CS 2LC3
Logical Reasoning for Computer Science

Tutorial 2

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September 23, 2022

Outline

- Announcements / Reminders
- Syntax and evaluation of Boolean expressions
- Satisfiability, validity, and duality
- Modeling English propositions
- Equivalence and true
- Negation, inequivalence and false
- Disjunction
- Conjunction

Announcements

- ❖Assignment 1 is posted on the course website and it is due on October 3 (Monday), 2022, 23:59 via Avenue.
- ❖The submission avenue option is already activated on the avenue.

Instructions: For all assignments, the students must submit their solutions to

Avenue → Assessments → Assignment #

Students can solve the exercises on paper, use a smartphone app called [CamScanner](), **convert their entire solution into a single PDF file, and submit it to Avenue.**
You can also use an iPad or any other software to write your assignment and as long as you convert it into a readable PDF file.
The maximum upload file size is 2 GB in Avenue for each submission.

Please make sure that the final PDF file is readable.

Students, who wish to use Microsoft word and do not have Microsoft Word on their computer, are suggested to use google document editor (Google Docs**). This online software allows you to convert your final file into a PDF file.**

There will be a mark deduction for not following the submission instruction.

Please first finish the assignment on your local computer and, at the end, **and attach your solution as a PDF file.**

You will have an unlimited number of submissions until the deadline.

Before we can answer exercise 2.1, we need to understand the following operators from the textbook.

Operator = is ***conventional equality***
Operator $\equiv$ is **equivalence** (second name of =)
Operator $\neq$ is ***conventional inequality***
Operator $\not\equiv$ **inequivalence** or ***xor***
Operator $\vee$ is called ***disjunction*** or ***or***
Operator $\wedge$ is called ***conjunction*** or ***and***
Operator $\Rightarrow$ is called ***implication***
Operator $\Leftarrow$ is called ***consequence*** *or* ***antecedent***

Page 27 of textbook fully explains the above operator, lets review page 27 of the textbook.

$b = c$	is read as	"b equals c"
$b \equiv c$	is read as	"b equivales c"
$b \neq c$	is read as	"b differ from c"
$b \not\equiv c$	is read as	"b differ from c"
$b \vee c$	is read as	"b or c"
$b \wedge c$	is read as	"b and c"
$b \Rightarrow c$	is read as	"b implies c"
$b \Leftarrow c$	is read as	"c follows b"

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If we were given $n = false$ and $p = t$rue, what is the value of the final expression?

$$(n \Rightarrow p)$$

Before we can answer exercise 2.1, we nee to understand the following table from the textbook.

$(n \Rightarrow p)$ is true

$n = f$ and $p = t$

			$\vee$	$\Leftarrow$		$\Rightarrow$		$\equiv$ $=$	$\wedge$	**nand**	$\not\equiv$ $\neq$					**nor**	
t	t	t	t	t	t	t	t	t	t	f	f	f	f	f	f	f	f
t	f	t	t	t	t	f	f	f	f	t	t	t	t	f	f	f	f
f	t	t	t	f	f	t	t	f	f	t	t	f	f	t	t	f	f
f	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f

Note: *true* is abbreviated by t and *false* by f

The above table is from Chapter 2, page 26 of textbook.

Given

m: $false$

n: $true$

What does the following evaluates to?

$$m \equiv n$$

$$m \equiv n$$

If m: false

n: true

$m \equiv n$ is false

If m: false

n: false

$m \equiv n$ is true

Syntax and evaluation of Boolean expressions

2.1 Each line below contains an expression and two states *S0* and *S1* (using t for *true* and f for *false*). Evaluate the expression in both states.

		state *S0*				state *S1*			
	expression	m	n	p	q	m	n	p	q
(f)	$m \vee (n \Rightarrow p)$	t	f	t	t	t	t	f	t
(j)	$(m \equiv n) \wedge (p \Rightarrow q)$	f	t	f	t	t	t	f	f

Syntax and evaluation of Boolean expressions

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		state *S0*				state *S1*			
	expression	m	n	p	q	m	n	p	q
(f)	$m \vee (n \Rightarrow p)$	t	f	t	t	t	t	f	t

In S0, $m \vee (n \implies p)$ evaluates to $t \vee (f \implies t)$.

$(f \implies t) \equiv t$ and $t \vee t \equiv t$, therefore $t \vee (f \implies t) \equiv t$.

