

# Homework 4 - Proofs (Spring 2021)

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## Question 1

**Direct Proof** to prove:

For all integers  $n$ , 3 divides  $(3n + 1)(3n + 2)(3n + 3)$ .

*Proof.* Let  $n \in \mathbb{Z}$

We are trying to prove  $3|(3n + 1)(3n + 2)(3n + 3)$

Let  $a \in \mathbb{Z}$

$$a = (3n + 1)(3n + 2)(3n + 3)$$

$$\equiv (27n^3 + 54n^2 + 33n + 6)$$

$$\equiv 3(9n^3 + 18n^2 + 11n + 2)$$

$$\frac{a}{3} = (9n^3 + 18n^2 + 11n + 2)$$

Since 3 is a factor of  $(27n^3 + 54n^2 + 33n + 6)$ , we have proved that for all integers  $n$ , 3 divides  $(3n + 1)(3n + 2)(3n + 3)$ .

□

## Question 2

**Proof by Contrapositive** to prove:

For any integer  $n$ , if  $3n + 1$  is even, then  $n$  is odd.

*Proof.* The **contrapositive** is:

If  $n$  is even, then  $3n + 1$  is odd

Let  $n \in \mathbb{Z}$

Let  $a \in \mathbb{Z}$

Let  $b \in \mathbb{Z}$

$n = 2a$ , since we assume  $n$  is even

$b = 3n + 1$ , if in fact  $3n + 1$  is odd

$$\equiv 3(2a) + 1$$

$$\equiv 6a + 1$$

$$\equiv 2(3a) + 1$$

Let  $c \in \mathbb{Z}$

$$c = 3a$$

$$b = 2(c) + 1$$

$$\equiv 2c + 1, \text{ the form of an odd integer}$$

Since  $b$  is now in the form of an odd integer, we have proven the contrapositive. We can conclude that for any integer  $n$ , if  $3n + 1$  is even, then  $n$  is odd.

□

### Question 3

**Proof of IFF** to prove:

Let  $x, y$  be integers, and prove that the product  $xy$  is odd if and only if  $x$  and  $y$  are both odd integers.

*Proof.* We must prove both:

if  $xy$  is odd, then  $x$  and  $y$  are both odd integers

if  $x$  and  $y$  are both odd integers, then  $xy$  is odd

For the first statement, we may prove the **contrapositive**:

if  $x$  and  $y$  are both **even** integers, then  $xy$  is even

Let  $n \in \mathbb{Z}$

$x = 2n$ , since we assume  $x$  is even

$y = 2n$ , since we assume  $y$  is even

$$xy = (2n)(2n)$$

$$\equiv 4n^2$$

$$\equiv 2(2n^2)$$

Let  $a \in \mathbb{Z}$

$$a = 2n^2$$

$$xy = 2a$$

$xy$  is now in the form of an even integer.

We have proven that if  $x$  and  $y$  are both **even** integers, then  $xy$  is even.

For the second statement, we may prove:

if  $x$  and  $y$  are both **odd** integers, then  $xy$  is odd

Let  $n \in \mathbb{Z}$

$x = 2n + 1$ , since we assume  $x$  is odd

$y = 2n + 1$ , since we assume  $y$  is odd

$$xy = (2n + 1)(2n + 1)$$

$$\equiv 4n^2 + 4n + 1$$

$$\equiv 2(2n^2 + 2n) + 1$$

Let  $k \in \mathbb{Z}$

$$k = 2n^2 + 2n$$

$$xy = 2k + 1$$

$xy$  is now in the form of an odd integer.

We have proven that if  $x$  and  $y$  are both **odd** integers, then  $xy$  is odd.  $\square$

## Question 4

**Proof by cases of:**

if  $n$  is an integer, then  $n^2 \geq n$

*Proof.* Let  $n \in \mathbb{Z}$

We have three possible cases since  $n \in \mathbb{Z}$ :

$n < 0$ ,  $n = 0$ , and  $n > 0$

**Case 1,**  $n < 0$

Let  $a \in \mathbb{Z}$   
 $(-a)^2 \geq (-a)$   
 $a^2 \geq -a$   
 $n = -a$   
 $n^2 \geq n$

**Case 2,**  $n = 0$

Let  $b \in \mathbb{Z}$   
 $(0)^2 \geq (0)$   
 $0 \geq 0$   
 $n = 0$   
 $n^2 \geq n$

**Case 3,**  $n > 0$

Let  $c \in \mathbb{Z}$   
 $(a)^2 \geq (a)$   
 $a^2 \geq a$   
 $n = a$   
 $n^2 \geq n$

□

## Question 5

**Proof by Counter Example** of:

For every prime number  $p > 2$ , there exists a natural number  $n$  such that  $p = 2^n - 1$

*Proof.* Consider the prime number 5:

There is no  $n \in \mathbb{N}$  such that  $5 = 2^n - 1$

$$5 = 2^n - 1$$

$$6 = 2^n$$

There does not exist  $n \in \mathbb{N}$  such that we can make this statement True.

□

## Question 6

**Proof by contradiction of:**

For all integers  $n \in \mathbb{Z}$ , if  $n^2$  is odd, then  $n$  is odd.

*Proof.* We can assume the negation, if this statement is False:

$n^2$  is odd and  $n$  is even

Let  $n \in \mathbb{Z}$

Let  $a \in \mathbb{Z}$

$n = 2a$ , since  $n$  is even

$$n^2 = (2a)^2$$

$$\equiv 4a^2$$

$$\equiv 2(2a^2)$$

Let  $b = 2a^2$

$n^2 = 2b$ , the form of an even integer

We have run into a contradiction since  $n$  is even **and**  $n^2$  is even, meaning that the opposite, our initial statement, is True.

□