The MIT bag-model

Glueball mass spectrum using the MIT bag-model

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Abstract

This thesis studies the MIT bag-model and derives important equations regarding this model. The masses of the hypothetical glueballs are determined using the MIT bag-model. Five different glueball masses are predicted and the lowest glueball mass is determined to be 0.961 GeV. This is in excellent agreement with similar predictions using the MIT bag-model. Lattice QCD and other nucleon models are compared to the MIT bag-model. Many recent studies using lattice QCD have found the lowest glueball mass to be $\sim$ 1.6 GeV. This result better suits newly discovered glueball candidates, indicating that our calculated mass is to low.

Sammanfattning

Denna avhandling studerar MIT bagmodellen och härleder relevanta ekvationer i modellen. En massberäkning av de hypotetiska klisterbollarna görs med hjälp av MIT bagmodellen. Fem olika klistebollar beräknas och lägsta klistebollsmassan fås till 0.961 GeV. Detta är i utmärkt överensstämmelse med liknande studier gjorda med MIT bagmodellen. Lattice QCD och andra nukleonmodeller jämförs med MIT bagmodellen. Många nya studier med lattice QCD har beräknat lägsta klisterbollsmassan till $\sim$ 1,6 GeV. Detta resultat passar bättre in på nyligen upptäckta klisterbollskandidater, vilket tyder på att vår beräknade massa är för låg.

Contents

1 Introduction

One of the most fundamental questions in the history of physics has always been the search of matters smallest components. If one were to take a piece of matter and half it, then half the smaller pieces and half the even smaller pieces and so on, would one ever reach a point where the matter is undividable? The idea of an atom, from the greek word átomos meaning undividable, was first coined by ancient Greek philosophers. Their idea of the atom as an elementary particle was purely philosophical and it was not until the early 19$^{\text{th}}$ century that the concept of an elementary particle was used in a more scientific way by John Dalton. Dalton used the concept of different atoms to describe the nature of different elements [1]. Since then Joseph John Thomson and Ernest Rutherford, in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century, discovered that the atom itself had smaller constituents, a nucleus and surrounding electrons [2, 3]. James Chadwick later complemented their model by the discovery of the neutron in 1932 [4].

It was only 32 years after the discovery of the neutron that the American physicists Murray Gell-Mann and George Zweig independently proposed the quark model in 1964 [5, 6]. The quark model suggests that the nucleons, the proton and the neutron, are not elementary particles at all but rather consist of three quarks. The idea was later strengthened in 1968 by the deep inelastic scattering experiments at Stanford Linear Acceleration Center (SLAC). It confirmed that the proton was built by smaller point-like object [7]. These ideas and discoveries led to extensive research in the field of particle physics and the development of the Standard Model.

As will be described in section 2.4 the concept of color charge in quarks makes them incredibly difficult to isolate. So far no isolated quark has ever been observed, yet they seem to move freely within the nucleons. These behaviors makes it very completated to determine the laws of motion for quarks. Different models have been proposed, one of which was developed at Massachusetts Institute of Technology (MIT) called the MIT bag-model. One of the main advantages of the MIT bag-model is its ability to quite accurately predict the masses of different hadrons.

Gluons are the carriers of the strong force. According to the theory these should attract each other via the strong force to create bound states of two or three gluons called *glueballs*. Glueballs are hypothetical particles that have not yet been observed. However, experiments conducted at particle accelerators have found scalar mesons that could be glueballs, but this has not been confirmed [8]. In this thesis the MIT bag-model will be used to calculate the mass of glueballs.

1.1 Purpose

The purpose of this thesis is to study the so-called MIT bag-model in the existing literature and to derive important equations regarding this model. Using the static spherical MIT bag-model we will calculate the mass spectrum of two and three gluon glueballs in a transverse electric and transverse magnetic mode.

1.2 The structure of this thesis

This sections objective is to outline the structure of this thesis.

The first section will focus on giving the reader enough background information to be able to fully fathom the result later on. It briefly explains the Standard model of particle physics, quarks and quantum chromodynamics to give the reader understanding for the need of different nucleon models. Then it treats some other nucleon models apart from the MIT bag-model. Lastly it will explain four vector algebra and natural units.

The result section will treat the MIT bag-model. First it will try to give an intuitive explanation for the color confinement in the hadrons. Sections 4.3, 4.4 and 4.5 are the key parts of the thesis. In section 4.3 we will derive all the important MIT bag-model equations using the Lagrangian density. In sections 4.4 and 4.5 we will calculate the mass spectrum of ground state glueballs in transverse electric and transverse magnetic modes using the MIT bag-model equations.

The discussion section begins with an analysis of the pros and cons with the MIT bag-model. A reflection on the approximations done in the thesis and how they may have altered our result is then given in section 5.1. There is a comparison between the MIT bag-model and other nucleon models and the discussion ends with recent research regarding glueballs.

Lastly we will end with a summary and conclusion section to summarize this thesis' results.

2 Background theory

Here the reader is given the elementary knowledge needed to understand the result and discussion sections of the thesis. The different concepts are only briefly explained and for further information regarding each subject, we refer to the mentioned references.

2.1 The Standard Model

The Standard Model (SM) is a model describing the elementary particles and forces. The beginning of what was to become the SM came in 1961 with a way to unify the electromagnetic and weak forces [9]. Many particles predicted by the SM has since then been discovered in particle accelerators, giving further credit to the model. It is today considered the most plausible explanation for the subatomic structure.

According to the SM there are four elementary forces and a finite number of elementary particles. The forces are gravity, electromagnetism and the strong and weak interactions. The SM does not yet have a mathematical description for gravity since it is so weak compared to the other elementary forces. A theoretical particle called graviton is proposed that would help incorporate gravity in the theory if found, but no such particle has yet been observed. Furthermore the electromagnetic and weak forces have been confirmed to be a unified force even though they appear very different at low energies. At high enough energies, as those in the beginning of the universe, these forces are actually one and the same. For this realization Sheldon Lee Glashow, Abdus Salam and Steven Weinberg were awarded the Nobel Prize in Physics 1979 [10].

The elementary particles are six leptons and six quarks, all of which have corresponding antiparticles, the force carriers and the Higgs Boson. The particles are shown in figure 1.

Particles not affected by the strong interaction are called leptons and they come in three generations. Each generation has one particle with electric charge -1 and a matching neutrino. The neutrinos are electrically neutral. The six leptons are:

- The electron and the electron neutrino.
- The muon and the muon neutrino.
- The tau and the tau neutrino.

All six leptons are spin-$\frac{1}{2}$ particles. The electron, the muon and the tau are all identical except for their difference in mass, the electron being the lightest and the tau the heaviest. Each lepton has a corresponding antilepton. The antilepton is identical to the lepton except that it carries opposite charge, the most famous being the antielectron, called the positron. The leptons are depicted in green in figure 1.

The force carriers are the gluons, the photon and the Z and W bosons, carriers of the strong force, electromagnetism, and the weak force respectively. Both the Z and the W bosons are carriers of the weak force. There is only one photon and one Z boson but there are eight different gluons and two W bosons. All force carriers are bosons, meaning they have integer spin. The force carriers travel between all interacting objects. In the same way as photons have to travel from the sun to the earth for us to feel the sun's warmth, gluons must travel between quarks for them to interact via the strong interaction [11]. The force carriers are depicted in red in figure 1.

The Higgs boson is a a boson with no spin and no color or electric charge. It is the Higgs boson that gives all the other particles their masses. The discovery of the Higgs boson is the latest confirmation of the many particles predicted by the SM [12]. The Higgs boson is yellow in figure 1.

Of greatest importance to this thesis are the strong interaction, the quarks and the gluons. They are all explained in greater detail in the following section.

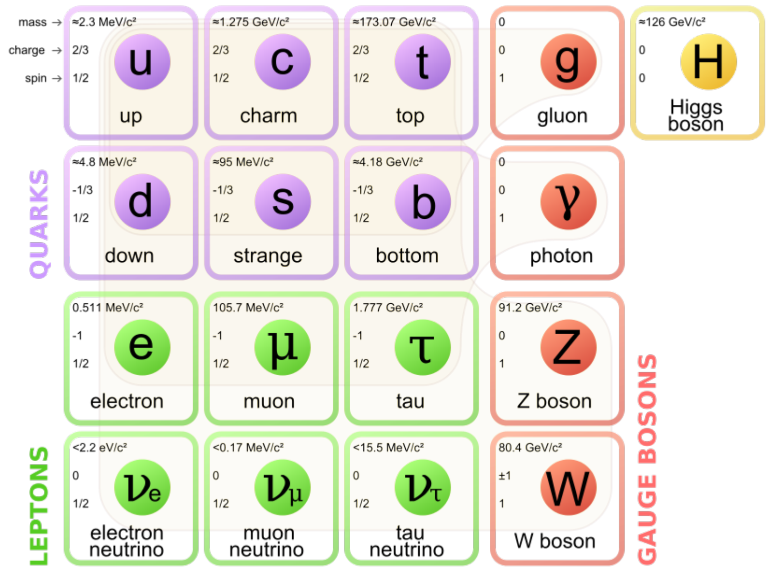


Figure 1: The elementary particles according to the SM. The values in the upper left corner of each box tells the mass, electrical charge and spin of the particle. The figure has been adopted from reference [13].

