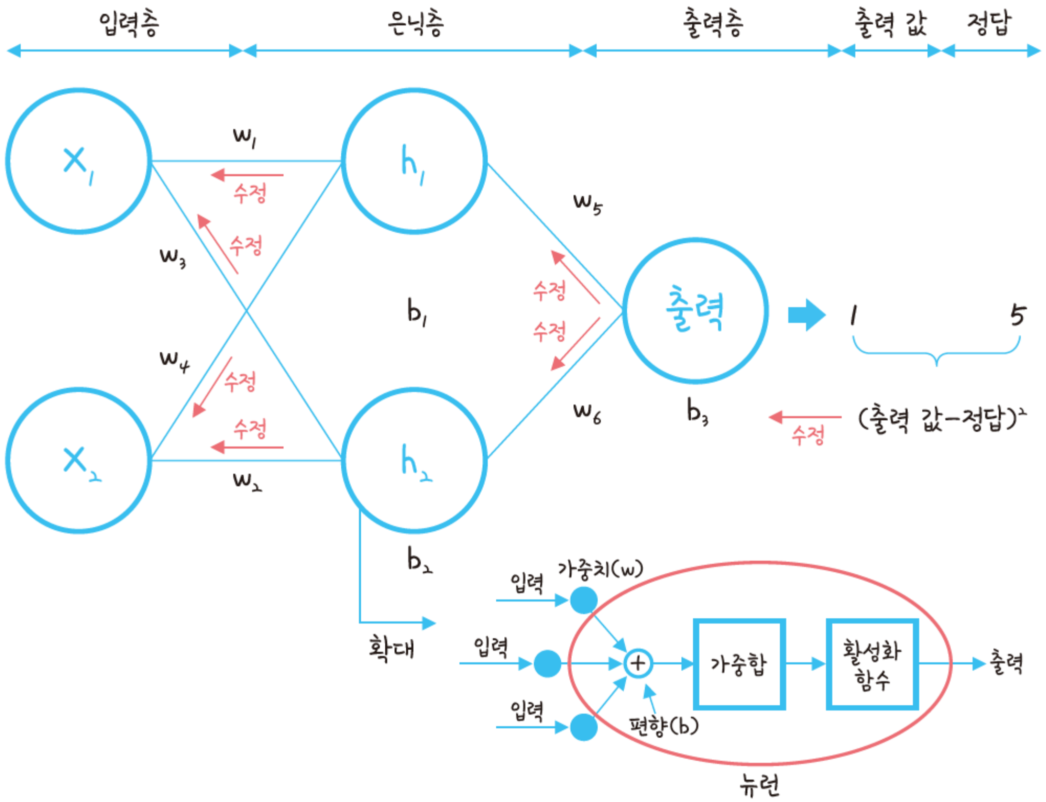
Limit and Continuity of Functions

Limit, maximum, minimum of a function

❖ Differential

Need for differential in AI

- In AI, differential is utilized in backpropagation
- To be precise, it is used to control the values of weights and biases



Limit, maximum, minimum of a function

❖ Differential

- **Input layer**: the layer on which the data is entered
- **Hidden layer**: located between the input and output layers and used to find discriminant boundaries in complex classification problems
- **Output layer**: determines output by calculating the activation function value
- **Weight**: A factor that controls the influence of each signal on the outcome, meaning that the larger the weight, the more important the signal is
- **Weighted sum**: the product of both the input value (x) and the weight (w) plus the value plus the bias (b)
- **Bias**: A constant in addition to the weighted sum, which controls the final output from a single neuron through the activation function
- **Activation function**: The result of the weighted sum is outputted 1 or 0 and sent to the next neuron, where the function that determines 0 and 1 is the activation function
  - Sigmoid, ReLU, etc

Limit, maximum, minimum of a function

MATHEMATIC

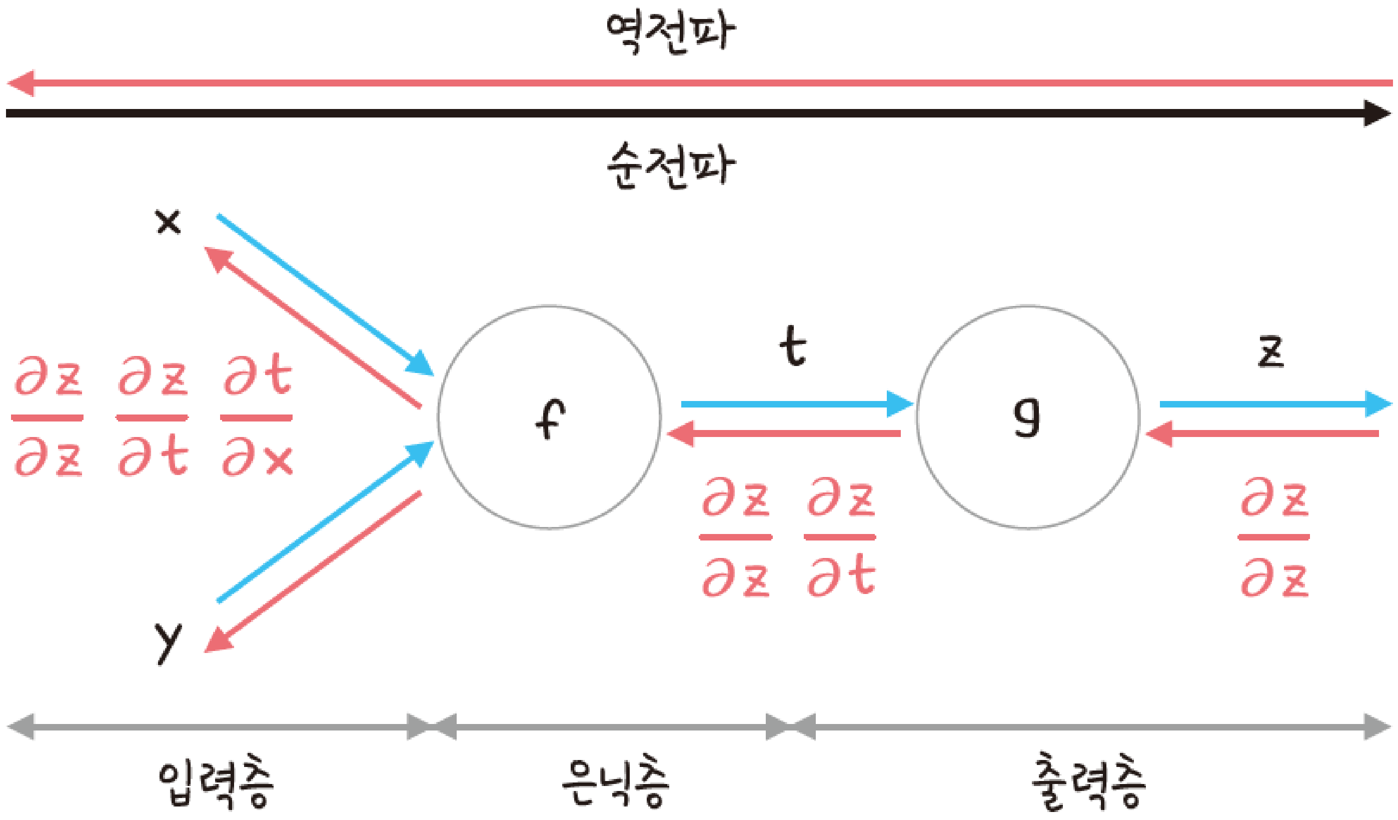
❖ Differential

Differential in Mathematics

- Differential refers to the slope at a point (originally differential refers to the slope between two points, but the distance between the two points is so close to zero that it is expressed as the slope at a point)
- That is, the slope at a point (a, f (a)) above the function f(x) and says f'(a) (read as f prime a)

Limit, maximum, minimum of a function

❖ **Differential for Backpropagation**



Limit, maximum, minimum of a function

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❖ Differential for Backpropagation

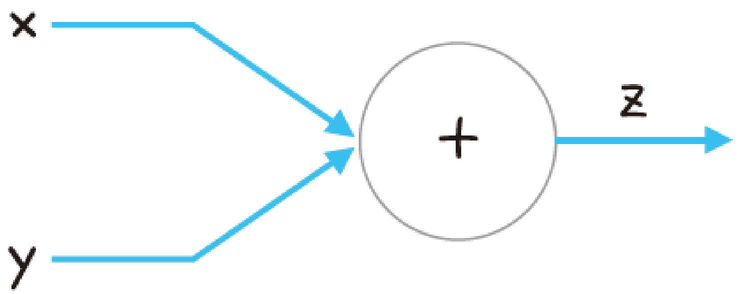
- The backpropagation is said to correct the weight immediately before it in the direction of reducing the error
- When correcting the weight, the partial differential value of $y = g(f(x))$ calculated from the net propagation is multiplied by the error and transmitted to the downstream node (the hidden layer)
- The reason for using **partial differential** at this time is that it does not need to consider all the weight values given to numerous nodes, only the connected weights need to be considered
- This is because using the chain law, even if there are many hidden layers between the output layer and the input layer, the slope can be calculated with a simple differential

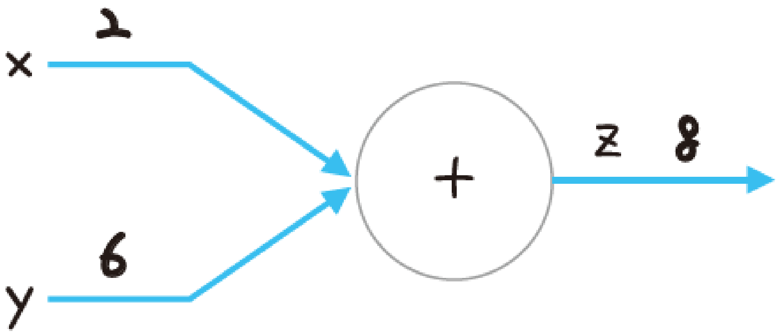
❖ Differential for Backpropagation

Backpropagation calculation

Node backpropagation of addition

- The calculation graph for the expression z = x + y is as follows
- The forward propagation calculation for z = x + y when x = 2, y = 6 is as follows





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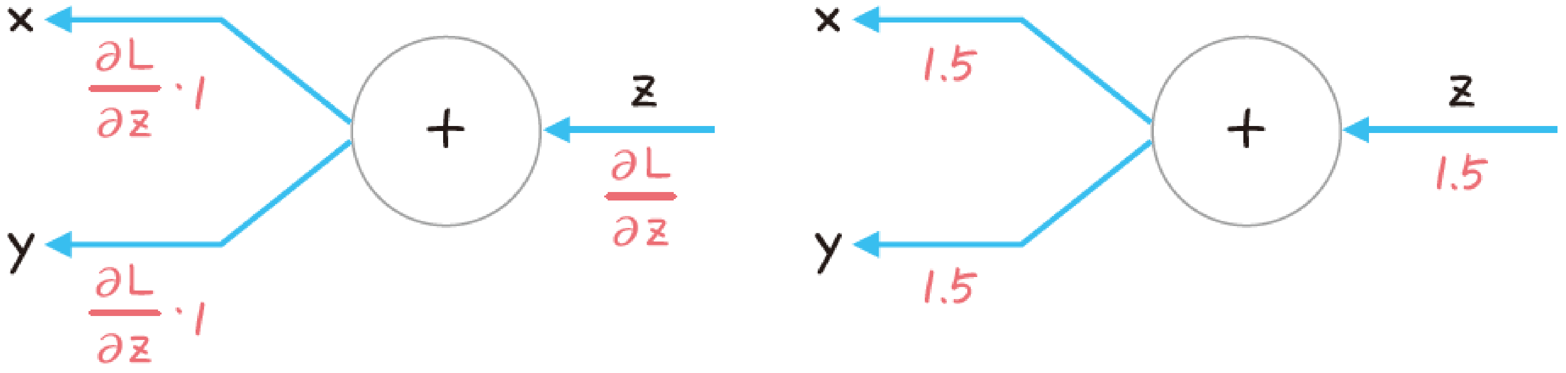
❖ Differential for Backpropagation

- Differentiable (L is considered the final output value)

$$z = x + y$$

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = 1$$



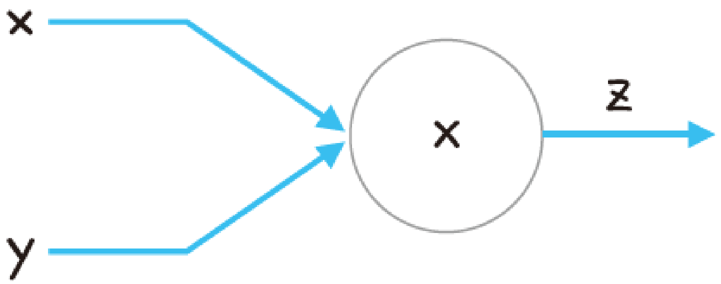
- Assuming that a value of 1.5 is input at upstream (output), the backpropagation of the addition node propagates the input value to the next node

