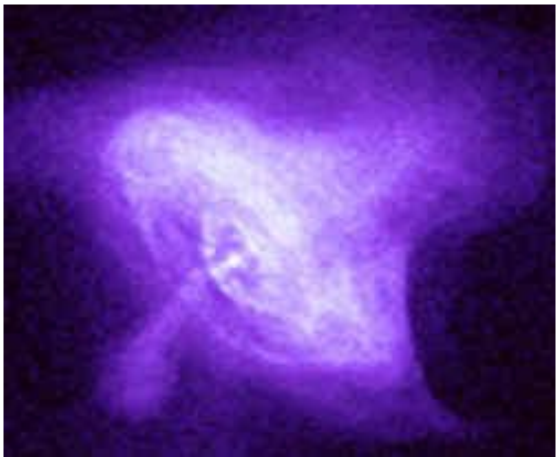
PHYS 3033 GENERAL RELATIVITY PART V
Chapter 13
General relativistic stellar models

13.1 The Tolman-Oppenheimer-Volkoff equation



In the older astrophysical theories, the Newtonian equations of gravity and mechanics were used. This was apparently justified, since **in many cases the crucial factor** $\frac{GM}{c^2 R}$ **was really small as compared to unity.** When, however, **this factor would approach unity, the use of the general theory of relativity seems imperative.** The **general relativistic hydrostatic equilibrium equation is referred to as the Tolman-Oppenheimer-Volkoff equation** (TOV).

The derivation of the TOV equation is based on the following assumptions[1]:

a) **Matter constitutes a perfect fluid** with energy-momentum tensor

$$T_i^k = (\rho c^2 + p)u_i u^k - p\delta_i^k ,$$

where ρ and p are the pressure and mass-density of the fluid, respectively, and u^i is the four-velocity vector of the fluid

b) **The star is spherically symmetric**; this assumption allows us to write the metric inside the star in the form

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where r is the radial co-ordinate and the domains of the spatial co-ordinates r , θ and φ are $0 \le r \le \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$.

c) **The system is in equilibrium**. This assumption leads to **two conditions**. First, it follows that **the metric coefficients are independent of the time** t, so that ν and λ are functions of r only, $\nu = \nu(r)$, $\lambda = \lambda(r)$. Second, **the velocity has no space-like components in the chosen metric**, so that

$$u^0 = e^{-\frac{\nu}{2}},\ u^1 = u^2 = u^3 = 0 .$$

With the use of these values it follows that **the energy-momentum tensor has only diagonal components non-vanishing**,

$$T_i^k = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}.$$

d) **The metric tensor inside the star is determined by the Einstein field equations**,

$$R_i^k - \frac{1}{2} R \delta_i^k = \frac{8\pi G}{c^4} T_i^k .$$

With the use of these assumptions it follows that the gravitational field inside a massive star is determined by the following system of equations:

$$\frac{\lambda' e^{-\lambda}}{r} - \frac{(e^{-\lambda} - 1)}{r^2} = \frac{8\pi G}{c^2} \rho , \qquad (1)$$

$$\frac{\nu' e^{-\lambda}}{r} + \frac{\left(e^{-\lambda} - 1\right)}{r^2} = \frac{8\pi G}{c^4} p , \qquad (2)$$

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r} \right) = \frac{8\pi G}{c^4} p . \qquad (3)$$

However, it is convenient to replace Equation (3) by the relation $T_{i;k}^k = 0$, which holds identically for all systems, and which gives

$$p' = -\left(\rho c^2 + p\right)\frac{\nu'}{2} . \qquad (4)$$

Exercise. Prove equation (4).

