Homework 7: Discrete Mathematics Spring 2021 – Functions and more proofs

Due Sunday March 28 @11:59:00pm

Show all your work to receive full credit
Please note that we will make this a homework quiz in Gradescope

1. **Definitions**

   Please read the following statements carefully. Which statements are correct? Briefly justify your answer.

   1. Let f be a function. If every element in the codomain has a pre-image, f must be an onto function.
   2. Let f be a function. If every element in the domain has an image, f must be an onto function.
   3. An onto function means that every element in the codomain must have a unique pre-image.
   4. A one-to-one function means that every element in the codomain must have a unique pre-image.
   5. A one-to-one function means that no two elements in the domain map to the same element in the codomain.
   6. Let f be a function. If the domain is larger than the codomain, f can't be a one-to-one function.

2. **Onto, and one-to-one functions**

   Function on power sets

   Let the set $A = \{1, 2, 3, 4, 5, \cdots, 10\}$

   Let's define the function g on the power set of A, $P(A)$ as follows:

   $$g : P(A) \to \mathbb{Z}$$

   defined by

   $$g(X) = \begin{cases} 0 & \text{if } |X| \text{ is even} \\ 1 & \text{if } |X| \text{ is odd} \end{cases}$$

   1. Calculate $g(\{\emptyset\})$, $g(\{1, 2, 3\})$, and $g(\{1, 2, 3, 4\})$.
   2. Prove that g is not onto.
   3. Prove that g is not one-to-one.
   4. Propose a change in the **co-domain** of g to make the function onto.

3. **Onto, and one-to-one functions**

   Hamming distance

   In computer science, the hamming distance (named after Richard Hamming) is a function used to count the number of flipped bits in a fixed-length binary lists. It is used in error correction and detection in information theory and other applications.

   Let's define the Hamming distance function H between two binary lists of size 6.

   $$H : \{0, 1\}^6 \times \{0, 1\}^6 \to \mathbb{N} \cup \{0\}$$

   defined for any two binary lists of size 6 by:

   $$H(l_1, l_2) = \text{the number of positions where } l_1 \text{ and } l_2 \text{ differ}$$

   We will simplify writing lists in $\{0, 1\}^6$ by removing the parentheses and the commas. For example, the list (0,0,0,0,0,1) will be written as 000001.

1. Calculate $H(000010, 000011)$, and $H(000010, 100001)$.
2. Prove that H is not onto.
3. Prove that H is not one-to-one.
4. Propose a change in the **domain** of H to make the function onto.

4. **Bijection, inverse functions and composition**

   Consider two functions:

$$f : \mathbb{Z} \to \mathbb{Z} \text{ defined by } f(n) = 3n - 2$$
$$g : \mathbb{R} \to \mathbb{R} \text{ defined by } g(x) = 3x - 2$$

   (a) Prove that f is not onto.

   (b) Prove that g is a bijection.

   (c) Find the inverse of g.

   (d) Calculate $g^{-1} \circ g$ and $g \circ g^{-1}$.

   (e) What can you say about the result of composing a function with its inverse $(g \circ g^{-1})$ and composing the inverse of a function with itself $(g^{-1} \circ g)$?

5. **Pigeon Hole Principle and Generalized Pigeon Hole Principle**

   (a) A drawer has ten blue, ten white, and ten red socks. Without looking at them you pull some socks out. What is the least number of socks you need to pull to ensure you get TWO pairs of matching socks? Justify your answer.

   Hint: Use Pigeon Hole Principle (PHP).

   (b) Prove that in a group of 85 students, the family name of at least four students must start with the same letter. There are 26 letters in the English alphabet.

   Hint: Use the Generalized Pigeon Hole Principle (GPHP).

6. **Proof by induction**

   Using induction, prove the following proposition.

   For all natural numbers $n \geq 2$,

$$n^2 \geq n + 2$$

   Hint: When trying to prove an inequality, it can be helpful to find some c such that $a \leq c \leq b$ so that $a \leq b$ follows by transitivity of inequalities.

   Please show clearly, your base case, inductive hypothesis and inductive step.

7. **Proof by smallest counter example**

   Let n be a positive integer. Prove by **smallest counter example** that:

$$\forall n > 0 \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

   Proof by strong induction will be covered in the next homework in the context of number theory.