CS 2LC3
Logical Reasoning for Computer Science

Tutorial 1

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Outline

- Announcements / Reminders
- Contact information, course website
- Textual substitution
- Reasoning using Leibniz rule
- Inference rules
- Solution to all suggested exercises for Chapter 1.

Announcements

- We sent an email from avenue
  - Please make sure that you can receive emails from avenue.
    - You must add your email address to avenue in order to receive notifications by email from avenue.
  - Please check your spam folder of your email.
  - The link to the course website is posted on the avenue.
    - http://www.cas.mcmaster.ca/~cs2lc3/
- We post lecture slides before any lectures.
- We post lecture recording the same day.
- We post tutorial slides, and recording at the end of the week.
- During the tutorial we discuss the textbook exercises.

McMaster University
Computing & Software

Course Information

Communications

Course Material

Term Marks

Academic Dishonesty

Logical Reasoning for Computer Science
COMP SCI 2LC3

Term 1, Fall 2022

Course Information

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Course website: http://www.cas.mcmaster.ca/~cs2lc3

Lecture Notes (copies of transparencies) will be on the website a few days after a class.

Lectures:

McMaster University
Computing & Software

Logical Reasoning for Computer Science
COMP SCI 2LC3

Term 1, Fall 2022

Course Material

- Chapter 1 Final solutions

Assignments:

Due Dates:

Tutorial Notes:

Tutorial Notes 1

Lecture Dates:

Lecture Slides

Lecture Notes 1

Lecture Notes 2

Lecture Notes 3

Lecture Videos:

September 12, 2022

September 13, 2022

Contact information

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- Please write your name and student ID when sending an e-mail.
- Please send mails from your McMaster address.

Textual substitution

1.1 Perform the following textual substitutions, Be careful with parenthesization and remove unnecessary parentheses.

b)

$$x + y \cdot x\ [x := b + 2]$$

$$x + y \cdot x\,[x := b + 2]$$

Note that because there are <u>no parentheses we only apply to the *x* on the right.</u>

Textual substitution has a higher precedence than any operator listed in the precedence table on the inside front cover. Consequently, in the first case below, the substitution is performed only on subexpression y. In the second case, parentheses are used to indicate that the substitution is being applied to $z + y$, rather than to y alone.

$$z + y[z, y := 5, 6] \text{ is } z + 6$$
$$(z + y)[z, y := 5, 6] \text{ is } 5 + 6$$

$$x + y \cdot x\,[x \coloneqq b + 2]$$

Step 1: $\mathrm{x} + y \cdot (b + 2)$

Final answer, the remove unnecessary parentheses

$$x + y \cdot (b + 2)$$

Does anyone know why we have := instead of = ?

Textual substitution

1.1 Perform the following textual substitutions, Be careful with parenthesization and remove unnecessary parentheses.

c)

$$(x + y \cdot x)[x := b + 2]$$

$$(x + y \cdot x)[x := b + 2]$$

$$(x + y \cdot x)[x := b + 2]$$

Step 1: $(b + 2) + y \cdot (b + 2)$

Final answer, the remove unnecessary parentheses

$$b + 2 + y \cdot (b + 2)$$

1.1 Perform the following textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

(d) $(x + x \cdot 2)[x := x \cdot y]$

$$(x + x \cdot 2)[x := x \cdot y]$$

$$(x + x \cdot 2)[x := x \cdot y]$$

Step 1: $(x \cdot y) + ((x \cdot y) \cdot 2)$

Final answer, the remove unnecessary parentheses

$$x \cdot y + x \cdot y \cdot 2$$

1.2 Perform the following simultaneous textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

(b) $x + y \cdot x[x, y := b + 2, x + 2]$

$$x + y \cdot x[x, y := b + 2, x + 2]$$

Consider $E = x$, and $F = y$, we then have:

$$E + F \cdot E[x, y := b + 2, x + 2]$$

Since textual substitution has higher precedence than any operator, apply it only on E.

Step 1: $x + y \cdot (b + 2)$

Final answer, the remove unnecessary parentheses

$$x + y \cdot (b + 2)$$

1.2 Perform the following simultaneous textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

(c) $(x + y \cdot x)[x, y := b + 2, x + 2]$

$$(x + y \cdot x)[x, y := b + 2, x + 2]$$

Step 1: $\big((b+2) + (x+2)\cdot(b+2)\big)$

Final answer, the remove unnecessary parentheses

$$b + 2 + (x + 2) \cdot (b + 2)$$

1.2 Perform the following simultaneous textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

(d) $(x + x \cdot 2)[x, y := x \cdot y, x \cdot y]$

$$(x + x \cdot 2)[x, y := x \cdot y, x \cdot y]$$

Step 1:$((x \cdot y) + x \cdot y \cdot 2)$

Final answer, the remove unnecessary parentheses

$$x \cdot y + x \cdot y \cdot 2$$

1.2 Perform the following simultaneous textual substitutions. Be careful with parenthesization and remove unnecessary parentheses.

