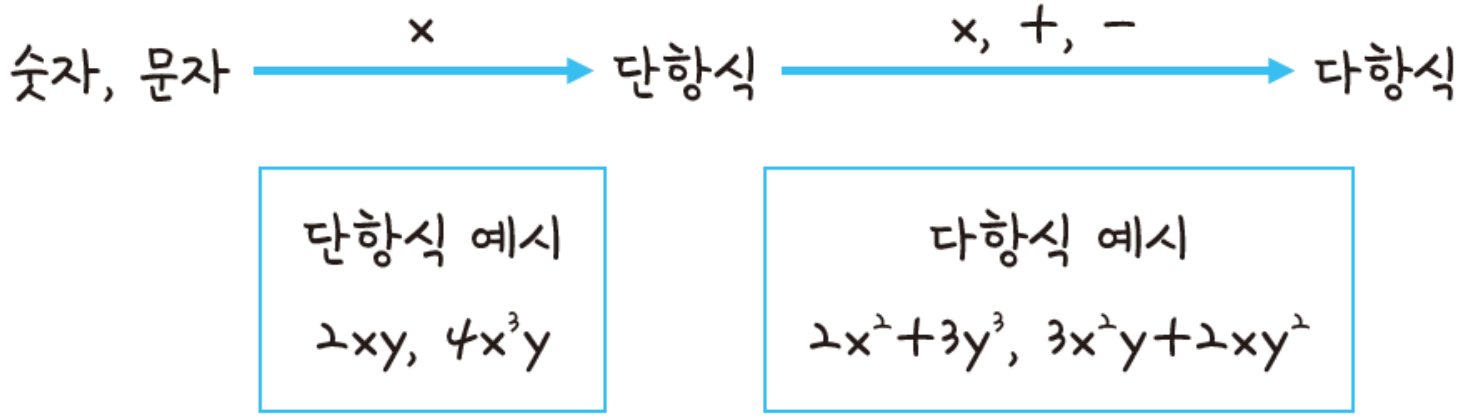
Polynomial and Geometry & Exponential/Logarithmic Functions

Polynomial

❖ Polynomial and monomial

- Monomial is an expression consisting only of multiplication of numbers and letters
- For example, an expression such as xy or $3x^2$ is called monomial
- Polynomial is an expression consisting of addition and subtraction of a monomial
- For example, $2xy + 4x + 2y - 2$ or $2x^2 + 5x + c$ is called polynomial



Polynomial

❖ Polynomial calculation

The laws of calculation of polynomials

- The laws of calculation of polynomials include commutative law, combination law, distributive law
- When there are three polynomials A, B and C, the following law holds
  - Commutative law : $A + B = B + A$, $AB = BA$
  - Combination law : $(A + B) + C = A + (B + C)$
  - Distributive law : $(A + B)C = AC + BC$

- Example
  - $A = x^2 + 2x + 1$, $B = 2x^2 + 4x – 5$
  - According to commutative law, $(x^2 + 2x + 1) + (2x^2 + 4x - 5) = (2x^2 + 4x - 5) + (x^2 + 2x + 1)$ is established

Polynomial

❖ Polynomial calculation

Polynomial calculation

Addition and subtraction of polynomial

- It is most important to find similar terms for addition and subtraction of polynomial
- First, find terms with the same degree and calculate them among the coefficients
- If there are parentheses, first solve the parentheses, calculate them, and sort them in descending order
- **Similar terms (Like terms)**

Meaning two or more terms with the same variables and exponent

(1) Parentheses based on similar terms

(2) Calculate similar terms and sort the in descending order

Polynomial

❖ Polynomial calculation

- Example
  - A = $x^2 + 2x + 1$, B = $2x^2 + 4x - 5$, C = $2x + 1$
  - Calculate A + (B - C)

$$A + (B - C)$$

$$= x^2 + 2x + 1 + ((2x^2 + 4x - 5) - (2x + 1))$$

$$= x^2 + 2x + 1 + (2x^2 + 2x - 6)$$

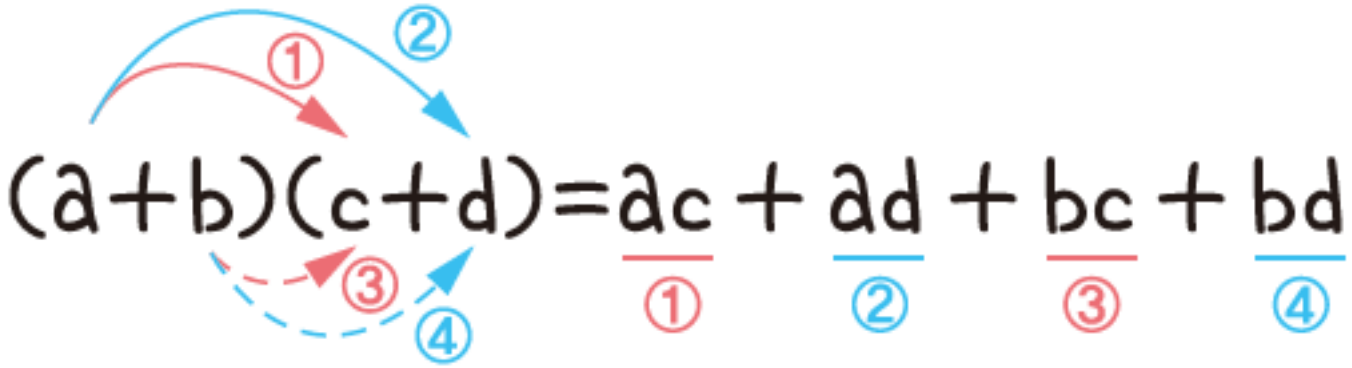
$$= 3x^2 + 4x - 5$$

Polynomial

❖ Polynomial calculation

Multiplication of polynomial

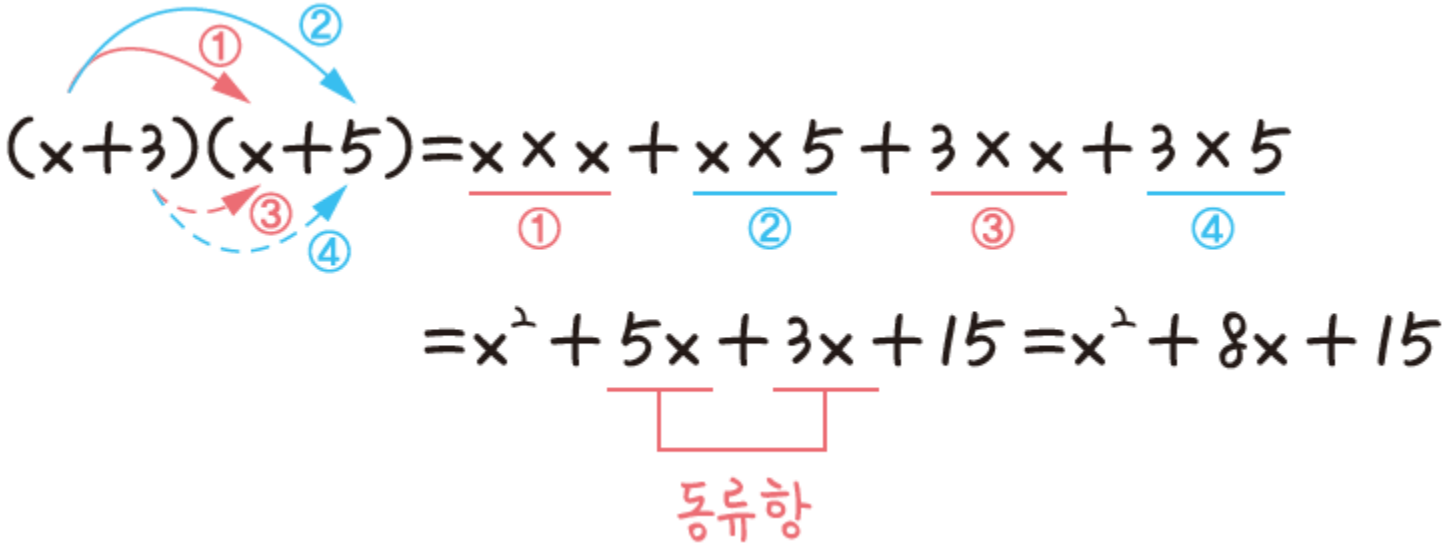
- Multiplication of polynomials is expanded using the distributive law, and if there is a similar term in the expansion equation, the similar terms are gathered together and listed



Polynomial

❖ Polynomial calculation

- Example
  - Polynomial and polynomial multiplication of (x +3)(x + 5)



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❖ Polynomial calculation

연습 문제

다항식 $x^2+2xy+y^2-2x-3y-3$에 대해 다음 물음에 답하세요.