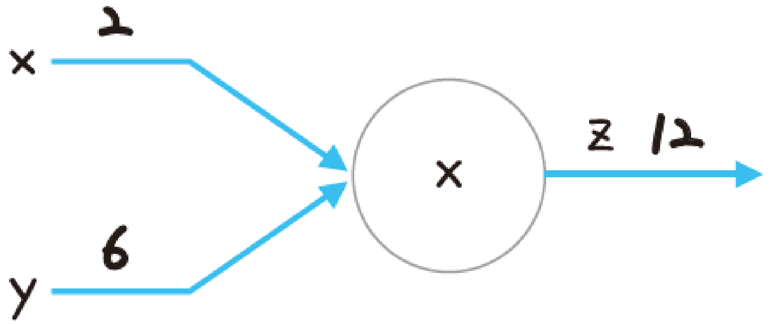
Limit, maximum, minimum of a function

❖ Differential for Backpropagation

Node backpropagation of multiplication

- The calculation graph for z = xy expression is shown on the left
- The forward propagation calculation for z = xy when x = 2, y = 6 is equal to the right





Limit, maximum, minimum of a function

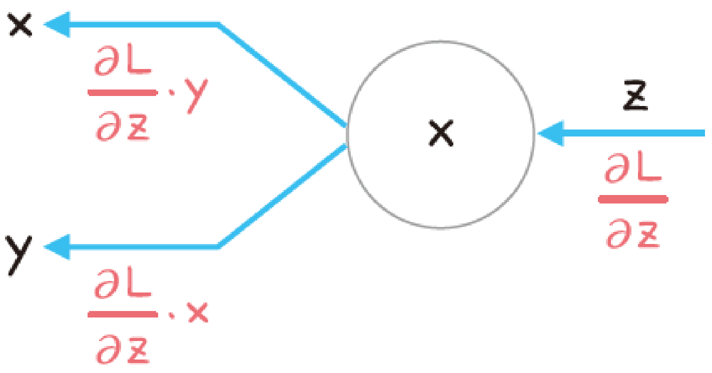
MATHEMATIC

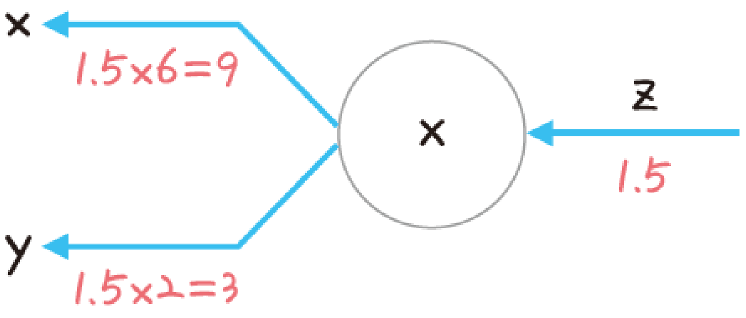
❖ Differential for Backpropagation

- Differential is possible as shown on the left
- Assuming that a value of 1.5 is input from the upstream, the backpropagation of the multiplication node can be sent downstream (the hidden layer) by multiplying the upstream (output) value by the input signal of the forward propagation by the 'switched value'

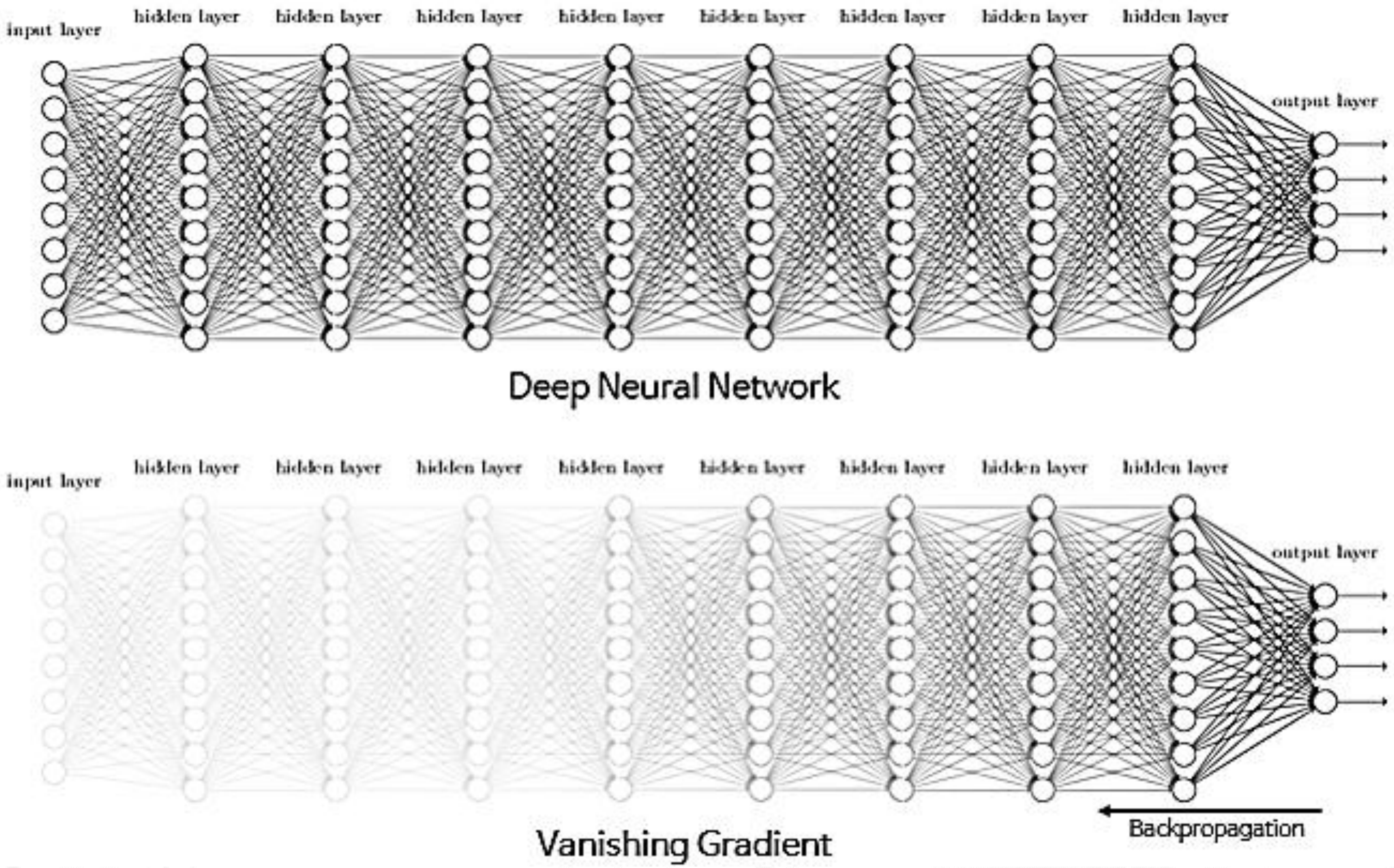
$$z = xy$$

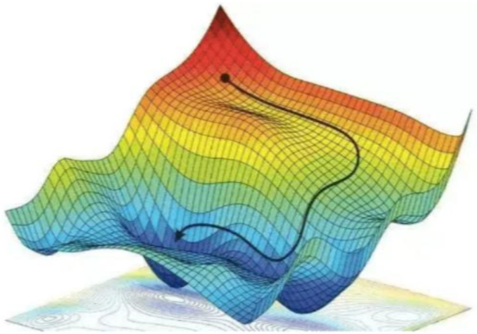
$$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x$$





Limit, maximum, minimum of a function





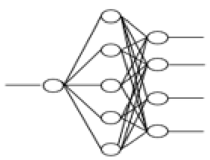
<그림 1> 인공 신경망의 비용 함수(Cost Function) 그래프

❖ Gradient descent

"Loss function(error)을 최소화 하는 신경망 모델의 파라미터를 찾는다 "

$$MSE = \frac{1}{N}\sum_{i=1}^{n}(\hat{y_i} - y_i)^2$$

Input : x



output : y

Loss function
$J = (y - t)^2$

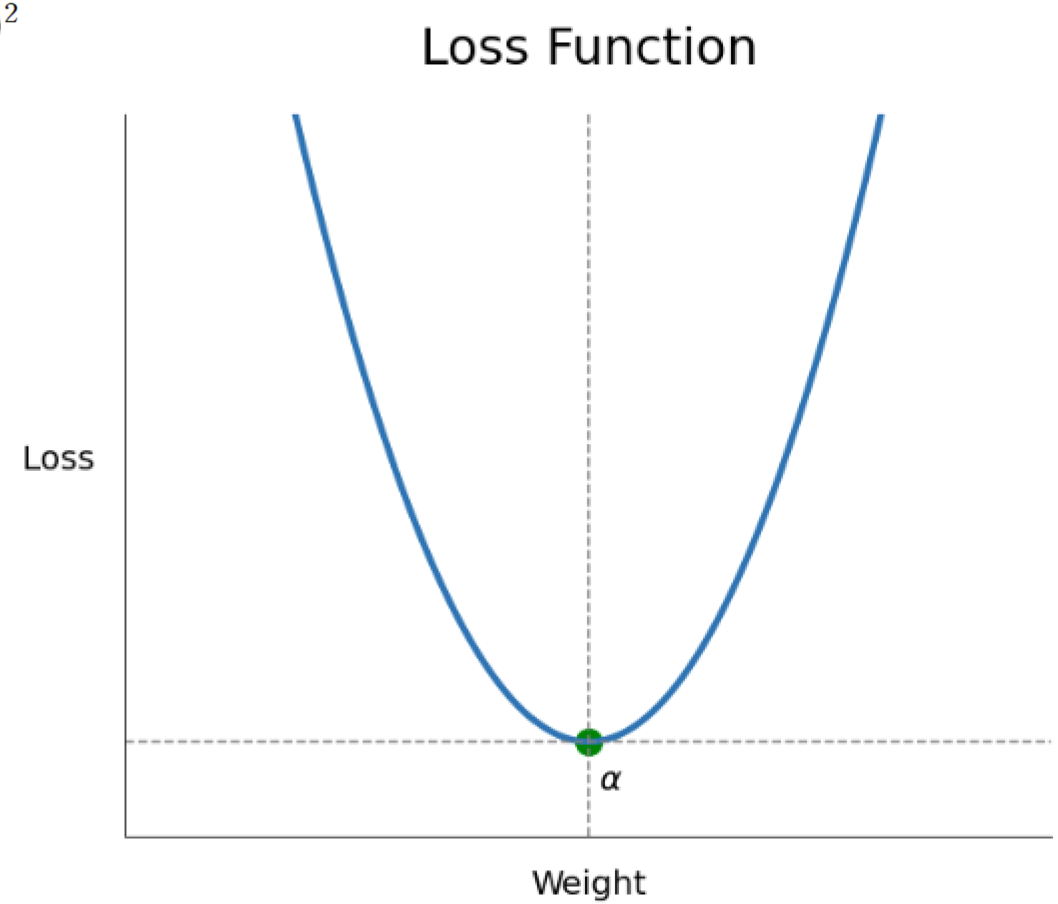
Target : T

1) 임의의 파라미터 값을 시작점으로 지정.
2) 해당 점에서의 목적함수 Gradient값을 구함.
3) 다음과 같이 파라미터를 업데이트 !

$$\boxed{x_{i+1} = x_i - \alpha\nabla f(x_i)}$$

목적함수 값이 특정 값으로 수렴할 때까지 위의 과정을 반복.

x_{i+1}: i+1번째 step의 파라미터 x의 값
x_i: i번째 step의 파라미터 x의 값
α: Learning rate or Step size (한 번에 어느 정도 이동할 것인지): 하이퍼파라미터
$\nabla f(x_i)$: 파라미터의 값이 xi일 때의 목적함수 f의 그래디언트 값 (방향 및 기울기)

