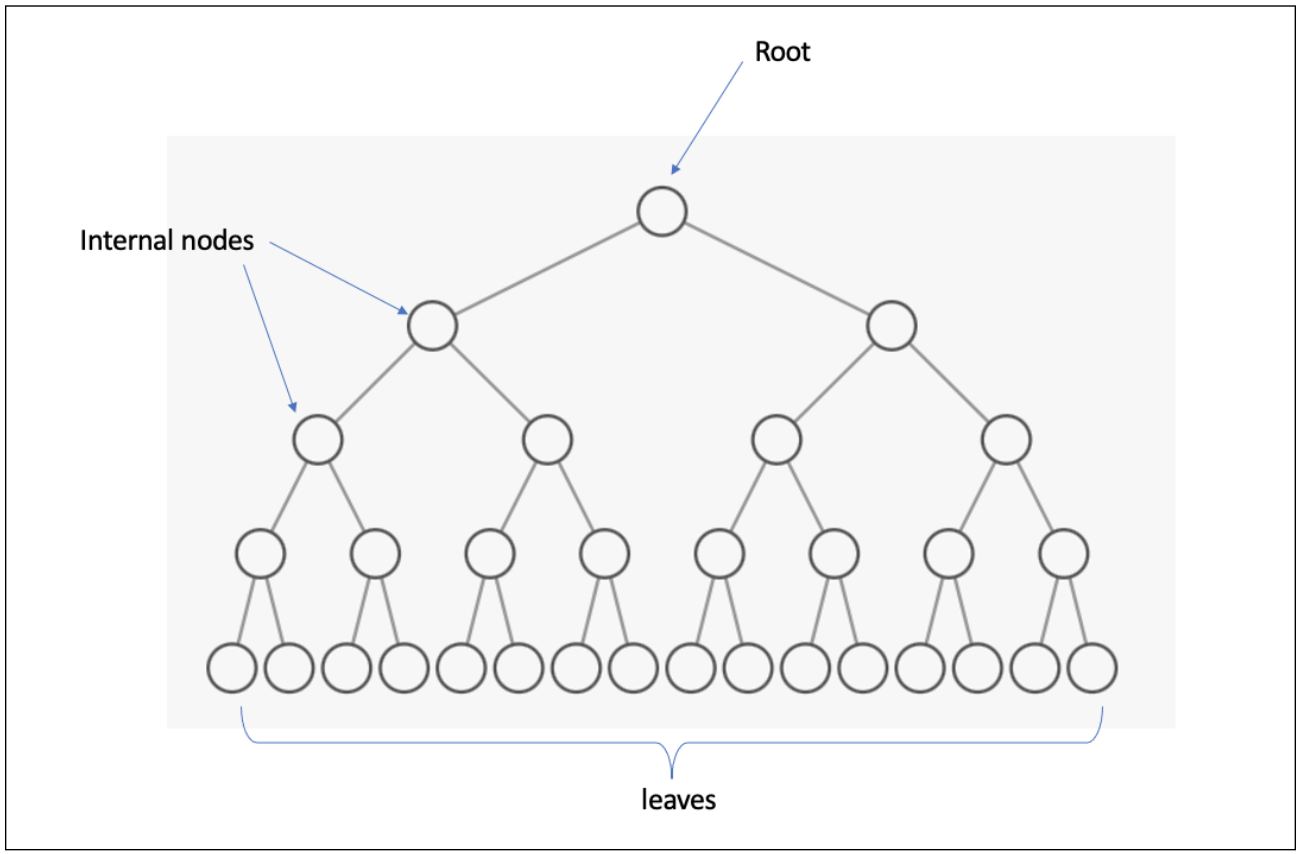
# Mergesort analysis: Memory

- How much memory does mergesort require?
  - Original input array = N.
  - Auxiliary array for merging = N.
  - Local variables: constant.
  - Function call stack: $\lg N$.
  - Total = 2N + O($\lg$ N).
- *How much memory do selection sort and insertion sort require?*
- In-place merger is complicated - see [Kronrud, 1969]

# Binary Trees

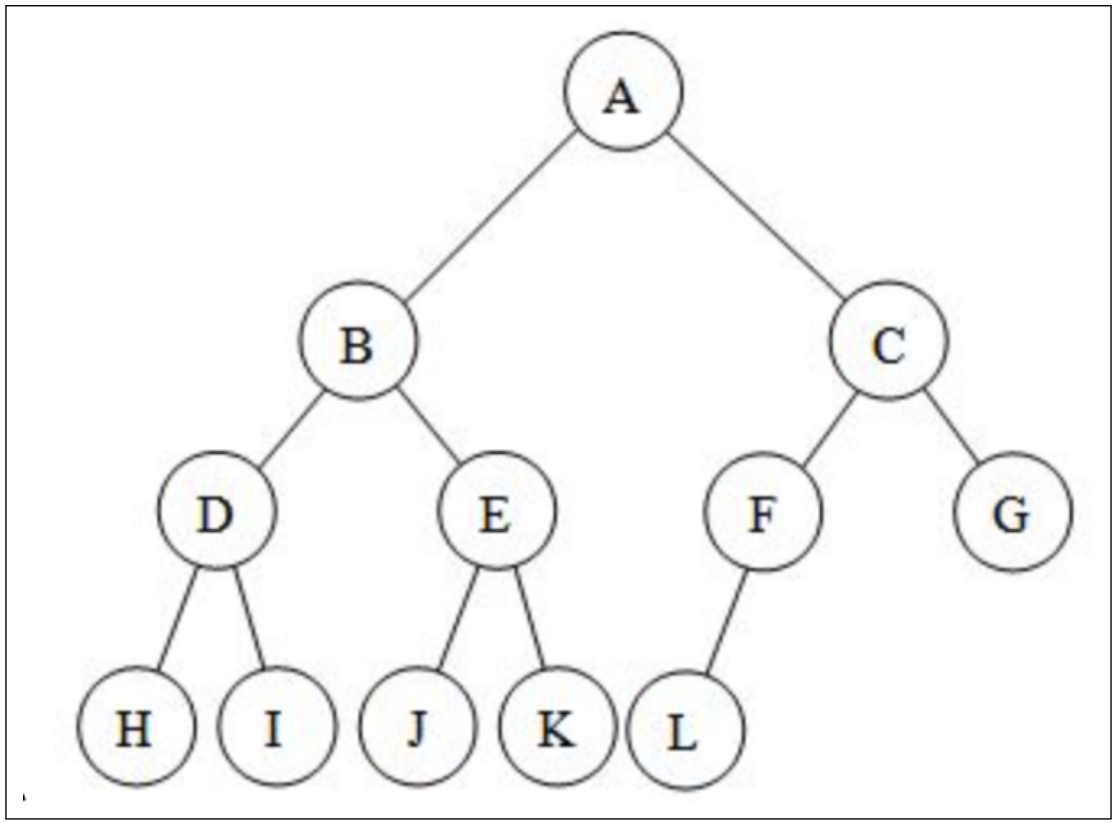
**Binary Tree:** A tree where each internal node has **at most two children**.