Equation (1) can easily be integrated to give

$$e^{-\lambda} = 1 - \frac{2GM(r)}{c^2 r} , \qquad (5)$$

where

$$M(r) = 4\pi \int_0^r \rho(r) r^2 dr \,. \tag{6}$$

Using equations (1)-(6) we obtain the **Tolman-Oppenheimer-Volkoff (TOV) equation, describing the hydrostatic equilibrium of an isotropic general relativistic static fluid sphere**:

$$\frac{dp(r)}{dr} = -\frac{\frac{G}{c^2}\left[\rho(r)c^2 + p(r)\right]\left[\frac{4\pi}{c^2} p(r) r^3 + M(r)\right]}{r^2\left(1 - \frac{2GM(r)}{c^2 r}\right)} \,. \tag{7}$$

Together with **the mass continuity equation**,

$$\frac{dM}{dr} = 4\pi\rho(r) r^2 \,, \tag{8}$$

the TOV equation gives the description of the general relativistic stars.

The system (7)-(8) must be integrated with **the boundary conditions**

$$M(0) = 0, \ \rho(R) = 0, \ p(R) = 0, \tag{9}$$

where R is the radius of the gaseous sphere, together with **an equation of state** of the form

$$p = p(\rho) \,. \tag{10}$$

For most of the physically relevant equations of state of the stellar matter **the integration of the TOV equation must be performed numerically**.

The use of the TOV equation and of the mass continuity equation is considerably simplified if one uses a dimensionless representation of the physical parameters. To obtain a dimensionless form of the TOV equation (13) and of the mass continuity equation, we shall introduce a dimensionless independent variable η and the dimensionless functions $\varepsilon(\eta)$ (energy density), $P(\eta)$ (pressure) and $m(\eta)$ (mass) by means of the transformations

$$r = a\eta \ , \ \rho = \rho_c \varepsilon(\eta) \ , \ p = \rho_c c^2 P(\eta) \ , \ M = M^* m(\eta) . \tag{11}$$

Here a is a scale factor (a characteristic length), ρ_c the central density of the star and M^* a characteristic mass.

Using the transformations (11) in equations (7) and (8) we obtain the following non-dimensional form of the mass continuity and TOV equations:

$$\frac{dm(\eta)}{d\eta} = \eta^2 \varepsilon(\eta), \tag{12}$$

$$\frac{dP(\eta)}{d\eta} = -\frac{\left[\varepsilon(\eta) + P(\eta)\right]\left[P(\eta)\eta^3 + m(\eta)\right]}{\eta^2\left(1 - \frac{2m(\eta)}{\eta}\right)}, \tag{13}$$

where we have taken

$$a^2 = \frac{c^2}{4\pi G \rho_c} \text{ and } M^* = 4\pi\rho_c a^3 = \frac{c^3}{\sqrt{4\pi G^3 \rho_c}} . \tag{14}$$

In the new variables, the equation of state becomes

$$P = P(\eta),$$

while the boundary conditions are given by

$$m(0) = 0 \ , \ \varepsilon\left(\eta_S\right) = 0 \ , \ P\left(\eta_S\right) = 0,$$

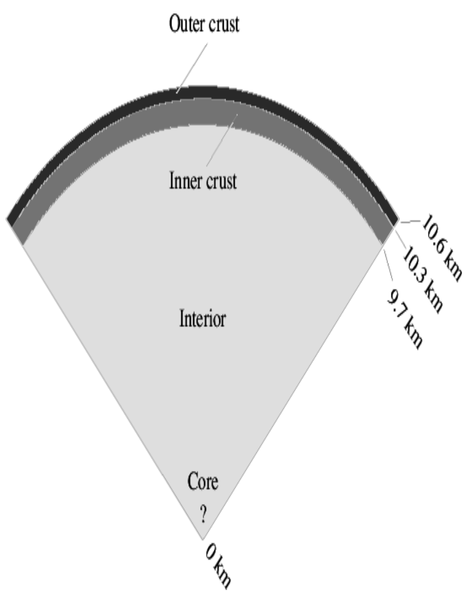
where $\eta_s = \frac{R}{a}$ is the value of the non-dimensional radial co-ordinate η at the surface of the star.