(e) $(x + y \cdot 2)[y, x := x \cdot y, x \cdot x]$

$$(x + y \cdot 2)[y, x := x \cdot y, x \cdot x]$$

Step 1:$((x \cdot x) + (x \cdot y) \cdot 2)$

Final answer, the remove unnecessary parentheses

$$x \cdot x + x \cdot y \cdot 2$$

Textual substitution

1.3 Perform the following textual substitutions, Be careful with parenthesization and remove unnecessary parentheses.

d)

$$(x + x \cdot 2)[x, y := y, x][x := z]$$

$$(x + x \cdot 2)[x, y := y, x][x := z]$$

$$(x + x \cdot 2)[x, y \coloneqq y, x][x \coloneqq z]$$

Step 1: $(y + y \cdot 2)\ [x \coloneqq z]$

Step 2: $(y + y \cdot 2)$

Final answer, the remove unnecessary parentheses

$$y + y \cdot 2$$

Textual substitution

1.3 Perform the following textual substitutions, Be careful with parenthesization and remove unnecessary parentheses.

e)

$$(x + x \cdot 2)[x, y := x, z][x := y]$$

$$(x + x \cdot 2)[x, y := x, z][x := y]$$

$$(x + x \cdot 2)[x, y \coloneqq x, z][x \coloneqq y]$$

Step 1: $(x + x \cdot 2)[x \coloneqq y]$

Step 2: $(y + y \cdot 2)$

Final answer, the remove unnecessary parentheses

$$y + y \cdot 2$$

Textual substitution

1.3 Perform the following textual substitutions, Be careful with parenthesization and remove unnecessary parentheses.

f)

$$(x + x \cdot y + x \cdot y \cdot z)[x, y := y, x][y := 2 \cdot \mathrm{y}]$$

$$(x + x \cdot y + x \cdot y \cdot z)[x, y := y, x][y := 2 \cdot \mathrm{y}]$$

$$(x + x \cdot y + x \cdot y \cdot z)[x, y \coloneqq y, x][y \coloneqq 2 \ \cdot y]$$

Step 1: $(y + y \cdot x + y \cdot x \cdot z)[y \coloneqq 2 \ \cdot y]$

Step 2: $((2 \cdot y) + (2 \cdot y) \cdot x + (2 \cdot y) \cdot x \cdot z)$

Final answer, the remove unnecessary parentheses

$$2 \cdot y + 2 \cdot y \cdot x + 2 \cdot y \cdot x \cdot z$$

Do not assume that multiplication or addition is commutative.

Reasoning with the Leibniz rule

1.5 Let X , Y , and Z be expressions and z a variable. Let E be an expression, which may or may not contain Z . Here is another version of Leibniz.

$$\textbf{Leibniz:}\quad \frac{Z = X,\ Z = Y}{E[z := X] = E[z := Y]} \quad .$$

Show that transitivity of $=$ follows from this definition.

For 1.5 We need to understand the next 3 slides from the textbook
Pages 11 to 13.

Leibniz's rule

(1.1) **Substitution:** $$\frac{E}{E[v := F]}$$

This rule asserts that if E is a theorem, then so is E with all occurrences of the variables of v replaced by the corresponding expressions of F. For example, if $x + y = y + x$ (this is E) is a theorem, Substitution allows us to conclude that $b + 3 = 3 + b$ (this is $E[x, y := b, 3]$) is also a theorem.

Here is another example. Suppose the expression $2 \cdot x/2 = x$ is a theorem. By Substitution (1.1), we can conclude that $(2 \cdot x/2 = x)[x := j]$, i.e. $2 \cdot j/2 = j$, is also a theorem.

It should be noted that an inference rule like Substitution (1.1) is really a scheme that represents an infinite set of rules —one rule for each combination of an expression E, list of variables v, and list of expressions F. For example, we can instantiate E, v, and F of Substitution (1.1) with $2 \cdot x/2 = x$, x, and $j + 5$, respectively, to obtain the inference rule

$$\frac{2 \cdot x/2 = x}{(2 \cdot x/2 = x)[x := j + 5]} \quad \text{or} \quad \frac{2 \cdot x/2 = x}{2 \cdot (j + 5)/2 = j + 5} \quad .$$

(1.2) **Reflexivity:** $x = x$

(1.3) **Symmetry** [4] **:** $(x = y) = (y = x)$

The third law for equality, *transitivity*, is given as an inference rule.