**Full Binary Tree:** A binary tree where each internal node has exactly two children.



# Binary Trees contd..

**Complete Binary Tree:** A complete binary tree is a binary tree in which every level, except possibly the last, is completely filled, and all nodes are as far left as possible.




https://web.cecs.pdx.edu/~sheard/course/Cs163/Doc/FullvsComplete.html

# Mergesort Time Complexity

**Proof. Sketch.**

The number of compares $C(N)$ to mergesort an array of length $N$ satisfies the recurrence:

$$C(N) \leq \underbrace{C(\lceil N/2 \rceil)}_{\text{left half}} + \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right half}} + \underbrace{N}_{\text{merge}} \quad \text{for } N > 1, \text{ with } C(1) = 0,$$

where $\lceil x \rceil$ is the smallest integer $\geq x$, i.e. $\lceil 1.5 \rceil = 2$, $\lceil 3.1 \rceil = 4$,
and $\lfloor x \rfloor$ is the biggest integer $\leq x$, i.e. $\lfloor 1.5 \rfloor = 1$, $\lfloor 3.1 \rfloor = 3$.
We solve the recurrence when N is a power of 2:

$$D(N) = 2D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

The result holds for all $N$, but general proof is a little bit messy. □

# Recursion Tree

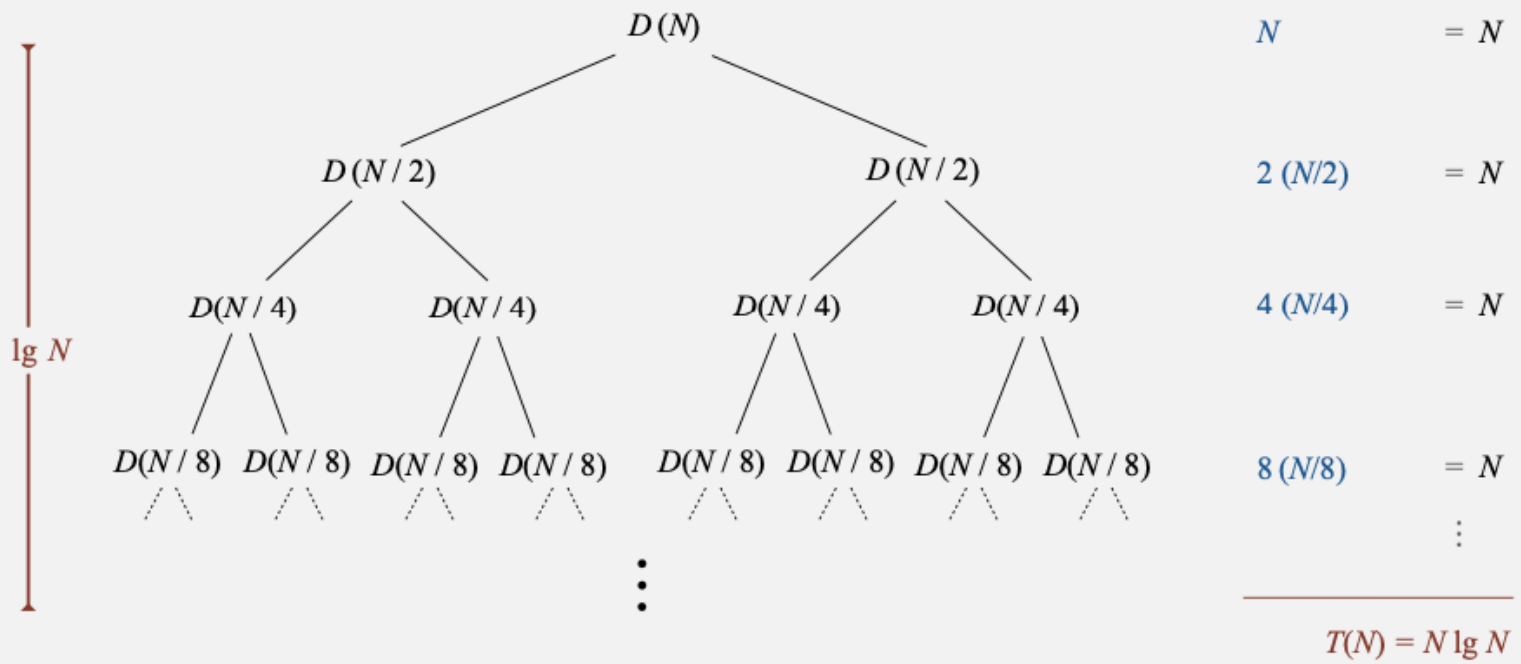
A **recurrance relation** is an equation that defines a sequence based on a rule that gives the next term as a function of the previous terms.