If we suppose that m is an increasing (non-decreasing) function of η, while ε and P are decreasing (non-increasing) function of the same argument, then from the transformations (15) it follows that

$$\varepsilon \in [1,0] \ , \ P \in \left[P_c = \frac{p_c}{\rho_c c^2}, 0 \right].$$

Exercise. Derive equations (12)-(14).

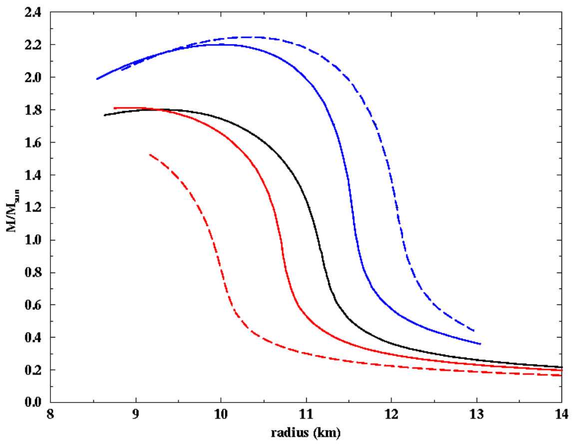
Before going to study some particular stellar models, let us see exactly how the numerical integration is performed if the equation of state $p = p(\rho)$ is known. As an input we take the value of the central density ρ_c.



This determines p_c, while p' and m vanish at $r = 0$. The mass continuity equation then determines m for an infinitesimal increase in r, and plugging in this value of m and ρ_c and p_c into the TOV equation determines the value of p', allowing a determination of p at the next step. For the p thus found the equation of state determines ρ and we go over the whole process once again to determine the values of the variables at the next step.

In this way, the computation of p, ρ and m for increasing values of r goes until we arrive at $p = 0$. We identify this as the boundary of the star.

The results in the model calculation may be represented in a number of ways. **We can have a plot of the mass M against the central density ρ_c or against the radius R.**



The general feature of these curves is the occurrence of a maximum mass at a finite value of the central density ($\rho_c \sim 10^{15}$ g/cm^3) and approximately 10 km radius. The maximum mass has a value ranging from $0.7\,M_\odot$ for a non-interacting neutron gas to a value of around $3\,M_\odot$, corresponding to the stiff equation of state $p = \rho c^2$.

13.2 Constant density stars

We consider now the structure and interior geometry of a **homogeneous gaseous sphere, that is, we suppose that the energy density is constant throughout the star**, $\rho = \rho_m = \rho_c = \text{constant}$. In this case we have $\varepsilon = 1$, $\forall \eta \in [0, \eta_S]$.

Using the boundary conditions

$$m(0) = 0\ ,\ P(0) = P_c\ ,$$

the system of differential equations (12)-(13) can be easily integrated, to give the exact solution

$$m(\eta) = \frac{\eta^3}{3},$$

$$P(\eta) = \frac{\frac{1}{3}(1+P_c) - \left[\frac{1}{3} + P_c\right]\left[1 - \frac{2}{3}\eta^2\right]^{\frac{1}{2}}}{\left[\frac{1}{3} + P_c\right]\left[1 - \frac{2}{3}\eta^2\right]^{\frac{1}{2}} - (1+P_c)}.$$

The radius $R = a\eta_S$ of the star can be obtained from the condition $P(\eta_S) = 0$ and is given by

$$R = a\frac{\left[6P_c(2P_c+1)\right]^{\frac{1}{2}}}{3P_c+1}.$$

In the dimensionless variables the total mass of the star is given by

$$m_S = \frac{1}{3}\eta_S^3,$$

or $M_S = \dfrac{4\pi}{3}\rho_c R_S^3$.