(1) x에 대해 내림차순으로 정리하세요.

(2) x에 대해 오름차순으로 정리하세요.

문제 풀이

x에 대해 내림차순 및 오름차순으로 정리하라고 했으므로 y는 상수 취급하여 풀어 줍니다(x가 아닌 변수는 상수 취급합니다).

(1) $x^2+2xy+y^2-2x-3y-3$

$=x^2+(2xy-2x)+y^2-3y-3$

$=x^2+2(y-1)x+y^2-3y-3$

따라서 내림차순은 $x^2+2(y-1)x+y^2-3y-3$입니다.

(2) (1)의 계산 결과로 나온 내림차순을 오름차순으로 바꾸어 줍니다.

따라서 오름차순은 $y^2-3y-3+2(y-1)x+x^2$입니다.

Polynomial

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- **Polynomial calculation Using Python**

❖ Angle of triangle

Length of the three sides of triangle and the size of the angle

- Triangle is divided into an acute triangle, right triangle, and obtuse triangle depending on the size of the angle

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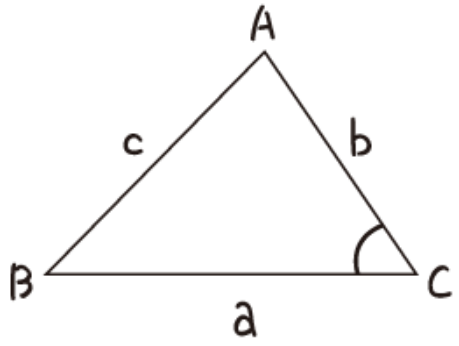
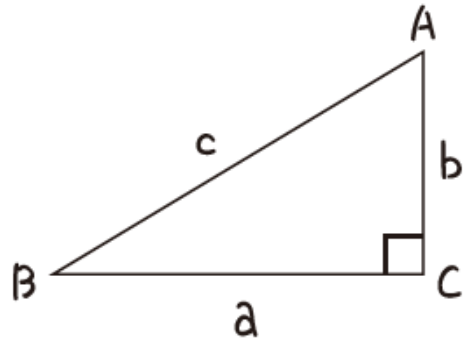
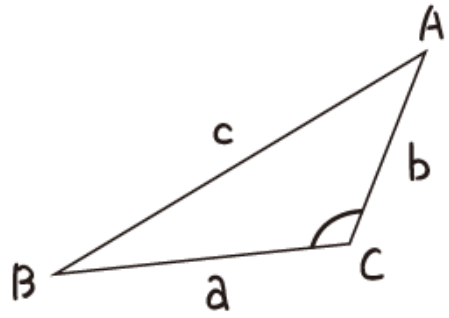
		
$C < 90^0$	$C = 90^0$	$C > 90^0$
$c^2 < a^2 + b^2$	$c^2 = a^2 + b^2$	$c^2 > a^2 + b^2$

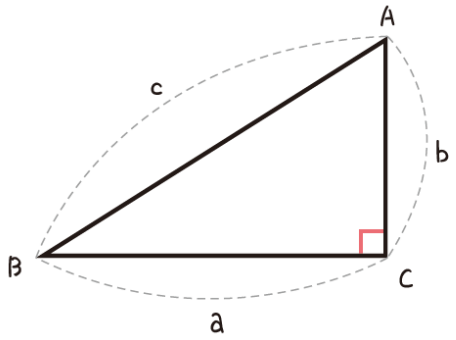
❖ Pythagorean theorem

- Right triangle ABC, if the lengths of the sides of each vertex are a, b, and c, then the square of the hypotenuse c is equal to the sum of the squares of the other two sides a, b
- In other words, Equation 5.1 is established when the following right triangles are present
- **Opposite side : a, b**

The opposite side of an angle

- **Hypotenuse : c**

In right triangle, it refers to the opposite side



피타고라스 정리

$a^2 + b^2 = c^2$ 수식 5.1

❖ Pythagorean theorem

- The inverse of the Pythagorean theorem
- If $a^2 + b^2 = c^2$ in triangles with lengths of three sides a, b, and c, then triangle is a right angle triangle with hypotenuse c

연습 문제

세 변의 길이가 3cm, $x$cm, 8cm인 삼각형이 있습니다. 이 삼각형이 직각삼각형일 때 x 값의 합을 구하세요.

문제 풀이

문제에서 직각삼각형이라고 했으니 피타고라스 정리를 이용할 수 있습니다.

(1) 세 변의 길이가 3, x, 8인 상태에서 8이 빗변의 길이라고 가정합니다(루트 값의 계산은 계산기를 이용합니다).

$3^2 + x^2 = 8^2$

$x^2 = 8^2 - 3^2$

$x^2 = 64 - 9 = 55$

$x = \sqrt{55} = 7.4$

(2) x가 빗변이라 가정하고 문제를 풀어 보세요(루트 값의 계산은 계산기를 이용합니다).

$3^2 + 8^2 = x^2$

$x^2 = 8^2 + 3^2 = 64 + 9 = 73$

$x = \sqrt{73} = 8.5$

(3) 빗변은 길이가 가장 길어야 하기 때문에 3은 빗변이 될 수 없습니다. 따라서 x 값의 합은 $7.4 + 8.5 = 15.9$가 됩니다.

❖ Pythagorean theorem using Python

- The inverse of the Pythagorean theorem
- If $a^2 + b^2 = c^2$ in triangles with lengths of three sides a, b, and c, then triangle is a right angle triangle with hypotenuse c

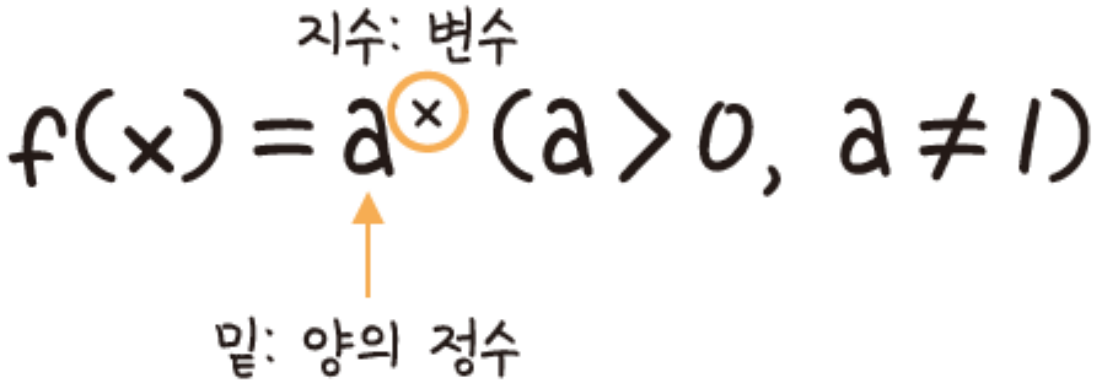
```
from sympy import symbols, Eq, simplify, solve

def prove_pythagorean_inverse(a_val, b_val, c_val):
    # 기호 정의
    a, b, c = symbols('a b c', positive=True, real=True)

    # 피타고라스 정리의 양변 구성
    lhs = a**2 + b**2
    rhs = c**2

    # 방정식 구성
    equation = Eq(lhs, rhs)

    # 실제 값 대입 후 확인
    is_true = equation.subs({a: a_val, b: b_val, c: c_val})

    print(f"좌변 (a² + b²) = {lhs.subs({a: a_val, b: b_val})}")
    print(f"우변 (c²)     = {rhs.subs({c: c_val})}")
    print(f"결과 방정식:  {is_true}")

    if is_true == True:
        print("이 삼각형은 직각삼각형입니다.")
    else:
        print("이 삼각형은 직각삼각형이 아닙니다.")

prove_pythagorean_inverse(3, 4, 5)
```