- A **recursion tree** is useful for visualizing what happens when such an equation is iterated, documenting not only the number of recursive calls, but also the amount of work done at each step.
- In a recursion tree, each node represents the size and cost of each subproblem.
- We sum the costs within each level of the tree to obtain a set of per-level costs, and then we sum all the per-level costs to determine the total cost of all levels of the recursion.

# Mergesort Recursion Tree

**Proposition.** If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

**Pf 1.** [assuming $N$ is a power of 2]



# Mergesort: Explanation of time complexity

Mergesort divides arrays into two equal parts at each step.

- The second level of the recursion tree contains $2^1$ subarrays of size $\frac{n}{2^1}$.
- The third level has $2^2$ subarrays of size $\frac{n}{2^2}$.

In general, assuming levels are zero-indexed, each level has $2^h$ arrays of size $\frac{n}{2^h}$.

- We are processing $2^h \cdot \frac{n}{2^h}$ items per level in the tree, so each level is $O(n)$, but *how many levels are there?*
- By construction, Mergesort creates complete binary trees.
- The number of levels in a full binary tree with $n$ nodes in it's deepest level is $\lg(n)$

Therefore, we have $n$ items being processed $lg(n)$ times, making Mergesort a $O(n \lg n)$ (linearithmic) algorithm!

# Solving Recurrance Equations

Consider the size of the sub arrays at each iteration.

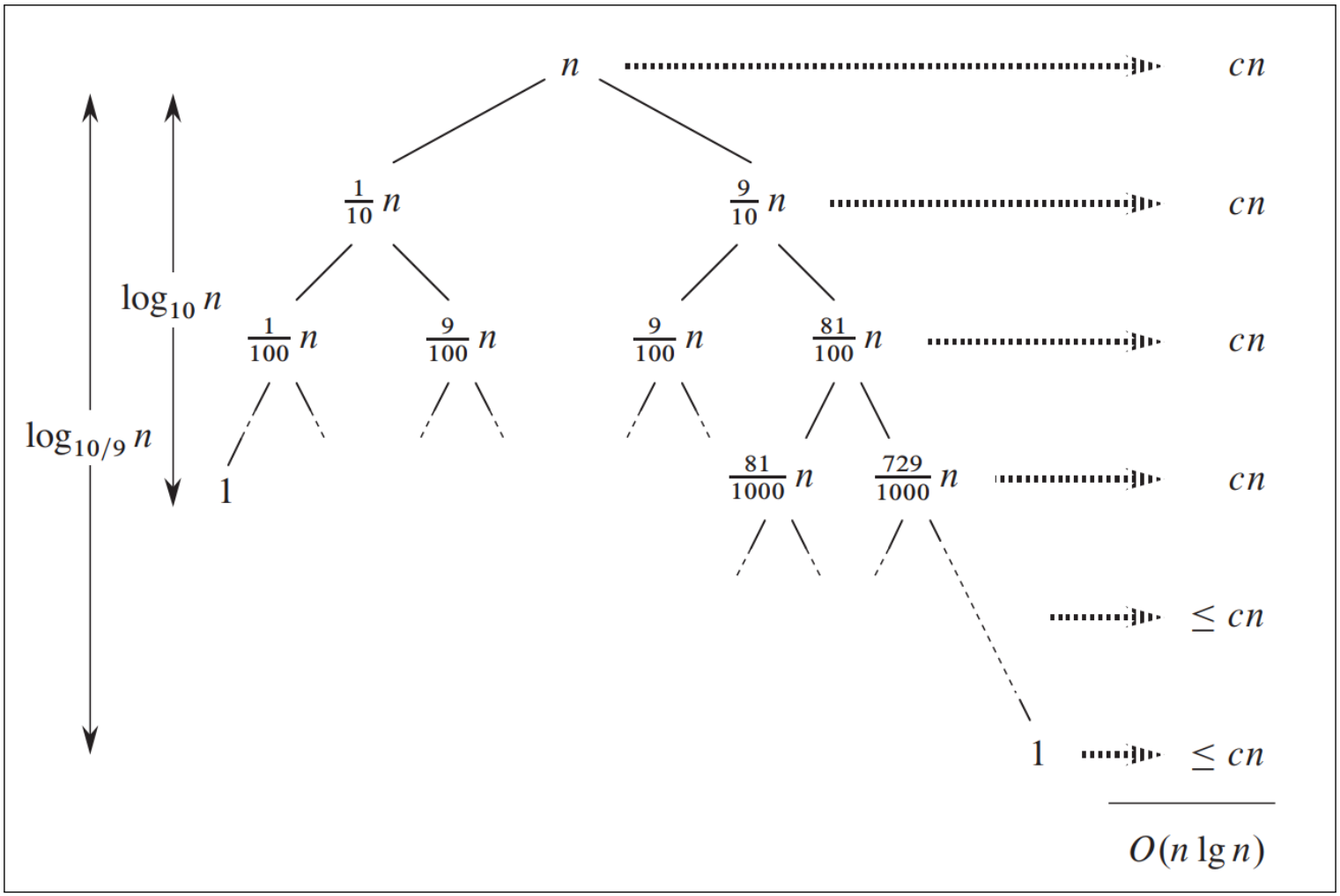
- If the algorithm creates **equal sub-arrays**, then every path of the recursion tree has the same height $h$.
- However, if the algorithm creates **unequal sub-arrays**, this property no longer holds. To determine the tree height we must examine the longest path.
- At each division, the sub-array with the most items will take longest to be sub-divided down to arrays of one item.
- Therefore, longest path always follows the largest sub-arrays, and this path is the height of the tree.