The radius R of the homogeneous stellar configuration relates to the gravitational (Schwarzschild) radius of the star $R_g = \dfrac{2GM_S}{c^2}$ (with M_S the total mass of the star), by the relation

$$R = \frac{(3P_c+1)^2}{2P_c\left[2(2P_c+1)\right]}R_g.$$

The values of P_c depend on the physically allowed upper limit for the pressure. If we consider the classical restriction of the general relativity,

$$p \leq \frac{\rho c^2}{3}, \tag{15}$$

as $p \geq 0$, it follows that in the case of the homogeneous sphere $P_c \in \left[0, \frac{1}{3}\right]$. For the sake of generality we can consider the more general restriction

$$p \leq \rho c^2, \tag{16}$$

the upper limit being the stiff equation of state for the hot nucleonic gas .

It is largely believed today that matter actually behaves in this manner at densities above about ten times nuclear, that is at densities greater than $10^{17}\,\mathrm{gcm}^{-3}$, namely at temperatures $T = \left(\frac{\rho}{\sigma}\right)^{\frac{1}{4}} > 10^{13}\,K$, where σ is the radiation constant. In this case it follows that $P_c \in [0,1]$.

For the classical restriction of the pressure given by equation (15) we find the following simple stability criterion for a homogeneous star in the presence of a cosmological constant:

$$R \geq \frac{9}{5} R_g .$$

For homogeneous configurations for which the more general restriction (16) holds, we obtain a stability criterion of the form

$$R \geq \frac{4}{3} R_g .$$

The expression of the metric tensor component $e^{-\lambda}$ inside the homogeneous star is given by:

$$e^{-\lambda} = 1 - \frac{1}{3} \frac{8\pi G \rho_m}{c^2} r^2 ,$$

while ν is given as a function of the dimensionless variable η by

$$e^{\nu(\eta)} = \frac{C}{\left(1 + P(\eta)\right)^2} = C\left[\left(\frac{1}{3} + P_c\right)\left(1 - \frac{2}{3}\eta^2\right)^{\frac{1}{2}} - \left(1 + P_c\right)\right]^2 , \qquad (17)$$

where C is a constant of integration.

For $r > R$ the geometry of the space-time is described by the Schwarzschild line element, given by

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2GM_S}{r}. \tag{18}$$

In order to match the interior metric in the star smoothly on the boundary surface $r = R$ with the exterior Schwarzschild line element, **we have to require the continuity of the gravitational potentials across that surface.** Matching equation (17) with the exterior Schwarzschild gravitational metric tensor component (18) at the boundary $r = R$ gives the value of the integration constant C in the form:

$$C = \frac{1 - \dfrac{2M_S G}{R}}{\left\{\left(\dfrac{1}{3} + P_c\right)\left[1 - \dfrac{2}{3}\left(\dfrac{R}{a}\right)^2\right]^{\frac{1}{2}} - (1 + P_c)\right\}^2}.$$

Hence, the problem of the structure of the homogeneous general relativistic fluid sphere is completely solved.

13.3 The upper mass limit of neutron stars

As the equation of state of the dense matter is not known for densities above the nuclear density ρ_{nucl} ($\sim 5 \times 10^{14}$ g cm^{-3}), we cannot integrate the TOV equation when such high densities occur. However, we can arrive at an upper mass limit[2] by adopting some very plausible assumptions:

a) As a first approximation, **we neglect rotation and magnetic fields, by assuming the star static and spherically symmetric**

b) the material of the star can be regarded as **a perfect fluid**, i.e. the tangential pressure vanishes

c) **the density is nonnegative**

d) **the condition** $\frac{dp}{d\rho} \geq 0$ **is satisfied everywhere inside the star**; otherwise, any decrease in volume locally, however small, would lead to a decrease in pressure of the material there and hence to further compression of the region. **The condition** $\frac{dp}{d\rho} \geq 0$ **is just the condition for local stability. Furthermore, since** p **is positive for ordinary densities, we assume that it will be positive for all densities under consideration**.

