Homework 11

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BIOENG 104 Biological Transport Phenomena | Aaron Streets

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1 Problem 1

In the homework, you calculated the velocity profile for pressure driven flow in a cylindrical pipe of length L and radius R, given a pressure difference of ΔP across the length of the pipe.

Assuming a steady-state condition, fully developed, laminar flow of an incompressible Newtonian fluid with viscosity μ, we found that the velocity in the direction of the pipe, v_z, was a function of the radial position r, and given by the following equation:

$$v_z(r) = \frac{R^2 \Delta P}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

(A) (*2 points*) Calculate the total force imparted on the length of this pipe from shear stress of fluid flow.

Since the fluid in the cylinder is Newtonian,

$$\tau_{rz} = \mu \frac{\mathrm{d}v_z}{\mathrm{d}r} = \mu \frac{R^2 \Delta P}{4\mu L}\frac{-2r}{R^2} = -\mu \frac{r\Delta P}{2\mu L} = -\frac{r\Delta P}{2L}$$

and $\tau_{rz} = \mathrm{d}F/\mathrm{d}A$ for differential area $\mathrm{d}A$ inside the cylinder (at $r = R$), then

$$F_{\text{tot}} = \int_{\mathcal{A}} \tau_{rz}|_{r=R}\,\mathrm{d}A = \int_{\mathcal{A}} -\frac{R\Delta P}{2L}\,\mathrm{d}A = -\frac{R\Delta P}{2L}\int_{\mathcal{A}} \mathrm{d}A = -\frac{R\Delta P}{2L}(2\pi RL) = \boxed{-\pi R^2 \Delta P}$$

since the inside surface $\mathcal{A}$ of the cylinder has area $2\pi RL$.

(B) (*2 points*) Calculate the volumetric flow rate Q, through the pipe, with units $[\mathrm{m}^3\,\mathrm{s}^{-1}]$.

By definition, the volumetric flow rate is

$$Q = \int_{\mathcal{A}} (\mathbf{v} \cdot \hat{\mathbf{n}})\, \mathrm{d}A = \int_{\mathcal{A}} v_z\, \mathrm{d}A$$

where $\mathcal{A}$ refers to the cross-sectional area of a cylinder. Since $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ in cylindrical coordinates,

$$\begin{aligned} Q &= \int_0^{2\pi} \int_0^R v_z (r\,\mathrm{d}r\,\mathrm{d}\theta) = \int_0^{2\pi} \int_0^R \frac{R^2 \Delta P}{4\mu L} \left(r - \frac{r^3}{R^2} \right) \mathrm{d}r\,\mathrm{d}\theta = \frac{R^2 \Delta P}{4\mu L} \int_0^{2\pi} \left[\frac{1}{2} r^3 - \frac{r^4}{4R^2} \right]_0^R \mathrm{d}\theta \\ &= \frac{R^2 \Delta P}{4\mu L} (2\pi) \left(\frac{1}{2} R^2 - \frac{1}{4} R^2 \right) = \frac{\pi R^2 \Delta P}{2\mu L} \frac{1}{4} R^2 = \boxed{\frac{\pi R^4 \Delta P}{8\mu L}} \end{aligned}$$

(C) (*2 points*) Consider a second pipe with the same radius R, and pressure difference of ΔP, but a length $L_2 = 2L$. What is the relationship between the new volumetric flow rate Q_2 and Q, the volumetric flow rate you calculated in the previous question (1B).

We have

$$Q_2 = \frac{\pi R^4 \Delta P}{8\mu (2L)} = \frac{1}{2} \frac{\pi R^4 \Delta P}{8\mu L} = \boxed{\frac{1}{2} Q}$$

(D) (*2 points*) Consider a third pipe with the same length L, and pressure difference of ΔP, as the first pipe, but a radius $R_3 = 2R$. What is the relationship between the new volumetric flow rate Q_3 and Q, the volumetric flow rate you calculated in question (1B).

We have

$$Q_3 = \frac{\pi (2R)^4 \Delta P}{8\mu L} = 16 \frac{\pi R^4 \Delta P}{8\mu L} = \boxed{16Q}$$

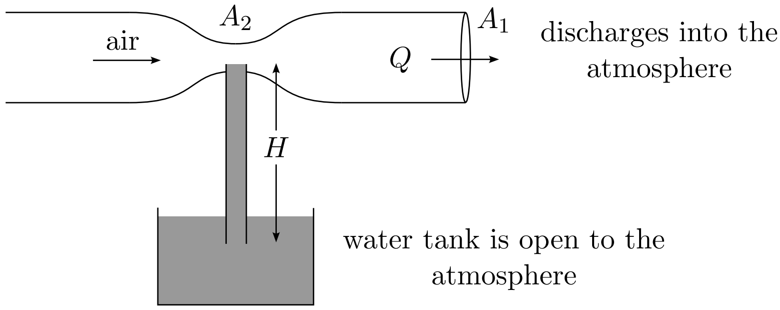
(E) (*2 points*) In electronic circuits, Ohm's law, $\Delta V = I R_E$, gives the relationship between the voltage difference ΔV, across a conducting wire of electrical resistance R_E, and the current through that wire I. An analogous equation in fluid dynamics gives the relationship between a pressure difference across a channel ΔP, the fluidic resistance of that channel R_f, and the volumetric flow rate Q. This relationship is $\Delta P = Q R_f$. What is the fluidic resistance R_f, of the first pipe in this problem?