2.2 The quark hypothesis

Particles that experience the strong force are called hadrons. Amongst the hadrons are the proton, the neutron and the mesons. During the 20th century more and more hadrons were discovered by scientists. Between 1950 and 1975 hundreds were found and there seemed to be some sort of underlaying symmetry. Furthermore, as the leptons behaved like point like objects, the hadrons had a measurable size, about $\sim 10^{-13}$ cm. These were good indications that there was more to the story.

Trying to see a grander picture the hadrons were first sorted into multiplets. Each multiplet had particles that had roughly the same mass and were identical in all other aspect except for electric charge. The multiplets could consist of one to four particles, for example the proton and the neutron made up a multiplet of two. These multiplets was then during the 1960s organized into supermultiplets. There were eight different quantum numbers within these supermultiplets and this has led to the quite popularized term "eightfold way" [11]. The mathematics behind the eightfold way are governed by the Lie group $SU(3)$. A more detailed description of how Lie group mathematics works is beyond the realms of this text, but can be studied further in, for example, *Analysis on Lie Groups* by Jacques Faraut [14].

The supermultiplets later led to the quark hypothesis in 1964. The hypothesis was proposed independently by Gell-Mann and Zweig, and the name *quark* proposed by Gell-Mann stuck. The idea of the quark hypothesis was quite simple. There were three different quarks, u, d and s, and three corresponding antiquarks. These could form three different types of hadrons: the mesons made of a quark and an antiquark, the baryons made of three quarks and the antibaryons made of three antiquarks. The simplicity was appealing and these combinations could account for all the hadrons known at the time. Experiments conducted at the Stanford Linear Acceleration Center (SLAC) could in the late 1960s confirm that the proton and neutron contained electrically charged, point-like particles, which greatly contributed to the credibility of the theory [11].

2.3 Quarks

Today, scientist have confirmed that there are not three, but six different types of quarks called flavors. The flavors are up, down, strange, charm, top and bottom/beauty, with symbols u, d, s, c, t and b respectively. The last three to be discovered, c, b and t, have significantly higher mass than the other three and are therefore unstable and hard to detect. These six quarks and their antiparticles make up the hadrons: the mesons and the baryons. As these make up electrically charged particles with integer charge, the quarks themselves have fractional charge. The u, c and b quarks have charge $+\frac{2}{3}$ and d, c and t quarks have charge $-\frac{1}{3}$. The proton is made of two u quarks and one d quark giving it at total charge of $+1$ and the neutron is made of one u and two d quarks making it neutral.

All quarks are spin-$\frac{1}{2}$ particles. That means the mesons, consisting of a quark and an antiquark, are bosons with either spin 0, if the spin of the quarks are in opposite direction or 1 if the spins are aligned. The baryons can have two different values of the spin: $\frac{1}{2}$ or $\frac{3}{2}$. Here comes an interesting contradiction: baryons consisting of three identical quarks have been observed, for instance the delta plus plus (Δ^{++}) particles made of three u quarks and with a total spin of $\frac{3}{2}$. This seems to violate the Pauli principle. How can this be? This is were the concept of color charge is introduced.

All quarks have color charge. (The term color is arbitrary and has nothing to do with visual colors). All six flavors can be charged with one of the three different colors, red, green or blue. The corresponding antiquarks have anticolors. Just as the electromagnetic force attract particles of different electric charge, so does the strong interaction attract particles of different color. The strong interaction has a very short range, $\sim 10^{-13}$ cm, but is 100 times more powerful than the electromagnetic interaction at that distance. The strong interaction acts in such a way as to make all hadrons color neutral or "white". That is why the only possible combination of hadrons are the mesons with one quark and one antiquark of the same color, or the baryons and antibaryons with three quarks or antiquarks of different colors [11].

2.4 Quantum chromodynamics (QCD)

The only noticeable effects in the our daily world are the those of electromagnetism and gravity, since these forces have an infinite range. Thinking of a force, we imagine something able to change an objects velocity, be it speed or direction. When looking at the extremely small scales of elementary particles it is necessary to leave the macroscopic Newtonian mechanics behind and work with quantum mechanics instead. In quantum mechanics a force is rather an interaction between particles. Only gravity has not yet been studied in the SM because it is so much weaker than the other forces, so on a subatomic level its effects are miniscule.

The other three forces are described with gauge theories. A gauge theory is a mathematical description of how the force behaves. In gauge theory the concept of symmetry is essential. Symmetry arises when the solutions to a given set of equations remains the same even though some characteristic of the system is changed. If a characteristic can be changed with a varying amount in all points in space and the set of equations are still valid the system is said to have local symmetry. According to the gauge theories the forces arise so that local symmetry still holds when characteristics are changed. As an example one might think of a rubber ball: it may be deformed (i.e. a change of a characteristic), but that deformation gives rise to forces trying to regain the balls original shape.

The gauge theory of the strong force is called quantum chromodynamics (QCD). Just as the gauge theories for the other elementary forces, QCD has force carriers. These are called gluons and are equivalent to the photon of electromagnetic interaction. Colored quarks interact through exchange of gluons. There are eight different types of gluons and they are in themselves colorcharged [11]. This leads to the theoretical existence of glueballs, hypothetical object of two or three gluons attracted to each other via their colorcharge.

To get QCD to match all experiments have shown to be quite difficult. This is mostly due to the nature of the colorcharge. Gravity and electromagnetic interaction both decrease with distance. This is not the case with the strong interaction. As no single quark has ever been observed in isolation, it is thought that the strong interaction increases with greater distance. That makes it extremely difficult to detect an unpaired quark. However, all experiments indicate that within the hadrons the quarks can move quite freely, without distracting interference between one and other. This state is known as asymptotic freedom, as the quarks asymptotically reaches free particle states the closer they get to each other. As the laws of motion for the quarks are not known, several models have been proposed [15]. Some of the most prominent are described in the section below.

2.5 Different nucleon models

Often when modeling complicated systems physicists use a method called perturbation theory. Perturbation theory is a technique where one first uses an easier system with a known solution. Then to model the real system one adds small disturbances, so called perturbations. The problems simplifies significantly using such a technique. However, as the irregular disturbances are so great in QCD, perturbation is unfortunately not a viable method. This, together with the odd behavior of increasing strength with distance, has made it very difficult to find a good model.

2.5.1 Lattice QCD

One of the more successful models was formulated by Kenneth G. Wilson of Cornell University in 1974 [16]. He proposed that time and space could be seen as discrete points on a lattice. The lattice is four-dimensional: three dimensions of space and one dimension of time. Time and space would be quantized on this lattice. Taking a step in one of the space directions would move you a discrete distance in that direction, and taking a step in the time direction would move you a discrete distance in time. However all experiments indicate that space-time is indeed continuous and the idea is that one would make the mesh finer and finer and in the mathematical limit of infinitesimal spacing, one would regain continuous space-time. The complete theory is quite complicated and can be studied in greater detail in chapter 3 of *Particles and Forces* [11]. One thing worth mentioning is a great breakthrough for the model that came in the late 1970s. It had been shown that the field between two charges was confined to a certain finite number of points on the lattice. This was good in the sense of QCD, as it would explain why the color charge was confined within the hadrons. However this was proven to be the case for all different fields, and this would indicate that the electric magnetic field between two electric charges would also be confined, as we know from experience to be untrue. The theory was that if the mesh was made finer and finer, hopefully the gauge theory of quantum electrodynamics (QED) would show that the electromagnetic field would stretch out to all points on the lattice, and therefore regain its infinite range. With very tedious and time consuming numerical methods, scientist were able to confirm that as the mesh became finer, the electromagnetic field got infinite range as opposed to the color field that continued to stay confined. Through this discovery, the lattice model gained support from many physicists [11].

2.5.2 The bag-model of Bogoliubov

Another model was suggested by the russian theoretical physicist Nikolay Bogoliubov. His idea was to give the quarks an enormous mass. This would confine the quarks by making them unable to move. However this would seem to contradict the asymptotical freedom observed at very close range. He solved this by confining the quarks within a spherical cavity of radius R in which they feel an attractive field of strength m. The quarks masses were also set to m, and then one let $m \to \infty$. This resulted in Bogoliubov's bag-model, where the quarks could move freely inside the bag but were completely confined within it. Although Bogoliubov's model was a very simple one, it still gave some accurate predictions [17]. The MIT bag-model is a continuation of this simple bag-model and will be thoroughly explained in section 4.

