

Homework 4 - Proofs (Spring 2021)

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Question 1

Direct Proof to prove:

For all integers n , 3 divides $(3n + 1)(3n + 2)(3n + 3)$.

Proof. Let $n \in \mathbb{Z}$

We are trying to prove $3|(3n + 1)(3n + 2)(3n + 3)$

Let $a \in \mathbb{Z}$

$$a = (3n + 1)(3n + 2)(3n + 3)$$

$$\equiv (27n^3 + 54n^2 + 33n + 6)$$

$$\equiv 3(9n^3 + 18n^2 + 11n + 2)$$

$$\frac{a}{3} = (9n^3 + 18n^2 + 11n + 2)$$

Since 3 is a factor of $(27n^3 + 54n^2 + 33n + 6)$, we have proved that for all integers n , 3 divides $(3n + 1)(3n + 2)(3n + 3)$.

□

Question 2

Proof by Contrapositive to prove:

For any integer n , if $3n + 1$ is even, then n is odd.

Proof. The **contrapositive** is:

If n is even, then $3n + 1$ is odd

Let $n \in \mathbb{Z}$

Let $a \in \mathbb{Z}$

Let $b \in \mathbb{Z}$

$n = 2a$, since we assume n is even

$b = 3n + 1$, if in fact $3n + 1$ is odd

$$\equiv 3(2a) + 1$$

$$\equiv 6a + 1$$

$$\equiv 2(3a) + 1$$

Let $c \in \mathbb{Z}$

$$c = 3a$$

$$b = 2(c) + 1$$

$$\equiv 2c + 1, \text{ the form of an odd integer}$$

Since b is now in the form of an odd integer, we have proven the contrapositive. We can conclude that for any integer n , if $3n + 1$ is even, then n is odd.

□

Question 3

Proof of IFF to prove:

Let x, y be integers, and prove that the product xy is odd if and only if x and y are both odd integers.

Proof. We must prove both:

if xy is odd, then x and y are both odd integers

if x and y are both odd integers, then xy is odd

For the first statement, we may prove the **contrapositive**:

if x and y are both **even** integers, then xy is even

Let $n \in \mathbb{Z}$

$x = 2n$, since we assume x is even

$y = 2n$, since we assume y is even

$$xy = (2n)(2n)$$

$$\equiv 4n^2$$

$$\equiv 2(2n^2)$$

Let $a \in \mathbb{Z}$

$$a = 2n^2$$

$$xy = 2a$$

xy is now in the form of an even integer.

We have proven that if x and y are both **even** integers, then xy is even.

For the second statement, we may prove:

if x and y are both **odd** integers, then xy is odd

Let $n \in \mathbb{Z}$

$x = 2n + 1$, since we assume x is odd

$y = 2n + 1$, since we assume y is odd

$$xy = (2n + 1)(2n + 1)$$

$$\equiv 4n^2 + 4n + 1$$

$$\equiv 2(2n^2 + 2n) + 1$$

Let $k \in \mathbb{Z}$

$$k = 2n^2 + 2n$$

$$xy = 2k + 1$$

xy is now in the form of an odd integer.

We have proven that if x and y are both **odd** integers, then xy is odd. \square

Question 4

Proof by cases of:

if n is an integer, then $n^2 \geq n$

Proof. Let $n \in \mathbb{Z}$

We have three possible cases since $n \in \mathbb{Z}$:

$n < 0$, $n = 0$, and $n > 0$

Case 1, $n < 0$

Let $a \in \mathbb{Z}$
 $(-a)^2 \geq (-a)$
 $a^2 \geq -a$
 $n = -a$
 $n^2 \geq n$

Case 2, $n = 0$

Let $b \in \mathbb{Z}$
 $(0)^2 \geq (0)$
 $0 \geq 0$
 $n = 0$
 $n^2 \geq n$

Case 3, $n > 0$

Let $c \in \mathbb{Z}$
 $(a)^2 \geq (a)$
 $a^2 \geq a$
 $n = a$
 $n^2 \geq n$

□

Question 5

Proof by Counter Example of:

For every prime number $p > 2$, there exists a natural number n such that $p = 2^n - 1$

Proof. Consider the prime number 5:

There is no $n \in \mathbb{N}$ such that $5 = 2^n - 1$

$$5 = 2^n - 1$$

$$6 = 2^n$$

There does not exist $n \in \mathbb{N}$ such that we can make this statement True.

□

Question 6

Proof by contradiction of:

For all integers $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Proof. We can assume the negation, if this statement is False:

n^2 is odd and n is even

Let $n \in \mathbb{Z}$

Let $a \in \mathbb{Z}$

$n = 2a$, since n is even

$$n^2 = (2a)^2$$

$$\equiv 4a^2$$

$$\equiv 2(2a^2)$$

Let $b = 2a^2$

$n^2 = 2b$, the form of an even integer

We have run into a contradiction since n is even **and** n^2 is even, meaning that the opposite, our initial statement, is True.

□