# Solving Recurrance Equations II

- To determine the height of this longest path, we formulate an equation $n/k^h = 1$, where $k$ is the fraction by which $n$ is divided at each level, to get the largest sub array.
- Computing the time taken at each level and multiplying it with $h$ will give you the order of the total complexity of the algorithm.

# Solving A Particular Recurrance Equation

Consider $T(n) = T(n/10) + T(9n/10) + cn$:



# Solving a Particular Recurrance Equation

The equation $T(n) = T(n/10) + T(9n/10) + cn$ is divided unevenly.

- The largest sub-arrays dictate tree height.
- At the second level, the largest subarray is of size $\frac{9n}{10}$.
- At the next level, the largest subarray is of size $\frac{9^2 n}{10^2}$, and so on.
- At maximum depth, $\frac{n}{(10/9)^h} = 1$, where $h$ is our maximum tree depth.
- Solving the above equation yields

$$h = \log_{10/9} n$$

- Each level of the tree still takes $cn$, so with $h$ levels the total time taken is $O(n \log_{10/9} n)$.

# Mergesort: Bottom-up

- Pass through array, merging as we go to double size of sorted subarrays.
- Keep performing the passes and merging subarrays, until you do a merge that encompasses the whole array.

**Bottom-up mergesort**

```
public class MergeBU
{
   private static Comparable[] aux;      // auxiliary array for merges

   // See page 271 for merge() code.

   public static void sort(Comparable[] a)
   {  // Do lg N passes of pairwise merges.
      int N = a.length;
      aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)        // sz: subarray size
         for (int lo = 0; lo < N-sz; lo += sz+sz) // lo: subarray index
            merge(a, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

# Mergesort: Bottom-up Java

| | | a[i] | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| | | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| sz = 1 | | | | | | | | | | | | | | | | | |
| | merge(a, 0, 0, 1) | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E |
| | merge(a, 2, 2, 3) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| | merge(a, 4, 4, 5) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| | merge(a, 6, 6, 7) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E |
| | merge(a, 8, 8, 9) | E | M | G | R | E | S | O | R | E | T | X | A | M | P | L | E |
| | merge(a, 10, 10, 11) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| | merge(a, 12, 12, 13) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E |
