Homework 10

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BIOENG 104 Biological Transport Phenomena | Aaron Streets

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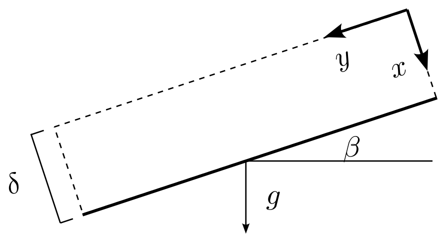
1 Problem 1

A thin film (thickness δ) of an incompressible Newtonian liquid (viscosity μ) flows down an inclined plane (with angle β) due to gravity. The top of the film is in contact with air. Beginning with the Navier-Stokes equation and ignoring the entry region and any edge effects, derive the steady-state flow profile $v_y(x)$.

Hints: Use the coordinate system in the figure. You can assume the thickness does not change in time or space. Because the top surface is in contact with air, you can assume there is no shear stress at the air/liquid interface.



We assume the following:

- the fluid is Newtonian with incompressible flow (as given) that is also laminar
- the system is at steady-state with no edge effects (as given)
- the thickness δ is not a function of space nor time (from hint)
- there is zero shear stress at the air–liquid interface (from hint)
- there is no flow in the x and z-directions
- the system has zero pressure gradient and no electric field

The Navier–Stokes equation for incompressible Newtonian fluids is the following:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} + \sigma \mathbf{E} \tag{1.1}$$

where the fluid has density ρ, pressure p, viscosity μ, and charge density σ, and a gravitational field $\mathbf{g}$ as well as an electric field $\mathbf{E}$ is applied to it. We can simplify this equation immediately from the steady-state assumption ($\partial \mathbf{v}/\partial t = 0$), the assumption of constant pressure ($\boldsymbol{\nabla} p = 0$), and the lack of electric field ($\mathbf{E} = \mathbf{0}$).

$$\rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \tag{1.2}$$

Taking the y-directional component of Equation (1.2), we get

$$\rho \left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \tag{1.3}$$

We can reduce this differential equation using the assumption of fully-developed flow ($\partial v_y / \partial y = 0$) and the assumption that there is no flow along the x or z directions ($v_x = v_z = 0$). Additionally, the velocity does not change along the z-direction ($\partial v_y / \partial z = 0$).

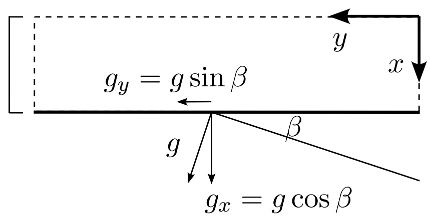
$$\rho \left(\cancel{v_x \frac{\partial v_y}{\partial x}} + \cancel{v_y \frac{\partial v_y}{\partial y}} + \cancel{v_z \frac{\partial v_y}{\partial z}} \right) = \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}} \right) + \rho g_y$$

$$\mu \frac{\partial^2 v_y}{\partial x^2} = -\rho g_y \tag{1.4}$$

Integrating with respect to x twice, we get

$$v_y = \iint -\frac{\rho g_y}{\mu} \, \mathrm{d}x \, \mathrm{d}x = -\frac{\rho g_y}{2\mu} x^2 + Ax + B \tag{1.5}$$

where $A, B \in \mathbb{R}$ are constants. To solve for A and B, we must use two boundary conditions. Let's tilt the diagram so that the film is fully horizontal.



Boundary conditions:
(1) $v_y(x = \delta) = 0$
(2) $\tau_{xy}(x = 0) = 0$

Boundary condition (1) is a consequence of the no-slip condition, and boundary condition (2) follows from the assumption that there is no shear stress at the air–liquid interface. We can reexpress BC (2) in terms of more familiar quantities by exploiting the Newtonian property.

$$\tau_{xy}(x = 0) = \mu \left. \frac{\partial v_y}{\partial x} \right|_{x=0} = 0 \implies \left. \frac{\partial v_y}{\partial x} \right|_{x=0} = 0$$

Thus, from Equation (1.5), we can find the value of A.

$$\frac{\partial v_y}{\partial x} = -\frac{\rho g_y}{\mu}x + A \implies \left.\frac{\partial v_y}{\partial x}\right|_{x=0} = \boxed{A = 0}$$