The fluidic resistance R_f is

$$R_f = \frac{\Delta P}{Q} = \frac{\Delta P}{\pi R^4 \Delta P / (8\mu L)} = \boxed{\frac{8\mu L}{\pi R^4}}$$

2 Problem 2

(*10 points*) Using the Bernoulli equation, calculate the flow rate of air Q ($\mathrm{m^3\,s^{-1}}$) required to draw the water up the entire height H in terms of A_1, A_2, the density of water, and the density of air.



Recall that the Bernoulli equation states that $\frac{1}{2}\rho v_1^2 + p_1 + \rho g z_1 = \text{constant}$.

Let ρ_{water} be the density of water, and p_0 represent atmospheric pressure. Applying the Bernoulli equation to the traversal up the vertical pipe, we have

$$\frac{1}{2}\rho_{\text{water}} v_0^2 + p_0 + \rho_{\text{water}} g(0) = \frac{1}{2}\rho_{\text{water}} v_0^2 + p_1 + \rho_{\text{water}} g H$$

Here, the velocity of the water v_0 is the same on both sides of the equation because the cross-sectional area is constant (this feeds into the continuity equation). This means we can find the pressure at the top of the vertical pipe to be

$$p_1 = p_0 - \rho_{\text{water}} g H \tag{2.1}$$

Applying Bernoulli's equation to the large horizontal pipe, we have

$$\frac{1}{2}\rho_{\text{air}} v_1^2 + p_0 = \frac{1}{2}\rho_{\text{air}} v_2^2 + p_1$$

Here, we don't have a change in height, so we can leave the height terms out, and the end of the horizontal pipe is exposed to the atmosphere. So, the pressure on the right side of the equation is ρ_0. Rearranging,

$$v_1^2 - v_2^2 = \frac{2}{\rho_{\text{air}}}(p_1 - p_0)$$

Since the cross-sectional area changes from A_1 to A_2, we can use the continuity equation and solve for v_1.

$$\begin{aligned} A_1 v_1 &= A_2 v_2 \\ v_1^2\left[1 - \left(\frac{A_1}{A_2}\right)^2\right] &= \frac{2}{\rho_{\text{air}}}(p_1 - p_0) \\ v_1^2 &= \frac{2(p_0 - p_1)}{\rho_{\text{air}}[(A_1/A_2)^2 - 1]} \end{aligned}$$

Plugging in Equation (2.1),

$$v_1^2 = \frac{2\rho_{\text{water}} gH}{\rho_{\text{air}}[(A_1/A_2)^2 - 1]} = \frac{A_2^2}{A_1^2 - A_2^2} \frac{\rho_{\text{water}}}{\rho_{\text{air}}} 2gH$$

and thus

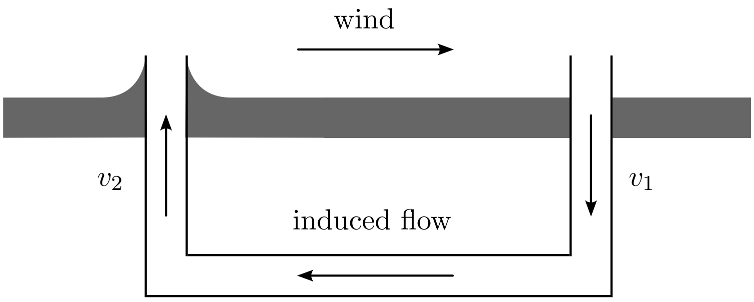
$$Q = A_1 v_1 = \boxed{\sqrt{\frac{A_1^2 A_2^2}{A_1^2 - A_2^2} \frac{\rho_{\text{water}}}{\rho_{\text{air}}} 2gH}}$$

3 Problem 3

(*10 points*) Prairie dog burrows are elaborate and have entrances at different heights. A protruding mound forces wind to flow more rapidly across it than along the rest of the ground. One effect of this is that differences in wind velocities generate pressures which ventilate the burrow.

In the diagram, neglecting differences in height, if the velocity over the mound (left) is 1.0 m/s and the velocity over the hole (right) is 0.75 m/s, what is the pressure difference generated to drive flow through the burrow?

Justify the direction given for the induced flow in the diagram.



We are given $z_1 \approx z_2$, $v_1 = 0.75\,\mathrm{m\,s^{-1}}$, and $v_2 = 1.0\,\mathrm{m\,s^{-1}}$ (1 represents inlet, 2 represents outlet). Also, the density of air is $\rho = 1.2\,\mathrm{kg\,m^{-3}}$. By the Bernoulli equation,

$$\begin{aligned}
\frac{1}{2}\rho v_1^2 + p_1 + \cancel{\rho g z_1} &= \frac{1}{2}\rho v_2^2 + p_2 + \cancel{\rho g z_2} \\
p_2 - p_1 &= \frac{1}{2}\rho(v_1^2 - v_2^2) \\
&= \frac{1}{2}(1.2\,\mathrm{kg\,m^{-3}})[(1.0\,\mathrm{m\,s^{-1}})^2 - (0.75\,\mathrm{m\,s^{-1}})^2] \\
&= \boxed{-0.26\,\mathrm{Pa}}
\end{aligned}$$

The fluid flows in the direction shown in the diagram because the pressure difference is negative from the right to the left. Fluids favorably flow from areas of higher pressure to those of lower pressure.