CS 2LC3
Logical Reasoning for Computer Science

Tutorial 3

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Outline

- Announcements / Reminders
- Table of Precedence for Boolean expressions
- Theorems and Axioms from Chapter 3.
- Implication

Announcements

- Hint File on the course website.
- All the slides for the tutorials will be posted on the course website.
- We are trying to extend the deadline for the assignment to October 13th, however this is not confirmed yet.

Before we can answer exercise 2.1, we nee to understand the following table from the textbook.

$(n \Rightarrow p)$ is true

$n = f$ and $p = t$

			$\vee$	$\Leftarrow$		$\Rightarrow$		$\equiv$ $=$	$\wedge$	nand	$\not\equiv$ $\neq$					nor	
t	t	t	t	t	t	t	t	t	t	f	f	f	f	f	f	f	f
t	f	t	t	t	t	f	f	f	f	t	t	t	t	f	f	f	f
f	t	t	t	f	f	t	t	f	f	t	t	f	f	t	t	f	f
f	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f	t	f

Note: *true* is abbreviated by t and *false* by f

The above table is from Chapter 2, page 26 of textbook.

Table of Precedences

(a)	$[x := e]$ (textual substitution)	(highest precedence)
(b)	$.$ (function application)	
(c)	unary prefix operators: $+ \quad - \quad \neg \quad \# \quad \sim \quad \mathcal{P}$	
(d)	$**$	
(e)	$\cdot \quad / \quad \div \quad$ **mod** **gcd**	
(f)	$+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$	
(g)	$\downarrow \quad \uparrow$	
(h)	$\#$	
(i)	$\triangleleft \quad \triangleright \quad \hat{}$	
(j)	$= \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad \mid$	(conjunctional, see page 29)
(k)	$\vee \quad \wedge$	
(l)	$\Rightarrow \quad \Leftarrow$	
(m)	$\equiv$	(lowest precedence)

All nonassociative binary infix operators associate to the left, except $**$, $\triangleleft$, and $\Rightarrow$, which associate to the right.

The operators on lines (j), (l), and (m) may have a slash / through them to denote negation —e.g. $b \not\equiv c$ is an abbreviation for $\neg(b \equiv c)$.

Page 2 of textbook.

In Arithmetic, multiplication and division has higher precedence than addition and negation.

$$13 + 4 \times 2 = ?$$

What would be the answer 34 or 21

$$13 + 8 =$$
$$= 21$$

$$(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Longrightarrow b)$$

Lets assume b is false

c is true

First we would start we the parenthesis

$$(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Longrightarrow b)$$

$$(false) \equiv (true) \wedge (false)$$

We which operator, do we start first?

?

Based on the table of precedence ∧ has higher Precedences than ≡

$$(false) \equiv (true) \wedge (false)$$

$$(false) \equiv (false)$$

$$True$$

2.2 Write truth tables to compute values for the following expressions in all states.

(h) $(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$

The final solution for 2.2h) should look like this if you answer it correctly.
But you must fill the entire table for receive the mark for this part.

If you the table of precedence ∧ has higher precedence than ≡

(h)

b	c	$b \equiv c$	$b \Rightarrow c$	$c \Rightarrow b$	$(b \Rightarrow c) \wedge (c \Rightarrow b)$	$(b \equiv c) \equiv (b \Rightarrow c) \wedge (c \Rightarrow b)$
t	t					t
t	f					t
f	t					t
f	f					t

3.1 Preliminaries

A *calculus* is a method or process of reasoning by calculation with symbols.[1] This chapter presents a *propositional* calculus. It is so named because it is a method of calculating with boolean expressions that involve propositional variables (see page 33). We call our propositional calculus *equational logic* **E**.

One part of **E** is a set of *axioms*, which are certain boolean expressions that define basic manipulative properties of boolean operators. As an example, for operator $\vee$, the axiom $p \vee q \equiv q \vee p$ indicates that $\vee$ is symmetric in its two operands, i.e. the value of a disjunction is unchanged if its operands are swapped.

The other part of our propositional calculus consists of three inference rules: Leibniz (1.5), Transitivity (1.4), and Substitution (1.1). We repeat them here, as a reminder, formulated in terms of identifiers that will typically be used in this chapter: $P, Q, R, \ldots$ for arbitrary boolean expressions and $p, q, r, \ldots$ for boolean variables.

Leibniz: $$\frac{P = Q}{E[r := P] = E[r := Q]}$$

Transitivity: $$\frac{P = Q,\ Q = R}{P = R}$$

Substitution: $$\frac{P}{P[r := Q]}$$

Preliminaries Of Propositional Calculus.

What is Propositional Calculus (equational logic E)?

Propositional calculus is method of calculating with Boolean expressions that involve propositional variables.

What are **axioms**?

In propositional logic**,** a set of ***axioms***, are certain Boolean expressions that define basic manipulative properties of Boolean operators.

Example,

For operator $\vee$,

The axiom $p \vee q \equiv q \vee p$ indicates that $\vee$ is **symmetric** in its two operands, i.e. the value of a disjunction is unchanged if its operands are swapped.

In other words, An **axiom**, or assumption is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments.

What is a theorem in propositional calculus?

Propositional logic may be studied through a formal system in which formulas of a formal language may be interpreted to represent propositions.
A system of axioms and inference rules allows **certain formulas** to be **derived**.
These derived formulas are called theorems and may be interpreted to be true propositions.
A constructed sequence of such formulas is known as a **derivation** or **proof** and the **last formula** of the sequence is the theorem. The derivation may be interpreted as proof of the proposition represented by the theorem.

