Lecture 13: Optimization

Algorithm

Jeong-Hun Kim

Table of Contents

Part 1

WHAT IS THE OPTIMIZATION?

What is the optimization?

- Optimization
  - Techniques for achieving minimal loss or maximum gain
  - Most common type of problem encountered in daily life
  - Keywords:
    - Minimum
    - Maximum

  - In other words,
    - Process of obtaining the best possible solution

What is the optimization?

- Example)
  - Find the value of x that minimizes $f(x)$

$$\min_{x \in X} f(x)$$

    - Three components:
      - Objective function: $f(x)$
      - Feasible set: X
      - Parameter: x
      - Constraint condition: $x \in X$

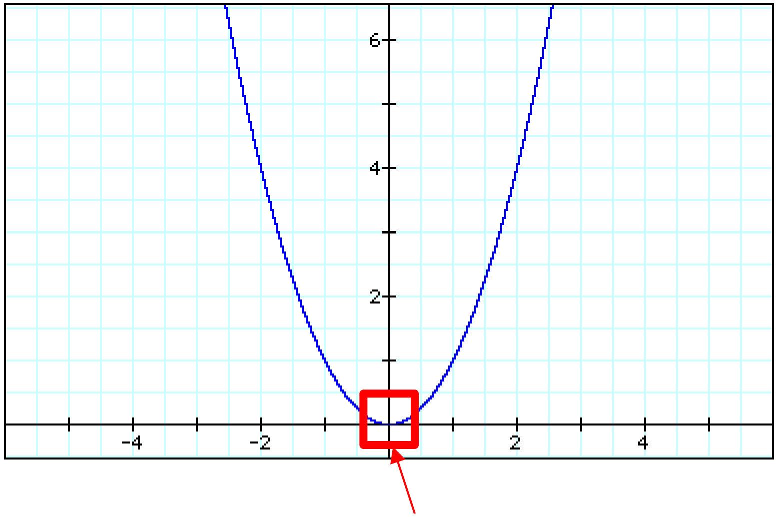
What is the optimization?

- Solution
  - Feasible solution
    - All solutions that satisfy the constraints
  - Optimal solution
    - Solution that best aligns with the objective of the problem

What is the optimization?

- Example)
  - Let $X = [-2, -1, 0, 1, 2]$ and $f(x) = x^2$. For the following optimization problem:

$$\min_{x \in X} f(x)$$



Optimal solution

What is the optimization?

- Objective function
  - Objective of the problem
  - The function to be optimized
  - Also known as a loss function or cost function

  - Thus, to solve an optimization problem, it must be reduced to a maximization or minimization problem

What is the optimization?

- Types of optimization problem
  - Unconstrained optimization problem
    - Optimization problems without constraint conditions
  - Constrained optimization problem
    - Optimization problems with constraint conditions

Part 2

LINEAR PROGRAMMING

Linear programming

- Linear programming
  - A mathematical model where constraints and objectives are expressed as linear relationships
  - E.g.,
    - Allocation model
    - Blending model
    - Scheduling model

Linear programming

- Allocation model
  - A problem of assigning values to variables to best fit the objective of the problem
  - Constraints may exist

Linear programming

❖ Example

National forestland needs to be allocated with many prescriptions or land management policies

i : analysis area number

j : prescription number, e.g.: timbering, grazing, or wilderness

s_i : size of analysis area i

p_{ij} : *NPV per acre in area i with j

Maximize NPV

*NPV: Net Present Value

Linear programming

❖ Example (cont'd)

Forest service allocation

t_{ij} : projected timber yield in area i with j

g_{ij} : projected grazing capability in area i with j

w_{ij} : wilderness index rating in area i with j

Constraints

At least 40 million board ft of timber, 5,000 animal unit months of grazing, and keeping average wilderness index to at least 70

Analysis Area, i	Acres, s_i (000)'s	Prescription, j	NPV, (per acre) $p_{i,j}$	Timber, (per acre) $t_{i,j}$	Grazing, (per acre) $g_{i,j}$	Wilderness Index, $w_{i,j}$
5	212	1	105	40	0.05	60
		2	460	32	0.08	60
		3	120	0	0	70
6	98	1	490	105	0.02	35
		2	55	25	0.03	50
		3	180	0	0	75

Linear programming

- Example (cont'd)

Let, x_{ij} = # thousands of acre for (i, j)

Objective function

$\max \sum_i \sum_j p_{ij} x_{ij}$,

s.t. $\sum_j x_{ij} \leq s_i$ for $\forall\ i$, $\sum_i \sum_j t_{ij} x_{ij} \geq 40000$, $\sum_i \sum_j g_{ij} x_{ij} \geq 5$, $\frac{1}{\sum_i s_i} \sum_i \sum_j w_{ij} x_{ij} \geq 70$, $x_{i,j} \geq 0$, $\forall\ i, j$

Linear programming

- Blending model
  - Used in finding the optimal conditions for material mixtures

- Example

	Composition (%)				Available (kg)	Cost (kr/ kg)
	Carbon	Nickel	Chromium	Molybdenum		
First scrap	0.80	18	12	—	75	16
Second scrap	0.70	3.2	1.1	0.1	250	10
Third scrap	0.85	—	—	—	Unlimited	8
Fourth scrap	0.40	—	—	—	Unlimited	9
Nickel	—	100	—	—	Unlimited	48
Chromium	—	—	100	—	Unlimited	60
Molybdenum	—	—	—	100	Unlimited	53
Minimum blend	0.65	3.0	1.0	1.1		
Maximum blend	0.75	3.5	1.2	1.3		

Mixing at the minimum cost while satisfying constraints on the amounts of specific components

Linear programming

❖ Example (cont'd)

Let, $x_j = \#$ kilos of ingredient j included in the charge, r_{ij} : ingredient j of composition i.