Exponential function

❖ Exponential function and graph

Exponential function

- Exponential function is one that has an unknown x in the exponent, i.e., $f(x) = a^x$ $(a > 0, a \neq 1)$ 형태로 나타낼 수 있는 함수



Exponential function

❖ Exponential function and graph

Base of exponential function

- Base must be a positive integer 밑은 양의 정수여야 한다.
- Positive number greater than zero, not 1 양수는 0보다 크고, 1이 아니다.
- If base is 1, f(x) is always 1 no matter what value it is 만약 밑이 1이면 항상 f(x) = 1 이다.
- The base can be divided into two types
  - $0 < a < 1$
  - $a > 1$

a != 1

Exponential function

❖ Exponential function and graph

- Exponential functions and logarithmic functions are mathematics used in the backpropagation of error in AI
- In Python, we can write exponential functions using the pow(), sqrt(), exp() functions in the math library

함수	설명
math.pow(x,y)	x의 y 제곱
math.sqrt(x)	x의 제곱근
math.exp(x)	e(자연상수, 2.718281828459045)의 x 제곱

Exponential function

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❖ Exponential function and graph

In [15]:

```
# 파이썬 math 라이브러리를 호출합니다
import math
```

In [16]:

```
# 파이썬에서 제공하는 지수함수를 유형별로 실행합니다
# 2의 5 제곱
pow = math.pow(2, 5)
print("pow 결과: ", pow)
```

Exponential function

MATHEMATIC

❖ Exponential function and graph

```
# 2의 제곱근
sqrt = math.sqrt(2)
print("sqrt 결과: ", sqrt)

# e의 2 제곱
exp = math.exp(2)
print("exp 결과: ", exp)
```

```
pow 결과: 32.0
sqrt 결과: 1.4142135623730951
exp 결과: 7.38905609893065
```

Exponential function

❖ Exponential function and graph

연습 문제

다음 중 지수함수가 아닌 것을 고르세요.

(1) $y = x^2$ X

(2) $y = 2^x$ O

(3) $y = -3^x$ X

(4) $y = 1^x$ X

Exponential function

❖ **Exponential function and graph**

문제 풀이

(1) 지수에 문자가 있어야 하므로 지수함수가 아닙니다.

(2) 지수에 문자가 있고 밑이 0보다 크므로 지수함수입니다.

(3) 밑은 마이너스일 수 없으므로 지수함수가 아닙니다.

(4) 밑은 1이 아니어야 하므로 지수함수가 아닙니다.

Exponential function

❖ Exponential function and graph

Exponential function graph

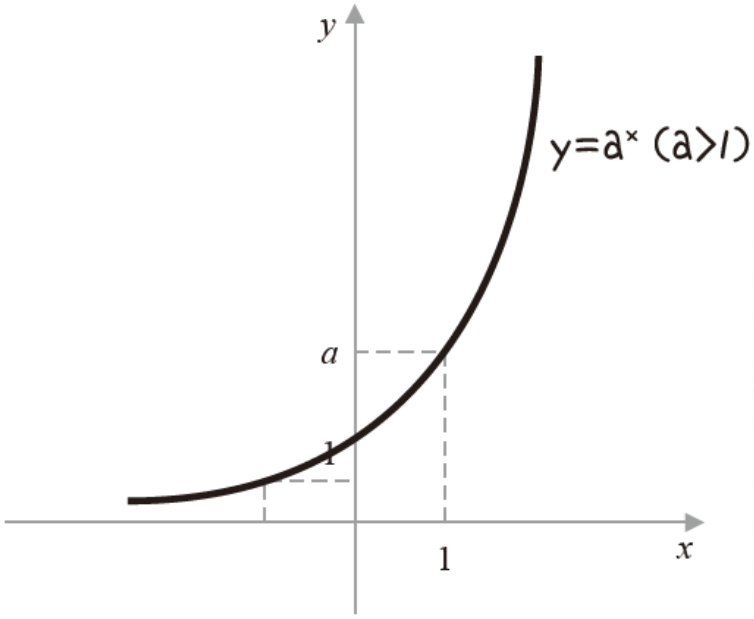
- Exponential function : $y = a^x$
- Since ‘a’ is bigger than -, substitute an arbitrary number 2 and draw graph according to the change in x

x	⋯	–2	–1	0	1	2	⋯
$y = a^x$	⋯	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	⋯

Exponential function

❖ Exponential function and graph

- As the exponent increases, the y value increases, and as the exponent decreases, the y value decreases, but it is bigger than 0
- This means that the larger the exponent x, the larger the y, resulting in a graph facing up to the right as follows
- As the exponent x decreases, y also decreases, resulting in a graph that approaches zero infinitely



Exponential function

❖ Exponential function and graph

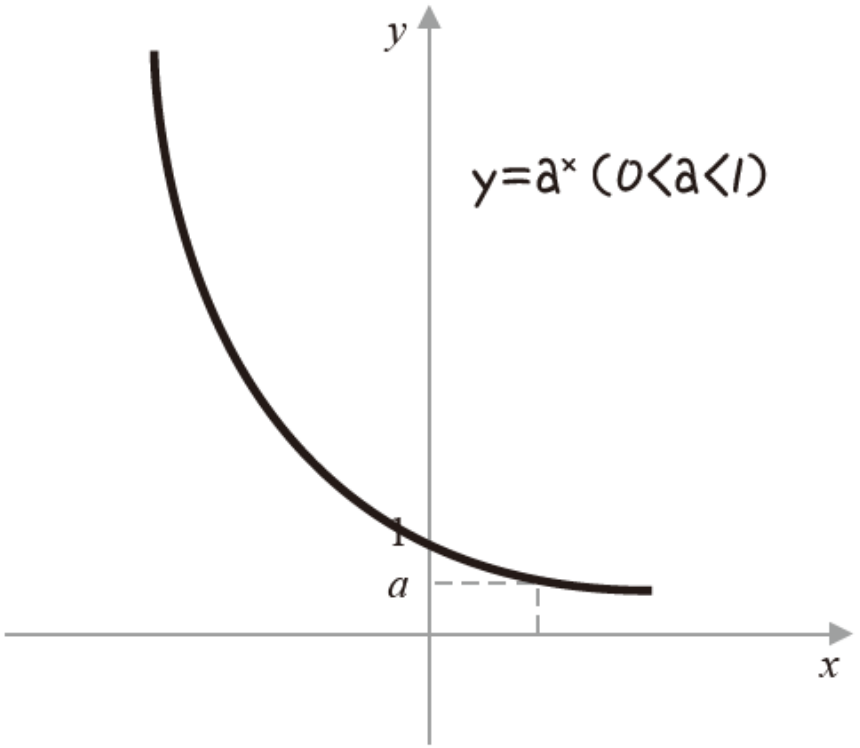
- Let’s look at the case $0 < a < 1$
- Let's substitute an arbitrary number $\frac{1}{2}$ into a and then change x

x	⋯	–2	–1	0	1	2	⋯
$y = a^x$	⋯	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	⋯