| | merge(a, 14, 14, 15) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | E | L |
| sz = 2 | | | | | | | | | | | | | | | | | |
| | merge(a, 0, 1, 3) | E | G | M | R | E | S | O | R | E | T | A | X | M | P | E | L |
| | merge(a, 4, 5, 7) | E | G | M | R | E | O | R | S | E | T | A | X | M | P | E | L |
| | merge(a, 8, 9, 11) | E | G | M | R | E | O | R | S | A | E | T | X | M | P | E | L |
| | merge(a, 12, 13, 15) | E | G | M | R | E | O | R | S | A | E | T | X | E | L | M | P |
| sz = 4 | | | | | | | | | | | | | | | | | |
| | merge(a, 0, 3, 7) | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P |
| | merge(a, 8, 11, 15) | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X |
| sz = 8 | | | | | | | | | | | | | | | | | |
| | merge(a, 0, 7, 15) | A | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

Trace of merge results for bottom-up mergesort

# Min Top-Down Comparisons

Proposition: Top-down mergesort uses between $1/2N \lg N$ and $N \lg N$ comparisons.

Let the total number of comparisons be $C(N)$:

$$C(N) \leq \underbrace{C(\lceil N/2 \rceil)}_{\text{left recursion}} + \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right recursion}} + \underbrace{cN}_{\text{comparisons at this level}}$$

- The smallest number of comparisons made by $\textsc{Merge}$ is $\frac{N}{2}$.
  - If one sub-array contains all the smallest elements, we only walk that array before appending the other.
- If you sort a sorted array, this would be the case at every level.
- Solving the above recursion with $c = \frac{1}{2}$ yields $C(N) = 1/2N \lg N$.

# Max Top-down Comparisons

- Similarly, the maximum number of comparisons is made when both sub-arrays must be fully examined.
- If the input array is of size $N$, then at most $N$ comparisons are made.
- If the above happens at every level of the recursion, the total number of comparisons at each level is at most $N$.
- By solving the same recurrance equation with $c = 1$, we get $C(N) = N \lg N$.

# Max Top down Accesses

Proposition. Top-down mergesort uses at most $6N \lg N$ array accesses to sort an array of length $N$.

- Each merge uses at most 6N array accesses
  - $2N$ to copy the sub arrays initially
  - $2N$ to put the values back (in order)
  - At most $N$ comparisons, each accessing two array elements ($2N$)
- Hence the total number of array accesses after solving the recurrence is most $6N \lg N$.

# Min/Max Bottom Up Comparisons/Accesses

Proposition. Bottom-up mergesort uses between $1/2N \lg N$ and $N \lg N$ compares and at most $6N \lg N$ array accesses to sort an array of length $N$.