What are theorems of propositional calculus?

A theorem of our propositional calculus is either

(i) An axiom,
(ii) The conclusion of an inference rule whose premises are theorems, or
(iii) A Boolean expression that, using the inference rules, is proved equal to an axiom or a previously proved theorem.

All Theorems of the propositional calculus are posted on the last pages of the textbook.

Theorems of the Propositional Calculus

Equivalence and true

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and false

(3.8) **Axiom, Definition of $false$:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \equiv \neg p \equiv q$

(3.10) **Axiom, Definition of $\not\equiv$:** $(p \not\equiv q) \equiv \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of $false$:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \equiv (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \equiv p \equiv q \not\equiv r$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction

(3.35) **Axiom, Golden rule:** $p \wedge q \equiv p \equiv q \equiv p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) **Definition of $\equiv$:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

Implication

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) **Definition of implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Definition of implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv true$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

The formats for the proofs

The Format for writing proofs was covered in chapter 1.5.

Equivalence and true

Equivalence is associative. This property is formalized as a manipulative property by the following axiom.

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \;\equiv\; (p \equiv (q \equiv r))$

Associativity allows us to be informal and insert or delete pairs of parentheses in sequences of equivalences, just as we do with sequences of **additions**

$$w + x + y + z$$

is equivalent to

$$w + (x + y) + z$$

Hence, we can write

$$p \equiv q \equiv r$$

instead of

$$p \equiv (q \equiv r)$$

$$(p \equiv q) \equiv r$$

Keeping axiom (3.1) in mind, we express the second axiom, **symmetry**, without parentheses.

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

this axiom is called *symmetry* by imagining parentheses as follows:

$$(p \equiv q) \equiv (q \equiv p)$$

We now give our first proof, of the following theorem:

$$p \equiv p \ \equiv \ q \equiv q$$

Remember that the axiom of associativity allows us to parenthesize an expression such as (3.2) in several ways.

In the following proof, we parenthesize (3.2) as

$$(p \equiv q \equiv q) \equiv p,$$

so that, using Leibniz, we can replace

$p \equiv q \equiv q$ in an expression by p.

$$p \ \equiv \ p \equiv q \equiv q$$

$=$ ⟨Symmetry of $\equiv$ (3.2) —replace $p \equiv q \equiv q$ by p⟩

$p \ \equiv \ p$

$=$ ⟨Symmetry of $\equiv$ (3.2) —replace first p by $p \equiv q \equiv q$⟩

$p \equiv q \equiv q \ \equiv \ p$

Since the final expression is axiom (3.2), and since, by the definition of theorem on page 42.

Any expression that is proved to equal to an axiom is a **theorem**.

The **first expression** has been proved to be a **theorem**.

Equivalence

Axiom "Definition of ≡": $(p \equiv q) = (p = q)$
Axiom (3.1) "Associativity of ≡": $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$
Axiom (3.2) "Symmetry of ≡": $(p \equiv q) \equiv (q \equiv p)$
Axiom (3.3) "Identity of ≡": $true \equiv (q \equiv q)$
Theorem (3.4): $true$
Theorem (3.5) "Reflexivity of ≡": $p \equiv p$

Negation and Inequivalence

Axiom (3.8) "Definition of `false`": $false \equiv \neg\, true$
Axiom (3.9) "Commutativity of ¬ with ≡" "Distributivity of ¬ over ≡": $\neg (p \equiv q) \equiv (\neg p \equiv q)$
Axiom (3.10) "Definition of ≢": $(p \not\equiv q) \equiv \neg (p \equiv q)$
Theorem (3.11) "¬ connection": $\neg p \equiv (q \equiv (p \equiv \neg q))$
Theorem (3.12) "Double negation": $\neg (\neg p) \equiv p$
Theorem (3.13) "Negation of `false`": $\neg\, false \equiv true$

(3.3) Axiom, Identity of $\equiv$: $true \equiv q \equiv q$

The constant symbol

true is an abbreviation for $\boldsymbol{q \equiv q}$

using a constant symbol is reasonable because

the value of $\boldsymbol{q \equiv q}$ does not depend on **the value of** $\boldsymbol{q}$.

(3.3) **Axiom, Identity of** $\equiv$: $true \equiv q \equiv q$

We call *true* the *identity* of $\equiv$ because, as can be seen from the axiom of symmetry and (3.3),

$$q \equiv q$$

Lets represent $q \equiv q$ by p

and

$$p = (true \equiv p)$$

$$(true \equiv p) = p$$

$$(p \equiv true) = p\,.$$

(3.4) $\quad true$

To show that $true$ is a theorem, we show that it equivales axiom (3.3):

$$
\begin{array}{ll}
 & true \\
= & \langle \text{Identity of } \equiv \text{ (3.3), with } q := true \,\rangle \\
 & true \equiv \boxed{true} \\
= & \langle \text{Identity of } \equiv \text{ (3.3) —replace the second } true \,\rangle \\
 & true \equiv q \equiv q \qquad \text{—Identity of } \equiv \text{ (3.3)}
\end{array}
$$

Axioms Identity (3.3) and Symmetry (3.2) imply that occurrences of "$\equiv true$" (or "$true \equiv$") in an expression are redundant.

Thus, $Q \cancel{\equiv true}$

may be replaced by Q

in any expression without changing the value of the expression.

Therefore, we usually eliminate such occurrences unless something (e.g. symmetry) encourages us to leave them in.