In S1, $m \vee (n \implies p)$ evaluates to $t \vee (t \implies f)$.
$(t \implies f) \equiv f$ and $t \vee f \equiv t$, therefore $t \vee (t \implies f) \equiv t$.

Syntax and evaluation of Boolean expressions

2.1 Each line below contains an expression and two states *S0* and *S1* (using t for *true* and f for *false*). Evaluate the expression in both states.

		state *S0*				state *S1*			
	expression	m	n	p	q	m	n	p	q
(j)	$(m \equiv n) \wedge (p \Rightarrow q)$	f	t	f	t	t	t	f	f

In S0, $(m \equiv n) \wedge (p \implies q)$ evaluates to $(f \equiv t) \wedge (f \implies t)$.
$(f \equiv t) \equiv f$, $(f \implies t) \equiv t$, and $f \wedge t \equiv f$ therefore $(f \equiv t) \wedge (f \implies t) \equiv f$.
In S1, $(m \equiv n) \wedge (p \implies q)$ evaluates to $(t \equiv t) \wedge (f \implies f)$.
$(t \equiv t) \equiv t$, $(f \implies f) \equiv t$, and $t \wedge t \equiv t$ therefore $(t \equiv t) \wedge (f \implies f) \equiv t$.

What if there was no parentheses?

Each line below contains an expression and two states *S0* and *S1* (using *t* for *true* and *f* for *false*). Evaluate the expression in both states.

	state *S0*				state *S1*			
expression	m	n	p	q	m	n	p	q
$m \equiv n \wedge p \Rightarrow q$	f	t	f	t	t	t	f	f

To answer this question we must understand the table of precedence on page 2 of the textbook.

Table of Precedences

(a)	$[x := e]$ (textual substitution)	(highest precedence)
(b)	$.$ (function application)	
(c)	unary prefix operators: $+ \quad - \quad \neg \quad \# \quad \sim \quad \mathcal{P}$	
(d)	$**$	
(e)	$\cdot \quad / \quad \div$ **mod** **gcd**	
(f)	$+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$	
(g)	$\downarrow \quad \uparrow$	
(h)	$\#$	
(i)	$\triangleleft \quad \triangleright \quad \hat{}$	
(j)	$= \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad \mid$	(conjunctional, see page 29)
(k)	$\vee \quad \wedge$	
(l)	$\Rightarrow \quad \Leftarrow$	
(m)	$\equiv$	(lowest precedence)

All nonassociative binary infix operators associate to the left, except $**$, $\triangleleft$, and $\Rightarrow$, which associate to the right.

The operators on lines (j), (l), and (m) may have a slash / through them to denote negation —e.g. $b \not\equiv c$ is an abbreviation for $\neg(b \equiv c)$.

From page 2 of textbook.

Syntax and evaluation of Boolean expressions

2.2 Write truth tables to compute values for the following expressions in all states.

(e) $\neg b \Rightarrow (b \vee c)$
(f) $\neg b \equiv (b \vee c)$

Syntax and evaluation of Boolean expressions

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(e) $\neg b \Rightarrow (b \vee c)$

b	c	$\neg b$	$b \vee c$	$\neg b \Rightarrow (b \vee c)$
t	t	f	t	t
t	f	f	t	t
f	t	t	t	t
f	f	t	f	f

Syntax and evaluation of Boolean expressions

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(f) $\neg b \equiv (b \vee c)$

b	c	$\neg b$	$b \vee c$	$\neg b \equiv (b \vee c)$
t	t	f	t	f
t	f	f	t	f
f	t	t	t	t
f	f	t	f	f

Satisfiability, validity, and duality

2.3 Write the duals P_D for each of the following expressions P.

(e) $\neg false \Rightarrow b \vee c$

(h) $(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$

(2.2) **Definition.** The *dual* P_D of a boolean expression P is constructed from P by interchanging occurrences of

$true$ and $false$,
$\wedge$ and $\vee$,
$\equiv$ and $\not\equiv$,
$\Rightarrow$ and $\not\Leftarrow$, and
$\Leftarrow$ and $\not\Rightarrow$.