(1.4) **Transitivity:** $\dfrac{X = Y,\ Y = Z}{X = Z}$

We read this inference rule as: from $X = Y$ and $Y = Z$, conclude $X = Z$. For example, from $x+y = w+1$ and $w+1 = 7$ we conclude, by Transitivity (1.4), $x + y = 7$. As another example, on page 4, we gave a proof that $(e = m \cdot c^2) = (e/c^2 = m)$. It is Transitivity that allows us to conclude that the first expression $e = m \cdot c^2$ equals the third, then equals the fourth, and finally equals the fifth expression, $e/c^2 = m$.

A fourth law of equality was articulated by Gottfried Wilhelm Leibniz, some 350 years ago (see Historical Note 1.2). In modern terminology, we paraphrase Leibniz's rule as follows.

Two expressions are equal in all states iff replacing one by the other in any expression E does not change the value of E (in any state).

A consequence of this law can be formalized as an inference rule (see also Exercise 1.4):

(1.5) **Leibniz:** $$\frac{X = Y}{E[z := X] = E[z := Y]}$$

Variable z is used in the conclusion of (1.5) because textual substitution is defined for the replacement of a variable but not for the replacement of an expression. In one copy of E, z is replaced by X, and in the other copy, it is replaced by Y. Effectively, this use of variable z allows replacement of an instance of X in $E[z := X]$ by Y.

Here is an example of the use of Leibniz (1.5). Assume that $b+3=c+5$ is a theorem. We can conclude that $d+b+3=d+c+5$ is a theorem, by choosing X, Y, and E of Leibniz, as follows.

1.5 Let X, Y, and Z be expressions and z a variable. Let E be an expression, which may or may not contain Z. Here is another version of Leibniz.

$$\textbf{Leibniz:} \quad \frac{Z = X, \ Z = Y}{E[z := X] = E[z := Y]} \ .$$

We can conclude $X = Y$

Show that transitivity of $=$ follows from this definition.

Textbook solution:

In the modified inference rule, choose E to be z itself.

Then the expression below the line is

$$X = Y .$$

Thus, from $Z = X$ and $Z = Y$

we can conclude that $X = Y$.

Inference rules

Exercise 1.6

1.6 Inference rule Substitution (1.1) stands for an infinite number of inference rules, each of which is constructed by instantiating expression E, list of variables v, and list of expressions F with different expressions and variables. Show three different instantiations of the inference rule, where E is $x < y \vee x \geq y$.

or

We need to come up with three instantiations (examples) of the interference rule.

Inference rules

Exercise 1.6

1.6 Inference rule Substitution (1.1) stands for an infinite number of inference rules, each of which is constructed by instantiating expression E, list of variables v, and list of expressions F with different expressions and variables. Show three different instantiations of the inference rule, where E is $x < y \vee x \geq y$.

Simply substitute different values for x and y.

x:=a+b
y:=5

$$\frac{x < y \vee x \geq y}{a+b < 5 \vee a+b \geq 5}$$

x:=1
y:=2

$$\frac{x < y \vee x \geq y}{1 < 2 \vee 1 \geq 2}$$

x:=16-y
y:=x

$$\frac{x < y \vee x \geq y}{16-y < x \vee 16-y \geq x}$$

In $\vee$, **as long as one part is true ,** ***it hold.***

For 1.7 we need to understand the next slide

(1.5) **Leibniz:** $$\frac{X = Y}{E[z := X] = E[z := Y]}$$

Variable z is used in the conclusion of (1.5) because textual substitution is defined for the replacement of a variable but not for the replacement of an expression. In one copy of E, z is replaced by X, and in the other copy, it is replaced by Y. Effectively, this use of variable z allows replacement of an instance of X in $E[z := X]$ by Y.

Here is an example of the use of Leibniz (1.5). Assume that $b+3 = c+5$ is a theorem. We can conclude that $d+b+3 = d+c+5$ is a theorem, by choosing X, Y, and E of Leibniz, as follows.

$$\begin{array}{llll} X: & b+3 & \quad E: & d+z \\ Y: & c+5 & \quad z: & z \end{array}$$

1.7 Inference rule Leibniz (1.5) stands for an infinite number of inference rules, each of which is constructed by instantiating E, X, and Y with different expressions. Below, are a number of instantiations of Leibniz, with parts missing. Fill in the missing parts and write down what expression E is. Do not simplify.