(3.5) Reflexivity of $\equiv$: $p \equiv p$

$$
\begin{array}{ll}
 & p \equiv p \\
= & \langle \text{Identity of } \equiv \ (3.3), \text{ with } q := p \rangle \\
 & true
\end{array}
$$

The Reflexive Property of Equality states *that a value is equal to itself*

3.3 Negation, inequivalence, and false

We introduce three axioms. The first defines *false*; the first and second together define negation, $\neg$; and the third defines inequivalence, $\not\equiv$.

(3.8) **Axiom, Definition of *false*:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \;\equiv\; \neg p \equiv q$

(3.10) **Axiom, Definition of $\not\equiv$:** $(p \not\equiv q) \;\equiv\; \neg(p \equiv q)$

Theorems relating $\equiv$, $\not\equiv$, $\neg$, and *false*

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of** *false* **:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \ \equiv \ \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of** $\not\equiv$ **:** $(p \not\equiv q) \ \equiv \ (q \not\equiv p)$

(3.17) **Associativity of** $\not\equiv$ **:** $((p \not\equiv q) \not\equiv r) \ \equiv \ (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \ \equiv \ (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \ \ \equiv \ \ p \equiv q \not\equiv r$

3.4 Disjunction

The disjunction operator $\vee$ is defined by the following five axioms.

(3.24) **Axiom, Symmetry of** $\vee$**:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of** $\vee$**:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency**[7] **of** $\vee$**:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of** $\vee$ **over** $\equiv$**:**

$$p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

Distributivity (3.27) can be viewed in two ways, much like distributivity of $\cdot$ over $+$. Replacing the LHS of (3.27) by the RHS could be called "multiplying out"; replacing the RHS by the LHS, "factoring".

Theorems concerning $\vee$

(3.29) **Zero[8] of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Basic properties of $\wedge$

(3.36) **Symmetry of** $\wedge$**:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of** $\wedge$**:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of** $\wedge$**:** $p \wedge p \equiv p$

(3.39) **Identity of** $\wedge$**:** $p \wedge true \equiv p$

(3.40) **Zero of** $\wedge$**:** $p \wedge false \equiv false$

(3.41) **Distributivity of** $\wedge$ **over** $\wedge$**:**

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

Theorems relating $\wedge$ and $\vee$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$

(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:**

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

(3.46) **Distributivity of $\wedge$ over $\vee$:**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Implication

We now define and investigate two final operators, implication $\Rightarrow$ and consequence $\Leftarrow$.

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

Because of the similarity of $\Rightarrow$ and $\Leftarrow$, we give only theorems that involve $\Rightarrow$; corresponding ones for $\Leftarrow$ follow immediately from (3.58).

The first thing to note about implication is that it can be written in many ways. Besides the next three theorems, other ways of rewriting implication are given in Exercises 3.44–3.46. Theorem (3.59) or (3.60) is sometimes used as the definition of implication.

Implication

Rewriting implication

(3.59) **Definition of Implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Definition of Implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

Proof of theorem (3.59), $p \Rightarrow q \equiv \neg p \vee q$.

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv \boxed{p \vee q \equiv q}$

$$\begin{array}{ll} & p \Rightarrow q \\ = & \quad \langle \text{Definition of implication } (3.57) \rangle \\ & \boxed{p \vee q \equiv q} \end{array}$$

(3.32) $p \vee q \equiv \boxed{p \vee \neg q} \equiv p$

$$\begin{array}{ll} = & \quad \langle (3.32),\ p \vee q \equiv p \vee \neg q \equiv p \text{, with } p, q := q, p \rangle \\ & \boxed{\neg p \vee q} \end{array}$$

Proof of the law of the Contrapositive (3.61), $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.59) Definition of implication: $p \Rightarrow q \equiv \boxed{\neg p \vee q}$

$$
\begin{array}{ll}
 & \neg q \Rightarrow \neg p \\
= & \langle \text{Implication (3.59)} \rangle \\
 & \boxed{\neg\neg q \vee \neg p}
\end{array}
$$

(3.12) Double negation: $\boxed{\neg\neg p \equiv p}$

$$
\begin{array}{ll}
= & \langle \text{Double negation (3.12)} \rangle \\
 & \boxed{q} \vee \neg p
\end{array}
$$

(3.59) Definition of implication: $\boxed{p \Rightarrow q} \equiv \neg p \vee q$

$$
\begin{array}{ll}
= & \langle \text{Implication (3.59)} \rangle \\
 & \boxed{p \Rightarrow q}
\end{array}
$$

Proof of theorem (3.60), $p \Rightarrow q \equiv p \wedge q \equiv p$

$$
\begin{array}{ll}
& p \Rightarrow q \\
= & \langle \text{Definition of implication } (3.57) \rangle \\
& \boxed{p \vee q \equiv q} \\
= & \langle \text{Golden rule } (3.35) \rangle \\
& \boxed{p \wedge q \equiv p}
\end{array}
$$

(3.57) Axiom, Definition of Implication: $p \Rightarrow q \equiv \boxed{p \vee q \equiv q}$

(3.35) Axiom, Golden rule : $\boxed{p \wedge q \equiv p} \equiv q \equiv p \vee q$

(3.59) Proof $p \implies q \equiv \neg p \equiv q$:

$$
\begin{aligned}
& p \implies q \\
= & \text{<Use axiom (3.57) >} \\
& p \vee q \equiv q \\
= & \text{<Use theorem (3.32) >} \\
& \boxed{\neg p \vee q}
\end{aligned}
$$

(3.57) Axiom, Definition of Implication: $p \Rightarrow q \equiv \boxed{p \vee q \equiv q}$