Exponential function

❖ Exponential function and graph

- That is, the smaller the exponent, the larger the y value, and the larger the exponent, the smaller the y value
- As the exponent x becomes larger, y becomes smaller, so it is down to the right as follows



Exponential function

❖ Exponential function and graph

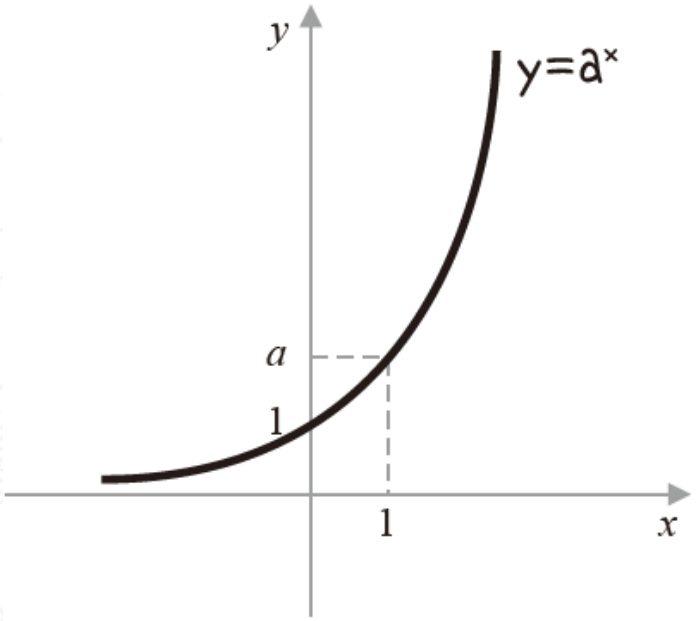
- At this time, the two exponential functions whose base is reciprocal are symmetric with respect to the y-axis
- For example, the values of $2^{-2} = \frac{1}{4}$ and $(\frac{1}{2})^2 = \frac{1}{4}$ are the same
- In other words, if the base is reciprocal and the exponent x has the opposite sign, then the y value is the same

Exponential function

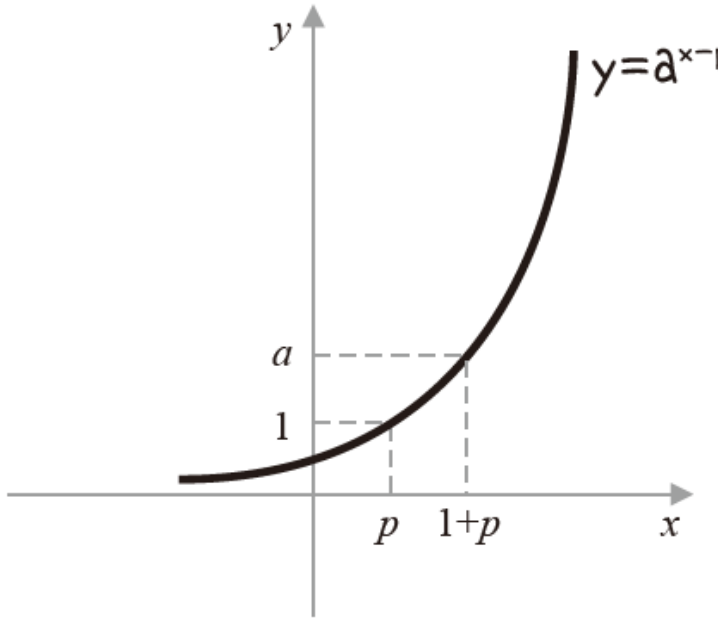
❖ Parallel and symmetric transposition of exponential function

Parallel transposition of exponential function

- Because the exponential function $y = a^x$ is parallel, the graph will not change shape, only the position will change
- That is, if you move the original point f(x, y) = 0 by (p, q), then f(x - p, y - q) = 0