Given:

(b) $$\frac{2 \cdot y + 1 = 5}{x + (2 \cdot y + 1) \cdot w = ?}$$

$$E = x + z.w$$

$$? = E[z := 5]$$

$$\boldsymbol{x + 5 \cdot w}$$

Final solution:

$$\frac{2 \cdot y + 1 = 5}{x + (2 \cdot y + 1) \cdot w = x + 5 \cdot w} \qquad (\text{using } E = x + z \cdot w)$$

1.7 Inference rule Leibniz (1.5) stands for an infinite number of inference rules, each of which is constructed by instantiating E, X, and Y with different expressions. Below, are a number of instantiations of Leibniz, with parts missing. Fill in the missing parts and write down what expression E is. Do not simplify.

Given:

(c) $$\frac{x+1=y}{3\cdot(x+1)+3\cdot x+1=\,?}$$

$$E = 3\cdot z + 3\cdot x + 1$$

$$? = E = [z := y]$$

$$3\cdot y + 3\cdot x + 1$$

Final solution:

$$\frac{x+1=y}{3\cdot(x+1)+3\cdot x+1=3\cdot y+3\cdot x+1}$$ (using $E = 3\cdot z+3\cdot x+1$)

Inference rules

1.8 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the expressions $E[z := X]$ and hints $X = Y$ below, write the resulting expression $E[z := Y]$. There may be more than one correct answer.

$E[z := X]$	hint $X = Y$	E	E[z:=Y]
$x + y + w$	$x = b + c$		
$x + y + w$	$b \cdot c = y + w$		
$x \cdot (x + y)$	$x + y = y + x$		

1.8 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the expressions $E[z := X]$ and hints $X = Y$ below, write the resulting expression $E[z := Y]$. There may be more than one correct answer.

$E[z := X]$	hint $\boxed{X} = \boxed{Y}$	E	E[z:=Y]
$x + y + w$	$\boxed{x} = \boxed{b + c}$	$z + y + w$	$b + c + y + w$

Explanation:

$$X = x$$
$$Y = b + c$$
$$E[z := X] = x + y + w$$
$$= X + y + w$$

Then

$$E = z + y + w$$

Then

$$E[z := Y] = Y + y + w$$
$$= b + c + y + w$$

Inference rules

1.8 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the expressions $E[z := X]$ and hints $X = Y$ below, write the resulting expression $E[z := Y]$. There may be more than one correct answer.

$E[z := X]$	hint $X = Y$	E	E[z:=Y]
$x \cdot (x + y)$	$x + y = y + x$	$x \cdot z$	$x \cdot (y + x)$

$$X = x + y$$
$$Y = y + x$$
$$E[z \coloneqq X] = x \cdot (x + y)$$
$$= x \cdot (X)$$

$$E = x \cdot (z)$$
$$E[z \coloneqq Y] = x \cdot Y$$
$$E[z \coloneqq Y] = x \cdot (y + x)$$

Inference rules

1.8 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the expressions $E[z := X]$ and hints $X = Y$ below, write the resulting expression $E[z := Y]$. There may be more than one correct answer.

	$E[z := X]$	hint $X = Y$	E	E[z:=Y]
	$x + y + w$	$x = b + c$	$z + y + w$	$b + c + y + w$
1.8 b)	$x + y + w$	$b \cdot c = y + w$	$x + y + w$	$x + b \cdot c$
	$x \cdot (x + y)$	$x + y = y + x$		

1.8 b) The textbook solution is incorrect.

Inference rules

1.9 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the following pair of expressions $E[z := X]$ and $E[z := Y]$, identify a hint $X = Y$ that would show them to be equal and indicate what E is.

$E[z := X]$	$E[z := Y]$	E	Hint X=Y
$x + y + w + x$	$x + y \cdot w + x$		
$x \cdot y \cdot x$	$(y + w) \cdot y \cdot x$		
$x \cdot y \cdot x$	$y \cdot x \cdot x$		

Inference rules

1.9 The purpose of this exercise is to reinforce your understanding of the use of Leibniz (1.5) along with a hint in proving two expressions equal. For each of the following pair of expressions $E[z := X]$ and $E[z := Y]$, identify a hint $X = Y$ that would show them to be equal and indicate what E is.

$E[z := X]$	$E[z := Y]$	E	Hint X=Y
$x + y + w + x$	$x + y \cdot w + x$	$x + z + x$	$y + w = y \cdot w$
$x \cdot y \cdot x$	$(y + w) \cdot y \cdot x$	$z \cdot y \cdot x$	$x = y + w$
$x \cdot y \cdot x$	$y \cdot x \cdot x$	$z \cdot x$	$x \cdot y = y \cdot x$

Inference rules

1.10 In Sec. 1.3, we stated that the four laws Reflexivity (1.2), Symmetry (1.3), Transitivity (1.4), and Leibniz (1.5) characterized equality. This statement is almost true. View $=$ as a function $eq(x, y)$ that yields a value *true* or *false* . There is one other function that, if used in place of eq in the four laws, satisfies all of them. What is it?