- The number of passes through the array is precisely $\lg N$.
- For each pass, by the same argument made as in the case of the top-down mergesort approach, the number of array accesses is exactly $6N$ and the number of compares is at most $N$ and no less than $N/2$.
- Therefore, bottom-up mergesort uses between $1/2N \lg N$ and $N \lg N$ compares and at most $6N \lg N$ array accesses to sort an array of length $N$.

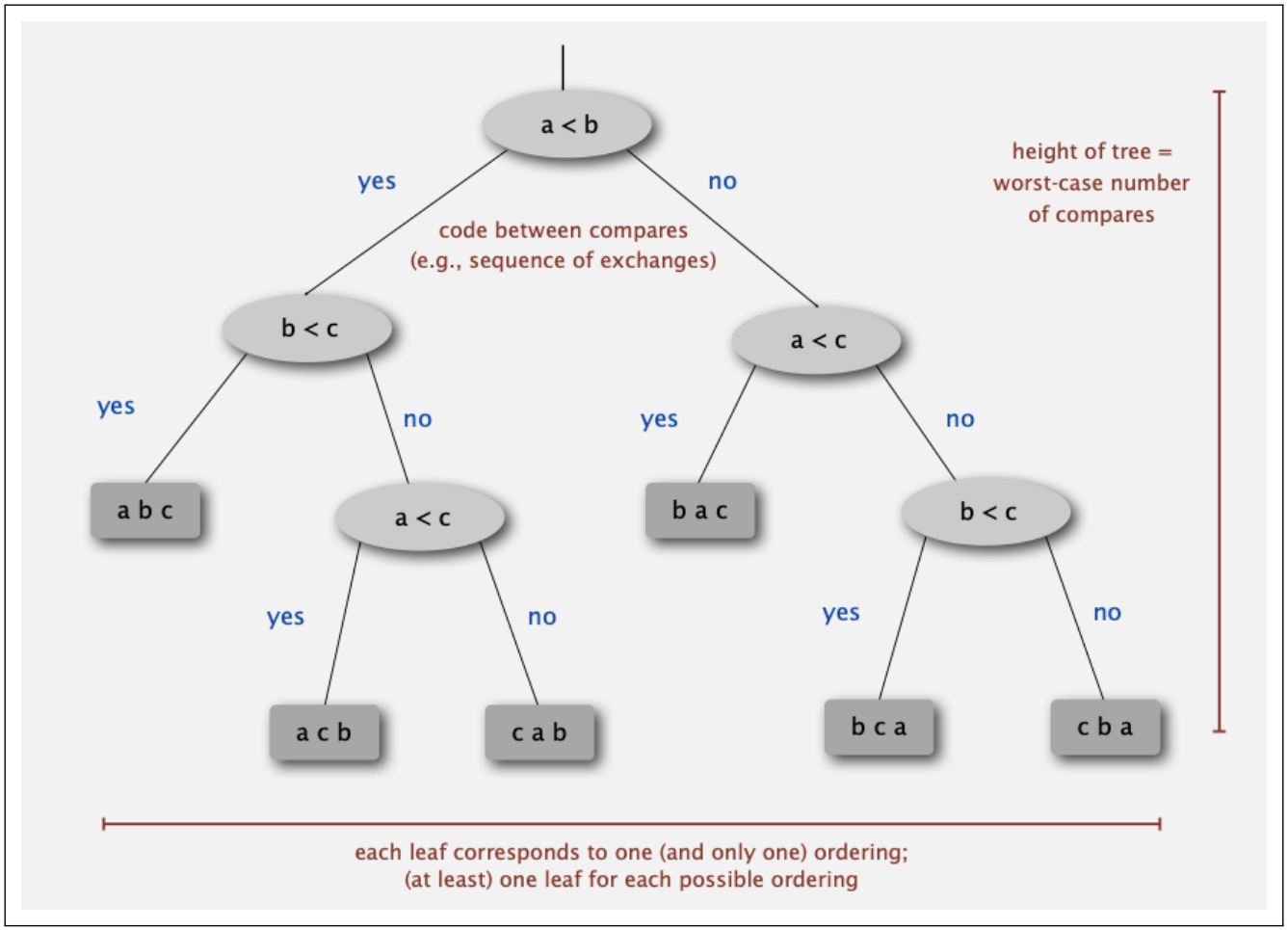
# Practicality!

A few small improvements

- Mergesort has too much overhead for tiny subarrays. Therefore, use insertion sort arrays of $< 10$ items.
- You can quickly check if two arrays are already sorted using $a[mid] \leq a[mid + 1]$. If so, nothing more needs to be done. Stop if already sorted; that is, do not call merge if the sub arrays are sorted. This can be done by a simple check

# Lower Bound for Sorting: Decision Tree

Recall that there are $n!$ permutations of $n$ elements. The comparison decisions which result in these permuations can be represented as a decision tree.



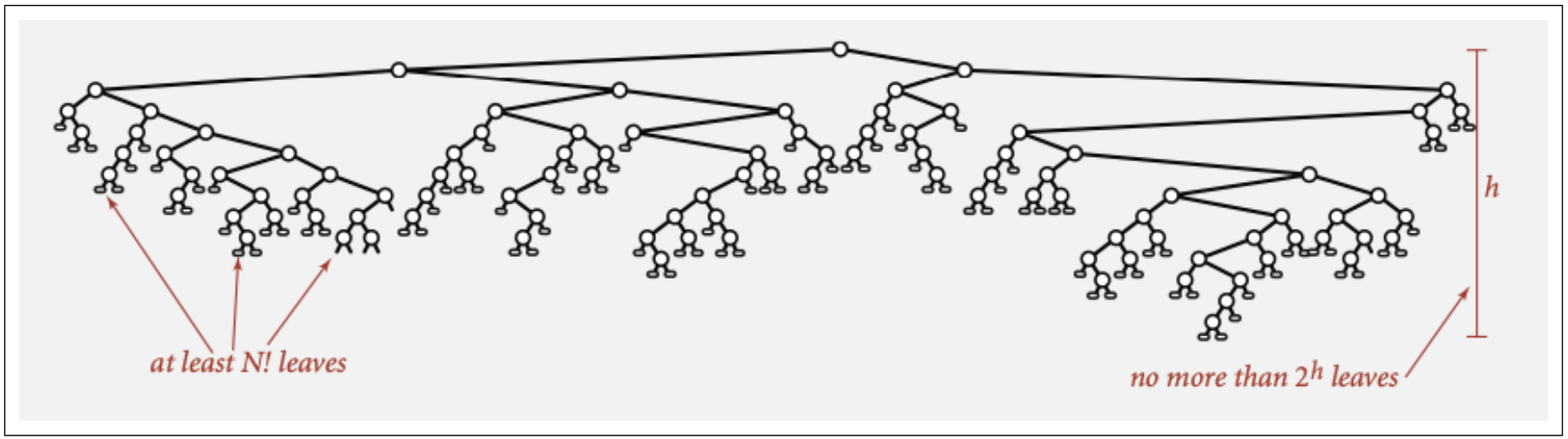
# Lower Bound for Sorting

Proposition: Any compare-based sorting algorithm must use at least $N \lg N$ compares; that is, $T(N) = \Omega(N \lg N)$.

**Proof Sketch:**

- Assume array consists of $N$ distinct values $a_1$ through $a_N$.
- Worst case dictated by height $h$ of decision tree.
- Binary tree of height $h$ has at most $2^h$ leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.

# Lower Bound for Sorting



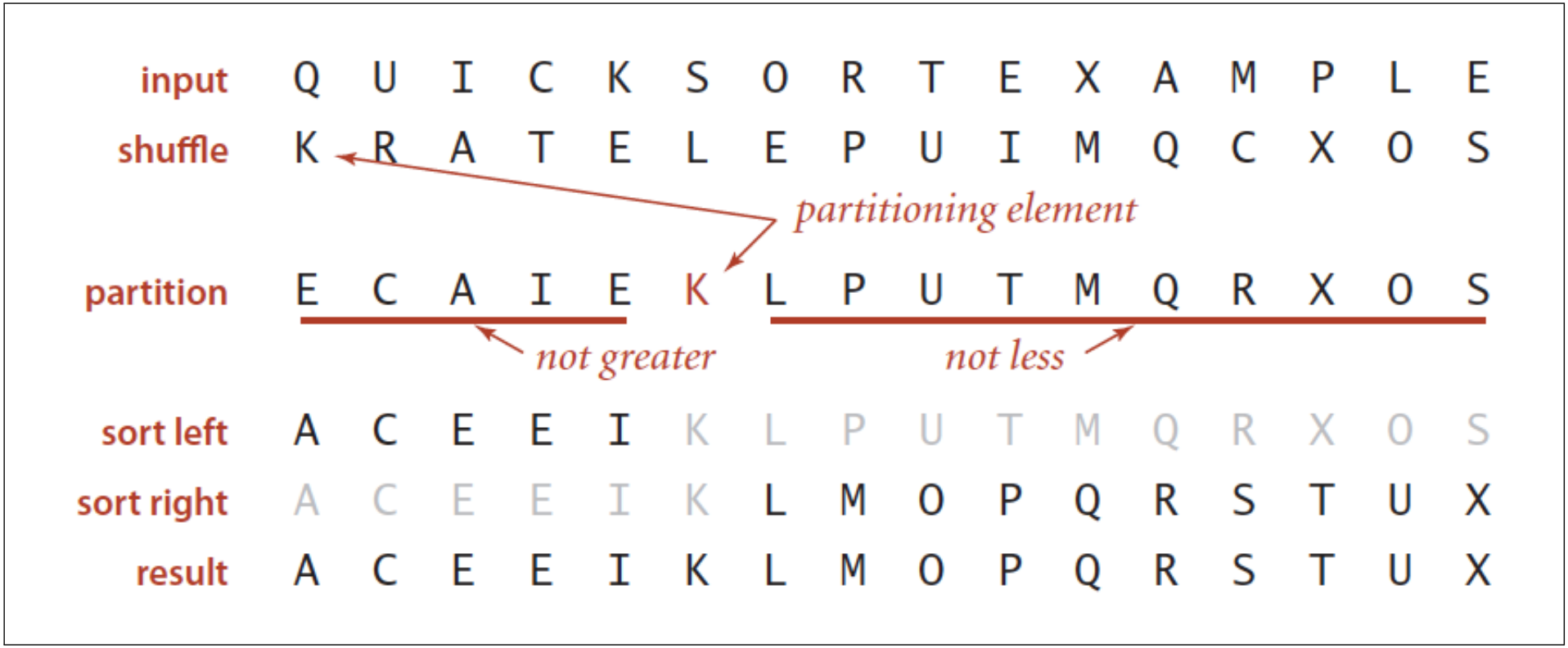

- $2^h \geq N! \Rightarrow h \geq \lg(N!)$, by Stirling's approximation we have $\lg(N!) \approx N \lg N$ `https://en.wikipedia.org/wiki/Stirling%27s_approximation`
- Hence $h \geq N \lg N$. Therefore, any compare-based sorting algorithm must use at least $N \lg N$ compares in the worst-case.

# Quicksort - Idea

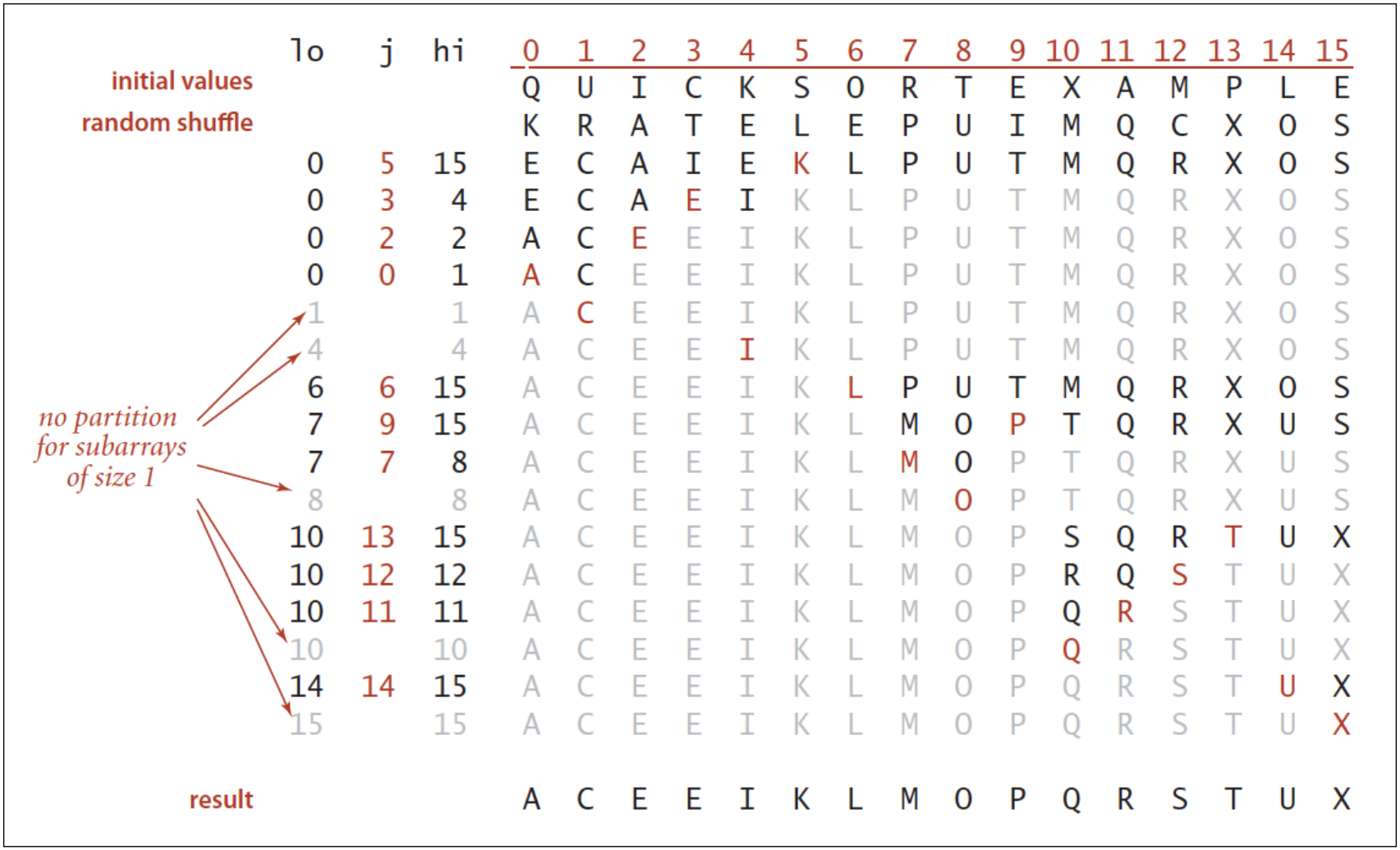
Quicksort was rated one of the top 10 algorithms of the 20th century. Here's the overall approach:

- Shuffle the array to eliminate dependence on input.
- Select first element of array as **pivot**.
- Create two sub arrays from remaining elements via in-place swapping.
  - One containing all the elements less than (or equal to) the pivot.
  - The other containg all the elements greater (or equal to) than the pivot.
- Recurse on both subarrays.
- Profit!

# Quicksort Visualized



# Quicksort Full Trace



# Quicksort Java I

**ALGORITHM 2.5** Quicksort