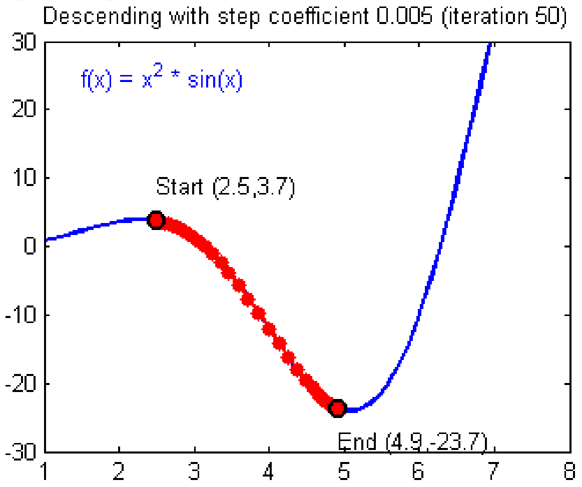


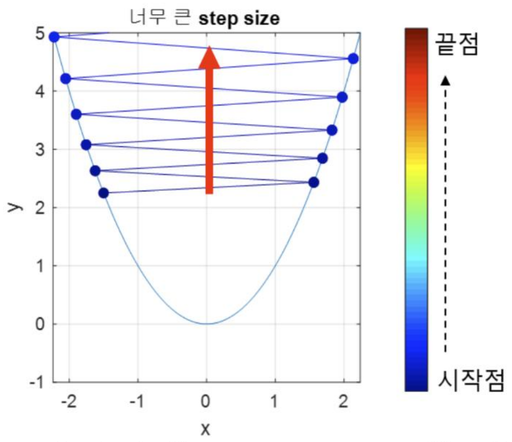
Limit, maximum, minimum of a function

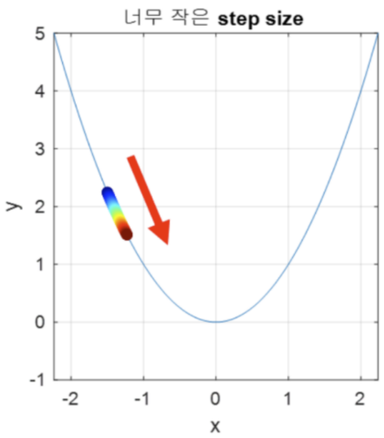
❖ Limitation of gradient descent

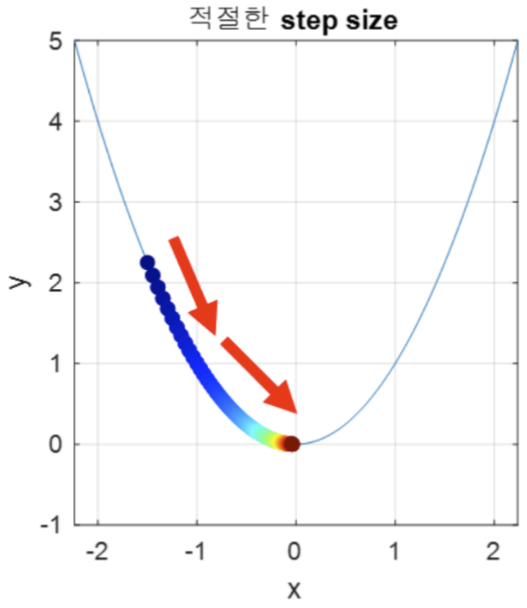
Step size = Learning rate

- **Learning rate**
  - It is a constant of how much changes are made when learning once, and it is a very important value in AI because the accuracy may vary depending on the learning rate value





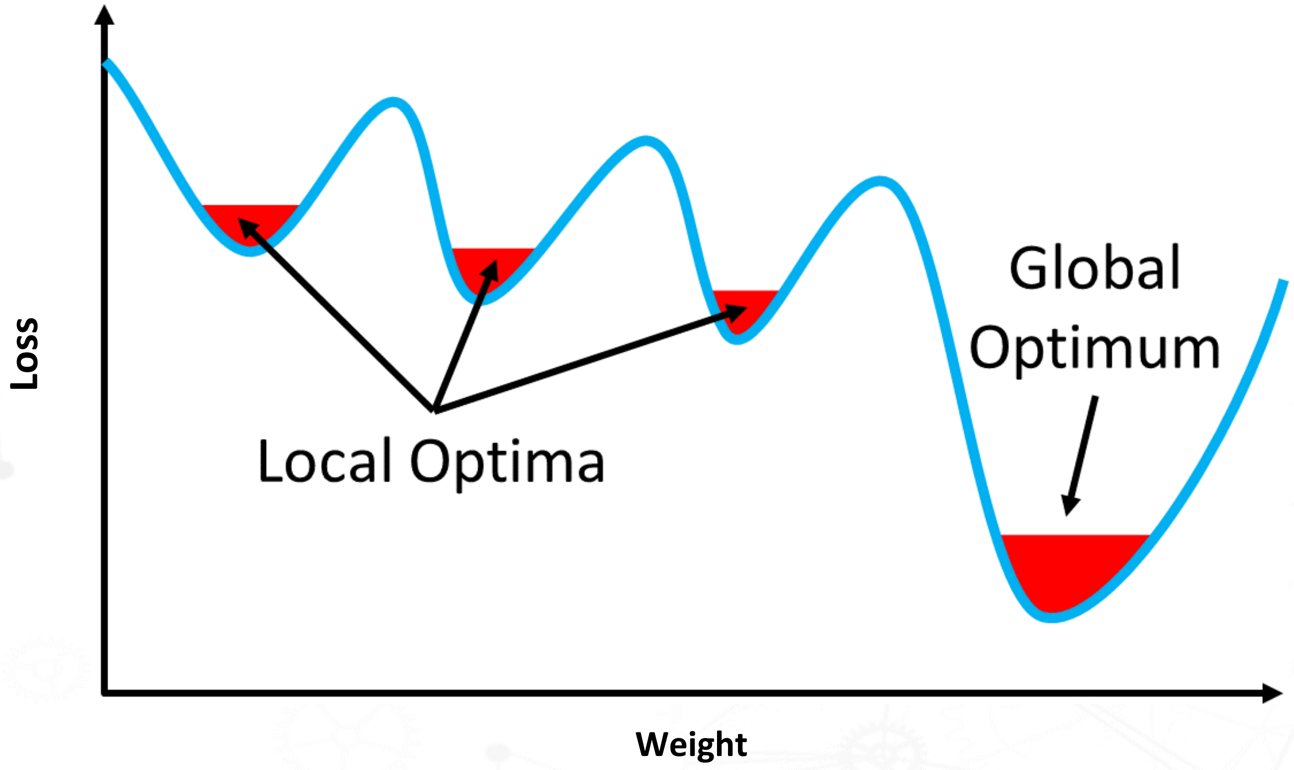




Limit, maximum, minimum of a function

❖ Limitation of gradient descent

Local minima



MATHEMATIC

❖ Differential

- f ‘(a) varies in meaning as follows

f '(a) = slope at the point a

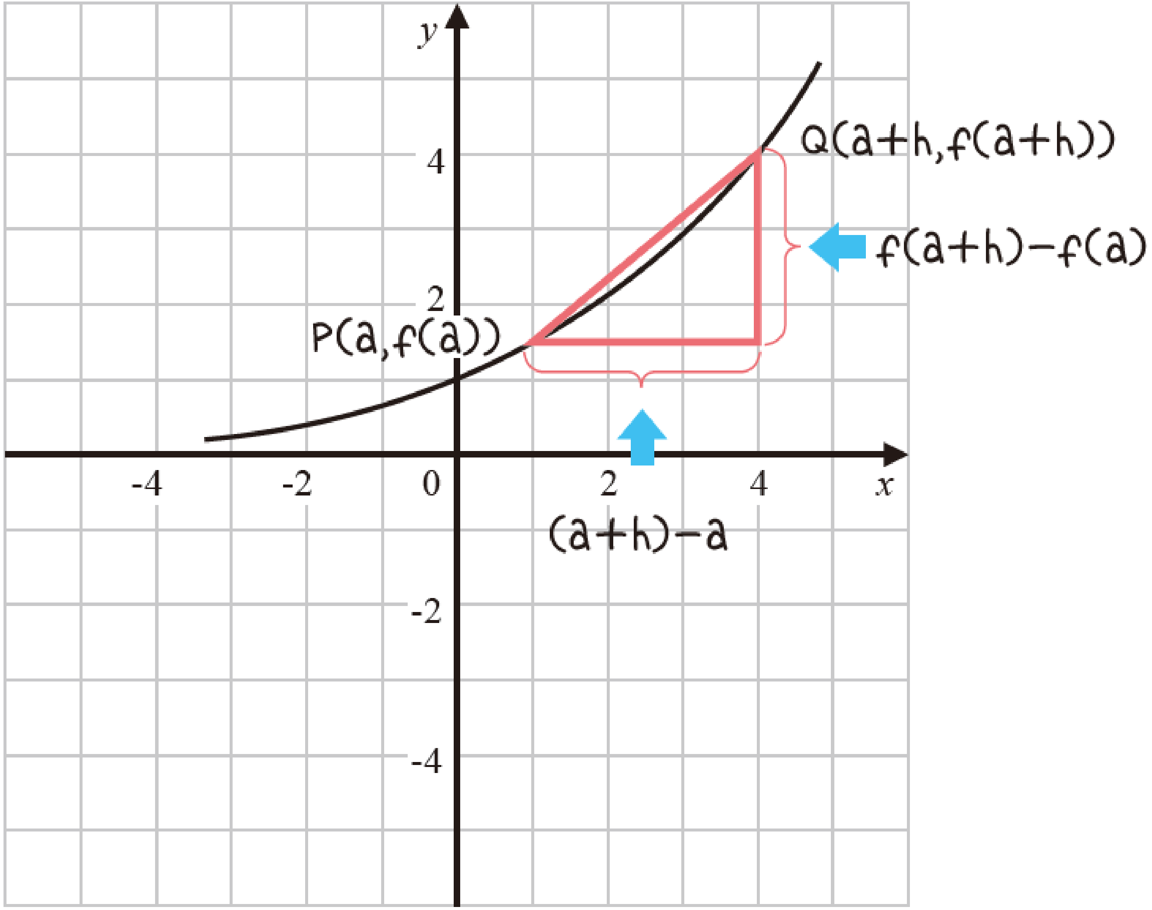
= slope of the tangent line in a

= differential value in a

= differential coefficient in a

❖ Differential

- Differential is expressed in coordinates of the graph as follows



Limit, maximum, minimum of a function

❖ Differential

- If there is a point P(a, f (a)) above the function f(x), and there are two points Q(a + h, f (a + h)) that move h from a to the right (→) and up (↑), the slope between the points P and Q can be expressed as Equation 7.1

$$\text{기울기} = \frac{y\text{의 증가량}}{x\text{의 증가량}} = \frac{f(a+h)-f(a)}{(a+h)-a} \qquad \text{수식 7.1}$$

Limit, maximum, minimum of a function

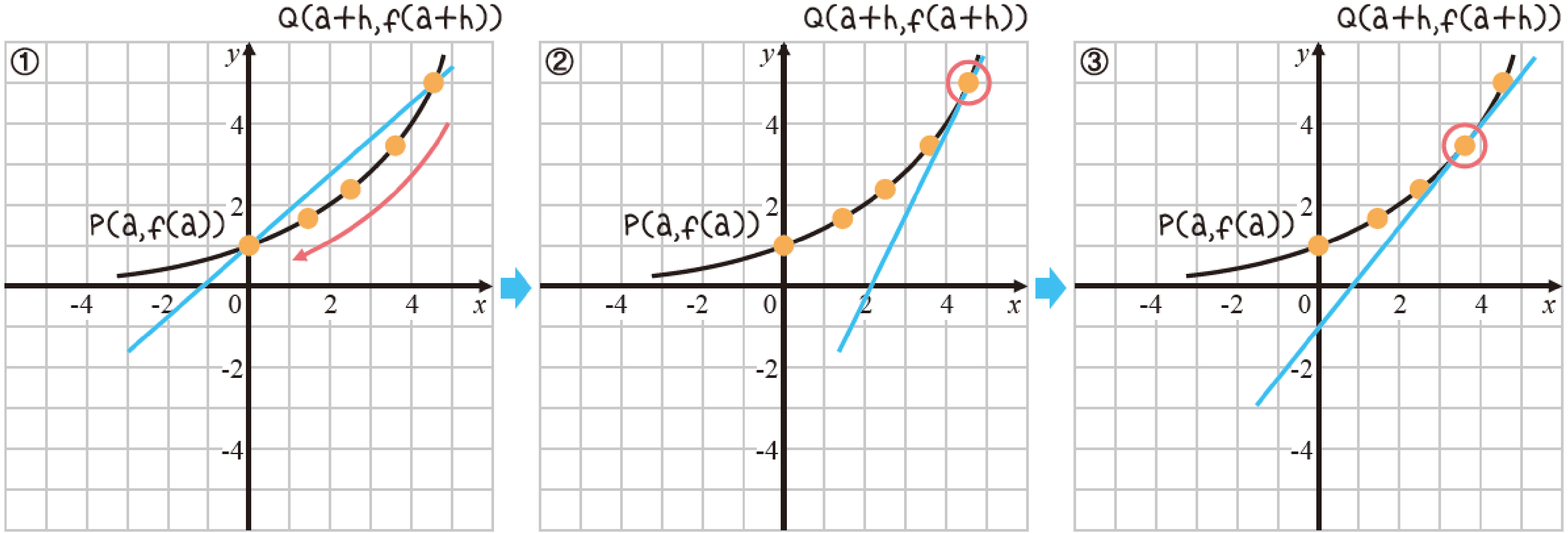
❖ Differential

- Differential is defined as the slope at a point
- It's the slope, but it's the slope as the increase in x approaches zero
- In Equation 7.1 the amount of increase in x is expressed as (a + h) - a), so it becomes a - a + h, and eventually the amount of increase in x becomes h
- If h is sent to 0, it can be expressed as Equation 7.2