Satisfiability, validity, and duality

2.3 Write the duals P_D for each of the following expressions P.

(e) $\neg false \Rightarrow b \vee c$

$\neg true \not\Leftarrow b \wedge c$

Satisfiability, validity, and duality

2.3 Write the duals P_D for each of the following expressions P.

(h) $(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$

$(b \not\equiv c) \not\equiv (b \not\Leftarrow c) \vee (c \not\Leftarrow b)$

Satisfiability, validity, and duality

2.4 For each expression $P \equiv Q$ below, write the expression $P_D \equiv Q_D$.

(g) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(h) $p \equiv q \equiv q \equiv p$

Satisfiability, validity, and duality

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(g) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(g) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(h) $p \equiv q \equiv q \equiv p$

P_D

(h) $p \not\equiv q \equiv q \not\equiv p$ Q_D

or $\boxed{p} \equiv \boxed{q \not\equiv q \not\equiv p}$

or $p \not\equiv q \not\equiv q \equiv p$

(h) $p \not\equiv q \equiv q \not\equiv p$

or $p \equiv q \not\equiv q \not\equiv p$

P_D or $\boxed{p \not\equiv q \not\equiv q} \equiv \boxed{p}$ Q_D

Q_D

(h) $\boxed{p \not\equiv q} \equiv \boxed{q \not\equiv p}$

or $p \equiv q \not\equiv q \not\equiv p$

P_D or $p \not\equiv q \not\equiv q \equiv p$

Satisfiability, validity, and duality

2.4 For each expression $P \equiv Q$ below, write the expression $P_D \equiv Q_D$.

(h) $p \equiv q \equiv q \equiv p$

(h) $p \not\equiv q \equiv q \not\equiv p$

or $p \equiv q \not\equiv q \not\equiv p$

or $p \not\equiv q \not\equiv q \equiv p$

Lets assume

p :$true$

q:$true$

Modeling English propositions

2.5 Translate the following English statements into boolean expressions.

(a) Whether or not it's raining, I'm going swimming.

(f) If it rains cats and dogs while I am going swimming, I'll eat my hat.

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Associate identifiers with the primitive subexpressions as follows.

r : It's raining
s : I'm going swimming
sc : It's raining cats
sd : It's raining dogs
eh : I'll eat my hat

Modeling English propositions

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r : It's raining
s : I'm going swimming
sc : It's raining cats
sd : It's raining dogs
eh : I'll eat my hat

The translations are then

(a) $r \vee \neg r \;\Rightarrow\; s$
(f) $sc \wedge sd \wedge s \Rightarrow eh$

Modeling English propositions

2.6 Translate the following English statements into boolean expressions.

(b) Exactly one of p and q is $true$.
(c) Zero, two, or four of p , q , r , and s are $true$.

Modeling English propositions

2.6 Translate the following English statements into boolean expressions.

(b) Exactly one of p and q is *true* .

(b) $\neg(p \equiv q)$, or $p \not\equiv q$

$$\neg p \equiv q$$

$$p \not\equiv q \longleftarrow (p \wedge \neg q) \vee (\neg p \wedge q) \equiv p \not\equiv q$$

Modeling English propositions

2.6 Translate the following English statements into boolean expressions.

(c) Zero, two, or four of p, q, r, and s are *true*.

$$p \equiv q \equiv r \equiv s$$

Solution 2: Or we can answer it the following way

$$(\neg p \wedge \neg q \wedge \neg r \wedge \neg s)$$
$$\vee$$
$$(p \wedge q \wedge \neg r \wedge \neg s)$$
$$\vee$$
$$(p \wedge r \wedge \neg q \wedge \neg)s$$

Solution 3:
Or we can answer it the following way

p	q	r	s	$P \equiv q \equiv r \equiv s$
F	F	F	F	T
F	F	F	T	F
F	F	T	F	F
F	F	T	T	T
F	T	F	F	F
F	T	F	T	T

we can determine without any additional formal manipulation that

$$false\ =\ false\ =\ false\ =\ true$$ is false,

because three (an odd number) of its equivalents are false

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Modeling English propositions

2.8 Translate the following English statement into a boolean expression. v is in $b[1..10]$ means that if v is in $b[11..20]$ then it is not in $b[11..20]$.

TABLE 2.3. TRANSLATION OF ENGLISH WORDS

and, but	becomes	$\wedge$
or	becomes	$\vee$
not	becomes	$\neg$
it is not the case that	becomes	$\neg$
if p then q	becomes	$p \Rightarrow q$
Means	becomes	$\equiv$
However	becomes	$\wedge$
;	becomes	$\wedge$

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Modeling English propositions

2.8 Translate the following English statement into a boolean expression. v is in $b[1..10]$ means that if v is in $b[11..20]$ then it is not in $b[11..20]$.