```
public class Quick
{
   public static void sort(Comparable[] a)
   {
      StdRandom.shuffle(a);          // Eliminate dependence on input.
      sort(a, 0, a.length - 1);
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);  // Partition (see page 291).
      sort(a, lo, j-1);              // Sort left part a[lo .. j-1].
      sort(a, j+1, hi);              // Sort right part a[j+1 .. hi].
   }
}
```

# Quicksort Java II

Partition returns the index of the pivot.

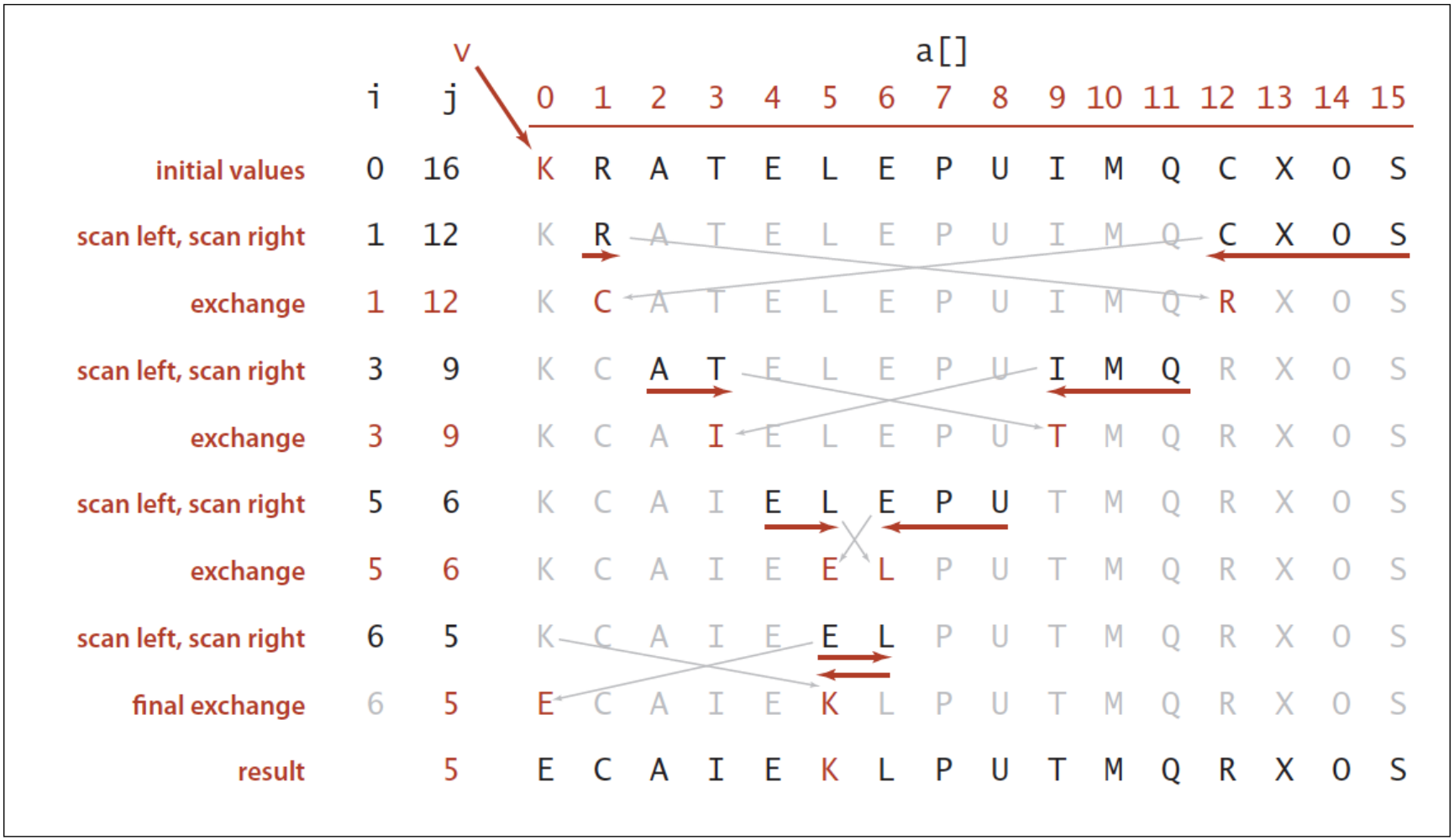
## Quicksort partitioning