(3.32) $p \vee q \equiv \boxed{p \vee \neg q} \equiv p$

(3.60) Proof $p \Rightarrow q \equiv p \wedge q \equiv p$:

$p \Longrightarrow q$

$=\langle$Use axiom (3.57)$\rangle$

$\boxed{p \vee q \equiv q}$

$=\langle$Use axiom (3.35)$\rangle$

$\boxed{p \wedge q} \equiv p$

(3.57) Axiom, Definition of Implication: $p \Rightarrow q \equiv \boxed{p \vee q \equiv q}$

(3.35) Axiom, Golden rule : $\boxed{p \wedge q} \equiv p \equiv q \equiv p \vee q$

(3.61) Proof $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$:

$p \Rightarrow q$

$=$< Use theorem (3.59) >

$\boxed{\neg p \vee q}$

$=$< Use $(\neg\neg q) \equiv q$ and symmetry of $\vee$ >

$\neg(\neg q) \vee \neg p$

$=$< Use theorem (3.59) >

$\neg q \Rightarrow \neg p$

(3.59) Definition of implication: $p \Rightarrow q \equiv \boxed{\neg p \vee q}$

(3.59) Definition of implication: $p \Rightarrow q \equiv \neg p \vee q$

Implication

(3.62) Proof $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$:

$p \Rightarrow (q \equiv r)$

$=<$Use theorem (3.60) $>$

$p \wedge (q \equiv r) \equiv p$

$=<$Use distributivity of $\wedge$ over $\equiv$ $>$

$p \wedge q \equiv p \wedge r \equiv p \equiv p$

$=<$Use identity $p \equiv p \equiv true$ twice $>$

$p \wedge q \equiv p \wedge r$

Proof of (3.63), $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv$

$p \Rightarrow q \equiv p \Rightarrow r$

$=$ $\langle$Implication (3.59), twice$\rangle$

$\neg p \vee q \equiv \neg p \vee r$

$=$ $\langle \vee$ distributes over $\equiv$ (3.27)$\rangle$

$\neg p \vee (q \equiv r)$

$=$ $\langle$Implication (3.59)$\rangle$

$p \Rightarrow (q \equiv r)$

Proof of theorem (3.64), $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.

$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

$=$ $\langle$Implication (3.59), twice$\rangle$

$\neg p \vee q \Rightarrow \neg p \vee r$

$=$ $\langle$Definition of implication (3.57)$\rangle$

$\neg p \vee q \vee \neg p \vee r \equiv \neg p \vee r$

$=$ $\langle$Idempotency of $\vee$ (3.26)$\rangle$

$\neg p \vee q \vee r \equiv \neg p \vee r$

$=$ $\langle$Distributivity of $\vee$ over $\equiv$ (3.27)$\rangle$

$\neg p \vee (q \vee r \equiv r)$

$=$ $\langle$Definition of implication (3.57)$\rangle$

$\neg p \vee (q \Rightarrow r)$

$=$ $\langle$Implication (3.59)$\rangle$

$p \Rightarrow (q \Rightarrow r)$

Implication

Proof of Shunting (3.65), $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$.

$$
\begin{array}{ll}
& p \wedge q \Rightarrow r \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& \neg(p \wedge q) \vee r \\
= & \quad \langle \text{De Morgan (3.47a)} \rangle \\
& \neg p \vee \neg q \vee r \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& \neg p \vee (q \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& p \Rightarrow (q \Rightarrow r)
\end{array}
$$

Proof of theorem (3.66), $p \wedge (p \Rightarrow q) \equiv p \wedge q$.

$$
\begin{array}{ll}
& p \wedge (p \Rightarrow q) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& p \wedge (\neg p \vee q) \\
= & \quad \langle \text{Absorption (3.44a)} \rangle \\
& p \wedge q
\end{array}
$$

Proof of theorem (3.67), $p \wedge (q \Rightarrow p) \equiv p$.

$$
\begin{array}{ll}
& p \wedge (q \Rightarrow p) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& p \wedge (\neg q \vee p) \\
= & \quad \langle \text{Absorption (3.43a)} \rangle \\
& p
\end{array}
$$

Proof of theorem (3.68), $p \vee (p \Rightarrow q) \equiv \mathit{true}$.

$$
\begin{array}{ll}
& p \vee (p \Rightarrow q) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
& p \vee \neg p \vee q \\
= & \quad \langle \text{Excluded middle (3.28)} \rangle \\
& \mathit{true} \vee q \\
= & \quad \langle \text{Zero of } \vee \text{ (3.29)} \rangle \\
& \mathit{true}
\end{array}
$$

Implication

Proof of theorem (3.69), $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$.

$$
\begin{array}{ll}
 & p \vee (q \Rightarrow p) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & p \vee \neg q \vee p \\
= & \quad \langle \text{Idempotency of } \vee \text{ (3.26)} \rangle \\
 & \neg q \vee p \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & q \Rightarrow p
\end{array}
$$

Proof of theorem (3.70), $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$.

$$
\begin{array}{ll}
 & p \vee q \Rightarrow p \wedge q \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & \neg(p \vee q) \vee (p \wedge q) \\
= & \quad \langle \text{De Morgan (3.47b)} \rangle \\
 & (\neg p \wedge \neg q) \vee (p \wedge q) \\
= & \quad \langle \text{Alternative definition of } \equiv \text{ (3.52)} \rangle \\
 & p \equiv q
\end{array}
$$

Implication

Proof of Reflexivity of $\Rightarrow$ (3.71), $p \Rightarrow p \equiv true$.