2.5.3 Chiral quark models and the Cloudy bag-model

Later in the thesis it will become evident that the MIT bag-model have some major flaws. The chiral quark models are extensions of the MIT bag-model where a pion field is incorporated outside the bag. As we shall see this will fix some of the MIT bag-model's flaws. One of these models is called the Cloudy bag-model and in section 5.2 we will compare the MIT bag-model to the Cloudy bag-model.

2.6 Four vector algebra

Throughout this text we will make common use of the four vector notation used in special relativity. A four vector is denoted with a greek letter, usually μ or ν, as super- or subscript:

$$x^\mu = (x^0, \mathbf{x}) = (t, x, y, z), \qquad x_\mu = (x^0, -\mathbf{x}) = (t, -x, -y, -z), \tag{1}$$

where t is the time component and x, y and z are the ordinary space components. Observe that the four vectors are indeed vectors but are not written in bold as ordinary vectors are. Equation (1) yields the scalar product

$$x \cdot x = x_\mu x^\mu = x^\mu x_\mu = t^2 - \mathbf{x} \cdot \mathbf{x} = t^2 - x^2 - y^2 - z^2. \tag{2}$$

The four-derivatives are

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right), \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right). \tag{3}$$

Defining the Minkowski tensor $g^{\mu\nu}$ as a 4×4 matrix that is only non zero on the diagonal. It is given by $g^{\mu\nu} = g_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$. Note the special property $g^{\mu\nu} = g_{\mu\nu}$. Following the rules of Einstein notation, any index appearing twice indicates a summation over that index [18, 15]. An example of how the Minkowski metric tensor works is that if $x^\mu = (t, x, y, z)$, then

$$x_\mu = g_{\mu\nu} x^\nu = (t, -x, -y, -z) \qquad \text{and} \qquad x^\mu = g^{\mu\nu} x_\nu = (t, x, y, z). \tag{4}$$

2.7 Natural units

Theoretical physicists often use what is called natural units. In natural units, the reduced Planck constant, $\hbar$, and the speed of light, c, and other physical constants are set to 1. This leads to some unfamiliar relationships between the ordinary units. As the speed of light equals 1,

$$1 \text{ s} = 2.998 \cdot 10^8 \text{ m}. \tag{5}$$

With the relation above $\hbar = 6.582 \cdot 10^{-16}$ eV·s $= 1.973 \cdot 10^{-7}$ eV·m. Putting $\hbar = 1$ then gives

$$1 \text{ m} = \frac{1}{1.973 \cdot 10^{-7} \text{eV}} \tag{6}$$

meaning that one over length has the unit energy. Einsteins famous formula $E = mc^2$ also simplifies to $E = m$, meaning that mass has the unit of energy [19]. Using the Planck-Einstein relation $E = \hbar\omega$ reduces to $E = \omega$, a relation that will be used in later on in the result. Natural units will be used throughout this text.

3 Method

Most of the time working with this thesis have been spent gathering and reading material to get an understanding of the subject. First we studied the referenced material, Bhaduri's Models of the Nucleon [15]. Looking into his references then gave a more detailed picture. Sites like `www.wikipedia.org` have been used to get a general overview and looking through their bibliography has also been helpful in the search for good references. Many references have been found using KTH library search, arXiv and different physics journals such as *Physics Letters* and *Physical Review*.

Getting the mass equation for the glueballs has been done following the reasoning in chapter 3.5 in Bhaduri's Models of the Nucleon. `MATLAB` has been used to get the numerical values and the plots for the mass equation. Figures 2 and 3 have been done using the software `Incscape`.

4 Result

4.1 The MIT bag-model

The MIT bag-model is an evolvement of Bogoliubov's bag-model discussed in section 2.5. The big difference is that the MIT bag-model has an additional term in its Lagrangian density. The term added is the bag constant B, which we shall see corresponds to the outwards pressure exerted on the bag. This simple correction has shown to greatly improve agreement between predictions and tabulated values of the masses of different quarks and hadrons.

4.2 Color confinement

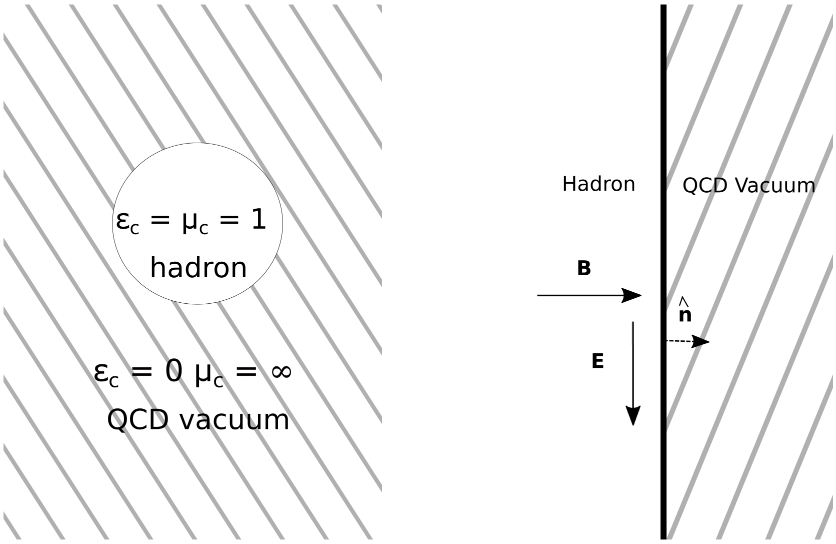
The bag-models, both that of Bogoliubov and MIT, have to explain the fact that a free color particle has never been observed. This section will give a brief description of their reasoning as to why that is.

In classical electromagnetism, the Coulumb potential between two charges gets weekend in a polarizable medium. If ϵ is the dielectric constant of the medium, the Coulomb potential gets screened by

$$V(r) = \frac{e^2}{4\pi\epsilon r}, \tag{7}$$

where e is the elementary charge and $|\mathbf{r}| = r$ is the spatial distance between the charges. Quantum electrodynamics (QED) is the gauge theory of electromagnetism. QED predicts that the photon, carrier of the electromagnetic force, screens itself even in a vacuum where there is no polarizable material. In classical physics $\epsilon = 1$ in a vacuum but due to this screening effect, QED yields $\epsilon > 1$. This makes the attraction grow weaker with increasing r and this is why the Coulumb force decreases with distance [15].

In QCD the quarks are color charged. This makes them subject to the same screening effect as the photons. However, they are also subject to what is called camouflage effect, and this effect overpowers the screening. Camouflage can be thought of as an "anti-screening" effect. This gives, for large enough r: $\epsilon_c < 1$ (where the subscript indicates color) and the attractive force is therefore increased with distance. As r tends to zero, the camouflage and screening effects weakens so $\epsilon_c \to 1$. From electrodynamics the relation



(a) The color permeability and permittivity of the bag-model.

(b) The color fields at the surface of the bag.

Figure 2: Visualization of the color fields in the bag-model.

between ϵ and μ is given by $\epsilon\mu = \frac{1}{c^2}$. In natural units, $c = 1$ results in $\epsilon\mu = 1$. Thus as ϵ_c tends to 1, so does μ_c [11].

The bag-model is an idealization of this phenomena. Inside the bag both μ_c and ϵ_c are equal to one and outside the bag $\epsilon_c = 0$ and $\mu_c = \infty$, see figure 2(a). If you exchange ϵ with μ and vice versa, this representation is analogous to the perfect conductor in classical electrodynamics, so the bag idealization is not very far fetched.

With the boundary condition of a perfect conductor in mind, $\hat{\mathbf{n}} \times \mathbf{E} = \mathbf{0}$ and $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$, the boundary condition for the color fields becomes

$$\hat{\mathbf{n}} \times \mathbf{B} = \mathbf{0}, \qquad \hat{\mathbf{n}} \cdot \mathbf{E} = 0. \tag{8}$$

From the equations above one can see that the color electric field is completely confined within the hadron, see figure 2(b). Observe that the boundary condition in equation (8) only applies to the color fields [15]. The boundary conditions for the quark fields, i.e. the wave function of the quarks, will be derived below.

4.3 Derivation of the MIT bag-model equations

This section will derive important equations for the quark fields using the Lagrangian density of the MIT bag-model. The derivation will follow the reasoning in Bhaduri's Models of the Nucleon [15]. These derivations will be pretty tedious so we will begin by briefly outlining the approach. The derivations have been divided into three different subsections:

- Starting with minimizing the action S given from field theory as the integral over the Lagrangian density, the Euler-Lagrange equations of motion are obtained.
- The specific Lagrangian density of the MIT bag-model is inserted. This yields the boundary conditions for the quark fields in the MIT bag-model.

- Using the energy momentum tensor and the conservation of energy and momentum, the value of the constant B in the Lagrangian density is derived.