From BC (1), Equation (1.5), and the fact that $A = 0$ we can state

$$v_y(x = \delta) = -\frac{\rho g_y}{2\mu}\delta^2 + B = 0 \implies \boxed{B = \frac{\rho g_y}{2\mu}\delta^2}$$

so our final velocity profile is

$$v_y = \frac{\rho g_y}{2\mu}(\delta^2 - x^2) = \boxed{\frac{\rho g \sin\beta}{2\mu}(\delta^2 - x^2)}$$

2 Problem 2

Two concentric cylinders with radii R_1 and R_2 are separated by a Newtonian fluid with viscosity μ, $R_1 < R_2$. The two cylinders are spinning in opposite directions. The inner cylinder (R_1) is spinning clockwise with an angular velocity of ω_1 and the outer cylinder (R_2) in spinning counter-clockwise with an angular velocity of ω_2.
(a) Solve for the velocity profile $V(r)$ between the two cylinders.

We take the following assumptions:

- the flow is laminar and fully developed,
- the system is at steady–state,
- the fluid is incompressible and Newtonian (as given),
- there is no radial nor axial flow,
- there is no pressure gradient, and
- the applied gravitational and electric fields are negligible

The Navier–Stokes equation for incompressible Newtonian fluids [Equation (1.1)] is

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V} = -\boldsymbol{\nabla}p + \mu\nabla^2\mathbf{V} + \rho\mathbf{g} + \sigma\mathbf{E}$$

The assumptions of steady-state, negligible gravitational/electric fields, and zero pressure gradient imply that $\partial\mathbf{V}/\partial t$, $\mathbf{g}$, $\mathbf{E}$, and $\boldsymbol{\nabla}p$ all equal zero.

$$\rho\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V} = \mu\nabla^2\mathbf{V}$$

The component of this reduced equation in the azimuthal (φ) direction is

$$\rho\left(V_r\frac{\partial V_\varphi}{\partial r}+\frac{V_\varphi}{r}\frac{\partial V_\varphi}{\partial \varphi}+\frac{V_rV_\varphi}{r}+V_z\frac{\partial V_\varphi}{\partial z}\right)$$
$$=\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V_\varphi}{\partial \varphi^2}-\frac{V_\varphi}{r^2}+\frac{2}{r^2}\frac{\partial V_r}{\partial \varphi}+\frac{\partial^2 V_\varphi}{\partial z^2}\right]$$

From the assumption of no radial nor axial flow, we have $V_r = V_z = 0$. From the assumption of laminar and fully developed flow, the velocity field does not vary with respect to φ (thus $\partial V_\varphi/\partial\varphi = 0$). Additionally, the velocity field does not vary with respect to z (thus $\partial V_\phi/\partial z = 0$).

$$\rho\left(\cancel{V_r\frac{\partial V_\varphi}{\partial r}}+\cancel{\frac{V_\varphi}{r}\frac{\partial V_\varphi}{\partial \varphi}}+\cancel{\frac{V_rV_\varphi}{r}}+\cancel{V_z\frac{\partial V_\varphi}{\partial z}}\right)$$
$$=\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\varphi}{\partial r}\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 V_\varphi}{\partial \varphi^2}}-\frac{V_\varphi}{r^2}+\cancel{\frac{2}{r^2}\frac{\partial V_r}{\partial \varphi}}+\cancel{\frac{\partial^2 V_\varphi}{\partial z^2}}\right]$$

which reduces to the simple differential equation:

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\varphi}{\partial r}\right)-\frac{V_\varphi}{r^2}\right]=0 \tag{2.6}$$

Using the product rule, we can reduce this down even further into a directly integrable form.

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rV_\varphi)}{\partial r}\right)&=-\frac{1}{r^2}\frac{\partial(rV_\varphi)}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rV_\varphi)}{\partial r}\right)\\&=-\frac{1}{r^2}\left(V_\varphi+r\frac{\partial V_\varphi}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(V_\varphi+r\frac{\partial V_\varphi}{\partial r}\right)\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\varphi}{\partial r}\right)+\cancel{\frac{1}{r}\frac{\partial V_\varphi}{\partial r}}-\frac{V_\varphi}{r^2}-\cancel{\frac{1}{r}\frac{\partial V_\varphi}{\partial r}}\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)-\frac{V_\varphi}{r^2}\end{aligned}$$

Thus, Equation (2.6) transforms into

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rV_\varphi)}{\partial r}\right)&=0\\\frac{1}{r}\frac{\partial(rV_\varphi)}{\partial r}&=A\\\frac{\partial(rV_\varphi)}{\partial r}&=Ar\\rV_\varphi&=\frac{1}{2}Ar^2+B\\V_\varphi&=\frac{A}{2}r+\frac{B}{r}\end{aligned} \tag{2.7}$$