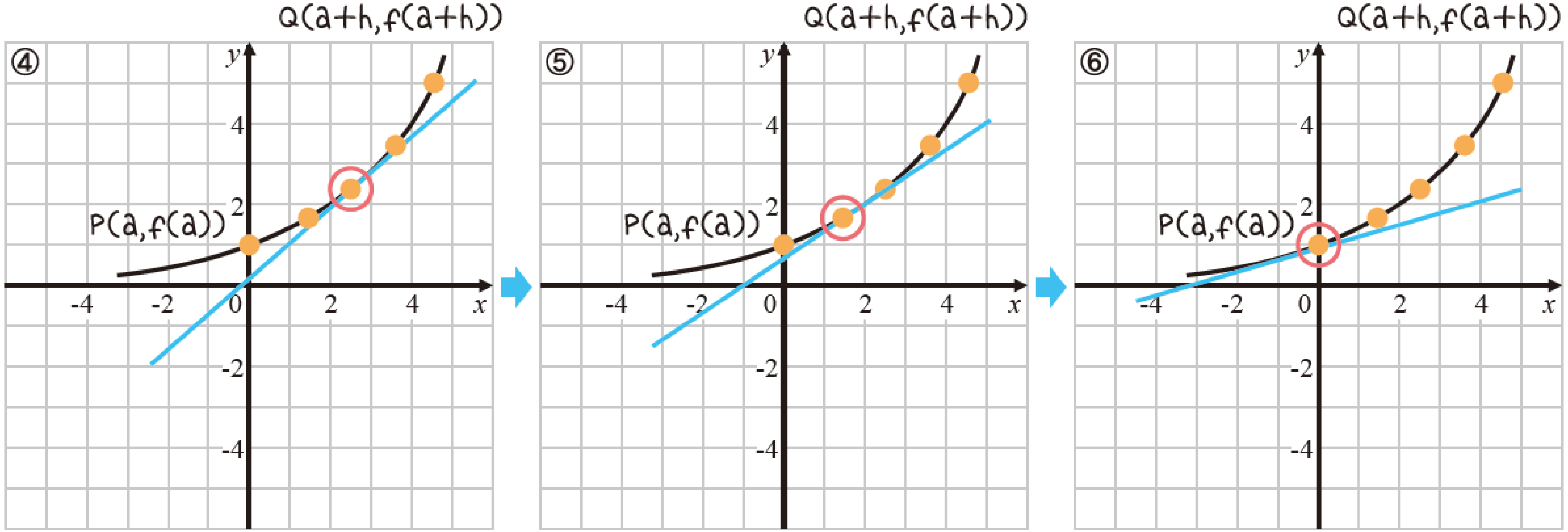
$$\lim_{x\text{ 증가량}\to 0} \frac{y\text{의 증가량}}{x\text{의 증가량}} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a} \qquad \text{수식 7.2}$$

❖ Differential

- If point Q approaches zero at point P, the slope changes as follows, eventually becoming the slope at point P



Limit, maximum, minimum of a function



Limit, maximum, minimum of a function

❖ Differential

- Eventually, it became a tangent at the point of P, and in other words, it becomes the slope of the tangent at the point of x = a
- This is the same as Equation 7.3

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = f'(a)$$

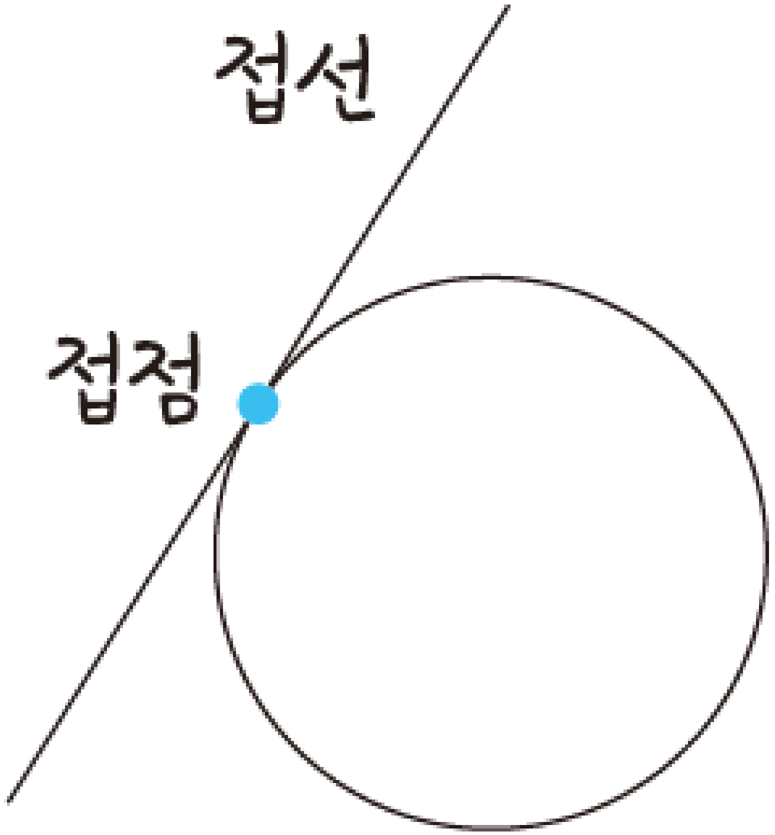
수식 7.3

Limit, maximum, minimum of a function

❖ Differential

Tangent, tangent line

- Tangent is a straight line that touches the curve at a point on the curve

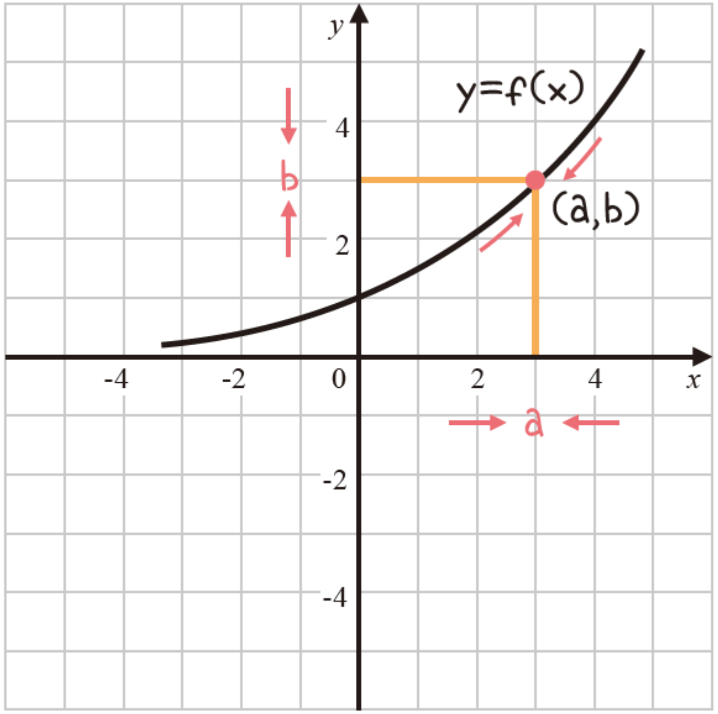


Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

Convergence

- In function f(x), when x takes a value different from a ($x \neq a$) and gets infinitely close to a, i.e., when the value of f(x) gets infinitely close to a constant value b, the function f(x) converges to b



Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- Here, b is called the limit of f(x) when the value of x is infinitely close to a, and is expressed as follows

$$x \to a \text{일 때}, f(x) \to b \text{ 또는 } \lim_{x \to a} f(x) = b$$

- x → a means
  - (1) If x and a are not the same (x =/ a), it means the state of being infinitely close to a
  - (2) The two cases of the left side limit (x → a - 0) and the right side limit (x → a + 0) are combined to be expressed as x → a

❖ **Limit of function** (convergence and divergence)

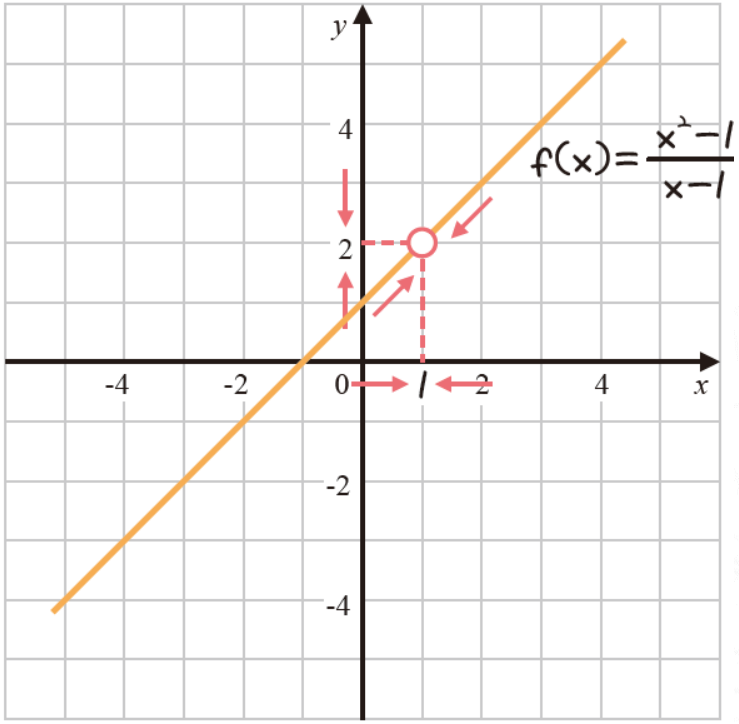
- In particular, the constant function f(x) = c (c is a constant) always outputs the same value c for all x values
- Regardless of a value, the following is established

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c$$

Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- For example, if x = 1 in function $f(x) = \frac{x^2-1}{x-1}$, the denominator becomes 0, so the function value f(1) is not defined at x = 1
- When x ≠ 1, $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1$, so if x has a non-1 value and approaches 1 infinitely, f(x) approaches 2



❖ Limit of function (convergence and divergence)

연습 문제

$x \to 2$일 때, 함수 $f(x) = \sqrt{x+2}$의 극한을 구하세요.