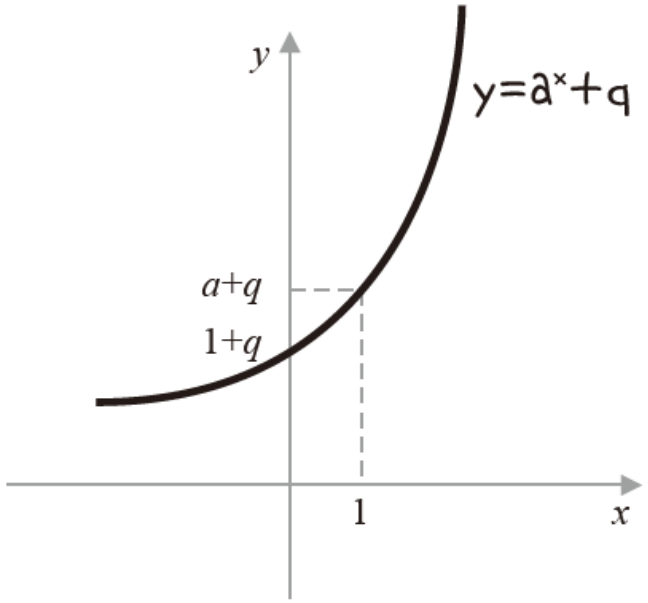


① $y=a^x$ 그래프

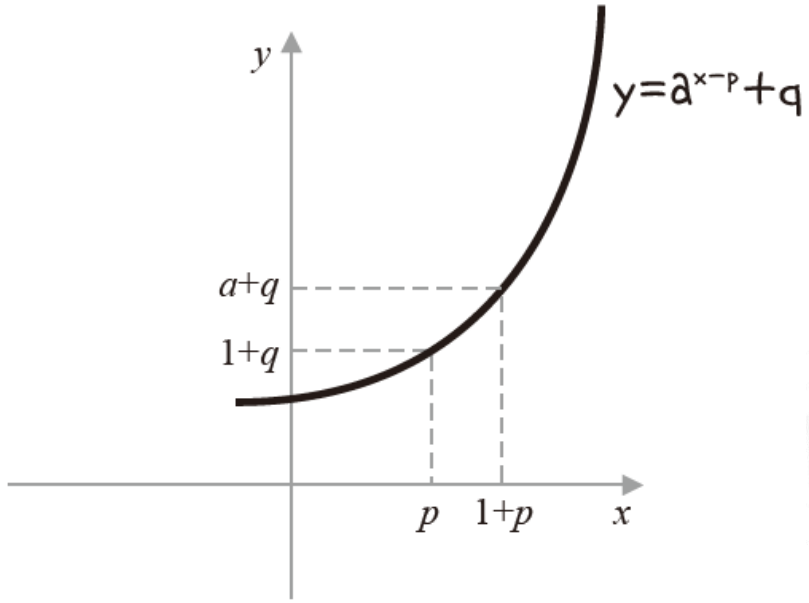


② $y=a^{x-p}$ 그래프

Exponential function



③ $y=a^x+q$ 그래프



④ $y=a^{x-p}+q$ 그래프

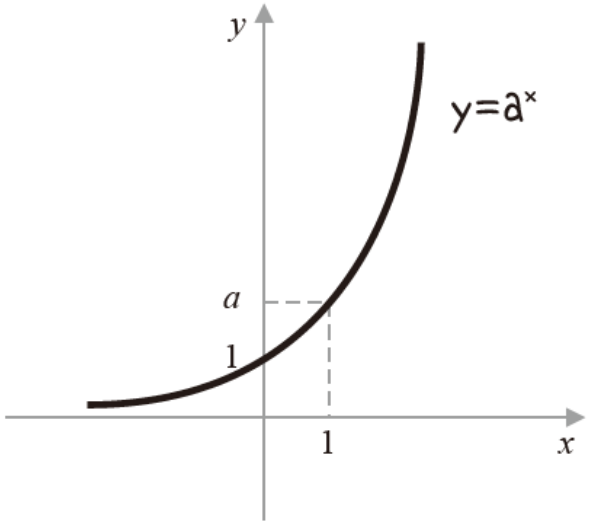
Exponential function

❖ Parallel and symmetric transposition of exponential function

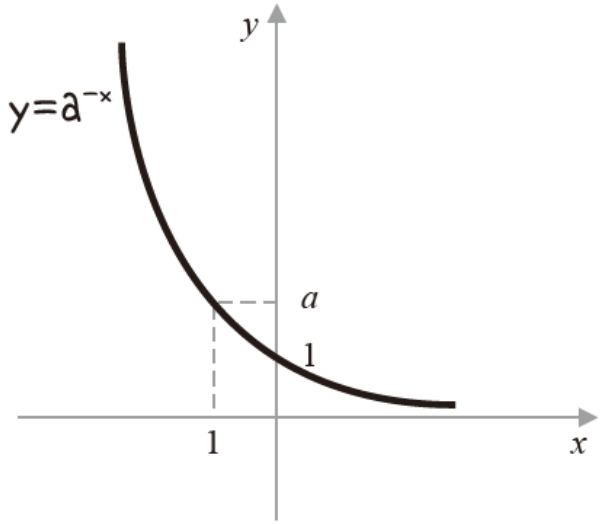
Symmetric transposition of exponential function

- Moving the exponential function $y = a^x$ symmetrically to the x-axis results in $-y = a^x$, $y = -a^x$
- Symmetric transposition on the y-axis becomes $y = a^{-x}$
- Symmetric transposition on the origin (0,0) becomes $-y = a^{-x}$

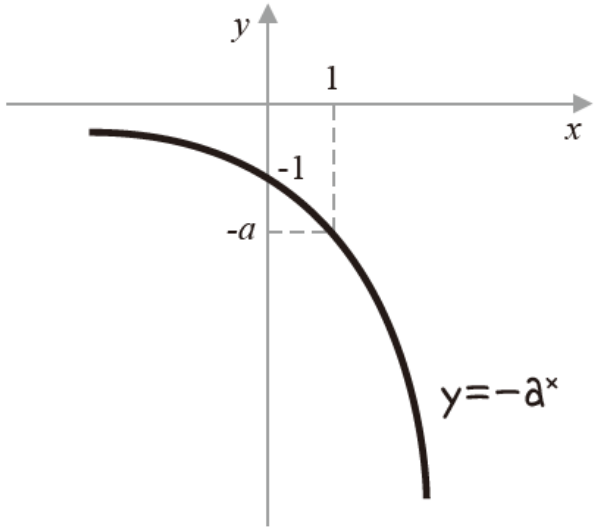
Exponential function



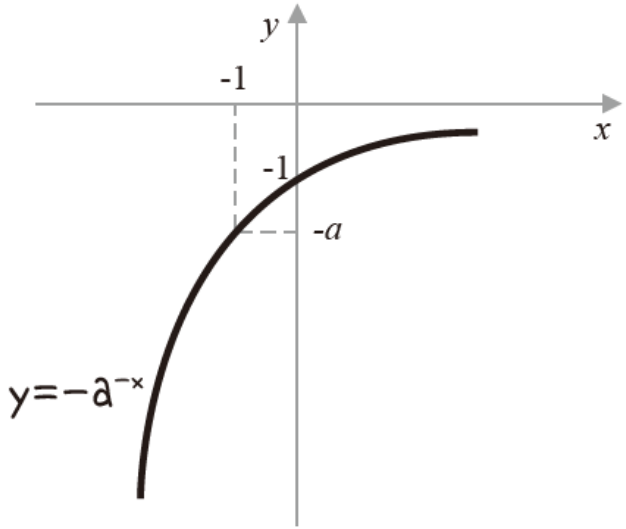
① $y=a^x$ 그래프



② $y=a^{-x}$ 그래프



③ $y=-a^x$ 그래프



④ $y=-a^{-x}$ 그래프

How could you draw the function in Python?

Exponential function

❖ Parallel and symmetric transposition of exponential function

- The graph of the exponential function does not meet the x-axis, but must pass through the y-axis and the point is 1 or -1

연습 문제

(1) $y = 2^x$ 그래프를 이용하여 함수 $y = 2^{x-3}$ 그래프를 그리세요.

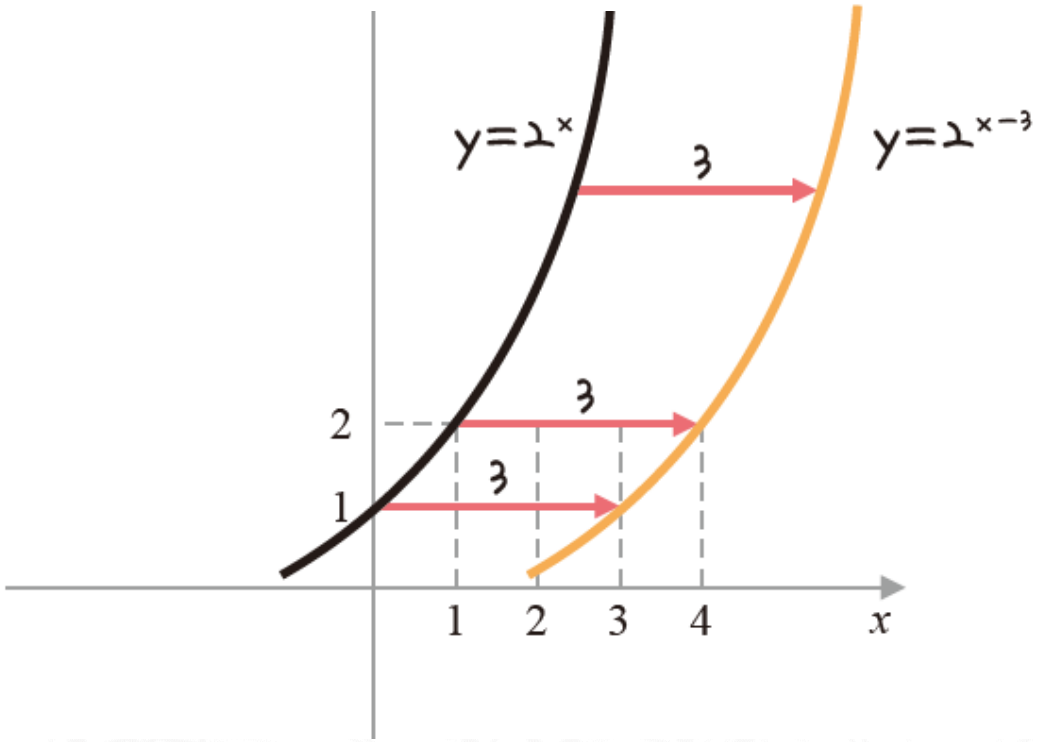
(2) $y = 2^{x-3}$ 그래프를 이용하여 $y = -2^{x-3}$ 그래프를 그리세요.