Let x: denote " v is in $b[1 \cdots 10]$ "
y: denote " v is in $b[11 \cdots 20]$ ".

$$x \equiv y \Rightarrow \neg y$$

Modeling English propositions

2.10 Solve the following puzzle. A certain island is inhabited by people who either always tell the truth or always lie and who respond to questions with a yes or a no. A tourist comes to a fork in the road, where one branch leads to a restaurant and the other does not. There is no sign indicating which branch to take, but there is an islander standing at the fork. What single yes/no question can the tourist ask to find the way to the restaurant?

Modeling English propositions

2.10 Solve the following puzzle. A certain island is inhabited by people who either always tell the truth or always lie and who respond to questions with a yes or a no. A tourist comes to a fork in the road, where one branch leads to a restaurant and the other does not. There is no sign indicating which branch to take, but there is an islander standing at the fork. What single yes/no question can the tourist ask to find the way to the restaurant?

Hint: Let p stand for "the islander at the fork always tells the truth" and let q stand for "the left-hand branch leads to the restaurant". Let E stand for a boolean expression such that, whether the islander tells the truth or lies, the answer to the question "Is E *true*?" will be yes iff the left-hand branch leads to the restaurant. Construct the truth table that E must have, in terms of p and q, and then design an appropriate E according to the truth table.

Hint:
Let

p stands for islander at the fork always tell the **truth**
q stands for “the left-hand branch leads to the restaurant
E stand for a Boolean expression such that, whether the islander tells the truth or lies ,the answer to the question “Is E true?”
Yes iff the left-hand branch leads to the restaurant

Construct the truth table that E must have, in terms of p and q and then design an appropriate E according to the truth table.

2.10 Solve the following puzzle. A certain island is inhabited by people who either always tell the truth or always lie and who respond to questions with a yes or a no. A tourist comes to a fork in the road, where one branch leads to a restaurant and the other does not. There is no sign indicating which branch to take, but there is an islander standing at the fork. What single yes/no question can the tourist ask to find the way to the restaurant?

p: The islander is truthful

q: Left Branch leads to the restaurant

p	q	E
T	T	T
T	F	F
F	T	F
F	F	T

Equivalence and true

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$

Associativity allows us to be informal and insert or delete pairs of parentheses in sequences of equivalences, just as we do with sequences of additions (e.g. $w + x + y + z$ is equivalent to $w + (x + y) + z$). Hence, we can write

$$p \equiv q \equiv r \quad \text{instead of} \quad p \equiv (q \equiv r) \quad \text{or} \quad (p \equiv q) \equiv r \quad .$$

Keeping axiom (3.1) in mind, we express the second axiom, symmetry, without parentheses.

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

You can see why this axiom is called *symmetry* by imagining parentheses as follows: $(p \equiv q) \equiv (q \equiv p)$.

We now give our first proof, of the following theorem:

$$p \equiv p \;\equiv\; q \equiv q \quad .$$

Remember that the axiom of associativity allows us to parenthesize an expression such as (3.2) in several ways. In the following proof, we parenthesize (3.2) as $(p \equiv q \equiv q) \equiv p$, so that, using Leibniz, we can replace $p \equiv q \equiv q$ in an expression by p .

$$\begin{array}{ll} & p \;\equiv\; p \equiv q \equiv q \\ = & \langle \text{Symmetry of } \equiv \text{ (3.2) —replace } p \equiv q \equiv q \text{ by } p \rangle \\ & p \;\equiv\; p \\ = & \langle \text{Symmetry of } \equiv \text{ (3.2) —replace first } p \text{ by } p \equiv q \equiv q \rangle \\ & p \equiv q \equiv q \;\equiv\; p \end{array}$$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

Equivalence and true

Two theorems

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Equivalence and true

Two theorems

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

(3.4)
$$\begin{array}{ll} & true \\ = & \langle \text{Identity of } \equiv \text{ (3.3), with } q := true \rangle \\ & true \equiv true \\ = & \langle \text{Identity of } \equiv \text{ (3.3) —replace the second } true \rangle \\ & true \equiv q \equiv q \qquad \text{—Identity of } \equiv \text{ (3.3)} \end{array}$$

Equivalence and true

Two theorems

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

(3.5) $p \equiv p$

$= \langle$ Use Identity (3.3) to replace the left p by $(true \equiv p)$ $\rangle$

$true \equiv p \equiv p$

Negation, inequivalence, and false

(3.8) **Axiom, Definition of** false **:** $\mathit{false} \equiv \neg \mathit{true}$

(3.9) **Axiom, Distributivity of** $\neg$ **over** $\equiv$ **:** $\neg(p \equiv q) \ \equiv\ \neg p \equiv q$