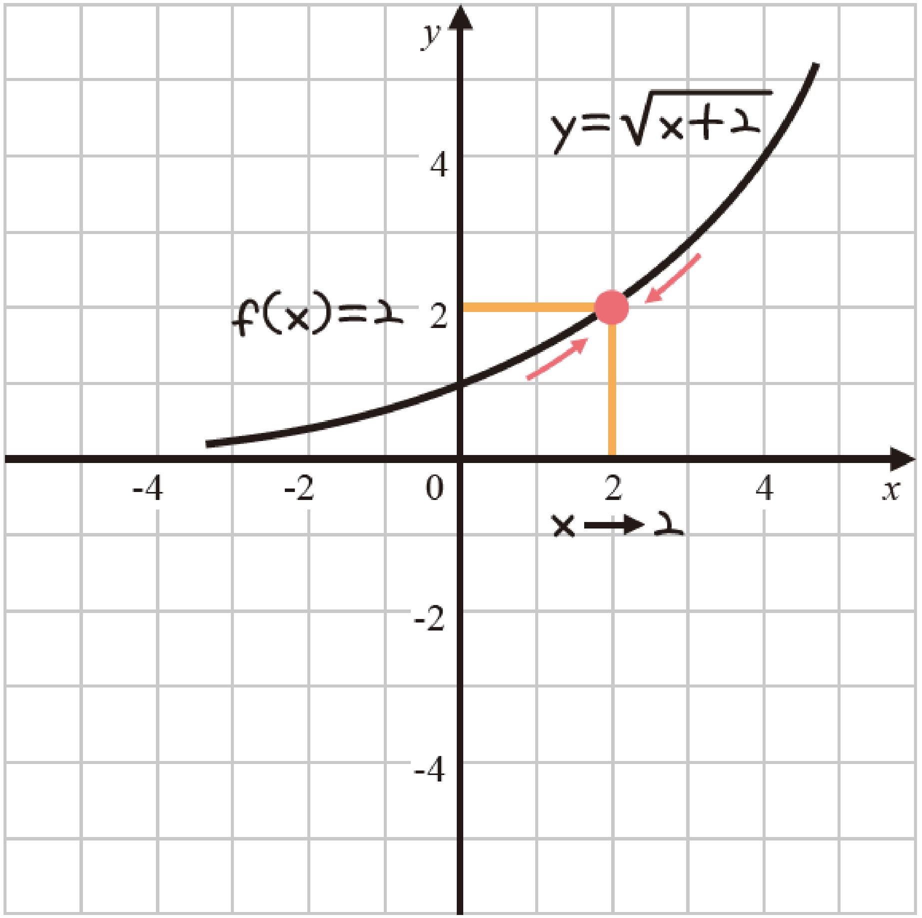
문제 풀이

그림 7–7의 그래프와 같이 $x \to 2$일 때, $\sqrt{x+2} \to 2$이므로 극한은 $\sqrt{2+2}$가 됩니다.

$$\therefore \lim_{x \to 2} f(x) = 2$$

Limit, maximum, minimum of a function

$y=\sqrt{x+2}$



Limit, maximum, minimum of a function

MATHEMATIC

❖ **Limit of function** (convergence and divergence)

Limit of function in ∞, -∞

- When x is positive in function f(x) and its absolute value increases infinitely, when the value of function f(x) approaches (converges) infinitely close to a certain value a, it is expressed as follows

$$x \to \infty \text{일 때}, f(x) \to a \text{ 또는 } \lim_{x\to\infty} f(x) = a$$

- When x is negative at f(x) and its absolute value increases infinitely, if the function f(x) value approaches (converges) infinitely close to a certain value a, it is expressed as follows

$$x \to -\infty \text{일 때}, f(x) \to a \text{ 또는 } \lim_{x\to-\infty} f(x) = a$$

Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- For example, let's find limits for $f(x) = \frac{2x-1}{x}$ when x → ∞ and x → -∞
- In $f(x) = \frac{2x-1}{x}$, the denominator and numerator multiplied by $\frac{1}{x}$ is as follows

$$f(x) = \frac{(2x \times \frac{1}{x}) - (1 \times \frac{1}{x})}{x \times \frac{1}{x}} = \frac{2 - \frac{1}{x}}{1} = 2$$

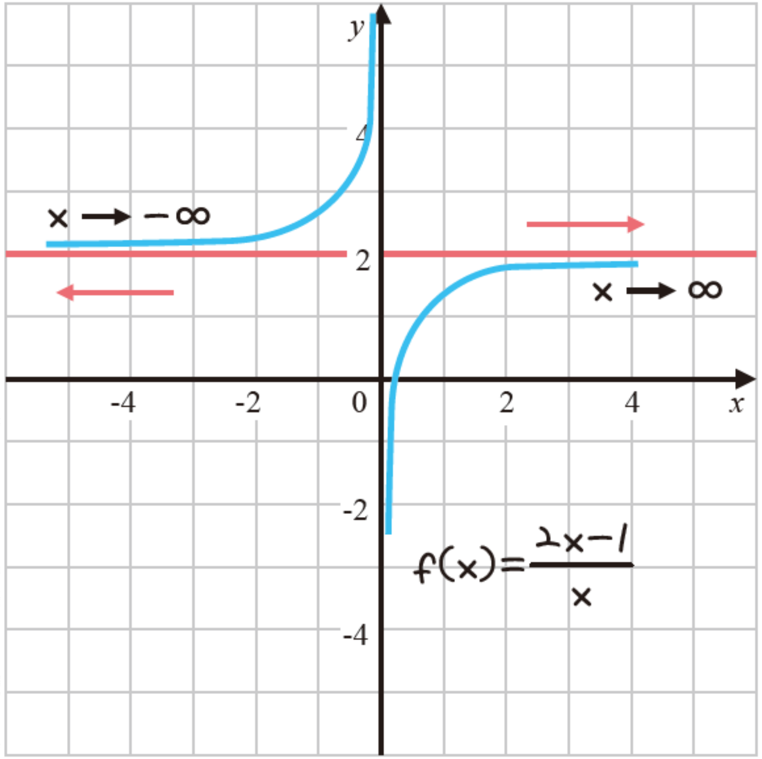
($\frac{1}{x}$ 은 0으로 수렴하기 때문에 0이 됩니다.)

Limit, maximum, minimum of a function

❖ **Limit of function** (convergence and divergence)

- According to the definition of infinity, if x → ∞ it becomes infinitely close to 2, if x → -∞ it becomes infinitely close to 2

$$\lim_{x\to\infty} f(x) = 2,\ \lim_{x\to-\infty} f(x) = 2$$



Limit, maximum, minimum of a function

❖ **Limit of function** (convergence and divergence)

Limit of function (divergence)

- Divergence of a function means that the value of the function in function f(x) grows infinitely without converging to any real value
- There is an infinite divergence of positive, an infinite divergence of negative, and oscillations in the divergence of a function
- Positive infinite divergence is expressed as follows: when the value of x in function f(x) approaches infinitely close to a, if the value of f(x) becomes infinitely large, then f(x) diverges to positive infinite

$$x \rightarrow a \text{일 때, } f(x) \rightarrow \infty \text{ 또는 } \lim_{x \rightarrow a} f(x) = \infty$$

Limit, maximum, minimum of a function

MATHEMATIC

❖ **Limit of function** (convergence and divergence)

- Divergence to negative infinity means that when the value of x in the function f(x) approaches infinitely close to a, if the value of f(x) is negative and the absolute value increases infinitely, then f(x) diverges to negative infinity

$$x \to a\text{일 때, } f(x) \to -\infty \text{ 또는 } \lim_{x \to a} f(x) = -\infty$$

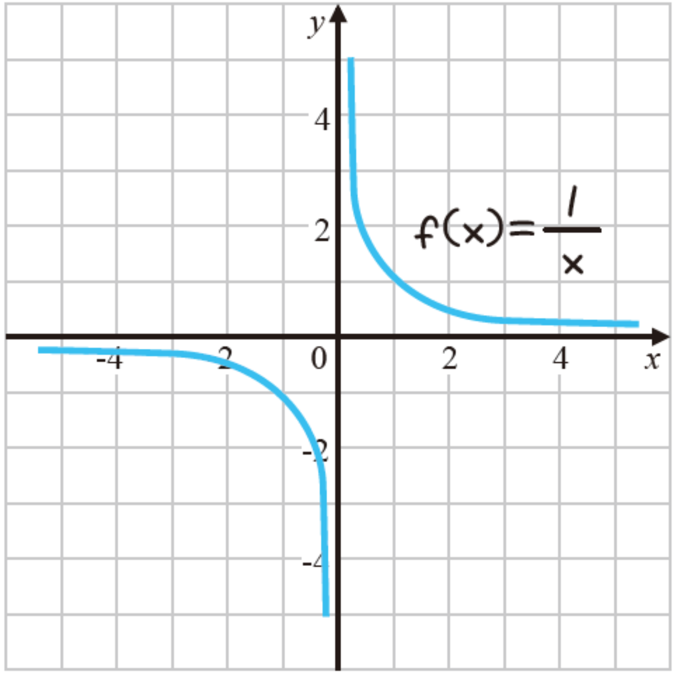
- $f(x) = \frac{1}{x}$ graphs extend to positive and negative infinity
- When x is infinitely close to +0, it becomes +∞, and when x is infinitely close to -0, it becomes -∞

$$\lim_{x \to +0} \frac{1}{x} = +\infty, \quad \lim_{x \to -0} \frac{1}{x} = -\infty$$

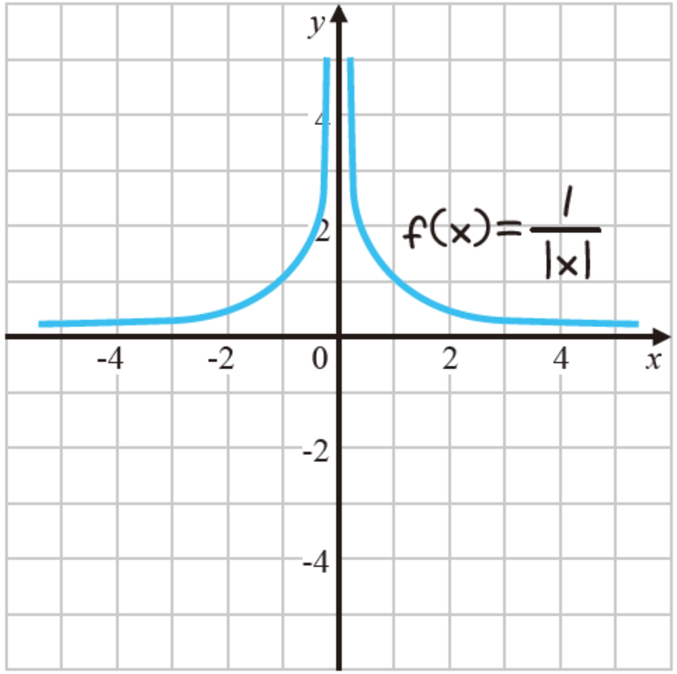
Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- Limit at x = 0 do not exist because the results (limit) for +0 and -0 do not match



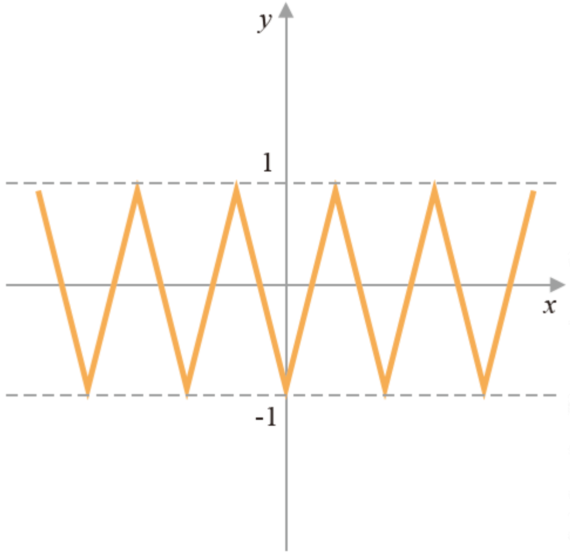
① $f(x)=\frac{1}{x}$ 그래프



② $f(x)=\frac{1}{|x|}$ 그래프

❖ Limit of function (convergence and divergence)

- x approaches infinitely close to +0, it becomes +∞, and x approaches infinitely close to -0, it becomes - ∞
- $\lim_{x \to +0} \frac{1}{|x|} = \infty$ and $\lim_{x \to -0} \frac{1}{|x|} = \infty$,so it can be expressed as $\lim_{x \to 0} \frac{1}{|x|} = \infty$
- Extreme value means the value at convergence, so this also does not converge, so 'no extreme value' is the answer
- For reference, the oscillation of the function is constantly vibrating as it approaches the limit



Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- Extremes can also be obtained from Python
- Limit and S functions must be imported from SymPy before extreme values can be obtained
- Extreme values can be obtained with the Limit function, and S functions can contain definitions of infinity
- For example, $\lim_{x \to \infty} \frac{1}{x}$

```
In [1]:
# 파이썬 SymPy 라이브러리에서 Limit, S, Symbol 클래스를 호출합니다
from sympy import Limit, S, Symbol

# x 변수를 생성하고, x가 0으로 한없이 가까워질 때의
# 1/x에 대한 극한값을 구합니다
x = Symbol('x')
Limit(1/x, x, S.Infinity).doit() # ①
```

Limit, maximum, minimum of a function

MATHEMATIC

❖ Limit of function (convergence and divergence)

Out [1]:

```
0
```

In [2]:

```
Limit(1/x, x, 0).doit() # ② 우극한값 구하기
```

Out [2]:

∞

In [3]:

```
Limit(1/x, x, 0, dir='-').doit() # ③ 좌극한값 구하기
```

❖ Limit of function (convergence and divergence)

Out [3]:

$-\infty$

① Limit은 기본적으로 인수를 세 개 전달합니다. 첫 번째는 극한을 구하는 함수 $\frac{1}{x}$, 두 번째는 x 변수, 세 번째는 x 변수가 다가가는 값입니다. 이때 무한대 $(x \to \infty)$를 정의하려고 S 함수(S.Infinity)를 사용합니다.