4.3.1 Derivation of the Euler-Lagrange equation

As in classical mechanics, the equations of motion are found when minimizing the action S. In field theory, S is given by

$$S = \int \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) d^4x. \tag{9}$$

Here $\mathcal{L}$ is the Lagrangian density, dependent on the field $\phi_i(x)$ and its first derivatives. The Lagrangian is a function summarizing the dynamics of a system. The Lagrangian density is like an ordinary Lagrangian that can vary in both space and time, and it must therefore be integrated over all space-time. By the principle of least action we want to minimize S. Setting $\delta S = 0$, i.e. minimizing S, leaves the following equation

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right] d^4x = 0. \tag{10}$$

Using integration by parts on the second term in the equation above yields

$$\int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) d^4x = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta \phi_i \right] - \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta \phi_i \right] d^4x. \tag{11}$$

Assuming that $\delta \phi_i$ vanishes at the endpoints, the first part of the right hand side is zero. Equation (10) is then reduced to

$$\delta S = \int \delta \phi_i \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right] d^4x = 0. \tag{12}$$

Ignoring the trivial field $\phi_i \equiv 0$ the expression in the brackets in the equation above has to equal zero at all points in space-time, hence

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0, \tag{13}$$

which are the Euler-Lagrange equations of motion that holds for all independent fields ϕ_i.

4.3.2 Using the Lagrangian density of the MIT bag-model

Now introducing the specific Lagrangian for the MIT bag-model as the following expression

$$\mathcal{L} = \left[\frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi) - B \right] \theta_v(x) - \frac{1}{2} \bar{\psi} \psi \Delta_s. \tag{14}$$

Here ψ and $\bar{\psi}$ are the wave function and the adjoint wave function of the quarks and γ^μ is the gamma matrix, also called Dirac matrix [20]. θ_v is a step-function that equals one inside the bag and zero outside the bag. Δ_s is given by the derivative of θ_v as is shown below

$$\theta_v = \theta(R - r), \qquad \frac{\partial \theta_v}{\partial x^\mu} = n_\mu \Delta_s, \qquad \Delta_s = \delta(R - r). \tag{15}$$

Here θ is the Heaveside function, δ is the Kronecker delta, n_μ is a unit vector normal to the surface and R is the radius of the bag. Inserting $\bar{\psi}$ as the field ϕ_i in equation (13) gives

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0. \tag{16}$$

(Using $\bar{\psi}$ instead of ψ yields the boundary condition for ψ). Inserting the Lagrangian density $\mathcal{L}$ from equation (14) into equation (16) and calculating each part separately yields the following expressions

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \bar{\psi}} &= \frac{\partial}{\partial \bar{\psi}} \left(\left[\frac{i}{2}(\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi) - B \right] \theta_v(x) - \frac{1}{2}\bar{\psi}\psi \Delta_s \right) \\ &= \Big(\frac{i}{2}\gamma^\mu \partial_\mu \psi \Big) \theta_v(x) - \frac{1}{2}\psi \Delta_s,\end{aligned} \tag{17}$$

and similarly

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = \partial_\mu \left[\Big(-\frac{i}{2}\gamma^\mu \psi \Big) \theta_v(x) \right] = \Big(-\frac{i}{2}\gamma^\mu \partial_\mu \psi \Big) \theta_v(x) - \frac{i}{2}\gamma^\mu n_\mu \psi \Delta_s. \tag{18}$$

Taking (17) and (18) and inserting them into (16) yields

$$\Big(\frac{i}{2}\gamma^\mu \partial_\mu \psi \Big) \theta_v(x) - \frac{1}{2}\psi \Delta_s - \left[\Big(-\frac{i}{2}\gamma^\mu \partial_\mu \psi \Big) \theta_v(x) - \frac{i}{2}\gamma^\mu n_\mu \psi \Delta_s \right] = 0, \tag{19}$$

which simplifies to

$$\Big(i\gamma^\mu \partial_\mu \psi \Big) \theta_v(x) + \frac{1}{2} \Big(i\gamma^\mu n_\mu \psi - \psi \Big) \Delta_s = 0. \tag{20}$$

From this, two different conditions are derived:

1. Inside the bag, $\Delta_s = 0$ and $\theta_v = 1$ results in
$$i\gamma^\mu \partial_\mu \psi = 0. \tag{21}$$

2. On the surface of the bag, $\Delta_s = \infty$ and $\theta_v = 0$ yields
$$\frac{1}{2} \Big(i\gamma^\mu n_\mu \psi - \psi \Big) \Delta_s = 0. \tag{22}$$
Since $\Delta_s = \infty$ the expression inside the parenthesis must be zero. This yields the boundary condition
$$i\gamma^\mu n_\mu \psi = \psi. \tag{23}$$
Identical calculation for the adjoint wave function ($\bar{\psi}$) shows that the surface boundary condition is given by
$$-i\gamma^\mu n_\mu \bar{\psi} = \bar{\psi}. \tag{24}$$

The probability density is obtain from $\bar{\psi}\psi$. By substituting first ψ and then $\bar{\psi}$ while using the boundary conditions above yields

$$\bar{\psi}\psi = i\bar{\psi}(\gamma^\mu n_\mu \psi) = -i(\bar{\psi}\gamma^\mu n_\mu)\psi, \tag{25}$$

on the bag surface. As the left- and righthand side are exactly the same but of opposite signs, this can only be true if $\bar{\psi}\psi = 0$. Since the probability density is zero at the bag surface, there is no quark current across the surface which means that the quarks are confined within the hadron.

In the next section the value of the bag constant B will be determined.

4.3.3 Deriving the value of the bag constant B

The purpose of this section is to determine the value of the bag constant B in the Lagrangian density. This will be done by starting with the energy momentum tensor, given by the expression below, and then utilizing the conservation of it. The energy momentum tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right)(\partial^\nu \psi) + (\partial^\nu \bar{\psi})\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right)\right]. \tag{26}$$

The big parenthesis in the second term in the brackets has already been calculated in the first step in (18). The same method also gives

$$\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = \frac{i}{2}(\bar{\psi}\gamma^\mu)\theta_v. \tag{27}$$

Inserting these expressions into the energy momentum tensor results in

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \frac{i}{2}\big(\bar{\psi}\gamma^\mu\partial^\nu\psi - (\partial^\nu\bar{\psi})\gamma^\mu\psi\big)\theta_v. \tag{28}$$

The conservation of energy and momentum says that $\partial_\mu T^{\mu\nu} = 0$. Simplifying the first term in the energy momentum tensor $-g^{\mu\nu}\mathcal{L}$ according to the four vector algebra from section 2.6 yields the following expression

$$-g^{\mu\nu}\mathcal{L} = -\frac{i}{2}\Big[(\bar{\psi}\gamma^\mu\partial^\nu\psi - (\partial^\nu\bar{\psi})\gamma^\mu\psi)\Big]\theta_v + g^{\mu\nu}B\theta_v + \frac{1}{2}g^{\mu\nu}\bar{\psi}\psi\Delta_s, \tag{29}$$

where the property $g^{\mu\nu}\partial_\mu = \partial^\nu$ has been used. The term in brackets on the right-hand side in the equation above is exactly the same as the term in parenthesis in $T^{\mu\nu}$, hence they cancel out. This leaves the conservation equation

$$\partial_\mu T^{\mu\nu} = \partial_\mu\big(g^{\mu\nu}B\theta_v + \frac{1}{2}g^{\mu\nu}\bar{\psi}\psi\Delta_s\big) = 0. \tag{30}$$

Once again using the properties of the Minkowski tensor $\partial_\mu g^{\mu\nu} = \partial^\nu$, equation (30) is reduced to

$$\partial^\nu(B\theta_v) + \frac{1}{2}\partial^\nu(\bar{\psi}\psi\Delta_s) = 0. \tag{31}$$

As B is the bag constant its derivative is zero. Equation (3) and (15) yields $\partial^\nu\theta_v = -n^\nu\Delta_s$ and hence

$$-Bn^\nu\Delta_s + \frac{1}{2}\partial^\nu(\bar{\psi}\psi)\Delta_s = 0, \tag{32}$$

and as $\partial^\nu\Delta_s = 0$, we are left with the following equation

$$Bn^\nu\Delta_s = \frac{1}{2}\partial^\nu(\bar{\psi}\psi)\Delta_s. \tag{33}$$

The quantity Δ_s is only non-zero at the boundary of the bag and thus

$$Bn^\nu = \frac{1}{2}\partial^\nu(\bar{\psi}\psi) \qquad \text{on the bag surface.} \tag{34}$$

The normal n^ν is space like, meaning that its time component is zero and therefore $n^\nu n_\nu = -1$ holds. Finally we arrive at

$$B = -n_\nu \frac{1}{2} \partial^\nu (\bar{\psi}\psi), \tag{35}$$

as the value for B on the bag's surface. In chapter 2.5 in Bhaduri [15] it is shown that the outward pressure exerted by a particle in an infinitely deep spherical cavity is

$$P = -\frac{1}{2}\frac{d}{dr}(\bar{\psi}\psi) \qquad \text{at } r = R. \tag{36}$$

If we approximate the bag as spherically symmetric, then $n_\nu \partial^\nu = \frac{d}{dr}$ and we see that

$$B = -\frac{1}{2}\frac{d}{dr}(\bar{\psi}\psi), \tag{37}$$

meaning that $B = P$, the pressure on the surface of the bag. In the specific Lagrangian density of the MIT bag-model B is negative. This means that the outward pressure on the boundary exerted by the confined particles within the bag, is balanced by the inward vacuum pressure B as can be seen in figure 3. When calculating the mass spectrum of glueballs in section 4.5, the fact that the pressure of the bag is equal to B will be used.