Constraints

Composition ratio lower limit L_t and upper limit U_t, blend weight: K

Objective function

$$\min \sum_j c_j x_j$$

subject to $\sum_j x_j \leq K$, $L_i \leq \frac{\sum_j r_{ij} x_j}{K} \leq U_i \ \forall\, i$, $0 \leq x_j \leq b_j \ \forall\, j$

Linear programming

- ❖ Scheduling model
  - Problem of obtaining an optimal schedule
  - Problem most commonly encountered in daily life

- ❖ Example
  - Full time with lunch time, part time no lunch
  - \$11/hr for R, \$12/hr for RN, 150% pay for Overtime, \$7, \$8 for PT

Start	Full-Time Shifts			Part-Time Shifts							
	11	12	13	11	12	13	14	15	16	17	18
11:00	R	—	—	R	—	—	—	—	—	—	—
12:00	R	R	—	R	R	—	—	—	—	—	—
13:00	R	R	R	R	R	R	—	—	—	—	—
14:00	R	R	R	R	R	R	R	—	—	—	—
15:00	—	R	R	—	R	R	R	R	—	—	—
16:00	R	—	R	—	—	R	R	R	R	—	—
17:00	R	R	—	—	—	—	R	R	R	R	—
18:00	RN	RN	RN	—	—	—	—	RN	RN	RN	RN
19:00	RN	RN	RN	—	—	—	—	—	RN	RN	RN
20:00	ON	RN	RN	—	—	—	—	—	—	RN	RN
21:00	—	ON	RN	—	—	—	—	—	—	—	RN

[a] R, regular duty; O, possible overtime; N, night differential.

Linear programming

- Example (cont'd)
  - Parameters
    - $r_h =$ workload at time period (shift start time) h
    - $K =$ number of machine (PC) availability, i.e., at most K workers
  - Constraints
    - No more than half the full-time employees on any shift work overtime
    - Total number of scheduled overtime not exceed 20 per day
    - Full time workers process 1/hr, and part-time workers process 0.8/hr

Linear programming

- Example (cont'd)
  - Variables
    - x_h = # full time workers beginning at h
    - y_h = # full time workers with overtime beginning at h
    - z_h = # part time workers beginning at h

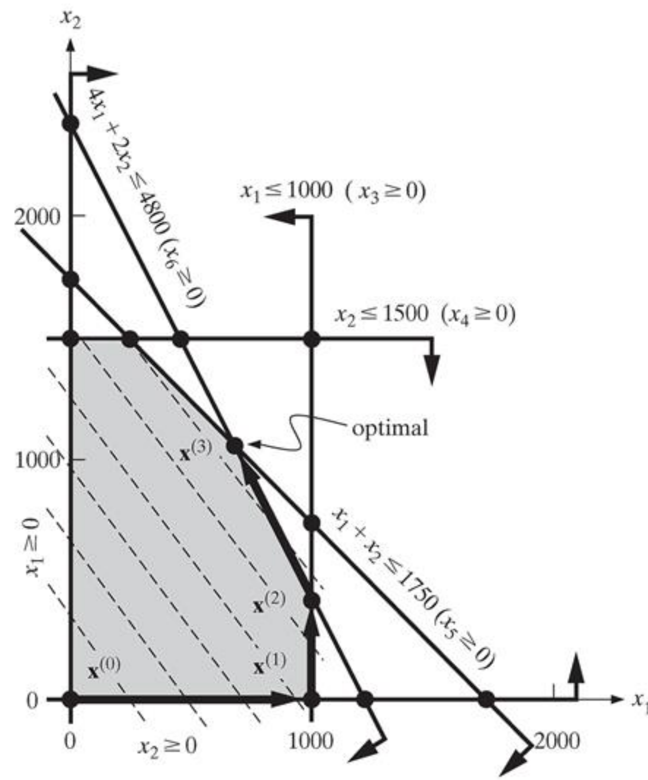
    (assuming x_h includes y_h)

