Homework 4 - Proofs (Spring 2021)

First Name, Last Name UNI

February 16, 2021

Question 1

Direct Proof to prove:
For all integers n, 3 divides $(3n+1)(3n+2)(3n+3)$.

Let $n \in \mathbb{Z}$
We are trying to prove $3|(3n+1)(3n+2)(3n+3)$

Let $a \in \mathbb{Z}$
$a = (3n+1)(3n+2)(3n+3)$
$\equiv (27n^3 + 54n^2 + 33n + 6)$
$\equiv 3(9n^3 + 18n^2 + 11n + 2)$

$\frac{a}{3} = (9n^3 + 18n^2 + 11n + 2)$

Since $\frac{a}{3}$ is equal to $(9n^3 + 18n^2 + 11n + 2)$, we have proved that for all integers n, 3 divides $(3n+1)(3n+2)(3n+3)$.

$\square$

Question 2

Proof by Contrapositive to prove:

For any integer n, if $3n + 1$ is even, then n is odd.

The **contrapositive** is:

If n is even, then $3n + 1$ is odd

Let $n \in \mathbb{Z}$
Let $a \in \mathbb{Z}$
Let $b \in \mathbb{Z}$

$n = 2a$, since we assume n is even

$b = 3n + 1$, if in fact $3n + 1$ is odd
$\equiv 3(2a) + 1$
$\equiv 6a + 1$
$\equiv 2(3a) + 1$

Let $c \in \mathbb{Z}$

$c = 3a$
$b = 2(c) + 1$
$\equiv 2c + 1$, the form of an odd integer

Since b is now in the form of an odd integer, we have proven the contrapositive. We can conclude that for any integer n, if $3n + 1$ is even, then n is odd.

$\square$

Question 3

Proof of IFF to prove:

Let x, y be integers, and prove that the product xy is odd if and only if x and y are both odd integers.

We must prove both:

if xy is odd, then x and y are both odd integers

if x and y are both odd integers, then xy is odd

For the first statement, we may prove the **contrapositive** of:

if x and y are both **even** integers, then xy is even

Let $n \in \mathbb{Z}$
$x = 2n$, since we assume x is even
$y = 2n$, since we assume y is even

$xy = (2n)(2n)$
$\equiv 4n^2$
$\equiv 2(2n^2)$

Let $a \in \mathbb{Z}$
$a = 2n^2$
$xy = 2a$

xy is now in the form of an even integer.
We have proven that if x and y are both **even** integers, then xy is even.

For the second statement, we may prove:

if x and y are both **odd** integers, then xy is odd

Let $n \in \mathbb{Z}$
$x = 2n + 1$, since we assume x is odd
$y = 2n + 1$, since we assume y is odd

$xy = (2n + 1)(2n + 1)$
$\equiv 4n^2 + 4n + 1$
$\equiv 2(2n^2 + 2n) + 1$

Let $k \in \mathbb{Z}$
$k = 2n^2 + 2n$
$xy = 2k + 1$

xy is now in the form of an odd integer.
We have proven that if x and y are both **odd** integers, then xy is odd.

$\square$

Question 4

Proof by cases of:
if n is an integer, then $n^2 \geq n$

Let $n \in \mathbb{Z}$

We have three possible cases since $n \in \mathbb{Z}$:
$n < 0$, $n = 0$, and $n > 0$

Case 1, $n < 0$

Let $a \in \mathbb{Z}$
$(-a)^2 \geq (-a)$
$a^2 \geq -a$
$n = -a$
$n^2 \geq n$

Case 2, $n = 0$

Let $b \in \mathbb{Z}$
$(0)^2 \geq (0)$
$0 \geq 0$
$n = 0$
$n^2 \geq n$

Case 3, $n > 0$

Let $c \in \mathbb{Z}$
$(a)^2 \geq (a)$
$a^2 \geq a$
$n = a$
$n^2 \geq n$

$\square$

Question 5

Proof by Counter Example of:

For every prime number $p > 2$, there exists a natural number n such that $p = 2^n - 1$

Consider the prime number 5:

There is no $n \in \mathbb{N}$ such that $5 = 2^n - 1$

$5 = 2^n - 1$
$6 = 2^n$

There does not exist $n \in \mathbb{N}$ such that we can make this statement True.

$\square$

Question 6

Proof by contradiction of:

For all integers $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

We can assume the negation, if this statement is False:

n^2 is odd and n is even

Let $n \in \mathbb{Z}$
Let $a \in \mathbb{Z}$

$n = 2a$, since n is even
$n^2 = (2a)^2$
$\equiv 4a^2$

We have run into a contradiction since n is even **and** n^2 is even, meaning that the opposite, our initial statement, is True.

$\square$