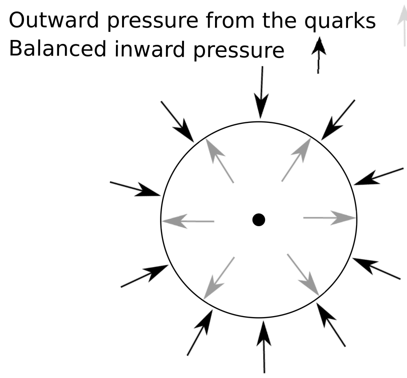


Figure 3: Pressure balance of the bag.

Pressure can be expressed in the units energy over volume, i.e $[E/V]$. Since the calculations and derivations are done in natural units, $B^{1/4}$ has the dimension of energy and thus also mass since they are the same in natural units [21].

4.4 Derivation of the eigenmodes of the color fields

This section will focus on deriving the eigenmodes equation of the color fields in transverse electric (TE) and transverse magnetic (TM) modes. In a TE mode $E_r = 0$ and in a TM mode $B_r = 0$. Later it shall be revealed that this gives the energy of the gluon color fields. The derivations in this section will mostly focus on those of a TE mode. The results in a TM mode will also be given but without the cumbersome derivations. The eigenmodes will be derived by using the Maxwell equations (38) and the boundary conditions for the color fields given in section 4.2. All calculations that follow are done by assuming a MIT bag-model for the glueballs and that the particles confined in the bag are not quarks but rather the gluons themselves.

To get an overview of the coming derivations we will once again begin by giving a brief outline of the approach. The section is divided into four subsections:

- In section 4.4.1 we will state the Maxwell equations and give the proper boundary conditions for the spherical MIT bag-model.
- In section 4.4.2 the Maxwell equation will be greatly simplified by working in a TE or TM mode.
- Section 4.4.3 will focus on solving a differential equation to get E_ϕ of the electric color field. B_ϕ will be given in the same way.
- Lastly in section 4.4.4 the eigenvalue equations for the eigenmodes is derived and an expression for the gluon color field energy is obtained.

4.4.1 Maxwell equations and the boundary conditions for the color fields

The classical Maxwell equations in electromagnetism are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \qquad (1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ Maxwell equations}). \tag{38}$$

The gluons have color charge and are therefore subject to the color fields. Referring to section 4.2 where the relation between the electromagnetic fields of QED and the color fields of QCD was discussed, the Maxwell equations for the color fields are given by changing $\mathbf{E} \to -\mathbf{B}$ and $\mathbf{B} \to \mathbf{E}$. If one inserts this in the Maxwell equation, one sees that they are in fact unchanged, hence the traditional Maxwell equation given in equation (38) actually governs the color fields as well. Therefore the color fields of the glueballs are also governed by the Maxwell equations. In this and in the forthcoming section $\mathbf{E}$ and $\mathbf{B}$ shall denote the electric and magnetic *color* fields of QCD.

All the calculations are done assuming the bag as static spherical. This assumptions is valid for low energy cases and is a necessity in the calculation of the glueball mass. In the spherical case the normal vector in the boundary conditions in equation (8) becomes $\hat{\boldsymbol{r}}$ and thus the boundary conditions becomes

$$\hat{\boldsymbol{r}} \times \mathbf{B}\big|_{r=R} = 0, \qquad \hat{\boldsymbol{r}} \cdot \mathbf{E}\big|_{r=R} = 0, \tag{39}$$

where R is the radius of the bag. Evaluating these equations gives

$$B_\theta = B_\phi = E_r = 0, \qquad \text{at } r = R. \tag{40}$$

The curl of a vector $\mathbf{A}$ in spherical polar coordinates is given by

$$\begin{aligned}\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\hat{\boldsymbol{\theta}} \\ + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned} \tag{41}$$

Assuming that $\mathbf{E}$ and $\mathbf{B}$ are monochromatic waves they have time dependence $\sim e^{-i\omega t}$, their time derivatives are

$$\frac{\partial \mathbf{E}}{\partial t} = i\omega \mathbf{E} = i\omega \Big[E_r \hat{\boldsymbol{r}} + E_\theta \hat{\boldsymbol{\theta}} + E_\phi \hat{\boldsymbol{\phi}} \Big]. \tag{42}$$

and

$$-\frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B} = -i\omega \Big[B_r \hat{\boldsymbol{r}} + B_\theta \hat{\boldsymbol{\theta}} + B_\phi \hat{\boldsymbol{\phi}} \Big]. \tag{43}$$

For simplicity only the ϕ-independent solutions will be considered , meaning that $\frac{\partial}{\partial \phi} = 0$.

4.4.2 Simplifications in a TE and TM mode

Now working in a TE mode means that the radial part of the electric color field is zero: $E_r = 0$. This leads to great simplifications in the Maxwell equations as will be seen in this section.

Starting by looking in the $\hat{\boldsymbol{r}}$-direction in the second Maxwell equation yields the following

$$\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta B_\phi)}{\partial\theta} - \frac{\partial B_\theta}{\partial \phi} \right] = i\omega E_r. \tag{44}$$

No ϕ-dependence and $E_r = 0$ gives B_ϕ as

$$\frac{\partial(\sin\theta B_\phi)}{\partial\theta} = 0 \qquad \Rightarrow \qquad B_\phi = \frac{g(r)}{\sin\theta} e^{-i\omega t} \tag{45}$$

after an integration with respect to θ. Observe that the unknown function g is not a function of ϕ as there is no ϕ-dependence. Next is the $\hat{\boldsymbol{\theta}}$-direction in the second Maxwell equation:

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{\partial(rB_\phi)}{\partial r} \right] = i\omega E_\theta \qquad \Rightarrow \qquad E_\theta = \frac{i}{\omega r}\frac{\partial(rB_\phi)}{\partial r}. \tag{46}$$

The $\hat{\boldsymbol{\phi}}$-direction in the first Maxwell equation gives

$$\frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta} \right] = -i\omega B_\phi. \tag{47}$$

With $E_r = 0$ and E_θ from (46) this becomes

$$-i\omega B_\phi = \frac{1}{r}\frac{\partial\left(r\frac{i}{\omega r}\frac{\partial(rB_\phi)}{\partial r}\right)}{\partial r} = \frac{i}{\omega r}\frac{\partial^2}{\partial r^2}(rB_\phi). \tag{48}$$

The above equation is easily rewritten by evaluating the derivative as

$$\frac{\partial^2 B_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial B_\phi}{\partial r} + \omega^2 B_\phi = 0, \tag{49}$$

which can be recognized as the characteristic differential equation for the spherical Bessel functions (with $l = 0$) [22]. The solution is given by $B_\phi = j_0(\omega r) f(\theta, t)$ where

$$j_0(\omega r) = \frac{\sin(\omega r)}{\omega r}. \tag{50}$$

Comparing this expression of B_ϕ with (45) the following relation is given

$$B_\phi = \frac{g(r)}{\sin\theta}e^{-i\omega t} = j_0(\omega r)f(\theta,t) \tag{51}$$

and it then follows that B_ϕ is

$$B_\phi = A\cdot\frac{j_0(\omega r)e^{-i\omega t}}{\sin\theta} \qquad \text{where } A \text{ is a constant.} \tag{52}$$

Lastly we insert B_ϕ into the expression of E_θ in equation (46):

$$E_\theta = \frac{i}{\omega r}\frac{\partial(rA\cdot\frac{j_0(\omega r)e^{-i\omega t}}{\sin\theta})}{\partial r} = \frac{Aie^{-i\omega t}}{\omega^2 r\sin\theta}\frac{\partial}{\partial r}\sin(\omega r) = \frac{Aie^{-i\omega t}}{\omega r\sin\theta}\cos(\omega r), \tag{53}$$

so $E_\theta \sim \frac{\cos(\omega r)}{r}$. But E_θ has to be limited at $r = 0$ and thus $A = 0$. Consequently, $E_\theta = B_\phi = 0$ in a TE mode.

Following the exact same reasoning in a TM mode, where $B_r = 0$, one gets $B_\theta = E_\phi = 0$.