1.10 In Sec. 1.3, we stated that the four laws Reflexivity (1.2), Symmetry (1.3), Transitivity (1.4), and Leibniz (1.5) characterized equality. This statement is almost true. View $=$ as a function $eq(x, y)$ that yields a value *true* or *false*. There is one other function that, if used in place of eq in the four laws, satisfies all of them. What is it?

Textbook solution:

- Let $f(x, y) = true$,
- No matter what x and y are,
- Using f for equality, the laws of **reflexivity**, $f(x, x)$, and **symmetry**, $f(f(x, y), f(y, x))$, hold.
- Further, the conclusions of inference rules Transitivity and Leibniz hold,
- Since they are just function applications
- $f(X, Z)$ and $f(E[z := X]), E[z := Y]$, which are **always true**.

Another solution:

The function $f(x, y) = true$, for all x and y satisfies all four laws because f is always true.

Hoare Triples (Assignment statement)

1.11 Using Definition (1.12) of the assignment statement on page 18, determine preconditions for the following statements and postconditions.

Statement	Postcondition R	Precondition R[x:=E]
$x := x - 1$	$x^2 + 2 \cdot x = 3$	
$x := x - 1$	$(x+1) \cdot (x-1) = 0$	
$y := x + y$	$y = x$	

(1.12) **Definition of assignment:** $\{R[x := E]\}\ \ x := E\ \ \{R\}$

As an example, consider the assignment $x := x + 1$ and postcondition $x > 4$. Thus, in definition (1.12) we would take E to be $x + 1$ and R to be $x > 4$. We conclude that a precondition for a valid triple is $(x > 4)[x := x + 1]$, which is $x + 1 > 4$.

Here are more examples of the use of definition (1.12).

$$\begin{array}{rll} \{x+1>5\} & x := x+1 & \{x>5\} \\ \{5 \neq 5\} & x := 5 & \{x \neq 5\} \\ \{x^2 > x^2 \cdot y\} & x := x^2 & \{x > x \cdot y\} \end{array}$$

1.11 Using Definition (1.12) of the assignment statement on page 18, determine preconditions for the following statements and postconditions.

	Statement	Postcondition	Precondition
(b)	$x := x - 1$	$x^2 + 2 \cdot x = 3$	$(x-1)^2 + 2 \cdot (x-1) = 3$

*Assuming **distributivity** of the product and **commutativity** (or **symmetry**), this can be simplified to* $x^2 = 4$

In the next slide we will explain ***distributivity, commutativity, symmetry.***

Distributive law, also called distributive property, in mathematics, the law relating the operations of multiplication and addition, stated symbolically as

$$a(b + c) = ab + ac;$$

that is, the monomial factor a is distributed, or separately applied, to each term of the binomial factor $b + c$, resulting in the product $ab + ac$.

In mathematics, a binary operation is **commutative** if changing the order of the operands does not change the result. It is a fundamental property of many binary operations, and many mathematical proofs depend on it. Most familiar as the name of the property that says something like

$$3 + 4 = 4 + 3$$

or

$$2 \times 5 = 5 \times 2,$$

the property can also be used in more advanced settings. The name is needed because there are operations, such as division and subtraction, that do not have it.
For example

$$3 - 5 \neq 5 - 3$$

Symmetry

$$(x = y) = (y = x)$$

1.11 Using Definition (1.12) of the assignment statement on page 18, determine preconditions for the following statements and postconditions.

	Statement	Postcondition	Precondition
(c)	$x := x - 1$	$(x + 1) \cdot (x - 1) = 0$	$(x - 1 + 1) \cdot (x - 1 - 1) = 0$ Simplify to $x \cdot (x - 2) = 0$

1.11 Using Definition (1.12) of the assignment statement on page 18, determine preconditions for the following statements and postconditions.

	Statement	Postcondition	Precondition
(d)	$y := x + y$	$y = x$	$x + y = x$ Simplify to $y = 0$

Using $x + y = x$ as the hint of the Leibniz rule, consider

$$E = -x + z.$$

This rule is from Leibniz rule

By Leibniz, we have $E[z := x + y] = E[z := x]$,

Therefore:

$$-x + x + y = -x + x$$

We add $-x$ to the both side of the equation

Thus:

$$0 + y = 0$$

Finally:

$$y = 0$$

Any Questions?