(3.10) **Axiom, Definition of** $\not\equiv$ **:** $(p \not\equiv q) \ \equiv\ \neg(p \equiv q)$

Negation, inequivalence, and false

Theorems relating $\equiv$, $\not\equiv$, $\neg$, and *false*

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of *false*:** $\neg \mathit{false} \equiv \mathit{true}$

(3.14) $(p \not\equiv q) \ \equiv\ \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv \mathit{false}$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \ \equiv\ (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \ \equiv\ (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \ \equiv\ (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \ \equiv\ p \equiv q \not\equiv r$

Negation, inequivalence, and false

(3.11) Proof $\neg p \equiv q \equiv p \equiv \neg q$:

$\neg p \equiv q$

$=$<Use axiom (3.9)>

$\neg(p \equiv q)$

$=$<Use axiom (3.2)>

$\neg(q \equiv p)$

$=$<Use axiom (3.9)>

$\neg q \equiv p$

$=$<Use axiom (3.2)>

$p \equiv \neg q$

(3.12) Proof $\neg\neg p \equiv p$:

$\neg\neg p$

$=$<Use axiom (3.3)>

$\neg\neg(true \equiv p)$

$=$<Use axiom (3.3) on true>

$\neg\neg(p \equiv p \equiv p)$

$=$<Use axiom (3.9)>

$\neg(\neg p \equiv p \equiv p)$

$=$<Use axiom (3.2)>

$\neg(p \equiv \neg p \equiv p)$

$=$<Use axiom (3.9)>

$\neg p \equiv \neg p \equiv p$

$=$<Use axiom (3.3)>

$true \equiv p$

$=$<Use axiom (3.3)>

p

(3.13) Proof $\neg false \equiv true$:

$\neg false$

$=$<Use axiom (3.8)>

$\neg\neg true$

$=$<Use theorem (3.12)>

$true$

(3.14) Proof $(p \not\equiv q) \equiv \neg p \equiv q$:

$(p \not\equiv q)$

$=$<Use axiom (3.10)>

$\neg(p \equiv q)$

$=$<Use axiom (3.9)>

$\neg p \equiv q$

Negation, inequivalence, and false

(3.15) Proof $\neg p \equiv p \equiv false$:

$\neg p \equiv p$

$=$<Use axiom (3.9)>

$\neg(p \equiv p)$

$=$<Use axiom (3.3)>

$\neg(true)$

$=$<Use axiom (3.8)>

$false$

(3.16) Proof $(p \not\equiv q) \equiv (q \not\equiv p)$:

$(p \not\equiv q)$

$=$<Use axiom (3.10)>

$\neg(p \equiv q)$

$=$<Use axiom (3.2)>

$\neg(q \equiv p)$

$=$<Use axiom (3.10)>

$(q \not\equiv p)$

(3.17) Proof $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$:

$((p \not\equiv q) \not\equiv r)$

$=$<Use theorem (3.14) and axiom (3.2)>

$((q \equiv \neg p) \not\equiv r)$

$=$<Use theorem (3.14)>

$(\neg(q \equiv \neg p) \equiv r)$

$=$<Use axiom (3.9)>

$(\neg q \equiv \neg p \equiv r)$

$=$<Use axiom (3.2) and axiom (3.1)>

$(\neg p \equiv (\neg q \equiv r))$

$=$<Use theorem (3.14)>

$(\neg p \equiv (q \not\equiv r))$

$=$<Use theorem (3.14)>

$(p \not\equiv (q \not\equiv r))$

(3.18) Proof $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$:

$((p \not\equiv q) \equiv r)$

$=$<Use theorem (3.14) and axiom (3.1)>

$(\neg p \equiv (q \equiv r))$

$=$<Use theorem (3.14)>

$(p \not\equiv (q \equiv r))$

Negation, inequivalence, and false

(3.19) Proof $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$:

$p \not\equiv q \equiv r$

$=$<Use axiom (3.10)>

$\neg(p \equiv q) \equiv r$

$=$<Use axiom (3.2) then axiom (3.9)>

$\neg q \equiv p \equiv r$

$=$<Use axiom (3.2) then axiom (3.9)>

$p \equiv \neg(q \equiv r)$

$=$<Use axiom (3.10)>

$p \equiv q \not\equiv r$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency[7] of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:**

$$p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

Disjunction

Theorems concerning $\vee$

(3.29) **Zero[8] of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Disjunction

(3.29) Proof $p \vee true \equiv true$:

$p \vee true$

$=$<Use axiom (3.3)>

$p \vee (p \equiv p)$

$=$<Use axiom (3.27)>

$p \vee p \equiv p \vee p$

$=$<Use axiom (3.26)>

$p \equiv p$

$=$<Use axiom (3.3)>

$true$

(3.30) Proof $p \vee false \equiv p$:

$p \vee false$

$=$<Use axiom (3.8) and axiom (3.3)>

$p \vee \neg(p \equiv p)$

$=$<Use axiom (3.9)>

$p \vee (\neg p \equiv p)$

$=$<Use axiom (3.27)>

$p \vee \neg p \equiv p \vee p$

$=$<Use axiom (3.28) and axiom (3.26)>

$true \equiv p$

$=$<Use axiom (3.3)>

p

Disjunction

Prove Distributivity of $\vee$ over $\vee$ (3.31), $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$.
The proof requires only the symmetry, associativity, and idempotency of $\vee$.

Proof of Distributivity of $\vee$ over $\vee$, (3.31), $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$.

$$
\begin{array}{ll}
 & (p \vee q) \vee (p \vee r) \\
= & \quad \langle \text{Associativity of } \vee \text{ (3.25)} \rangle \\
 & p \vee (q \vee p) \vee r \\
= & \quad \langle \text{Symmetry of } \vee \text{ (3.24)} \rangle \\
 & p \vee (p \vee q) \vee r \\
= & \quad \langle \text{Associativity of } \vee \text{ (3.25)} \rangle \\
 & (p \vee p) \vee (q \vee r) \\
= & \quad \langle \text{Idempotency of } \vee \text{ (3.26)} \rangle \\
 & p \vee (q \vee r)
\end{array}
$$

(3.25) Axiom, Associativity of $\vee$: $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.24) Axiom, Symmetry of $\vee$: $p \vee q \equiv q \vee p$

(3.25) Axiom, Associativity of $\vee$: $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) Axiom, Idempotency of $\vee$: $p \vee p \equiv p$

Disjunction

(3.32) Proof $p \vee q \equiv p \vee \neg q \equiv p$:

$p \vee q \equiv p \vee \neg q$

=<Use axiom (3.27)>

$p \vee (q \equiv \neg q)$

=<Use axiom (3.2) and axiom (3.9)>

$p \vee \neg(q \equiv q)$

=<Use axiom (3.3)>

$p \vee \neg(true)$

=<Use axiom (3.8)>

$p \vee false$

=<Use theorem (3.30)>

p

Conjunction

(3.35) **Axiom, Golden rule :** $p \wedge q \equiv p \equiv q \equiv p \vee q$

Conjunction

Basic properties of $\wedge$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge \mathit{true} \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge \mathit{false} \equiv \mathit{false}$

(3.41) **Distributivity of $\wedge$ over $\wedge$:**

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$$

(3.42) **Contradiction:** $p \wedge \neg p \equiv \mathit{false}$

Conjunction

Theorems relating $\wedge$ and $\vee$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$

(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:**

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:**

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Conjunction

(3.36) Proof $p \land q \equiv q \land p$:

$p \land q$
$= \langle$Use axiom (3.35)$\rangle$
$p \equiv q \equiv p \lor q$
$= \langle$Use axiom (3.2) and axiom (3.24)$\rangle$
$q \equiv p \equiv q \lor p$
$= \langle$Use axiom (3.35)$\rangle$
$q \land p$

(3.37) $(p \land q) \land r \equiv p \land (q \land r)$

$(p \land q) \land r$
$= \quad \langle$Golden rule (3.35)$\rangle$
$(p \equiv q \equiv p \lor q) \land r$
$= \quad \langle$Golden rule (3.35), with $p, q := (p \equiv q \equiv p \lor q), r\rangle$
$p \equiv q \equiv p \lor q \equiv r \equiv (p \equiv q \equiv p \lor q) \lor r$
$= \quad \langle$Distributivity of $\lor$ over $\equiv$ (3.27)$\rangle$
$p \equiv q \equiv p \lor q \equiv r \equiv p \lor r \equiv q \lor r \equiv p \lor q \lor r$
$= \quad \langle$Symmetry and associativity of $\equiv$ and $\lor\rangle$
$p \equiv q \equiv r \equiv p \lor q \equiv q \lor r \equiv r \lor p \equiv p \lor q \lor r$