$$\begin{array}{ll} & p \Rightarrow p \\ = & \quad \langle \text{Implication (3.59)} \rangle \\ & \neg p \vee p \\ = & \quad \langle \text{Excluded middle (3.28)} \rangle \\ & true \end{array}$$

Proof of Left identity of $\Rightarrow$ (3.73), $true \Rightarrow p \equiv p$.

$$\begin{array}{ll} & true \Rightarrow p \equiv p \\ = & \quad \langle \text{Implication (3.57)} \rangle \\ & true \vee p \\ = & \quad \langle \text{Zero of } \vee \text{ (3.29)} \rangle \\ & true \end{array}$$

Proof of Right zero of $\Rightarrow$ (3.72), $p \Rightarrow true \equiv true$.

$$\begin{array}{ll} & p \Rightarrow true \\ = & \quad \langle \text{Implication (3.59)} \rangle \\ & \neg p \vee true \\ = & \quad \langle \text{Zero of } \vee \text{ (3.29)} \rangle \\ & true \end{array}$$

Proof of theorem (3.74), $p \Rightarrow false \equiv \neg p$.

$$\begin{array}{ll} & p \Rightarrow false \\ = & \quad \langle \text{Implication (3.59)} \rangle \\ & \neg p \vee false \\ = & \quad \langle \text{Identity of } \vee \text{ (3.30)} \rangle \\ & \neg p \end{array}$$

Implication

Proof of theorem (3.75), $false \Rightarrow p \equiv true$.

$$\begin{array}{ll} & false \Rightarrow p \\ = & \quad \langle \text{Implication (3.57)} \rangle \\ & false \vee p \equiv p \quad \text{—which is Identity of } \vee \text{ (3.30)} \end{array}$$

Proof of theorem (3.76a), $p \Rightarrow p \vee q$.

$$\begin{array}{ll} & p \Rightarrow p \vee q \\ = & \quad \langle \text{Implication (3.60), with } q := p \vee q) \rangle \\ & p \wedge (p \vee q) \equiv p \quad \text{—which is Absorption (3.43a)} \end{array}$$

Proof of theorem (3.76b), $p \wedge q \Rightarrow p$.

$$\begin{array}{ll} & p \wedge q \Rightarrow p \\ = & \quad \langle \text{Definition of implication (3.57), with } p,q := p \wedge q, p \rangle \\ & p \vee (p \wedge q) \equiv p \quad \text{—which is Absorption (3.43b)} \end{array}$$

Proof of theorem (3.76c), $p \wedge q \Rightarrow p \vee q$.

$$\begin{array}{ll} & p \wedge q \Rightarrow p \vee q \\ = & \quad \langle \text{Implication (3.60)} \rangle \\ & p \wedge q \wedge (p \vee q) \equiv p \wedge q \\ = & \quad \langle \text{Absorption (3.43a), with } p,q := q,p \rangle \\ & p \wedge q \equiv p \wedge q \quad \text{—which is Identity of } \equiv \text{ (3.3)} \end{array}$$

Proof of (3.76d), $p \vee (q \wedge r) \Rightarrow p \vee q$.

$$\begin{array}{ll} & p \vee (q \wedge r) \Rightarrow p \vee q \\ = & \quad \langle \text{Distributivity of } \vee \text{ over } \wedge \text{ (3.45)} \rangle \\ & (p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \quad \text{—which is Strengthening (3.76b)} \\ & \quad \text{with } p,q := p \vee q, p \vee r \end{array}$$

Proof of (3.76e), $p \wedge q \Rightarrow p \wedge (q \vee r)$.

$$\begin{array}{ll} & p \wedge q \Rightarrow p \wedge (q \vee r) \\ = & \quad \langle \text{Distributivity of } \wedge \text{ over } \vee \text{ (3.46)} \rangle \\ & p \wedge q \Rightarrow (p \wedge q) \vee (p \wedge r) \quad \text{—which is Strengthening (3.76a)} \\ & \quad \text{with } p,q := p \wedge q, p \wedge r \end{array}$$

Proof of Modus ponens (3.77), $p \wedge (p \Rightarrow q) \Rightarrow q$.

$$\begin{array}{ll} & p \wedge (p \Rightarrow q) \Rightarrow q \\ = & \quad \langle (3.66),\ p \wedge (p \Rightarrow q) \equiv p \wedge q \rangle \\ & p \wedge q \Rightarrow q \quad \text{—which is Strengthening (3.76b)} \end{array}$$

Implication

Proof of theorem (3.78), $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.

$$
\begin{array}{ll}
 & (p \Rightarrow r) \wedge (q \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
 & (\neg p \vee r) \wedge (\neg q \vee r) \\
= & \quad \langle \text{Distributivity of } \vee \text{ over } \wedge \text{ (3.45)} \rangle \\
 & (\neg p \wedge \neg q) \vee r \\
= & \quad \langle \text{De Morgan (3.47b)} \rangle \\
 & \neg(p \vee q) \vee r \\
= & \quad \langle \text{Implication (3.59)} \rangle
\end{array}
$$

Proof of theorem (3.79), $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$.

$$
\begin{array}{ll}
 & (p \Rightarrow r) \wedge (\neg p \Rightarrow r) \\
= & \quad \langle \text{Case analysis (3.78), with } q := \neg p \rangle \\
 & p \vee \neg p \Rightarrow r \\
= & \quad \langle \text{Excluded middle (3.28)} \rangle \\
 & true \Rightarrow r \\
= & \quad \langle \text{Left identity of } \Rightarrow \text{ (3.73)} \rangle \\
 & r
\end{array}
$$