To solve for A and B, we need two boundary conditions. Since the inner cylinder is spinning clockwise at angular velocity ω_1 and the outside cylinder spins counterclockwise at angular velocity ω_2, then our boundary conditions are

Boundary conditions:
(1) $V_\varphi(r = R_1) = -\omega_1 R_1$
(2) $V_\varphi(r = R_2) = \omega_2 R_2$

following the convention that counterclockwise direction is positive. Our boundary conditions yield a system of equations:

$$\begin{cases} V_\varphi(r = R_1) = \dfrac{A}{2}R_1 + \dfrac{B}{R_1} = -\omega_1 R_1 & (1) \\ V_\varphi(r = R_2) = \dfrac{A}{2}R_2 + \dfrac{B}{R_2} = \omega_2 R_2 & (2) \end{cases}$$

Multiplying Equation (2) by R_2/R_1 and subtracting the result from Equation (1),

$$\begin{aligned} \frac{A}{2}\left(R_1 - \frac{R_2^2}{R_1}\right) &= -\omega_1 R_1 - \omega_2 \frac{R_2^2}{R_1} \\ \frac{A}{2}(R_1^2 - R_2^2) &= -\omega_1 R_1^2 - \omega_2 R_2^2 \\ \frac{A}{2} &= \frac{\omega_1 R_1^2 + \omega_2 R_2^2}{R_2^2 - R_1^2} \end{aligned}$$

and to find B, just substitute back to, e.g., Equation (2).

$$\begin{aligned} B &= \left(\omega_2 - \frac{A}{2}\right) R_2^2 \\ &= \left(\omega_2 + \frac{\omega_1 R_1^2 + \omega_2 R_2^2}{R_1^2 - R_2^2}\right) R_2^2 \\ &= \left(\frac{\omega_2 R_1^2 - \cancel{\omega_2 R_2^2} + \omega_1 R_1^2 + \cancel{\omega_2 R_2^2}}{R_1^2 - R_2^2}\right) R_2^2 \\ &= \frac{R_1^2 R_2^2(\omega_1 + \omega_2)}{R_1^2 - R_2^2} = -\frac{R_1^2 R_2^2(\omega_1 + \omega_2)}{R_2^2 - R_1^2} \end{aligned}$$

Plugging the values of $A/2$ and B back into Equation (2.7),

$$V_\varphi = \frac{\omega_1 R_1^2 + \omega_2 R_2^2}{R_2^2 - R_1^2} r - \frac{R_1^2 R_2^2(\omega_1 + \omega_2)}{R_2^2 - R_1^2} \cdot \frac{1}{r}$$

$$\boxed{V_\varphi = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left[\left(\frac{\omega_1}{R_2^2} + \frac{\omega_2}{R_1^2}\right) r - \frac{\omega_1 + \omega_2}{r}\right]}$$

(b) Compare the total force experienced by each cylinder.

Let the two cylinders have axial length L. By definition, the shear stress $\tau_{r\varphi}$ is the differential force in the φ-direction over a differential area of the r-face. This means we can find the forces F in the φ-direction by integrating over the inside face of the cylinder at $r = R_1$ and at $r = R_2$.

$$\tau_{r\varphi} = \frac{\mathrm{d}F_\varphi}{\mathrm{d}A_r} \implies F_{r\varphi} = \iint_{\text{face}} \tau_{r\varphi}\,\mathrm{d}A$$

To find the $\tau_{r\varphi}$, we utilize the Newtonian fluid property, which states that

$$\tau_{r\varphi} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \varphi}\right]$$

From one of our assumptions, we set $V_r = 0$. Additionally, the derivative of V_φ/r with respect to r is

$$\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right) = \frac{\partial}{\partial r}\left(\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left[\left(\frac{\omega_1}{R_2^2} + \frac{\omega_2}{R_1^2}\right) - \frac{\omega_1 + \omega_2}{r^2}\right]\right) = \frac{2}{r^3}\underbrace{\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}}_{\equiv \lambda}(\omega_1 + \omega_2)$$

so the shear stress $\tau_{r\varphi}$ is equal to

$$\tau_{r\varphi} = \frac{2\mu\lambda}{r^2}(\omega_1 + \omega_2)$$

where $\lambda = (R_1 R_2)^2/(R_2^2 - R_1^2)$. Integrating with respect to $\mathrm{d}A = R_2\,\mathrm{d}\varphi\,\mathrm{d}z$, we have

$$F|_{r=R_2} = \int_0^L \int_0^{2\pi} \tau_{r\varphi}|_{r=R_2} R_2\,\mathrm{d}\varphi\,\mathrm{d}z = \frac{2\mu\lambda}{R_2^2}(\omega_1 + \omega_2)(2\pi R_2 L) = \frac{4\pi\mu\lambda L}{R_2}(\omega_1 + \omega_2)$$