In order to simplify the discussion in the following we adopt a system of units so that $c = G = 1$. We introduce also the mean density of the star, $\bar{\rho}$, defined according to

$$\frac{4}{3}\pi\bar{\rho}r^3 \equiv m(r) = 4\pi\int_0^r \rho(r)r^2 dr \,,$$

and introduce a new variable $x = \int_0^r re^{\frac{\lambda}{2}}dr$, so that $dx = re^{\frac{\lambda}{2}}dr$.

By using the gravitational field equations (1)-(3) it is easy to show that the following two equations hold:

$$\frac{e^{-\frac{\nu+\lambda}{2}}}{r^2}\frac{d}{dr}\left[e^{-\frac{\lambda}{2}}r^2\frac{d}{dr}\left(e^{\frac{\nu}{2}}\right)\right] = 4\pi(\rho + 3p), \tag{19}$$

and

$$e^{-\lambda}\frac{\nu'}{r} = 8\pi\left(p + \frac{\bar{\rho}}{3}\right), \tag{20}$$

respectively. Eliminating p between equations (19) and (20) gives

$$e^{-\frac{\nu+\lambda}{2}} r \frac{d}{dr}\left[\frac{e^{-\frac{\lambda}{2}}}{r}\frac{d}{dr}\left(e^{\frac{\nu}{2}}\right)\right] = 4\pi(\rho - \bar{\rho}),$$

or, equivalently,

$$\frac{d^2}{dx^2}\left(e^{\frac{\nu}{2}}\right) = \frac{4\pi e^{\frac{\nu}{2}}}{r^2}(\rho - \bar{\rho}) \le 0\,. \tag{21}$$

The condition $(\rho - \bar{\rho}) \le 0$ can be proven as follows. From the TOV equation it is obvious that dp/dr is negative unless $1 - 2m/r$ becomes negative. However, such a situation which makes r a timelike coordinate is not consistent with the static condition. Hence $dp/dr < 0$, and consequently ρ is also a non-increasing function of r. Hence $\bar{\rho} \ge \rho$, the equality occurring in the case of uniform density. Inequality (21) indicates that $y \equiv \frac{d}{dx} e^{\frac{\nu}{2}}$ is in general a decreasing function of x and is constant only in the case of uniform density. Hence the total change of $e^{\frac{\nu}{2}}$ will be underestimated by taking the values of y at x and multiplying by the interval x, i.e.,

$$\left(e^{\frac{\nu}{2}}\right)_{x>0} - \left(e^{\frac{\nu}{2}}\right)_{x=0} \ge x\frac{dy}{dx},$$

or, since $\left(e^{\frac{\nu}{2}}\right)_{x=0} > 0$, we have

$$\frac{d}{dx} e^{\frac{\nu}{2}} \le \frac{\left(e^{\frac{\nu}{2}}\right)_x}{x}\,.$$

Restoring the variable r gives

$$\frac{e^{-\frac{\lambda}{2}}}{2r}\frac{d\nu}{dr} \leq \frac{1}{x}. \tag{22}$$

Again, we can set a lower bound to x. Note that for any $a > r$,

$$\frac{m(r)}{r} = \frac{4}{3}\bar{\rho}_r r^2 \geq \frac{4}{3}\bar{\rho}_a r^2 = \frac{m(a)}{a^3} r^2.$$

Hence

$$x_{r=a} = \int_0^a r\left(1-\frac{2m}{r}\right)^{-1/2} dr \geq \int_0^a r\left(1-\frac{2m_a}{a^3}r^2\right)^{-1/2} dr = \frac{a^3}{2m_a}\left[1-\left(1-\frac{2m_a}{a}\right)^{1/2}\right]. \tag{23}$$

Recall now the TOV equation,

$$\frac{dp}{dr} = -(\rho + p)\frac{\nu'}{2} = -\frac{(\rho + p)(4\pi pr^3 + m)}{r(r-2m)}.$$