Implication

Proof of Mutual implication (3.80), $(p \Rightarrow q) \land (q \Rightarrow p) \equiv p \equiv q$.

$$
\begin{array}{cl}
 & (p \Rightarrow q) \land (q \Rightarrow p) \\
= & \quad \langle \text{Implication (3.59), twice} \rangle \\
 & (\neg p \lor q) \land (\neg q \lor p) \\
= & \quad \langle \text{Golden rule (3.35)} \rangle \\
 & \neg p \lor q \lor \neg q \lor p \equiv \neg p \lor q \equiv \neg q \lor p \\
= & \quad \langle \text{Excluded middle (3.28); Zero of } \lor \text{ (3.29)} \rangle \\
 & true \equiv \neg p \lor q \equiv \neg q \lor p \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
 & \neg p \lor q \equiv \neg q \lor p \\
= & \quad \langle \text{(3.32), } p \lor q \equiv p \lor \neg q \equiv p \text{, twice} \rangle \\
 & p \lor q \equiv q \equiv q \lor p \equiv p \\
= & \quad \langle \text{Symmetry of } \equiv \text{ (3.2)} \rangle \\
 & p \equiv q
\end{array}
$$

Proof of Antisymmetry of $\Rightarrow$ (3.81), $(p \Rightarrow q) \land (q \Rightarrow p) \Rightarrow (p \equiv q)$.

$$
\begin{array}{cl}
 & (p \Rightarrow q) \land (q \Rightarrow p) \Rightarrow (p \equiv q) \\
= & \quad \langle \text{Mutual implication (3.80)} \rangle \\
 & (p \equiv q) \Rightarrow (p \equiv q) \\
= & \quad \langle \text{Reflexivity of } \Rightarrow \text{ (3.71)} \rangle \\
 & true
\end{array}
$$

Implication

Proof of Transitivity of $\Rightarrow$ (3.82a), $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

$$
\begin{array}{cl}
 & (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Implication (3.59), thrice} \rangle \\
 & (\neg p \vee q) \wedge (\neg q \vee r) \Rightarrow (\neg p \vee r) \\
= & \quad \langle \text{Implication (3.59)} \rangle \\
 & \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) \\
= & \quad \langle \text{De Morgan (3.47a) or (3.47b), three times} \rangle \\
 & (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee \neg p \vee r \\
= & \quad \langle \text{Double negation (3.12), twice, and regroup} \rangle \\
 & \neg p \vee (p \wedge \neg q) \vee (q \wedge \neg r) \vee r \\
= & \quad \langle \text{Distributivity of } \vee \text{ over } \wedge \text{ (3.45), twice} \rangle \\
 & ((\neg p \vee p) \wedge (\neg p \vee \neg q)) \vee ((q \vee r) \wedge (\neg r \vee r)) \\
= & \quad \langle \text{Excluded middle (3.28), twice} \rangle \\
 & (true \wedge (\neg p \vee \neg q)) \vee ((q \vee r) \wedge true) \\
= & \quad \langle \text{Identity of } \wedge \text{ (3.39), twice} \rangle \\
 & \neg p \vee \neg q \vee q \vee r \\
= & \quad \langle \text{Excluded middle (3.28)} \rangle \\
 & \neg p \vee true \vee r \\
= & \quad \langle \text{Zero of } \vee \text{ (3.29)} \rangle \\
 & true
\end{array}
$$

Implication

Proof of theorem (3.82b), $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

$$
\begin{array}{ll}
& (p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Mutual implication (3.80)} \rangle \\
& (p \Rightarrow q) \wedge (q \Rightarrow p) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
& (q \Rightarrow p) \Rightarrow ((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \\
= & \quad \langle \text{Transitivity of } \Rightarrow \text{ (3.82a), and any theorem equivales } \mathit{true} \rangle \\
& (q \Rightarrow p) \Rightarrow \mathit{true} \\
= & \quad \langle \text{Right zero of } \Rightarrow \text{ (3.72)} \rangle \\
& \mathit{true}
\end{array}
$$

Proof of theorem (3.82c), $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$.

$$
\begin{array}{ll}
& (p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Mutual implication (3.80)} \rangle \\
& (p \Rightarrow q) \wedge (q \Rightarrow r) \wedge (r \Rightarrow q) \Rightarrow (p \Rightarrow r) \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
& (r \Rightarrow q) \Rightarrow ((p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \\
= & \quad \langle \text{Transitivity of } \Rightarrow \text{ (3.82a), and any theorem equivales } \mathit{true} \rangle \\
& (r \Rightarrow q) \Rightarrow \mathit{true} \\
= & \quad \langle \text{Right zero of } \Rightarrow \text{ (3.72)} \rangle \\
& \mathit{true}
\end{array}
$$

Leibniz

(3.83) **Axiom, Leibniz:** $(e = f) \Rightarrow (E^z_e = E^z_f)$ $(E$ any expression$)$

Leibniz

Rules of substitution

(3.84) **Substitution:** (a) $(e = f) \land E_e^z \;\equiv\; (e = f) \land E_f^z$

(b) $(e = f) \Rightarrow E_e^z \;\equiv\; (e = f) \Rightarrow E_f^z$

(c) $q \land (e = f) \Rightarrow E_e^z \;\equiv\; q \land (e = f) \Rightarrow E_f^z$

Replacing variables by boolean constants

(3.85) **Replace by** *true* **:** (a) $p \Rightarrow E_p^z \;\equiv\; p \Rightarrow E_{true}^z$