② 극한값을 계산하려고 doit() 함수를 사용합니다.

③ 좌극한은 dir='-'를 포함해야 합니다.

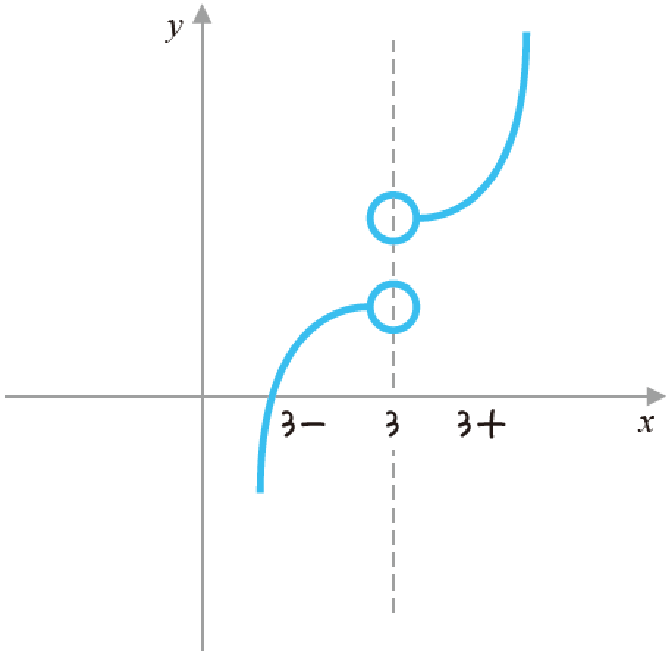
Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

Left, right side limit and conditions of limit value

Left, right side limit of functions

- Show two states where the x value approaches 3

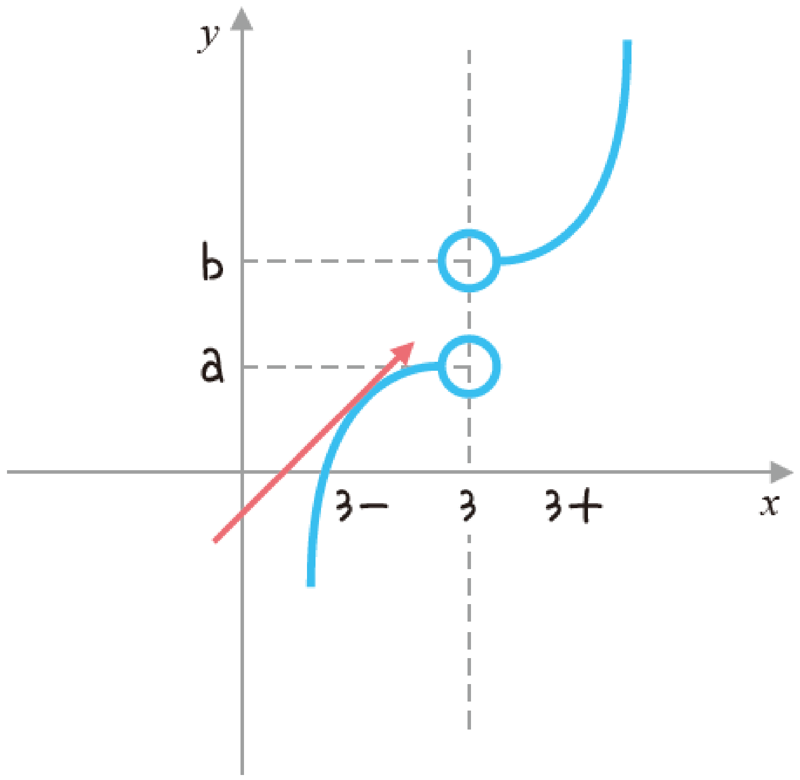


3보다 크면서 3으로 다가오는 상태를 3+로 표현하고, 3보다 작으면서 3으로 다가오는 상태를 3−로 표현합니다.

Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- When x is smaller than a in function f(x) and gets closer and closer to a, if there is a value at which f(x) gets closer, this is called the left side limit of the function at point a



Limit, maximum, minimum of a function

MATHEMATIC

❖ **Limit of function** (convergence and divergence)

- In function y = f(x), x approaches 3 with a value less than 3
- This means that when x → 3-, it can be expressed as follows

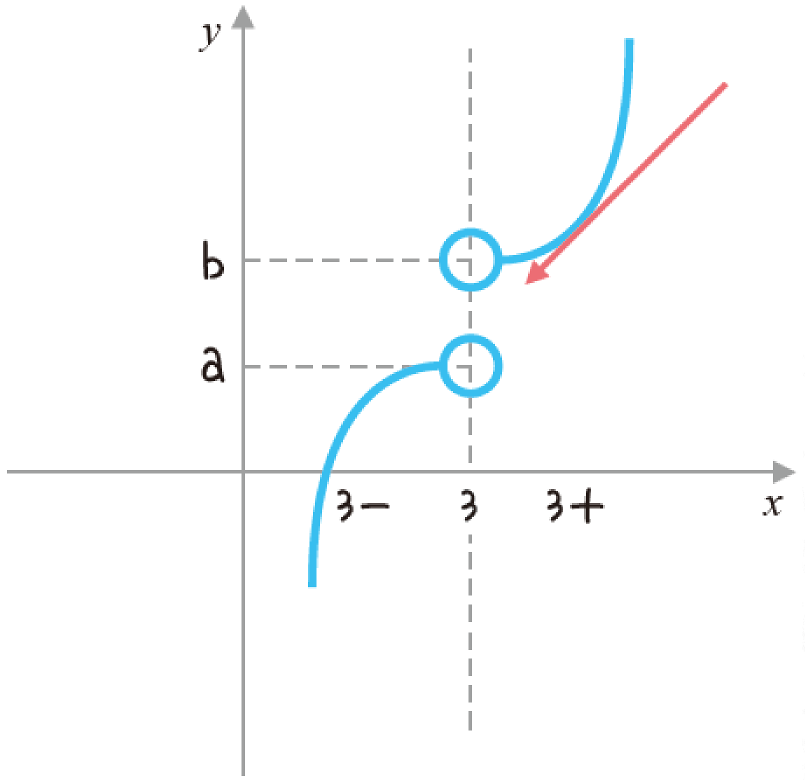
$$\lim_{x \to 3-} f(x) = a$$

- The left side limit at this time is a

Limit, maximum, minimum of a function

❖ Limit of function (convergence and divergence)

- When x is greater than a in function f(x) and gets closer and closer to a, if there is a value at which f(x) gets closer, this is called the right side limit of the function at point a



Limit, maximum, minimum of a function

- **Limit of function** (convergence and divergence)
  - In function y = f(x), x approaches 3 with a value greater than 3
  - This means that when x → 3+, it can be expressed as follows

$$\lim_{x \to 3+} f(x) = b$$

  - The right side limit at this time is b

Limit, maximum, minimum of a function

MATHEMATIC

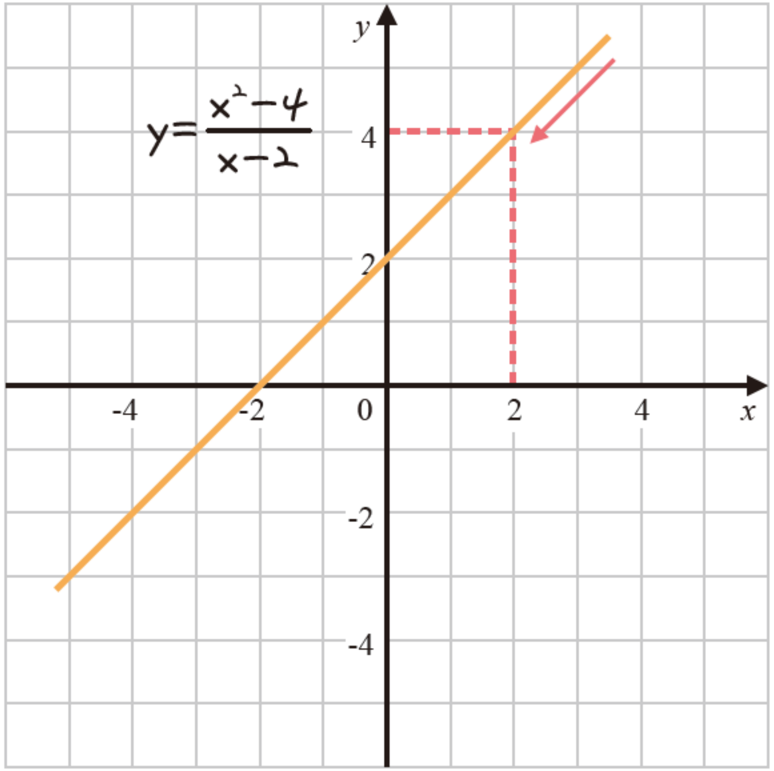
❖ **Limit of function** (convergence and divergence)

Condition of limit values

- The conditions under which the limit value will exist must be equal to the right and left side limits
- If, $\lim_{n \to a-} f(x) = \lim_{n \to a+} f(x) = a$(left and right side limits are the same), then $\lim_{n \to a} f(x) = a$

❖ **Limit of function** (convergence and divergence)

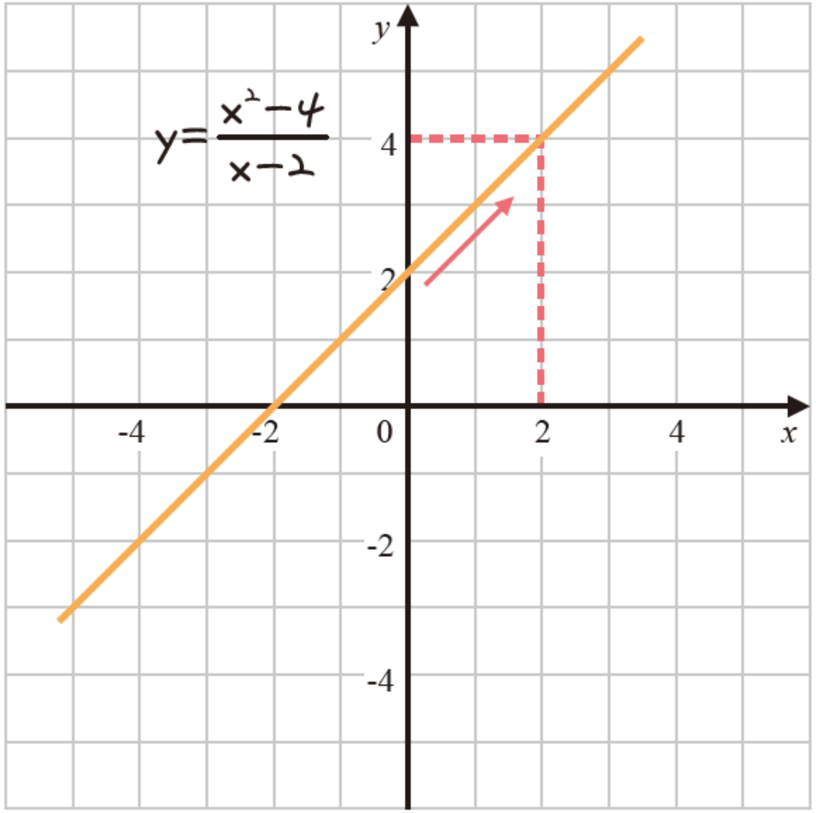
- Let’s find the right side limit and left side limit



Limit, maximum, minimum of a function

❖ **Limit of function** (convergence and divergence)