Exponential function

❖ Parallel and symmetric transposition of exponential function

문제 풀이

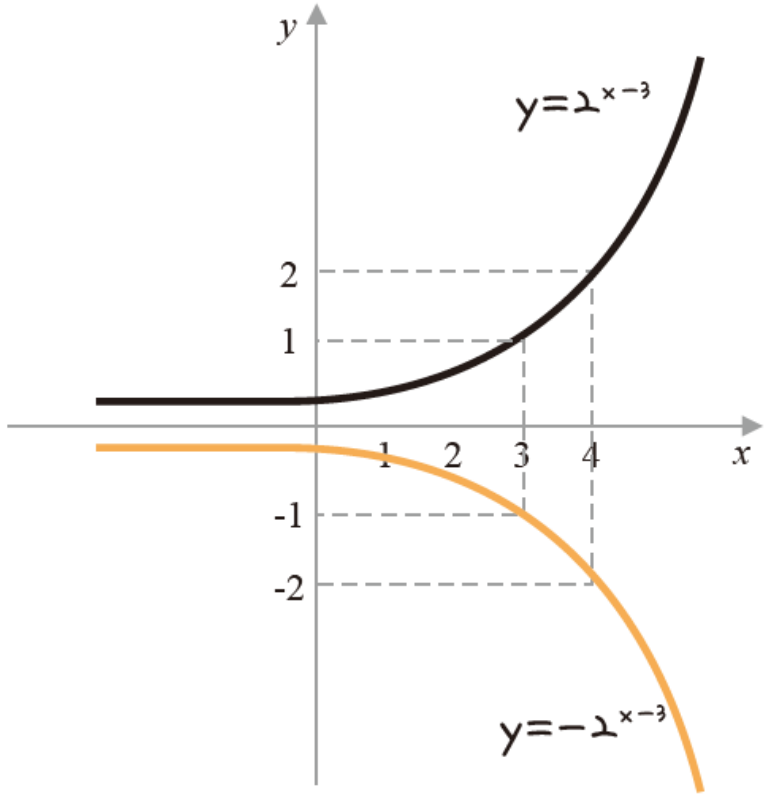
(1) $y=2^{x-3}$ 그래프는 지수함수 $y=2^x$ 그래프에서 x축 방향으로 3만큼 평행 이동한 것입니다. 따라서 $y=2^{x-3}$ 그래프는 다음과 같습니다.



Exponential function

❖ Parallel and symmetric transposition of exponential function

(2) $y=-2^{x-3}$ 그래프는 지수함수 $y=2^{x-3}$ 그래프에서 x축으로 대칭 이동한 것입니다. 따라서 $y=-2^{x-3}$ 그래프는 다음과 같습니다.



Exponential function

❖ Parallel and symmetric transposition of exponential function using Python

```
import numpy as np
import matplotlib.pyplot as plt

# 지수 함수 정의
def exponential(x, a=2):
    return a ** x

# 설정
x = np.linspace(-5, 5, 400)
a = 2  # 밑

# 원본
y_original = exponential(x, a)

# 평행이동
y_vertical = exponential(x, a) + 2        # y축 위로 2
y_horizontal = exponential(x - 2, a)      # x축 오른쪽으로 2

# 대칭이동
y_x_symmetry = -exponential(x, a)         # x축 대칭
y_y_symmetry = exponential(-x, a)         # y축 대칭
y_origin_symmetry = -exponential(-x, a)   # 원점 대칭

# 그래프 출력
plt.plot(x, y_original, label='Original')
plt.plot(x, y_vertical, label='Vertical +2')
plt.plot(x, y_horizontal, label='Horizontal +2')
plt.plot(x, y_x_symmetry, label='X-axis Symmetry')
plt.plot(x, y_y_symmetry, label='Y-axis Symmetry')
plt.plot(x, y_origin_symmetry, label='Origin Symmetry')

plt.title('Parallel and Symmetric Transpositions of Exponential Function')
plt.legend()
plt.grid(True)
plt.show()
```

Exponential function

❖ Parallel and symmetric transposition of exponential function

Logistic function

- Logistic functions are variations of exponential functions and are frequently used in regression or ANN
- Logistic regression is used when analysis results are categorical

점수	91~100점	81~90점	71~80점	70점 이하
그룹	A	B	C	D

Exponential function

❖ Parallel and symmetric transposition of exponential function

- For example, the form of automatically classifying groups based on student scores is called categorical analysis, and logistic regression is often used
- Logistic regression is analyzed using a logistic function

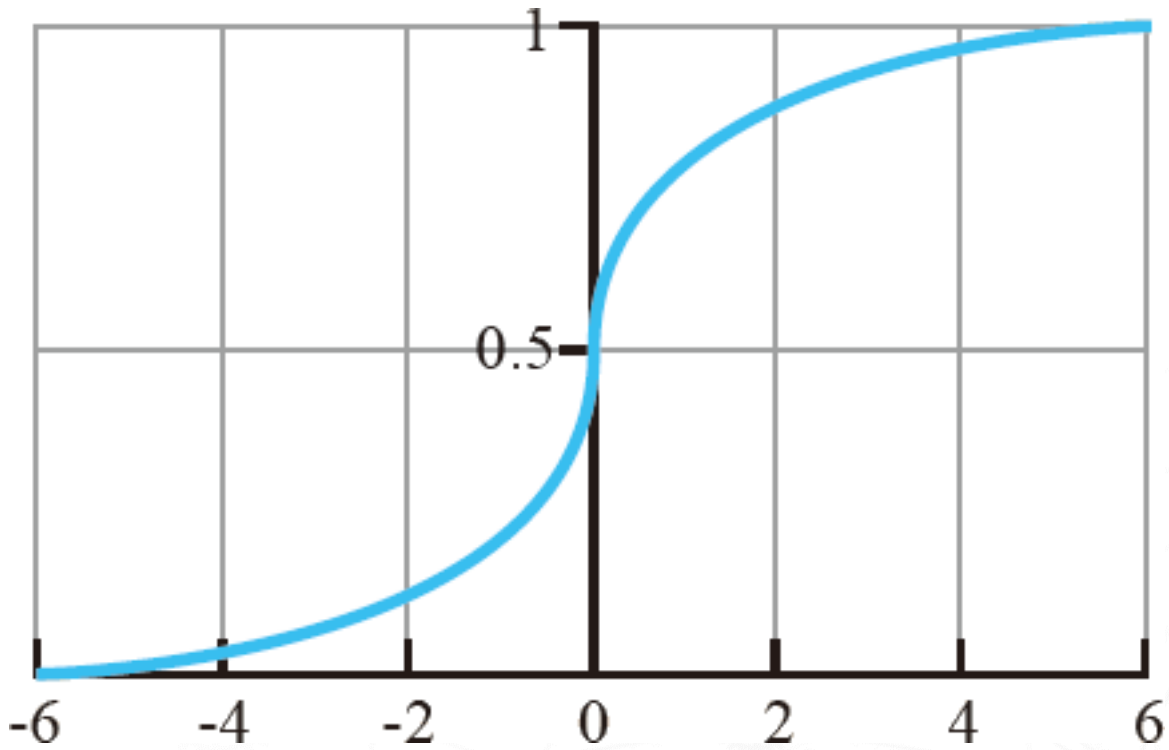
$$f(x) = \frac{1}{1+e^{-x}}$$

```
import numpy as np
import matplotlib.pyplot as plt

# 1. x 값의 범위를 설정
# -10부터 10까지 200개의 균등 분할 점을 생성
x = np.linspace(-6, 6, 200)

# 2. 시그모이드 함수 f(x)를 계산
y = 1 / (1 + np.exp(-x))

# 3. 그래프
plt.plot(x, y, label=r'$f(x)=\frac{1}{1+e^{-x}}$')
plt.title("Sigmoid Function")
plt.xlabel("x")
plt.ylabel("f(x)")
plt.grid(True)
plt.legend()
plt.ylim(-0.1, 1.1)
plt.show()
```

Exponential function

❖ Parallel and symmetric transposition of exponential function

- When using a logistic function, f(x) has a value between 0 and 1, independent of the value of x



미분 그래프: 본 함수의 변화량이기 때문에, 종모양으로 그래프가 나올 것임.