(b) $q \land p \Rightarrow E_p^z \;\equiv\; q \land p \Rightarrow E_{true}^z$

(3.86) **Replace by** *false* **:** (a) $E_p^z \Rightarrow p \;\equiv\; E_{false}^z \Rightarrow p$

(b) $E_p^z \Rightarrow p \lor q \;\equiv\; E_{false}^z \Rightarrow p \lor q$

(3.87) **Replace by** *true* **:** $p \land E_p^z \;\equiv\; p \land E_{true}^z$

(3.88) **Replace by** *false* **:** $p \lor E_p^z \;\equiv\; p \lor E_{false}^z$

(3.89) **Shannon:** $E_p^z \;\equiv\; (p \land E_{true}^z) \lor (\neg p \land E_{false}^z)$

Leibniz

Proof of theorem (3.84a), $(e = f) \wedge E_e^z \;\equiv\; (e = f) \wedge E_f^z$.

$$
\begin{array}{ll}
 & true \\
= & \quad \langle \text{Leibniz's rule (3.83)} \rangle \\
 & e = f \;\Rightarrow\; (E[z := e] \;\equiv\; E[z := f]) \\
= & \quad \langle \text{Implication (3.60)} \rangle \\
 & e = f \;\wedge\; (E[z := e] \;\equiv\; E[z := f]) \;\equiv\; e = f \\
= & \quad \langle \text{Distributivity of } \wedge \text{ over } \equiv \text{ (3.49)} \rangle \\
 & e = f \;\wedge\; E[z := e] \;\equiv\; e = f \;\wedge\; E[z := f]
\end{array}
$$

Proof of theorem (3.84b), $e = f \;\Rightarrow\; E_e^z \;\equiv\; e = f \;\Rightarrow\; E_f^z$.

$$
\begin{array}{ll}
 & e = f \;\Rightarrow\; E[z := e] \;\equiv\; e = f \;\Rightarrow\; E[z := f] \\
= & \quad \langle \text{Distributivity of } \Rightarrow \text{ over } \equiv \text{ (3.63)} \rangle \\
 & e = f \;\Rightarrow\; (E[z := e] \;\equiv\; E[z := f]) \text{ —Axiom Leibniz, (3.83)}
\end{array}
$$

Proof of theorem (3.84c), $q \wedge e = f \;\Rightarrow\; E_e^z \;\equiv\; q \wedge e = f \;\Rightarrow\; E_f^z$.

$$
\begin{array}{ll}
 & q \wedge e = f \;\Rightarrow\; E[z := e] \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
 & q \;\Rightarrow\; (e = f \;\Rightarrow\; E[z := e]) \\
= & \quad \langle \text{Substitution (3.84b)} \rangle \\
 & q \;\Rightarrow\; (e = f \;\Rightarrow\; E[z := f]) \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
 & q \wedge e = f \;\Rightarrow\; E[z := f]
\end{array}
$$

Leibniz

Proof of theorem (3.85a), $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := true]$.

$$
\begin{array}{ll}
 & p \Rightarrow E[z := p] \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
 & p = true \Rightarrow E[z := p] \\
= & \quad \langle \text{Substitution (3.84b)} \rangle \\
 & p = true \Rightarrow E[z := true] \\
= & \quad \langle \text{Identity of } \equiv \text{ (3.3)} \rangle \\
 & p \Rightarrow E[z := true]
\end{array}
$$

Proof of theorem (3.85b), $q \wedge p \Rightarrow E^z_p \equiv q \wedge p \Rightarrow E^z_{true}$.

$$
\begin{array}{ll}
 & q \wedge p \Rightarrow E[z := p] \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
 & q \Rightarrow (p \Rightarrow E[z := p]) \\
= & \quad \langle \text{(3.85a)}, \ p \Rightarrow E^z_p \equiv p \Rightarrow E^z_{true} \rangle \\
 & q \Rightarrow (p \Rightarrow E[z := true]) \\
= & \quad \langle \text{Shunting (3.65)} \rangle \\
 & q \wedge p \Rightarrow E[z := true]
\end{array}
$$

Leibniz

Proof of theorem (3.85a), $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := true]$.

$$
\begin{array}{ll}
& p \Rightarrow E[z := p] \\
= & \quad \langle \text{Identity of } \equiv \ (3.3) \rangle \\
& p = true \Rightarrow E[z := p] \\
= & \quad \langle \text{Substitution } (3.84b) \rangle \\
& p = true \Rightarrow E[z := true] \\
= & \quad \langle \text{Identity of } \equiv \ (3.3) \rangle \\
& p \Rightarrow E[z := true]
\end{array}
$$

Proof of theorem (3.85b), $q \wedge p \Rightarrow E^z_p \equiv q \wedge p \Rightarrow E^z_{true}$.

$$
\begin{array}{ll}
& q \wedge p \Rightarrow E[z := p] \\
= & \quad \langle \text{Shunting } (3.65) \rangle \\
& q \Rightarrow (p \Rightarrow E[z := p]) \\
= & \quad \langle (3.85a),\ p \Rightarrow E^z_p \equiv p \Rightarrow E^z_{true} \rangle \\
& q \Rightarrow (p \Rightarrow E[z := true]) \\
= & \quad \langle \text{Shunting } (3.65) \rangle \\
& q \wedge p \Rightarrow E[z := true]
\end{array}
$$

Leibniz

Proof of theorem (3.86a), $E[z := p] \Rightarrow p \quad \equiv \quad E[z := \mathit{false}] \Rightarrow p$.