4.4.3 Calculating E_ϕ and B_ϕ

Consequently, in a TE and TM mode there are only three non-zero equations left in the Maxwell relations. The three TE mode equations are given below. The TM mode equations are identical if $\mathbf{E}$ is exchanged with $\mathbf{B}$. The three equations are

$$\frac{1}{r}\frac{\partial(rE_\phi)}{\partial r} = i\omega B_\theta, \tag{54}$$

$$\frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta} = -i\omega B_r, \tag{55}$$

$$\frac{1}{r}\left[\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial\theta}\right] = i\omega E_\phi. \tag{56}$$

Solving for B_θ and B_ϕ in equations (54) and (55), and inserting these into (56) yields

$$\begin{aligned} i\omega E_\phi &= \frac{1}{r}\left[-\frac{\partial\big(r\frac{i}{\omega r}\frac{\partial(rE_\phi)}{\partial r}\big)}{\partial r} - \frac{\partial\big(\frac{i}{\omega r\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}\big)}{\partial\theta}\right] \\ &= \frac{i}{\omega r}\left[-\frac{\partial^2(rE_\phi)}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}\right)\right] \end{aligned} \tag{57}$$

which reduces to

$$\frac{1}{r}\frac{\partial^2(rE_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(\sin\theta E_\phi)}{\partial\theta}\right) + \omega^2 E_\phi = 0. \tag{58}$$

Calculating the two derivative terms and comparing with the Laplacian in spherical coordinates, it can be shown that the previous equation can be written more compactly as

$$\left(\nabla^2 + \omega^2 + \frac{1}{r^2\sin^2\theta}\right)E_\phi = 0. \tag{59}$$

Using the method of separation of variables, i.e. putting $E_\phi = f(r)h(\cos\theta)e^{-i\omega t}$, and inserting it to equation (59) it is possible to completely separate the r dependence and the θ dependence. Hence, they must both be constant. This gives two differential equations that can be solved separately.

The equation with the r-dependence gives $f(r) = j_l(\omega r)$ where $j_l(\omega r)$ are the previously mentioned spherical Bessel functions. The equation with the θ-dependence gives $h(\cos\theta) = L_l^1(\cos\theta)$, where $L_l^1(\cos\theta)$ are the associated Legendre Polynomials [22]. Consequently

$$E_\phi = E_0 L_l^1(\cos\theta) j_l(\omega r)e^{-i\omega t}. \tag{60}$$

The solution $l = 0$ is trivial because $L_0^1(\cos\theta) = 0$ so the lowest energy solution is given by $l = 1$, where $L_1^1(\cos\theta) = \sin\theta$. Hence E_ϕ is simply

$$E_\phi = E_0 \sin\theta j_1(\omega r)e^{-i\omega t} = E_0 \sin\theta\bigg(\frac{\sin(\omega r)}{(\omega r)^2} - \frac{\cos(\omega r)}{\omega r}\bigg)e^{-i\omega t} \tag{61}$$

where the definition of $j_1(\omega r)$ has been used in the last step.

In a TM mode all calculations for B_ϕ are identical and one ends up with the corresponding expression

$$B_\phi = B_0 \sin\theta\bigg(\frac{\sin(\omega r)}{(\omega r)^2} - \frac{\cos(\omega r)}{\omega r}\bigg)e^{-i\omega t}. \tag{62}$$

4.4.4 Deriving the eigenmode equations

Beginning in a TE mode by inserting E_ϕ into equation (54) yields

$$i\omega B_\theta = \frac{E_0\sin\theta}{r}\frac{\partial(rj_1(\omega r))}{\partial r}e^{-i\omega t} = \frac{E_0\sin\theta}{r}\bigg[j_1(\omega r) + r\frac{\partial(j_1(\omega r))}{\partial r}\bigg]e^{-i\omega t}. \tag{63}$$

It is now time to use the boundary condition $B_\theta = 0$ at $r = R$. This gives

$$\frac{E_0\sin\theta}{r}e^{-i\omega t}\bigg[j_1(\omega r) + r\frac{\partial(j_1(\omega r))}{\partial r}\bigg] = 0 \qquad \text{at } r = R, \tag{64}$$

meaning that the term inside the brackets has to be zero on the boundary of the bag. Inserting the definition of $j_1(\omega r) \equiv \frac{\sin(\omega r)}{(\omega r)^2} - \frac{\cos(\omega r)}{\omega r}$ and then evaluating the derivative yields the following expression

$$\frac{\sin(\omega R)}{\omega^2R^2} - \frac{\cos(\omega R)}{\omega R} + R\bigg(-\frac{2\sin(\omega R)}{\omega^2R^3} + \frac{2\cos(\omega R)}{\omega R^2} + \frac{\sin(\omega R)}{R}\bigg) = 0, \tag{65}$$

which simplifies to

$$\sin(\omega R)(1 - \omega^2R^2) = \omega R\cos(\omega R). \tag{66}$$

Finally we get arrive at eigenmode equation for the color fields of a gluon in a TE mode

$$\tan(\omega R) = \frac{\omega R}{1 - \omega^2R^2}. \tag{67}$$

The lowest non trivial solution is given by $\omega R = 2.744$.

Now working in a TM mode the boundary conditions remain the same. Looking back at equation (39), one recognizes that $B_\phi = 0$ at the boundary. This gives the condition

$$B_\phi = B_0 \sin\theta \left(\frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} \right) e^{-i\omega t} = 0. \tag{68}$$

Ignoring the trivial solution $B_0 = 0$ results in that the parenthesis must be zero

$$\frac{\sin(\omega R)}{(\omega R)^2} - \frac{\cos(\omega R)}{\omega R} = 0 \qquad \rightarrow \qquad \tan(\omega R) = \omega R. \tag{69}$$

Hence for a TM mode one get the eigenmode equation $\tan(\omega R) = \omega R$. The lowest non trivial solution is given by $\omega R = 4.493$.

To get the energy from the frequency the Planck-Einstein relation gives $E = \hbar\omega$, which in natural units simply is $E = \omega$ and thus

$$E_g = \frac{\zeta}{R}. \tag{70}$$

The subscript g denotes that it is the gluon color field energy [15] and ζ is given by

$$\zeta = \begin{cases} \zeta_{TE} = 2.744 & \text{in a TE mode} \\ \zeta_{TM} = 4.493 & \text{in a TM mode.} \end{cases} \tag{71}$$

4.5 Calculation of the glueballs mass spectrum

Now we have everything needed for an estimation of the lowest energy states of two and three gluon glueballs. The energy calculated in the previous section is the energy for the gluon color fields in the lowest TE and TM modes. It is the color field energy for a single gluon, hence it must be multiplied by the number of gluons in the glueball. This thesis will study five different constellations of glueballs. They will be referred to by the following notation: $(\mathrm{TE})^2$ meaning a two gluon glueball where both gluons are in a TE mode, (TETM) meaning a two gluon glueball where one of the gluons is in a TE mode and the other in a TM mode and so forth.

The gluon color field energy is not the only contribution to the mass of the glueball. There are various other energy contributions like gluon interactions, hyperfine interaction, centre of mass correction etc. that we choose to neglect due to their small significance and difficulty to account for. However, we will include the second and third greatest energy contributions (after the gluon color field energies) in our mass calculation: the potential energy due to the pressure in the bag and the zero-point energy. The assumption that the bag is static spherical means that it has a constant radius R. This considerably simplifies the calculation and the problem can be solved analytically [23]. It is under these assumptions that the strength of the MIT bag-model for calculating mass spectrum of hadrons is evident.

First we shall consider the potential energy due to the pressure in the bag. As the bag is under constant pressure it has potential energy equal to the pressure times the volume of the bag. Recalling that the bag constant is equal to the pressure, this energy contribution becomes

$$E_B = \frac{4\pi}{3} B R^3. \tag{72}$$

The third energy contribution we shall regard is the zero-point energy contribution. Even in the lowest possible ground state a quantum system has energy. In gauge theories the zero point energy is the energy of the vacuum, which is not considered to be empty space but rather the ground state of the fields [11]. Following the logic in *Masses and other parameters of the light hadrons* by Thomas DeGrand et al. [23] we shall account for this by adding the term $E_0 = -\frac{Z_0}{R}$ to our mass equation.