Logarithmic function

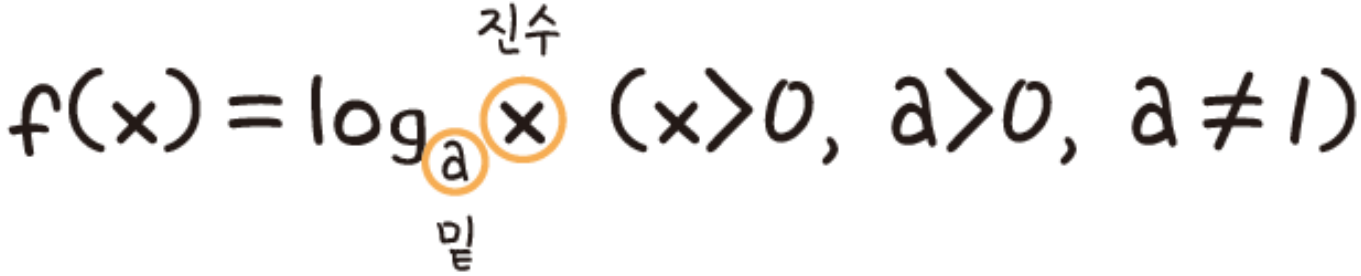
❖ Logarithmic functions and graphs

Logarithmic functions

- Logarithmic function is a function with an unknown x at the base or the antilog,

  i.e., $f(x) = \log_a x (x > 0, a > 0, a \neq 1)$

- Logarithmic function is an inverse function relationship with exponential function



Logarithmic function

❖ Logarithmic functions and graphs

- The base (a) of the logarithmic function must be positive and not 1, and x must be positive bigger than 0
- The nature of logarithmic function
  - $\log_a xy = \log_a x + \log_a y$
  - $\log_a(1) = 0$

Logarithmic function

MATHEMATIC

❖ Logarithmic functions and graphs

- Python allows you to calculate logarithmic functions as follows
- Use the math library to calculate log functions

In [17]:

```
# 파이썬 math 라이브러리를 호출합니다
import math


# log 값을 얻으려고 math 라이브러리의 log() 함수를 사용합니다
math.log(2,4) # 밑이 4고, 진수가 2인 로그: log42
```

```
0.5
```

In [18]:

```
math.log(4,2) # 밑이 2고 진수가 4인 로그: log24
```

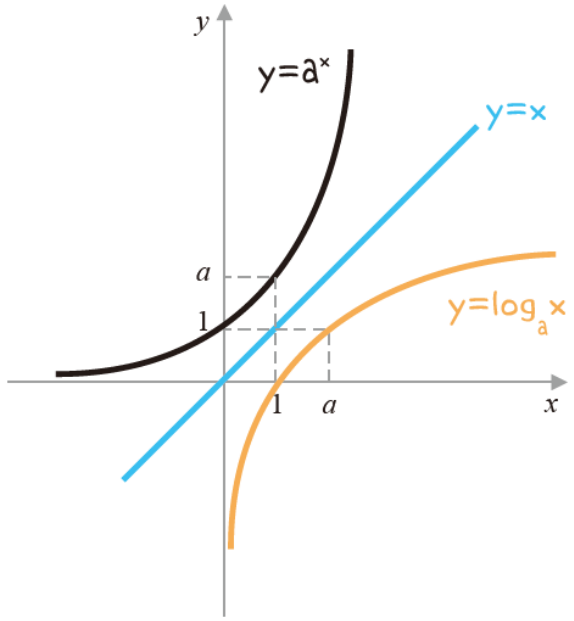
```
2.0
```

Logarithmic function

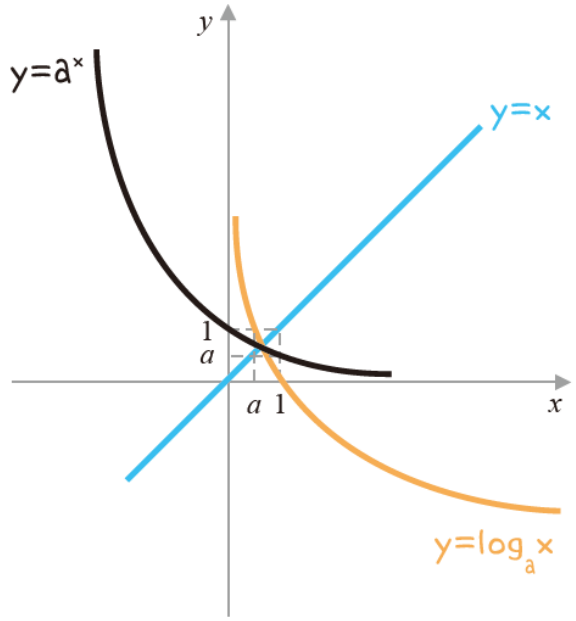
❖ Logarithmic functions and graphs

Logarithmic function

- Logarithmic function is the inverse function of the exponential function
- Because the inverse function is symmetric with y = x, the graph that moves the $y = a^x$ graph symmetric transposition with y = x is the graph of the logarithmic function $y = \log_a x$



① a>1일 때 $y=\log_a x$



② 0<a<1일 때 $y=\log_a x$

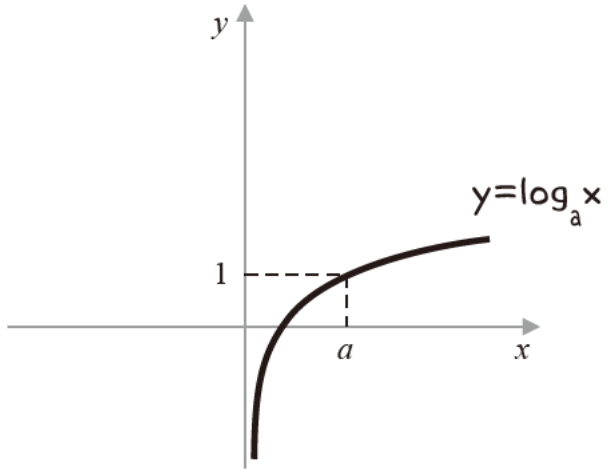
Logarithmic function

❖ Parallel and symmetric transposition of logarithmic function

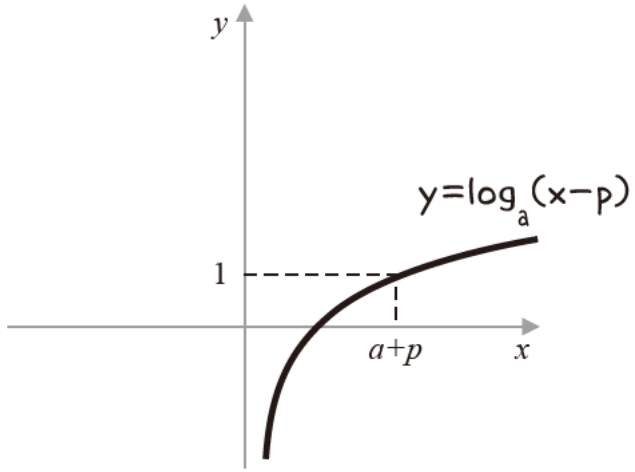
Parallel transposition of logarithmic function

- Parallel transposition of the logarithmic function $y = \log_a x$ does not changes the shape of the graph, but only the position