$$
\begin{array}{ll}
 & E[z := p] \Rightarrow p \\
= & \quad \langle \text{Contrapositive } (3.61) \rangle \\
 & \neg p \Rightarrow \neg E[z := p] \\
= & \quad \langle (3.15),\ \neg p \equiv p \equiv \mathit{false}\ ;\ \text{property of textual subst.} \rangle \\
 & p = \mathit{false} \Rightarrow (\neg E)[z := p] \\
= & \quad \langle \text{Substitution } (3.84b) \rangle \\
 & p = \mathit{false} \Rightarrow (\neg E)[z := \mathit{false}] \\
= & \quad \langle (3.15),\ \neg p \equiv p \equiv \mathit{false}\ ;\ \text{property of textual subst.} \rangle \\
 & \neg p \Rightarrow \neg E[z := \mathit{false}] \\
= & \quad \langle \text{Contrapositive } (3.61) \rangle \\
 & E[z := \mathit{false}] \Rightarrow p
\end{array}
$$

Proof of theorem (3.86b), $E_p^z \Rightarrow p \vee q \quad \equiv \quad E_{\mathit{false}}^z \Rightarrow p \vee q$.

$$
\begin{array}{ll}
 & E[z := p] \Rightarrow p \vee q \\
= & \quad \langle \text{Implication } (3.59) \rangle \\
 & \neg E[z := p] \vee p \vee q \\
= & \quad \langle \text{Implication } (3.59) \rangle \\
 & (E[z := p] \Rightarrow p) \vee q \\
= & \quad \langle (3.86a),\ E[z := p] \Rightarrow p \quad \equiv \quad E[z := \mathit{false}] \Rightarrow p \rangle \\
 & (E[z := \mathit{false}] \Rightarrow p) \vee q \\
= & \quad \langle \text{Implication } (3.59) \rangle \\
 & \neg E[z := \mathit{false}] \vee p \vee q \\
= & \quad \langle \text{Implication } (3.59) \rangle \\
 & E[z := \mathit{false}] \Rightarrow p \vee q
\end{array}
$$

Leibniz

Proof of theorem (3.87), $p \wedge E[z := p] \ \equiv \ p \wedge E[z := true]$.

$$\begin{array}{ll} & p \wedge E[z := p] \\ = & \quad \langle \text{Implication (3.60)} \rangle \\ & p \Rightarrow E[z := p] \ \equiv \ p \\ = & \quad \langle \text{Replace by true (3.85a)} \rangle \\ & p \Rightarrow E[z := true] \ \equiv \ p \\ = & \quad \langle \text{Implication (3.60)} \rangle \\ & p \wedge E[z := true] \end{array}$$

Proof of theorem (3.88), $p \vee E[z := p] \ \equiv \ p \vee E[z := false]$.

$$\begin{array}{ll} & p \vee E[z := p] \\ = & \quad \langle \text{Implication (3.57)} \rangle \\ & E[z := p] \Rightarrow p \ \equiv \ p \\ = & \quad \langle \text{Replace by } \mathit{false} \text{ (3.86a)} \rangle \\ & E[z := false] \Rightarrow p \ \equiv \ p \\ = & \quad \langle \text{Implication (3.57)} \rangle \\ & p \vee E[z := false] \end{array}$$

Leibniz

Proof of Shannon's theorem, (3.89).

$$
\begin{array}{ll}
 & (p \wedge E[z := true]) \vee (\neg p \wedge E[z := false]) \\
= & \quad \langle (3.15),\ \neg p \equiv p \equiv false \ \rangle \\
 & (p \wedge E[z := true]) \vee (p = false \ \wedge E[z := false]) \\
= & \quad \langle \text{Replace by true (3.87); Substitution (3.84a)} \rangle \\
 & (p \wedge E[z := p]) \vee (p = false \ \wedge E[z := p]) \\
= & \quad \langle (3.15),\ \neg p \equiv p \equiv false \ \rangle \\
 & (p \wedge E[z := p]) \vee (\neg p \wedge E[z := p]) \\
= & \quad \langle \text{Distributivity (3.46)} \rangle \\
 & (p \vee \neg p) \wedge E[z := p] \\
= & \quad \langle \text{Excluded middle (3.28); Identity of } \wedge \text{, (3.39)} \rangle \\
 & E[z := p]
\end{array}
$$

Any Questions?

Implication

Miscellaneous theorems about implication

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:**
$p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv true$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

Implication

Implication and boolean constants

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p \equiv true$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \equiv true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \equiv p$

(3.74) $p \Rightarrow false \equiv \neg p$

(3.75) $false \Rightarrow p \equiv true$

Weakening, strengthening, and Modus ponens

(3.76) **Weakening/strengthening:**
(a) $p \Rightarrow p \vee q$
(b) $p \wedge q \Rightarrow p$
(c) $p \wedge q \Rightarrow p \vee q$
(d) $p \vee (q \wedge r) \Rightarrow p \vee q$
(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$.

Implication

Forms of case analysis

(3.78) $(p \Rightarrow r) \land (q \Rightarrow r) \equiv (p \lor q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \land (\neg p \Rightarrow r) \equiv r$

Mutual implication and transitivity

(3.80) **Mutual implication:** $(p \Rightarrow q) \land (q \Rightarrow p) \equiv p \equiv q$

(3.81) **Antisymmetry**[11]**:** $(p \Rightarrow q) \land (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:** (a) $(p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \land (q \equiv r) \Rightarrow (p \Rightarrow r)$