Adding the contributions together, the mass equation is given by

$$M(R) = E_g + E_B + E_0 = \frac{n_{TE} \cdot \zeta_{TE} + n_{TM} \cdot \zeta_{TM}}{R} + \frac{4\pi}{3} BR^3 - \frac{Z_0}{R}, \tag{73}$$

where n_i is the number of gluons in the ith mode in the glueball. As nature will strive to minimize the energy, the ground state radius R_0 is given by

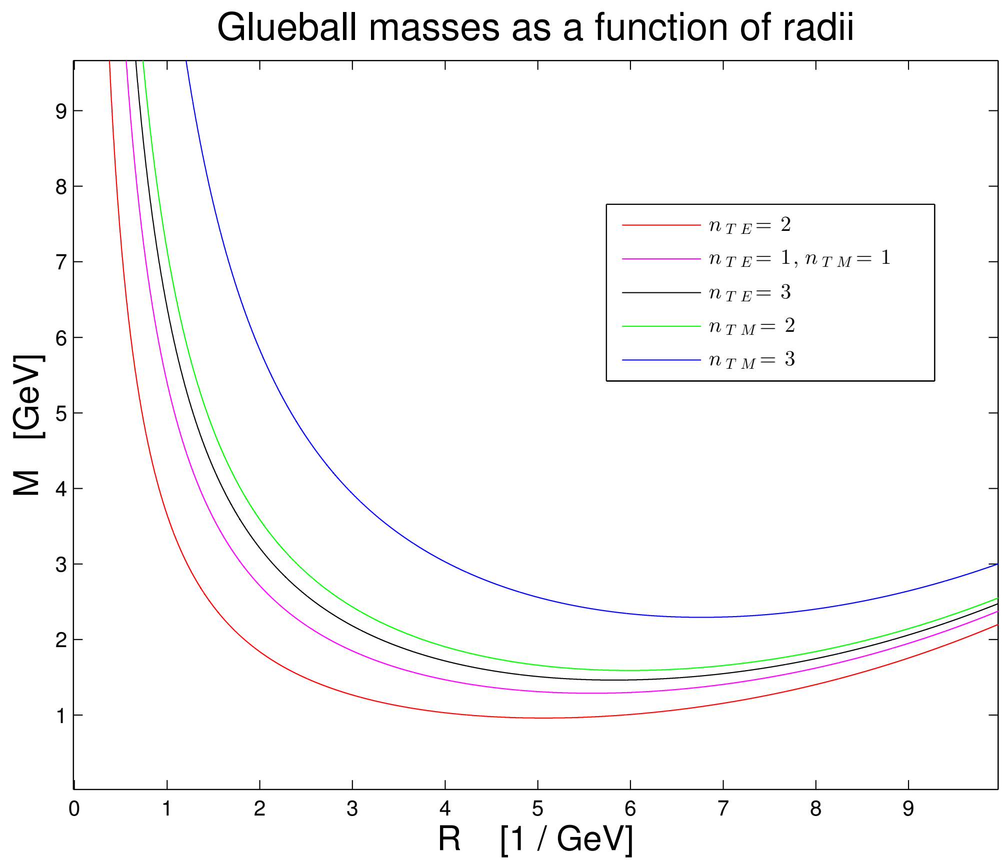


Figure 4: Glueball masses in the ground state as a function of radii with $B^{1/4} = 0.145$ GeV and $Z_0 = 1.84$. n_{TE} and n_{TM} represents the number of gluons in each mode.

$$\left.\frac{\partial M(R)}{\partial R}\right|_{R=R_0} = 0. \tag{74}$$

and the mass of the ground state glueball will be $M(R_0)$. The above equation gives the condition

$$-\frac{\alpha}{R_0^2} + 4\pi B R_0^2 = 0, \tag{75}$$

where $\alpha = n_{TE} \cdot \zeta_{TE} + n_{TM} \cdot \zeta_{TM} - Z_0$. Thus the radius of the glueball is

$$R_0 = \left(\frac{\alpha}{4\pi}\right)^{1/4} B^{-1/4}. \tag{76}$$

This inserted into the mass equation (80) gives the mass as only a function of the bag constant B

$$M = \left[\alpha \cdot \left(\frac{4\pi}{\alpha}\right)^{1/4} + \frac{4\pi}{3} \cdot \left(\frac{\alpha}{4\pi}\right)^{3/4}\right] B^{1/4}. \tag{77}$$

As the MIT bag-model works fine for calculations of various hadron masses using the same method we just did, the values of the constants Z_0 and B can be calculated by comparing the mass equation to already known hadron masses. By trying to fit many hadron masses as well as possible, DeGrand et al. obtained the constants as $B^{1/4} = 0.145$ GeV and $Z_0 = 1.84$ [23]. Inserting these values into equation (80) with the different n_{TE} and n_{TM} combinations and plotting in `MATLAB` yields the plot in figure 4. Finding the minimum values of the five curves gives the values in table 1. Inserting the value of B and Z_0 into (76) and (77) yields the same result as the `MATLAB` calculations. These results are in excellent agreement with the more detailed calculations by Robert L. Jaffe and Keith Johnson [24], indicating that our previous approximations are not to crude.

Table 1: The radii and masses for different ground state glueballs.

n_{tot}	n_{TE}	n_{TM}	R [GeV^{-1}]	R [fm]	M [GeV]
2	2	0	5.06	1.00	0.961
2	1	1	5.58	1.10	1.29
3	3	0	5.83	1.15	1.46
2	0	2	5.99	1.18	1.59
3	0	3	6.77	1.33	2.29

5 Discussion

The MIT bag-model is one of the models simulating the laws of motion for the quarks within the nucleon. One of its biggest advantages is its ability to fit masses of ground state baryons and mesons reasonably close to the tabulated values. That these masses are easily solved for and analytically solvable is of course great assets to the model. However, its predictions are not accurate for all masses. The mass of the pion is predicted to be 280 MeV, about twice its actually mass [25, 26]. The model's predictions of the masses of excited hadron states are also unsatisfactory.

Apart from this, the MIT bag-model have another serious drawback: the predicted radii of the hadrons are too big. The model predicts the radius for the bag to be about 1.1 fm giving a nucleon a volume of $\sim$ 6 fm^3. The measured volume of a nucleus is roughly 6 fm, meaning that the bags would be in contact with each other. This in turn means that the quarks wave functions would overlap, a phenomenon not observed in experiments [15]. The chiral quark models are extensions of the MIT bag-model that take consideration of this. It will be compared to the MIT bag-model in section 5.2.

5.1 Approximations in our calculations

During the calculation and derivations of the glueballs masses a lot of approximations have been done. First of all the bag was approximated as static spherical. The pressure is only isotropic in low energy state, meaning that for excited states the bag will not remain spherical, hence the bag will be deformed [15]. However, as this thesis only focuses on low energy states, this should not be a major issue.

Secondly, as mentioned in section 4.5, many energy contributions have been discarded. If one wanted to make the calculations more precise, one should first of all consider the gluon-gluon interaction. As the gluons are color charged, they attract each other via the color fields. This hyperfine splitting of energy levels gives a contribution to the mass term proportional to the coupling constant. The coupling constant, denoted α_s, is a measurement of how strongly a gluon is affected by the color fields and is given by $\alpha_s = \frac{g^2}{4\pi}$, where g is the color charge. This term can be added perturbatively to get a more exact result. We chose to ignore this term as Bhaduri mentions that the coupling constant is not really a constant, but rather dependent on the relative distance between the color charges making it more difficult to account for. It also gives only a small correction to the mass. Other energy contributions that have been neglected are centre of mass correction and color magnetic self energy due to their small significance and difficulty to account for.

The choice of the constants B and Z_0 have been taken from [23]. These values were obtained, as mentioned before, by trying to fit the mass equation to known masses of hadrons. This choice of constants are in no way unique and by doing another fit, new values will be obtained, though they will be relatively equal.

In section 4.4 the calculations have been done assuming ϕ-independence. Intuitively, this feels most plausible and since our result corresponds very well to other calculated values, this seems like a valid approximation.

5.2 Comparing the MIT bag-model to other nucleon models

This section will compare the MIT bag-model to the other nucleon models mentioned in section 2.5.

The bag-model of Bogoliubov had some serious faults. Even though it gave some decent predictions it was not very well suited for spectroscopy. Also the radius of the bag was inserted manually and was impossible to derive from the model. As a matter of fact, minimizing the energy of the bag gave $R \to \infty$. The MIT bag-model manages to eliminate these flaws. It is very well suited for mass spectrum calculations and the value of the radius is indeed derived mathematically, even if the derived radii are too large [17].

Contrary to the MIT bag-model where the results can be found analytically, lattice QCD is a model that is very dependent on numerical methods. Even if the theory itself is valid, it is hard to test it due to its need for extreme computational power. To get enough lattice points, a Monte Carlo simulation is often applied in lattice QCD, and this leads to statistical errors. As the theory implies an infinite lattice, which of course is impossible to simulate, one also get approximational errors and the numerical calculation are very time consuming. However, as today's computers are very rapidly getting more advanced, calculations using lattice QCD have become more and more common. Glueball masses have been calculated using lattice QCD. In a study done by Chen et al. [27] numerous different glue ball masses were calculated. A comparison between our results and their corresponding masses are shown in figure 5. As one does not have a confining boundary in lattice QCD, the glueballs are not in any specific transverse mode. Glueballs have specific quantum numbers in total angular momentum, parity and c-parity [15]. Therefore they are compared to the glueballs with the same quantum numbers in the measurements done by Chen et al. Hence the TE and TM written on the x-axis in figure 5 are for our calculations only. Observe that $(\mathrm{TM})^3$ has been left out due to us not finding the corresponding glueball and that the masses of $(\mathrm{TE})^2$ and $(\mathrm{TM})^2$ from Chen et al. are equal since they have identical quantum numbers. One can see that the results differ quite a bit and that lattice QCD in general predicts a much higher mass than the MIT bag-model. Many other recent studies have also predicted these higher values so there is reason to believe these are more accurate [28, 8].