- Left side limit



Limit, maximum, minimum of a function

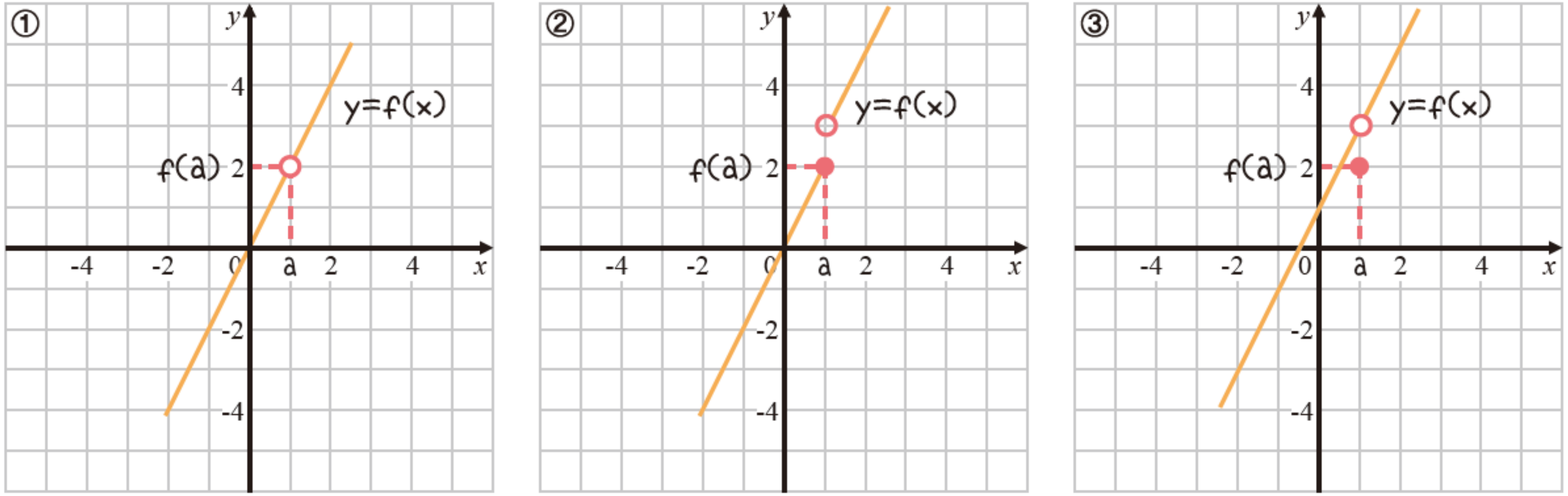
MATHEMATIC

❖ Limit of function (convergence and divergence)

- Graphically confirmed that the values of the right and left sides are the same for the function $\lim_{x \to 2} \frac{x^2-4}{x-2}$
- Mathematical solution
  - (1) If $(x^2 - 4)$ is factorized, it becomes (x + 2)(x - 2),
  - (2) $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} x + 2$
  - (3) Substituting 2 for x results in $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$
  - (4) So the limit value is 4

❖ Continuous of function

- **Continuous of function** means a function in which the graph is continuously connected without breaking
- To understand the meaning of continuous connectivity of graphs, let's first look at the concept of discontinuities in which functions are not continuous
- Discontinuous of function is a function broken at x = a



❖ Continuous of function

- All three are discontinuous (broken) graphs
- ① is a discontinuous graph because the function f(x) is not defined at x=a, ② has different values of the left and right side limits, ③ is a discontinuous graph because the $\lim_{x \to a} f(x)$ and f(a) are different
- If we think of these three discontinuities in reverse, it is easy to derive the conditions under which the function is continuous

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❖ Continuous of function

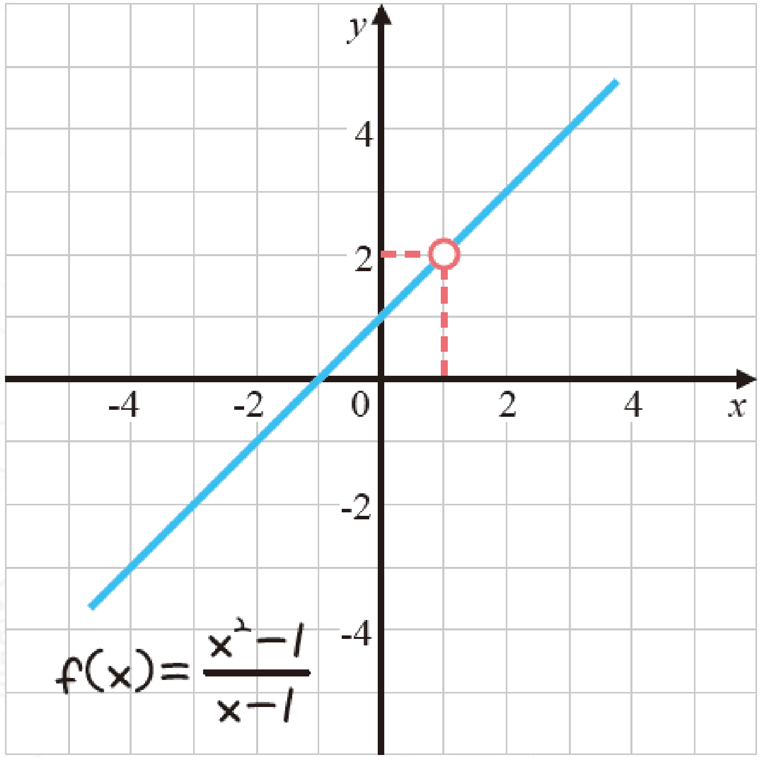
What does the expression ‘f(x) is not defined in x=a’ mean?

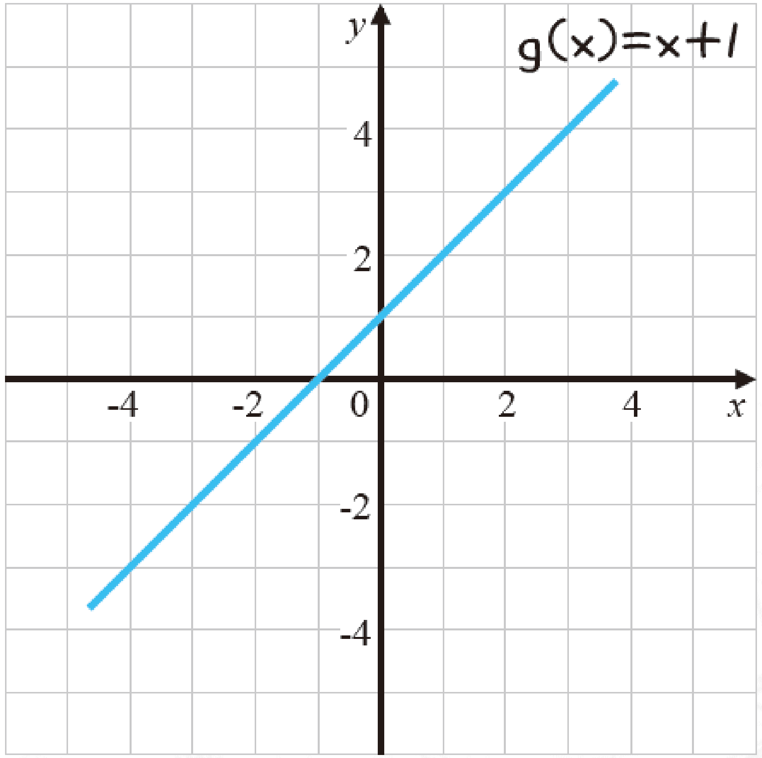
- To simplify the formula for graph expression for function f(x)

  - $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1$

- However, when x is 1, the denominator becomes 1 and the f(x) value becomes 0, so it must be $x \neq 1$
- At this time, $x \neq 1$ means that the function is not defined when x = 1 in function f(x), and on the graph, x = 1 is expressed as a circle with an empty hole
- Note that if the value is defined at x = 1, it is expressed in a circle filled with holes

Limit, maximum, minimum of a function





Limit, maximum, minimum of a function

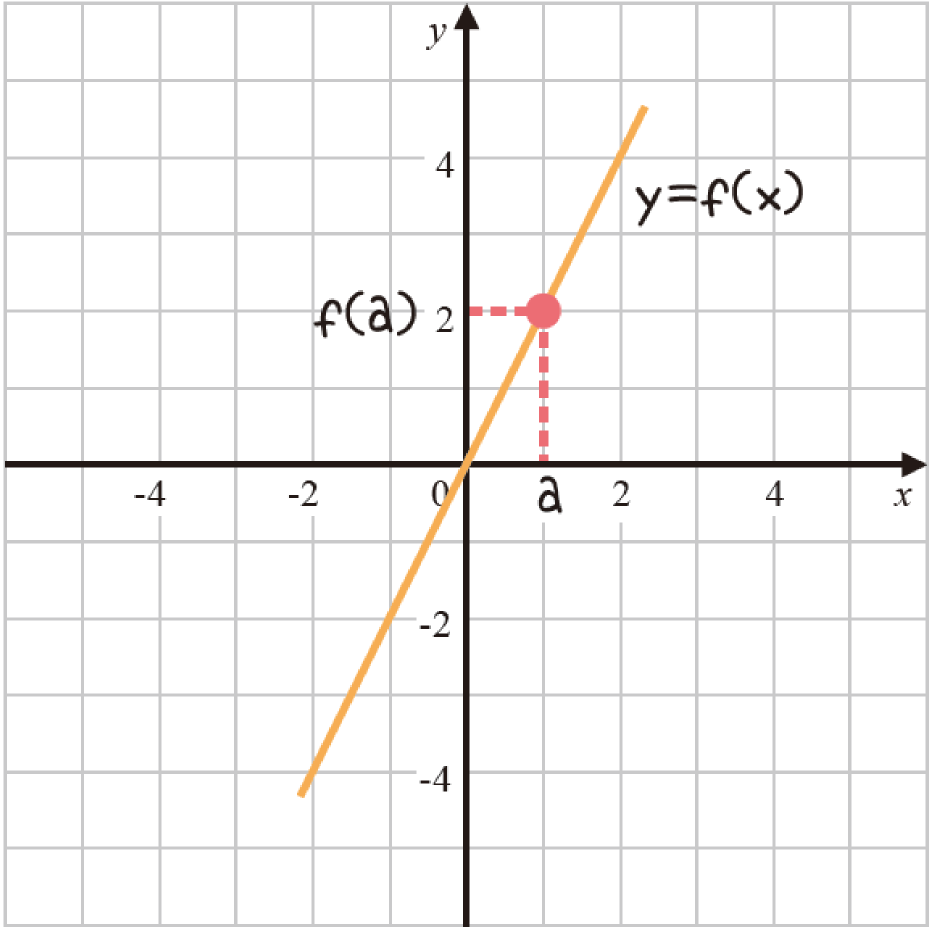
❖ Continuous of function

- The graph of the function continues unbroken at a point where x = 1 like g(x) is described as 'continuous at x = 1’
- If the function graph is broken at x = 1 like f(x), it is expressed as 'discontinuous at x = 1'

❖ Continuous of function

- The following are the conditions under which the function is continuous
  - (1) Function f(x) must be defined at x=a
  - (2) $\lim_{x \to a} f(x)$ must exist
  - (3) $\lim_{x \to a} f(x) = f(a)$

Limit, maximum, minimum of a function



❖ Continuous of function

- In summary, to satisfy the continuity of the function, the following formula must be satisfied

$$\lim_{x \to a} f(x) = f(a)$$

- This formula is expressed as 'function f(x) is continuous at x = a'

❖ Continuous of function

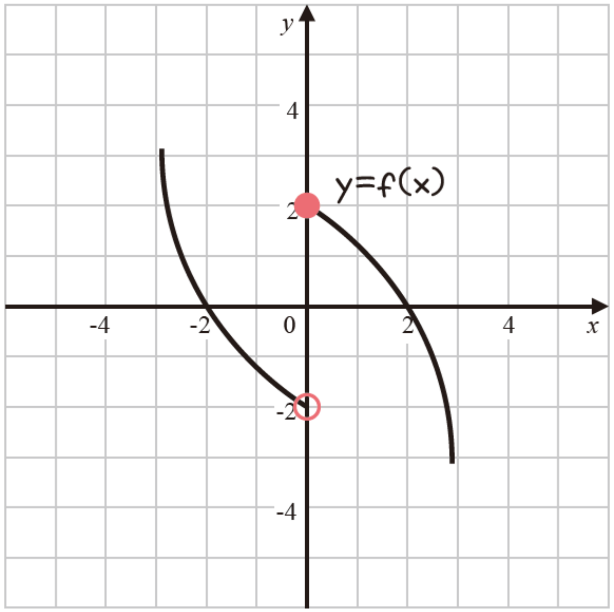
Example from a graph perspective

$$f(x) = \begin{cases} -x^2 + 2 & (x \geq 0) \\ x^2 - 2 & (x < 0) \end{cases}$$

- Let's find out if it's continuous or discontinuous when x = 0

$$\lim_{x \to 0+} f(x) = 2, \quad \lim_{x \to 0-} f(x) = -2$$

- Limit value do not exist because the left side limit and right side limit are different
- Function f(x) is discontinuous when x = 0