(3.55) $(p \land q) \land r \equiv$
$p \equiv q \equiv r \equiv p \lor q \equiv q \lor r \equiv r \lor p \equiv p \lor q \lor r$

$p \land (q \land r)$
$= \quad \langle$Symmetry of $\land$ (3.36)$\rangle$
$(q \land r) \land p$
$= \quad \langle$(3.55), with $p, q, r := q, r, p\rangle$
$q \equiv r \equiv p \equiv q \lor r \equiv r \lor p \equiv p \lor q \equiv q \lor r \lor p$
$= \quad \langle$Symmetry and associativity of $\equiv$ and $\lor\rangle$
$p \equiv q \equiv r \equiv p \lor q \equiv q \lor r \equiv r \lor p \equiv p \lor q \lor r$
$= \quad \langle$(3.55)$\rangle$
$(p \land q) \land r$

Conjunction

Use the Golden rule (3.35) for every expression containing a conjunction. (3.38) to (3.42) are all similar to disjunction proofs after the Golden rule is applied.

(3.43)(a) Proof $p \wedge (p \vee q) \equiv p$:

$p \wedge (p \vee q)$
=<Use axiom (3.35)>
$p \equiv (p \vee q) \equiv p \vee (p \vee q)$
=<Use axiom (3.25) and axiom (3.26)>
$p \equiv p \vee q \equiv p \vee q$
=<Use axiom (3.2)>
$p \equiv true$
=<Use axiom (3.2)>
p

(3.44)(a) Proof $p \wedge (\neg p \vee q) \equiv p \wedge q$:

$p \wedge (\neg p \vee q)$
=<Use axiom (3.35)>
$p \equiv (\neg p \vee q) \equiv p \vee (\neg p \vee q)$
=<Use axiom (3.25), axiom (3.28)>
$p \equiv (\neg p \vee q) \equiv true \vee q$
=<Use theorem (3.29) and axiom (3.3)>
$p \equiv (\neg p \vee q)$
=<Use theorem (3.32) with p,q:=q,p>
$p \equiv (p \vee q) \equiv q$
=<Use axiom (3.2) and axiom (3.35)>
$p \wedge q$

Conjunction

Use the Golden rule (3.35) for every expression containing a conjunction. (3.38) to (3.42) are all similar to disjunction proofs after the Golden rule is applied.

(3.45) Proof $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$:

$p \vee (q \wedge r)$

$= < \text{Use axiom (3.35)} >$

$p \vee (q \equiv r \equiv q \vee r)$

$= < \text{Use axiom (3.27)} >$

$p \vee q \equiv p \vee r \equiv p \vee (q \vee r)$

$= < \text{Use theorem (3.31)} >$

$p \vee q \equiv p \vee r \equiv (p \vee q) \vee (p \vee r)$

$= < \text{Use axiom (3.35)} >$

$(p \vee q) \wedge (p \vee r)$

(3.46) Proof $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$:

$(p \wedge q) \vee (p \wedge r)$

$= < \text{Use axiom (3.35)} >$

$(p \equiv q \equiv p \vee q) \vee (p \equiv r \equiv p \vee r)$

$= < \text{Use axiom (3.27)} >$

$(p \equiv q \equiv p \vee q) \vee p \equiv (p \equiv q \equiv p \vee q) \vee r \equiv (p \equiv q \equiv p \vee q) \vee (p \vee r)$

$= < \text{Use axiom (3.27) and axiom (3.25)} >$

$p \vee p \equiv q \vee p \equiv p \vee q \vee p \equiv p \vee r \equiv q \vee r \equiv p \vee q \vee r \equiv p \vee p \vee r \equiv q \vee p \vee r \equiv p \vee q \vee p \vee r$

$= < \text{Use axiom (3.26), axiom (3.24) and axiom (3.2)} >$

$p \equiv p \vee q \equiv p \vee q \equiv p \vee r \equiv q \vee r \equiv p \vee q \vee r \equiv p \vee r \equiv p \vee q \vee r \equiv p \vee q \vee r$

$= < \text{Use axiom (3.2) and axiom (3.3)} >$

$p \equiv true \equiv q \vee r \equiv p \vee q \vee r \equiv p \vee r \equiv p \vee r \equiv true$

$= < \text{Use axiom (3.2) and axiom (3.3)} >$

$p \equiv q \vee r \equiv p \vee q \vee r \equiv true \equiv true \equiv true$

$= < \text{Use axiom (3.3)} >$

$p \equiv q \vee r \equiv p \vee q \vee r$

$= < \text{Use axiom (3.25)} >$

$p \equiv q \vee r \equiv p \vee (q \vee r)$

$= < \text{Use axiom (3.35)} >$

$p \wedge (q \vee r)$

Conjunction

(3.47)(a) Proof $\neg(p \wedge q) \equiv \neg p \vee \neg q$:

$\quad \neg(p \wedge q)$

$= \langle$Use axiom (3.35)$\rangle$

$\quad \neg(p \equiv q \equiv p \vee q)$

$= \langle$Use axiom (3.9)$\rangle$

$\quad \neg p \equiv q \equiv p \vee q$

$= \langle$Use theorem (3.32) with $p, q := q, p \rangle$

$\quad \neg p \equiv \neg p \vee q$

$= \langle$Use theorem (3.32) with $p, q := \neg p, q \rangle$

$\quad \neg p \vee \neg q$

(3.48) Proof $p \wedge q \equiv p \wedge \neg q \equiv \neg p$:

$\quad p \wedge q \equiv p \wedge \neg q$

$= \langle$Use theorem (3.12)$\rangle$

$\quad \neg\neg(p \wedge q \equiv p \wedge \neg q)$

$= \langle$Use axiom (3.9) with $p, q := p \wedge q, p \wedge \neg q \rangle$

$\quad \neg(\neg(p \wedge q) \equiv p \wedge \neg q)$

$= \langle$Use axiom (3.2) and axiom (3.9)with $p, q := p \wedge \neg q, \neg(p \wedge q) \rangle$

$\quad \neg(p \wedge q) \equiv \neg(p \wedge \neg q)$

$= \langle$Use axiom (3.2) and axiom (3.9)with $p, q := p \wedge \neg q, \neg(p \wedge q) \rangle$

$\quad \neg(p \wedge q) \equiv \neg(p \wedge \neg q)$

$= \langle$Use theorem (3.47)(a) $\rangle$

$\quad \neg p \vee \neg q \equiv \neg p \vee q$

$= \langle$Use theorem (3.32) with $p, q := \neg p, q \rangle$

$\quad \neg p$

Conjunction

(3.49) Proof $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$:

$p \wedge (q \equiv r)$

$=$<Use theorem (3.12)>

$\neg\neg(p \wedge (q \equiv r))$

$=$<Use theorem (3.47)(a)>

$\neg(\neg p \vee \neg(q \equiv r))$

$=$<Use axiom (3.9)>

$\neg(\neg p \vee (\neg q \equiv r))$

$=$<Use axiom (3.27)>

$\neg(\neg p \vee \neg q \equiv \neg p \vee r)$

$=$<Use axiom (3.9) and theorem (3.47)(a)>

$p \wedge q \equiv \neg p \vee r$

$=$<Use theorem (3.32) with $p, q := r, p$ >

$p \wedge q \equiv p \vee r \equiv r$

$=$<Use axiom (3.3) twice>

$p \wedge q \equiv p \vee r \equiv r \equiv p \equiv p$

$=$<Use axiom (3.35)>

$p \wedge q \equiv p \wedge r \equiv p$

(3.50) Proof $p \wedge (q \equiv p) \equiv p \wedge q$:

$p \wedge (q \equiv p)$

$=$<Use theorem (3.49) with $p, q, r := p, q, p$ >

$p \wedge q \equiv p \wedge p \equiv p)$

$=$<Use theorem (3.38) >

$p \wedge q \equiv true$

$=$<Use axiom (3.3) >

$p \wedge q$

(3.51) Proof $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$:

$(p \equiv q) \wedge (r \equiv p)$

$=$<Use theorem (3.49) with $p, q, r := (r \equiv p), p, q$ >

$(r \equiv p) \wedge p \equiv (r \equiv p) \wedge q \equiv (r \equiv p)$

$=$<Use theorem (3.50)>

$p \wedge r \equiv (r \equiv p) \wedge q \equiv r \equiv p$

$=$<Use theorem (3.49)>

$p \wedge r \equiv r \wedge q \equiv p \wedge q \equiv q \equiv r \equiv p$

$=$<Use theorem (3.50)>

$p \wedge r \equiv q \wedge (r \equiv q) \equiv p \wedge q \equiv q \equiv r \equiv p$

$=$<Use theorem (3.49)>

$p \wedge (r \equiv q) \equiv q \wedge (r \equiv q) \equiv q \equiv r$

$=$<Use theorem (3.49) and symmetry of $\equiv$ and $\wedge$ >

$(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

Any Questions?