```
private static int partition(Comparable[] a, int lo, int hi)
{  // Partition into a[lo..i-1], a[i], a[i+1..hi].
   int i = lo, j = hi+1;              // left and right scan indices
   Comparable v = a[lo];              // partitioning item
   while (true)
   {  // Scan right, scan left, check for scan complete, and exchange.
      while (less(a[++i], v)) if (i == hi) break;
      while (less(v, a[--j])) if (j == lo) break;
      if (i >= j) break;
      exch(a, i, j);
   }
   exch(a, lo, j);       // Put v = a[j] into position
   return j;             // with a[lo..j-1] <= a[j] <= a[j+1..hi].
}
```

# Quicksort Partitioning

- First, we select the item we are partitioning on (v).
- We then set up a left and a right scan.
  - We scan from the right until we hit a value greater than v.
  - Then we scan from the left until we hit a value less than v.
  - we swap these two values and then continue the scan.
- The two scans will eventually meet in the middle, and that's where we put v.
- The two partitions are guaranteed to be lower/higher than v, but not guaranteed to be sorted, so we recursively sort them.

# Quicksort Partitioning Example



# Quicksort Analysis

- Using an auxillary array eliminates the need to perform the scan partition, simplifying the algorithm, but is not worth the cost.
- Shuffling is key for or performance guarantee. An alternate way to preserve randomness is to choose the pivot randomly.
- When the scan hits an element identical to the pivot, it is (counter-intuitively) best to stop the scan. This aids in avoiding quadratic running time in certain applications.

# Quicksort Performance Characteristics

Quick Sort is considered the fastest sorting when input is highly disordered and $(N > 15)$.

- Half of all cases execute **very** quickly.
- The inner loop is very short.
  - All we're doing is incrementing an index and comparing an array entry to a fixed value.
  - It is difficulty to have less going on in the inner loop.
  - Both mergesort and shellsort move data within their inner loops.

# Quicksort Best/Worst Cases

- Best case: Number of comparisons is $\approx N \lg N$; that is, $\Omega(N \lg N)$
  - Partitioning always creates equal size subarrays.
  - Recurrence relation is $T(n) = 2T(n/2) + cn$
  - Therefore, $T(n) \in \Omega(N \lg N)$.
- Worst case: Number of comparisons is $\approx 1/2N^2$; that is, $O(N^2)$
  - Partitioning always results in one partition of size zero.
  - The recurrence relation is $T(n) = T(n-1) + T(0) + cn$,
  - Therefore $T(n) \in O(N^2)$.

# Quicksort running time Analysis

- Average case: Expected number of comparisons is $\approx 1.39N \log N$; that is, $\Theta(N \log N)$.
- Although mergesort also has the same running time and quicksort does 39% more compares, quicksort is typically faster as it does much less data movement.

# Quicksort: Practical improvements

All the below are suggestions validated with refined math models and experiments.

- Best choice of pivot element = median.
- Even quicksort has too much overhead for tiny subarrays. Therefore, use insertion sort for $< 15$ items.
- There is a non-recursive version using stacks which always sorts the smaller half first.

# Sorting Summary

Running time estimates:

- Home pc executes $10^8$ comparisons/second.
- Supercomputer executes $10^{12}$ comparisons/second.

Insertion Sort ($N^2$)

| computer | thousand | million | billion |
|---|---|---|---|
| home | instant | 2.8 hours | 317 years |
| super | instant | 1 second | 1.6 weeks |

Mergesort (N log N)

| thousand | million | billion |
|---|---|---|
| instant | 1 sec | 18 min |
| instant | instant | instant |

Quicksort (N log N)

| thousand | million | billion |
|---|---|---|
| instant | 0.3 sec | 6 min |
| instant | instant | instant |