❖ Continuous of function

Example from a formula perspective

- Let’s find out whether the following function is continuous or discontinuous at x =2

$$f(x) = \begin{cases} \sqrt{x+2} & (x \geq 2) \\ x+1 & (x < 2) \end{cases}$$

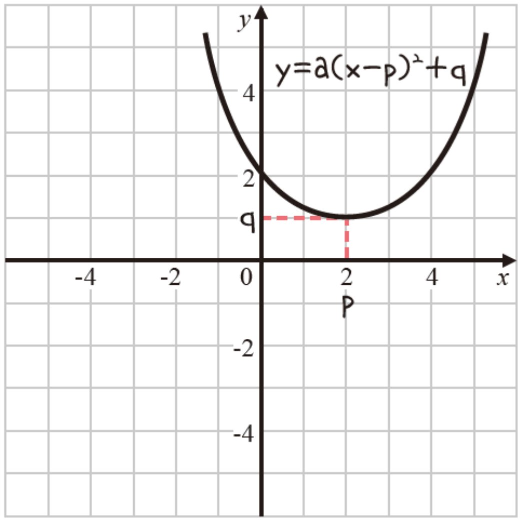
- Check $\lim_{x\to2} f(x) = f(2)$
  - (1) $\lim_{x\to2+} \sqrt{2+2} = \lim_{x\to2-} (x+1) = 2$
  - (2) f(2) = 2
  - (3) Function f(x) is continuous at x = 2 because it satisfied the $\lim_{x\to2} f(x) = f(2)$

Limit, maximum, minimum of a function

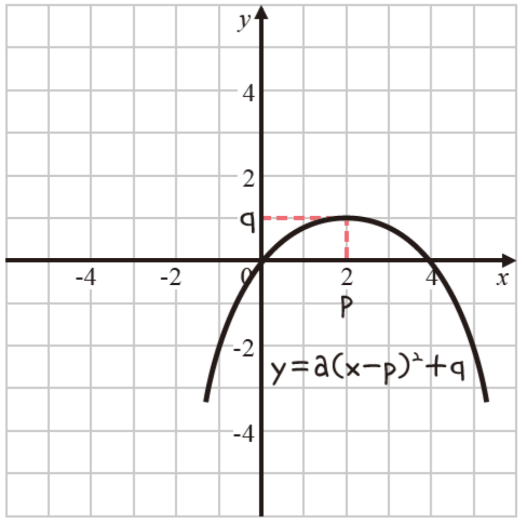
MATHEMATIC

❖ Maximum, minimum of function

- Generally, when expressing quadratic functions, the formula $y = a(x - p)^2 + q$ is used a lot
- Let's find out the maximum and the minimum
    - (1) When $a > 0$, there is no maximum as shown in ①, and the minimum is q
    - (2) When $a < 0$, maximum is q and there is no minimum as shown in ②



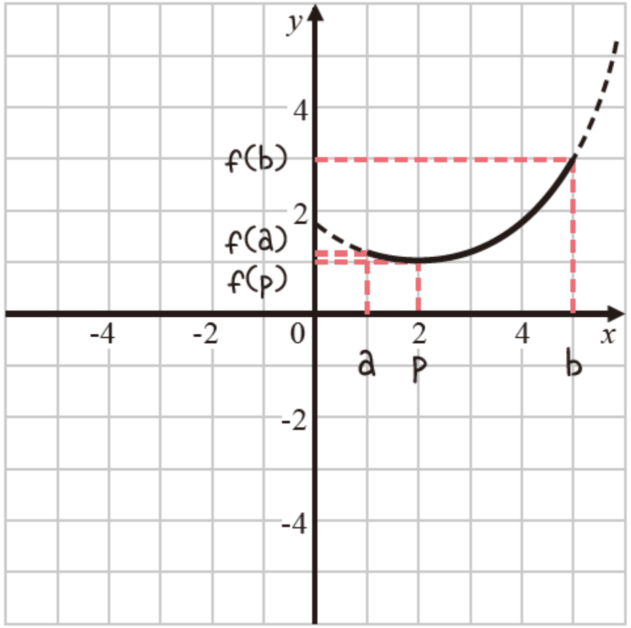
① a>0일 때 $y=a(x-p)^2+q$



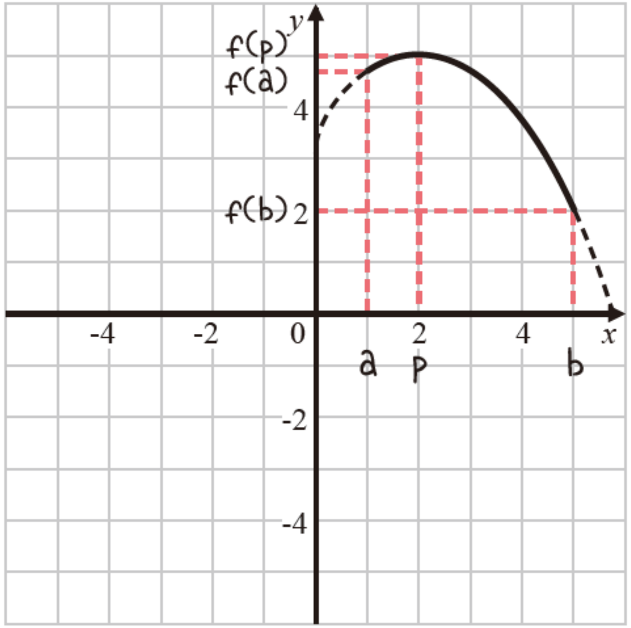
② a<0일 때 $y=a(x-p)^2+q$

❖ Maximum, minimum of function

- Let's find out the maximum and minimum of the quadratic function $y = a(x - p)^2 + q$ when $\{x \mid a \le x \le b\}$
  - (1) If the x-coordinate of the vertex is included in $a \le x \le b$
    - – The largest values of f(a), f(p), f(b) are the maximum, and the smallest values are the minimum



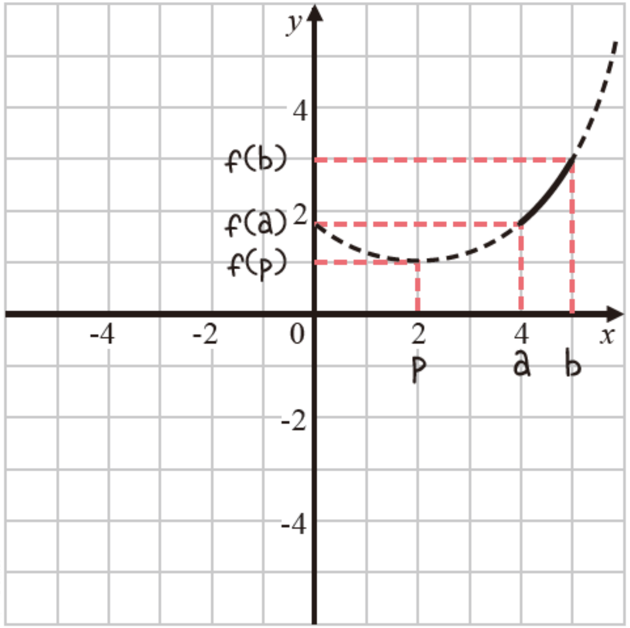
① a>0일 때 $y=a(x-p)^2+q$



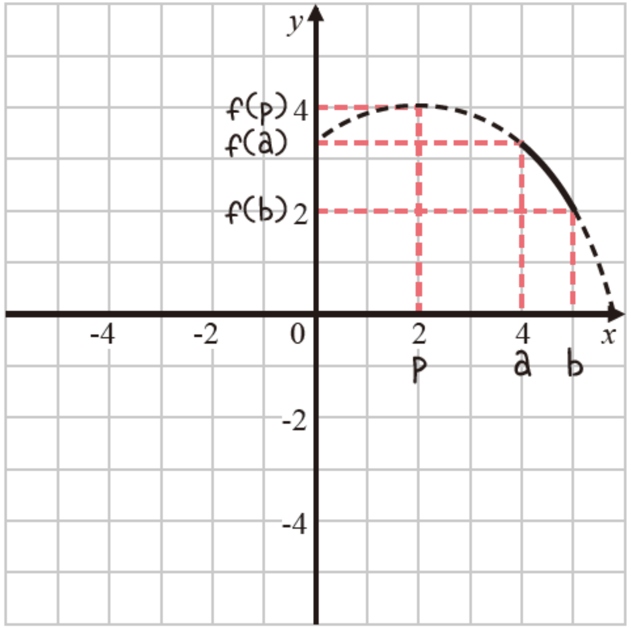
② a<0일 때 $y=a(x-p)^2+q$

❖ Maximum, minimum of function

- (2) If the x-coordinate of the vertex is not included in a ≤ x ≤ b
  - Larger values of f(a) and f(b) are the maximum, and smaller values are the minimum



① a>0일 때 $y=a(x-p)^2+q$



② a<0일 때 $y=a(x-p)^2+q$

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❖ Maximum, minimum of function

연습 문제

(1) 이차함수 $y = x^2 - 2x + 3$의 최댓값과 최솟값을 구하세요.

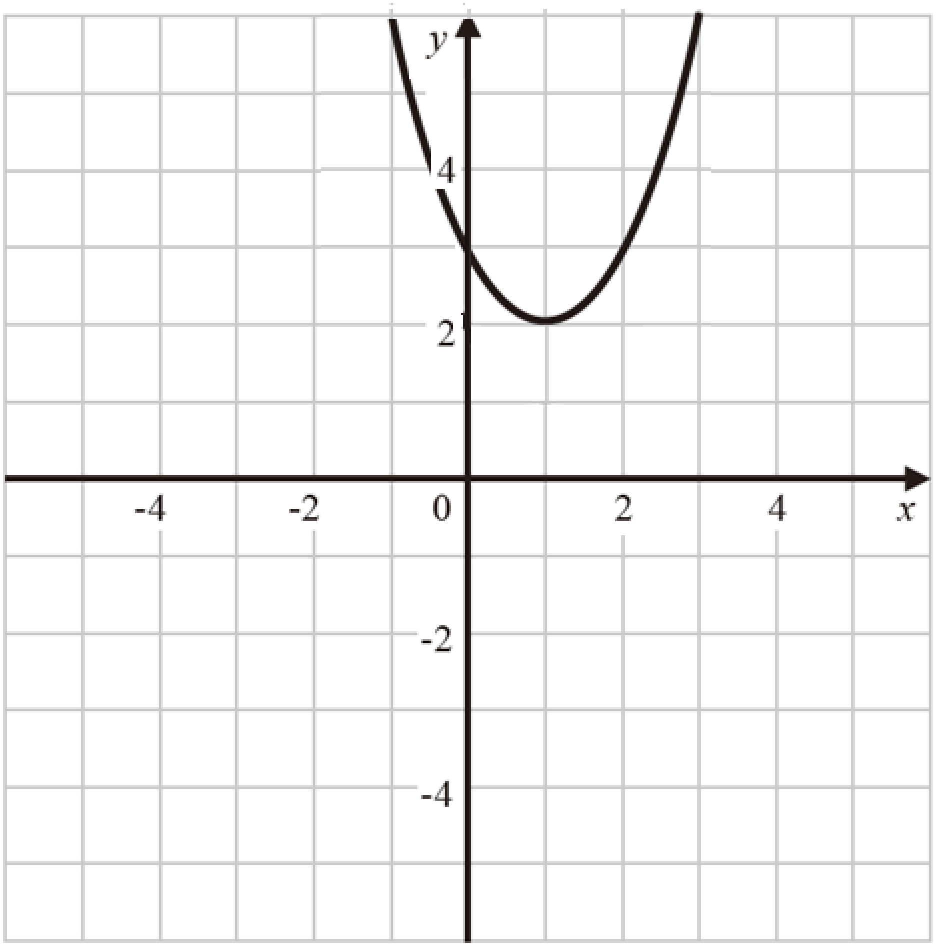
(2) x 값 범위가 $\{2 \le x \le 3\}$인 이차함수 $f(x) = x^2 - 2x + 3$의 최댓값과 최솟값을 구하세요.

문제 풀이

그래프를 이용하여 문제를 풀어 봅시다.

(1) $y = x^2 - 2x + 3$ 그래프는 다음과 같습니다.

Limit, maximum, minimum of a function



❖ **Maximum, minimum of function**

$y = x^2 - 2x + 3$을 인수분해하면 $y = (x-1)^2 + 2$가 됩니다. 따라서 그림 7−23과 같이 최댓값은 없으며, 최솟값은 1입니다.

(2) $f(x) = x^2 - 2x + 3$을 인수분해하면 $f(x) = (x-1)^2 + 2$가 됩니다. 또 x 값 범위가 $\{2 \le x \le 3\}$이므로 $f(2) = 3$, $f(3) = 6$입니다. 따라서 최댓값은 6, 최솟값은 3 입니다.