Similarly, since the shear stress at $r = R_1$ is equal to $2(\omega_1 + \omega_2)/R_1^2$, then

$$F|_{r=R_1} = \int_0^L \int_0^{2\pi} \tau_{r\varphi}|_{r=R_1} R_1\,\mathrm{d}\varphi\,\mathrm{d}z = \frac{2\mu\lambda}{R_1^2}(\omega_1 + \omega_2)(2\pi R_1 L) = \frac{4\pi\mu\lambda L}{R_1}(\omega_1 + \omega_2)$$

Plugging back in λ,

$$F|_{r=R_1} = 4\pi\mu L\frac{R_1 R_2^2}{R_2^2 - R_1^2}(\omega_1 + \omega_2) \qquad F_{r=R_2} = 4\pi\mu L\frac{R_1^2 R_2}{R_2^2 - R_1^2}(\omega_1 + \omega_2)$$

We can determine the ratio between the total forces experienced by each cylinder to be

$$\frac{F|_{r=R_2}}{F|_{r=R_1}} = \frac{R_1}{R_2}$$

meaning $F|_{r=R_2} = (R_1/R_2)F|_{r=R_1}$. Since R_1 is smaller than R_2, then the force on the outer cylinder is a scaled-down value of the force on the inner cylinder.

3 Group Project Assignment

Meet with your group and decide on one of the two proposals you have been brainstorming and start your project. Find 4 published manuscripts related to your application or model. Write a draft introduction to your final manuscript. Cite the 4 papers in your introduction. Your introduction should address the following questions:

- What is the biological system or biomedical device that you are modelling and why is this an interesting or important system?
- What work has been previously published that is most related to your question and where does this previous research fall short?
- What is the question you hope to answer with your model?

As of 2020, approximately 50 million patients are affected by Alzheimer's disease, the most common type of dementia.[1] Although Alzheimer's symptoms may be reduced by effective drug administration, the blood–brain barrier (BBB), which refers to the semipermeable membrane between the circulatory system and the brain, poses a significant challenge. Due to it being primarily composed of tightly knit endothelial cells, the BBB has drastically low permeability, lowering drug diffusion efficiency to the intended site. Previous studies (e.g. Sefidgar *et al.*, 2015) examined how the structure of capillary networks can affect drug delivery efficiency. Although it determined that a combination of avascular and vascular networks would produce irregularities in vessel thickness and pressure, the study leaves a gap in understanding how these parameters affect drug delivery.[2]

To determine optimal conditions for therapeutic drug delivery, we aim to construct a computational model of a small capillary network with adjustable geometries. This two-dimensional model will be a recreation of that featured in Sefidgar *et al.*, 2015 and will be composed of multiple layers to account for the heterogeneity of cell types throughout the brain. The model will adopt the governing equations and boundary conditions used in studies of drug transport across the BBB.[3] Furthermore, the representation of the multilayered BBB will mimic approaches that model passive transport properties of the BBB using a mesh to represent the cell density and standard mass flux equations defined in previous studies.[3,4] Using these governing equations and parameters, we can simulate the fluid transport of drugs throughout the network. By adjusting the geometric parameters of the model, such as vessel thickness and radii, we can obtain the geometry of the capillary network most ideal for fastest diffusion of the drug. For proof of principle, we additionally aim to use this model to determine the minimal concentration of Memantine, a drug commonly used to slow the progression of Alzheimer's, required for effective therapeutic results. The results of this numerical model can serve as a starting point for more efficacious treatments against neurodegenerative diseases. For example, the optimal geometries of the capillary network can help determine the optimal sites for drug injection within the brain, by comparing the measured geometric parameters with those obtained by our model. Furthermore, as it aims to minimize dosage requirements, our model may also reduce concerns of potential toxicities or side effects associated with the treatment drug of choice.

References

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[3] Sarafraz M, Nakhjavani M, Shigdar S, Christo FC, Rolfe B. Modelling of mass transport and distribution of aptamer in blood-brain barrier for tumour therapy and cancer treatment. Eur J Pharm Biopharm. 2022;173:121-31. Available from: https://doi.org/10.1016/j.ejpb.2022.03.004.

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