A flaw lattice QCD shares with the MIT bag-model is that chiral symmetry is broken. The chiral quark model manages to correct this flaw. It also adjusts the drawback of the large radius. The most prominent chiral quark model is the *Cloudy bag-model.* This model is like the MIT bag, but in this case the bag is surrounded by a pion cloud that can penetrate the bag. This incorporation leads to extra terms in the Lagrangian of the Cloudy bag-model. Evaluated this can be shown to decreases the radius of the bag to 0.82 fm, which is in good agreement with experiments [29].

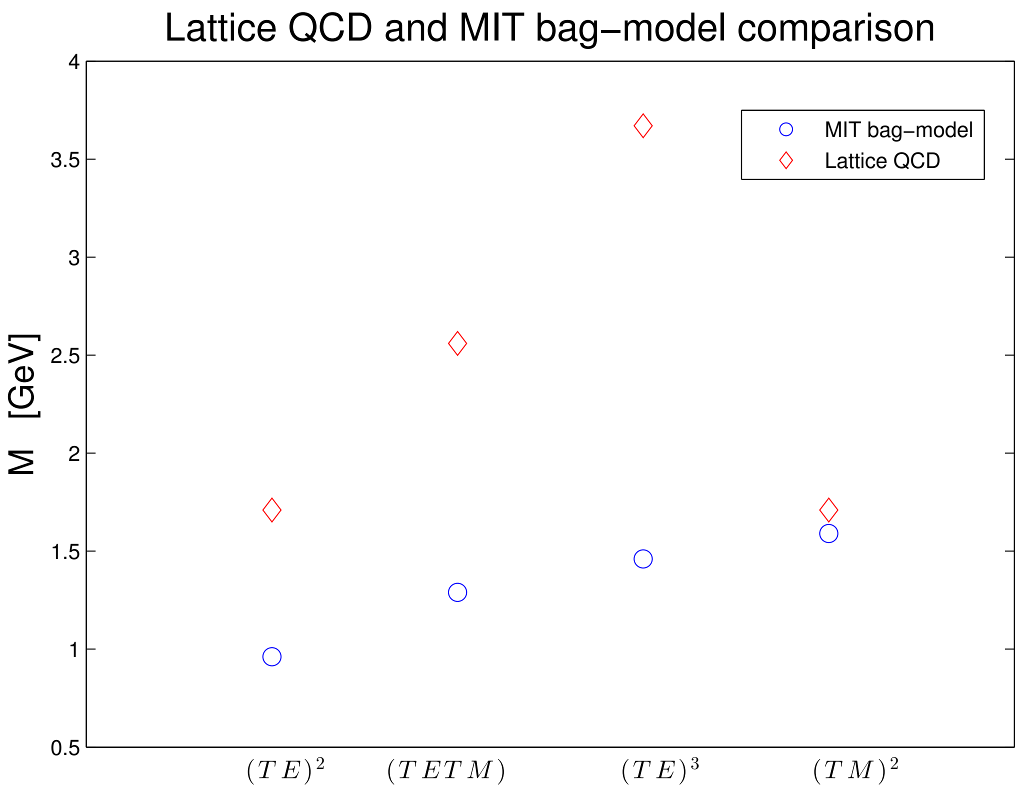


Figure 5: Our calculations of glueball masses using the MIT bag-model compared to calculations by Chen et al. [27] using lattice QCD.

5.3 Further research on glueballs

The MIT bag-model was developed in the 1970s and most of the work done around the model was during that time. All the references for the MIT bag-model used in this thesis were written in the 1970s and the 1980s. Since then, much work has been done. For instance, the mass of the lowest glueball state corresponding to our $(\text{TE})^2$ state have been calculated several times and found to be $\sim$1.6 GeV. The most popular method for mass estimation seems to be lattice QCD [30].

In recent times a lot of data has been gathered in the search for particles that could be the predicted glueballs. Many researchers work with decays and annihilation processes to study the daughter products created. In these decays they have been able to determine new particles that could be glueballs. Very recent studies done on the subject conclude that there are various experimentally discovered particles that are good candidates, but no one has yet been able to prove they are in fact a glueball. Three particles often mentioned are $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$. They are all scalar mesons and their masses are given in MeV in their respective parenthesis. In a study as recent as March 2015, Xiao-Gang He and Tzu-Chiang Yuan [28] show results that indicate that the most likely candidate is $f_0(1710)$, and that other experiments can be conducted that would greatly support this theory.

6 Summary and conclusion

The nature of color charge is that it grows stronger with distance and at close range the color charges seems to be unaffected by each other. To get a model that satisfies all experimental results have shown to be quite difficult. Hence various different models have been proposed, one of which is the MIT bag-model.

The idea with the MIT bag-model is that the color charges, be it quarks or gluons, are confined within a cavity. Within this cavity the color charges act like free particles and the boundary condition is that there is no color flow through the surface. The particles are then completely confined within the bag and this would explain why no free color charge has ever been experimentally observed.

The MIT bag-model was not the first bag-model proposed. A simpler model was suggested by Bogoliubov, but his model had some serious flaws most of which were corrected by the MIT bag-model. It did so by incorporating a bag constant B in the Lagrangian density, equation (14). The boundary conditions and the bag constant B were both derived in section 4.3. The boundary conditions were derived as follows:

- Starting with minimizing the action S, the Euler-Lagrange equations of motion for a field was obtained. Insertion of the specific Lagrangian density of the MIT bag-model resulted in the boundary conditions of the bag. These could be combined to give $\bar{\psi}\psi = 0$ at the surface of the bag meaning that there is no flow of color charge across the boundary.

The bag constant B was derived as follows:

- The energy momentum tensor and the conservation of energy and momentum led to the value of the bag constant $B = -\frac{1}{2}n_\nu\partial^\nu(\bar{\psi}\psi)$. In the static spherical approximation it was shown that $B = P$, where P is the pressure at the surface of the bag. This means that the outward pressure exerted by the confined color particles on the surface of the bag, is balanced by the inwards vacuum pressure B in the Lagrangian density.

Having derived the MIT bag-model equations, we started with the determination of the glueball masses. The MIT bag-model was used to model the confinement of the color charged gluons within the glueball. The Maxwell equations for the color fields together with the boundary conditions led to three equations after the simplifying assumption of TE and TM modes. These three equations were solved to give the ϕ-part of the fields as $E_\phi = E_0 L_l^1(\cos\theta) j_l(\omega r) e^{-i\omega t}$ and $B_\phi = B_0 L_l^1(\cos\theta) j_l(\omega r) e^{-i\omega t}$. Using the boundary conditions $B_\theta = B_\phi = 0$ at $r = R$ the lowest gluon color field energy were given as

$$E_g = \frac{\zeta}{R}. \tag{78}$$

where ζ is

$$\zeta = \begin{cases} \zeta_{TE} = 2.744 & \text{in a TE mode} \\ \zeta_{TM} = 4.493 & \text{in a TM mode.} \end{cases} \tag{79}$$

The gluon color field energy was only one of the components of the total mass. The other two energy components accounted for in this thesis were the potential energy E_B due to the pressure and the zero point energy E_0. Adding these three terms together gave the mass formula

$$M(R) = \frac{n_{TE} \cdot \zeta_{TE} + n_{TM} \cdot \zeta_{TM}}{R} + \frac{4\pi}{3} B R^3 - \frac{Z_0}{R}. \tag{80}$$

Using this formula, and numerical values of B and Z_0 taken from DeGrand et al. [23], resulted in the graph in figure 4. The minimum value of the curves yielded the mass and radii of the different glueballs represented in table 1. The lowest glueball mass was that of $(\mathrm{TE})^2$ and was calculated to be 0.961 GeV.

The discussion section focused on the approximations, comparison of the MIT bag-model to other nucleon models and a brief insight into the further research of glueballs. The approximations impact on the result were discussed and found to be mostly negligible, as our result was in great agreement with other articles doing the same mass calculations with the MIT bag-model. In the comparison we treated the bag-model of Bogoliubov, lattice QCD and the Cloudy bag-model to the MIT bag-model. The MIT bag-model have some great advantages. It is especially easy to work with for mass determination. The Cloudy bag-model is an extension of the MIT model that corrects the chiral symmetry as well as the drawback of the large radius. Lattice QCD is a model that requires many numerical calculations and very great computational power. In recent times, computers have improved greatly making the numerical calculation of lattice QCD simpler. This has led to lattice QCD being the most preferred method for mass estimation of glueballs.

The search for finding glueballs is still ongoing today. The most promising candidate is the scalar meson $f_0(1710)$. Articles published only months ago believe this may actually be the lowest state glueball. More research is needed to determine beyond doubt, the existence of glueballs. Proof of their existence would be a great contribution to the standard model of particle physics.

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