min	$90x_{11} + 91x_{12} + 92x_{13} + 18y_{11} + 18y_{12} + 28z_{11} + 28z_{12}$ $+ 28z_{13} + 28z_{14} + 29z_{15} + 30z_{16} + 31z_{17} + 32z_{18}$		(total pay)
s.t.	$x_{11} + z_{11}$	≤ 35	(11:00 machine)
	$x_{11} + x_{12} + z_{11} + z_{12}$	≤ 35	(12:00 machine)
	$x_{11} + x_{12} + x_{13} + z_{11} + z_{12} + z_{13}$	≤ 35	(13:00 machine)
	$x_{11} + x_{12} + x_{13} + z_{11} + z_{12} + z_{13} + z_{14}$	≤ 35	(14:00 machine)
	$x_{12} + x_{13} + z_{12} + z_{13} + z_{14} + z_{15}$	≤ 35	(15:00 machine)
	$x_{11} + x_{13} + z_{13} + z_{14} + z_{15} + z_{16}$	≤ 35	(16:00 machine)
	$x_{11} + x_{12} + z_{14} + z_{15} + z_{16} + z_{17}$	≤ 35	(17:00 machine)
	$x_{11} + x_{12} + x_{13} + z_{15} + z_{16} + z_{17} + z_{18}$	≤ 35	(18:00 machine)
	$x_{11} + x_{12} + x_{13} + z_{16} + z_{17} + z_{18}$	≤ 35	(19:00 machine)
	$y_{11} + x_{12} + x_{13} + z_{17} + z_{18}$	≤ 35	(20:00 machine)
	$y_{12} + x_{13} + z_{18}$	≤ 35	(21:00 machine)

Linear programming

❖ Simplex

- Most basic method for solving an LP problem
- Algorithm that determines the direction of movement for the input variables to find the optimal solution



Linear programming

❖ Example

$$
\begin{aligned}
& Max\ 12x_1 + 9x_2 \\
s.t.\ & x_1 \le 1{,}000 \\
& x_2 \le 1{,}500 \\
& x_1 + x_2 \le 1{,}750 \\
& 4x_1 + 2x_2 \le 4{,}800
\end{aligned}
$$

- How should an optimization problem like above one be solved?
  - It is difficult to solve intuitively

- So, how does the Simplex method solve it?
  - First, the inequalities are converted into equalities

Linear programming

- ❖ Example (cont'd)
  - How are the inequalities converted into equalities?
    - By using slack variables
      - Represents the surplus portion of the constraint ('>')

$$\max Z = 12x_1 + 9x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$$

subject to
$x_1 + S_1 = 1000,$
$x_2 + S_2 = 1500,$
$x_1 + x_2 + S_3 = 1750,$
$4x_1 + 2x_2 + S_4 = 4800,$
and $x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$

Linear programming

❖ Example (cont'd)

- Constructing the optimization cost table

Iteration-1		C_j	12	9	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio $\frac{X_B}{x_1}$
S_1	0	1000	(1)	0	1	0	0	0	$\frac{1000}{1} = 1000 \rightarrow$
S_2	0	1500	0	1	0	1	0	0	---
S_3	0	1750	1	1	0	0	1	0	$\frac{1750}{1} = 1750$
S_4	0	4800	4	2	0	0	0	1	$\frac{4800}{4} = 1200$
$Z = 0$		Z_j	0	0	0	0	0	0	
		$Z_j - C_j$	-12 ↑	-9	0	0	0	0	

Linear programming

- Example (cont'd)
  - Constructing the optimization cost table

Iteration-2		C_j	12	9	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio $\frac{X_B}{x_2}$
x_1	12	1000	1	0	1	0	0	0	---
S_2	0	1500	0	1	0	1	0	0	$\frac{1500}{1} = 1500$
S_3	0	750	0	1	-1	0	1	0	$\frac{750}{1} = 750$
S_4	0	800	0	(2)	-4	0	0	1	$\frac{800}{2} = 400 \rightarrow$
$Z = 12000$		Z_j	**12**	**0**	**12**	**0**	**0**	**0**	
		$Z_j - C_j$	0	-9 ↑	12	0	0	0	

Linear programming

❖ Example (cont'd)

- Constructing the optimization cost table

Iteration-3		C_j	12	9	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio $\frac{X_B}{S_1}$
x_1	12	1000	1	0	1	0	0	0	$\frac{1000}{1} = 1000$
S_2	0	1100	0	0	2	1	0	-0.5	$\frac{1100}{2} = 550$
S_3	0	350	0	0	(1)	0	1	-0.5	$\frac{350}{1} = 350 \rightarrow$
x_2	9	400	0	1	-2	0	0	0.5	---
$Z = 15600$		Z_j	**12**	**9**	**-6**	**0**	**0**	**4.5**	
		$Z_j - C_j$	0	0	-6 ↑	0	0	4.5	

Linear programming

- ❖ Example (cont'd)
  - Constructing the optimization cost table

Iteration-4		C_j	12	9	0	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4	MinRatio
x_1	12	650	1	0	0	0	-1	0.5	
S_2	0	400	0	0	0	1	-2	0.5	
S_1	0	350	0	0	1	0	1	-0.5	
x_2	9	1100	0	1	0	0	2	-0.5	
$Z = 17700$		Z_j	**12**	**9**	**0**	**0**	**6**	**1.5**	
		$Z_j - C_j$	0	0	0	0	6	1.5	

Questions?

SEE YOU NEXT TIME!