Using the TOV equation to eliminate ν' from (22) and the bound of x given by (23) we obtain

$$\frac{4\pi pr^3 + m}{r^3\sqrt{1-\frac{2m}{r}}} \leq \frac{2m}{r^3}\left[1-\left(1-\frac{2m}{r}\right)^{1/2}\right]^{-1}.$$

Simplifying the above inequality gives a quadratic expression in m/r:

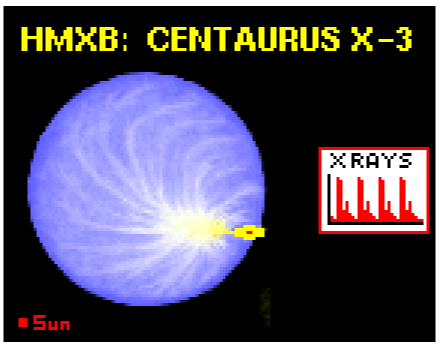
$$\frac{m}{r} \leq \frac{2}{9}\left[1-6\pi r^2 p + \left(1+6\pi r^2 p\right)^{1/2}\right].$$

Quite generally, we obtain a bound on M/R, where R is the boundary value of r, at which $p=0$ and $M \equiv m(r)$,

$$\frac{M}{R} \leq \frac{4}{9}. \tag{24}$$

Besides the inequality (24) there is another restriction as well, coming from the non-increasing nature of the density ρ

$$M \geq \frac{4}{3}\pi\rho R^3. \tag{25}$$



By combining equations (24) and (25) it follows

$M_{\max} = \frac{4}{9}\left(\frac{1}{3\pi\rho}\right)^{1/2} = 6.8\left(\frac{\rho_{nucl}}{\rho}\right) M_{\odot}$, where

$\rho_{nucl} = 2.8\times10^{14}$ g cm^{-3}. For $\rho = 5\times10^{14}$ g cm^{-3},

$M_{\max} \approx 5\ M_{\odot}$.

The maximum mass occurs for stars with uniform density and with $dp/d\rho \to \infty$ at the center. If we introduce the additional restriction $dp/d\rho \leq 1$(frequently referred as the causality condition, implying that violation of this condition would lead to the velocity of compressional waves exceeding the velocity of light and thus leading to a breakdown of causality), the value of the limiting mass comes down to about $M_{\max} \approx 3\ M_{\odot}$. This is the value usually taken as the theoretical upper bound of neutron star masses.

Notes

[1] The Tolman-Oppenheimer-Volkoff equation and its astrophysical implication are discussed in detail in most of the books devoted to the study of relativistic astrophysics, like N. K. Glendenning, Compact stars, nuclear physics, particle physics and general

relativity, Springer Verlag, New York, Berlin, Heidelberg 2000, A. K. Raychaudhury, S. Banerji and A. Banerjee, General relativity, astrophysics and cosmology, Springer Verlag, New York, Berlin, Hong Kong, 1992, Stuart L. Shapiro, Saul A. Teukolsky **,** Black holes, white dwarfs, and neutron stars : the physics of compact objects**,** Wiley, New York, 1983 and Ya. B. Zeldovich and I.D. Novikov, Stars and relativity**,** Dover Publications, Mineola, N.Y., 1996.

[2] The upper mass limit of neutron stars was derived first in W. A. Buchdahl, Phys. Rev. 116, 1027, 1959; this limit is also discussed in A. K. Raychaudhury, S. Banerji and A. Banerjee, General relativity, astrophysics and cosmology, Springer Verlag, New York, Berlin, Hong Kong, 1992, N. K. Glendenning, Compact stars, nuclear physics, particle physics and general relativity, Springer Verlag, New York, Berlin, Heidelberg 2000 and N. Straumann, General relativity and relativistic astrophysics**,** Springer, Berlin, 1984.