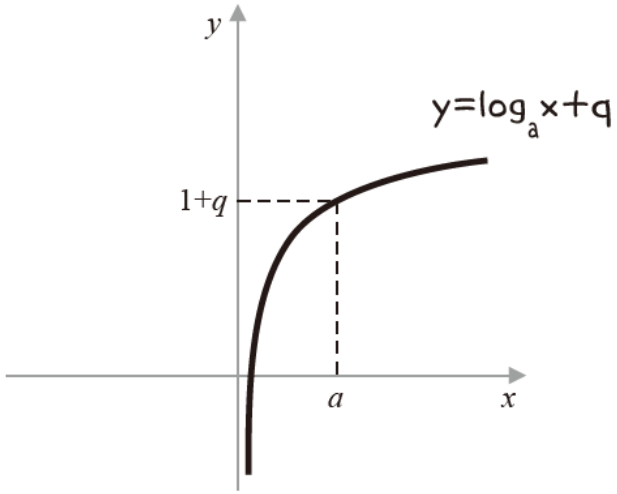
Logarithmic function



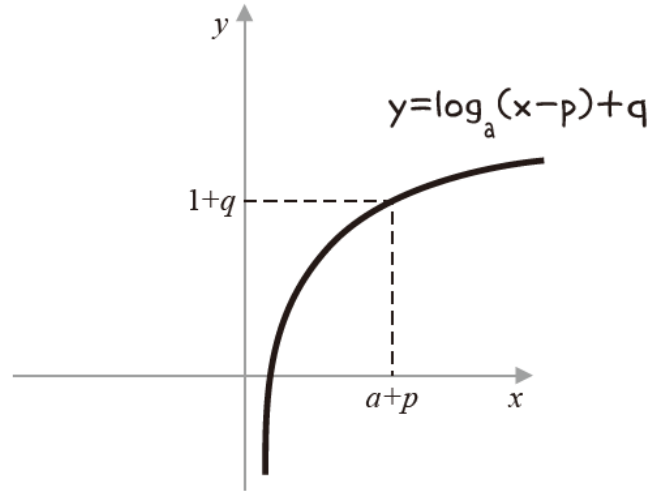
① $y=\log_a x$ 그래프



② $y=\log_a(x-p)$ 그래프



③ $y=\log_a x+q$ 그래프



④ $y=\log_a(x-p)+q$ 그래프

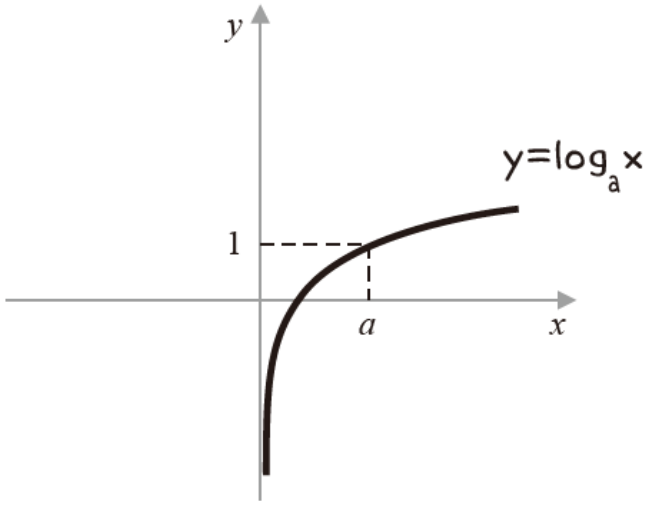
Logarithmic function

❖ Parallel and symmetric transposition of logarithmic function

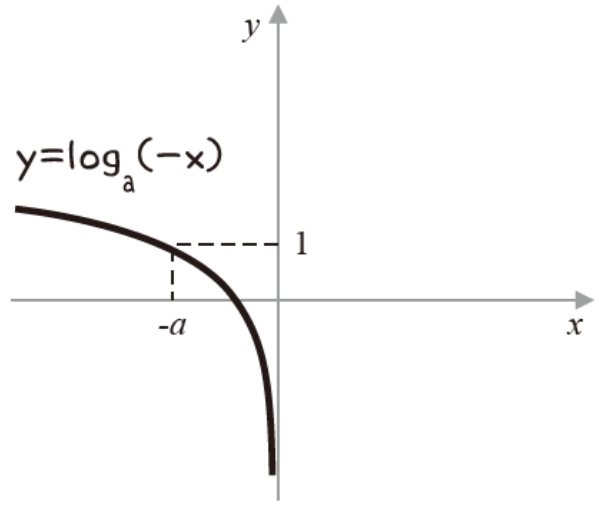
Symmetric transposition of logarithmic function

- Symmetric transposition of the logarithmic function $y = \log_a x$ are also considered symmetric transposition with the x-axis, y-axis, and origin

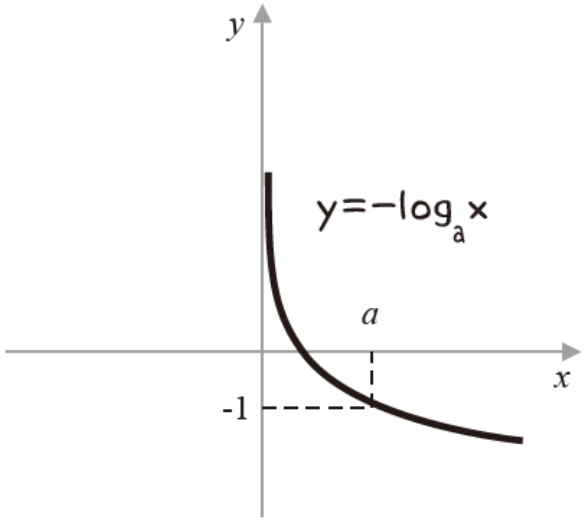
Logarithmic function



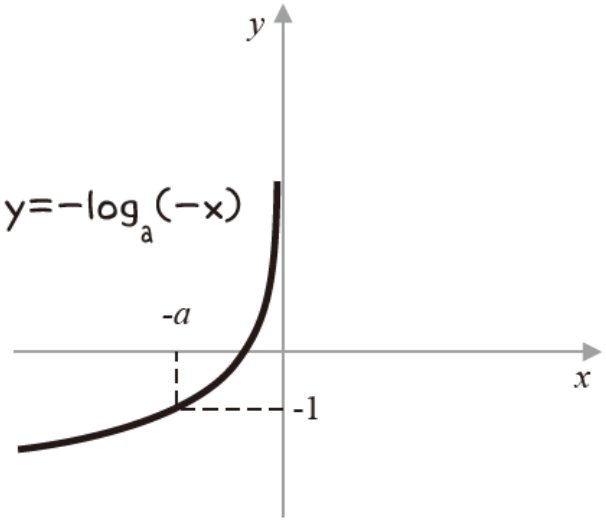
① $y=\log_a x$ 그래프



② $y=\log_a(-x)$ 그래프



③ $y=-\log_a x$ 그래프



④ $y=-\log_a(-x)$ 그래프

Logarithmic function

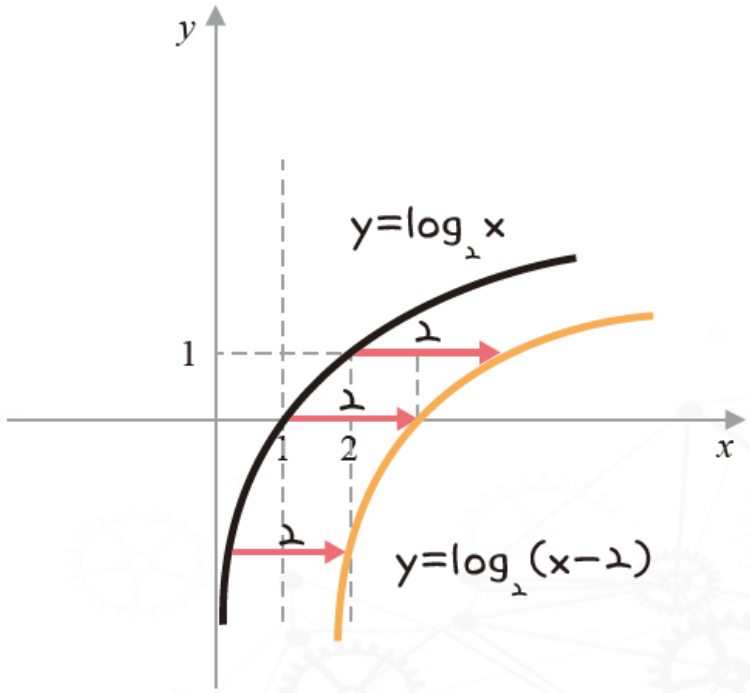
❖ Parallel and symmetric transposition of logarithmic function

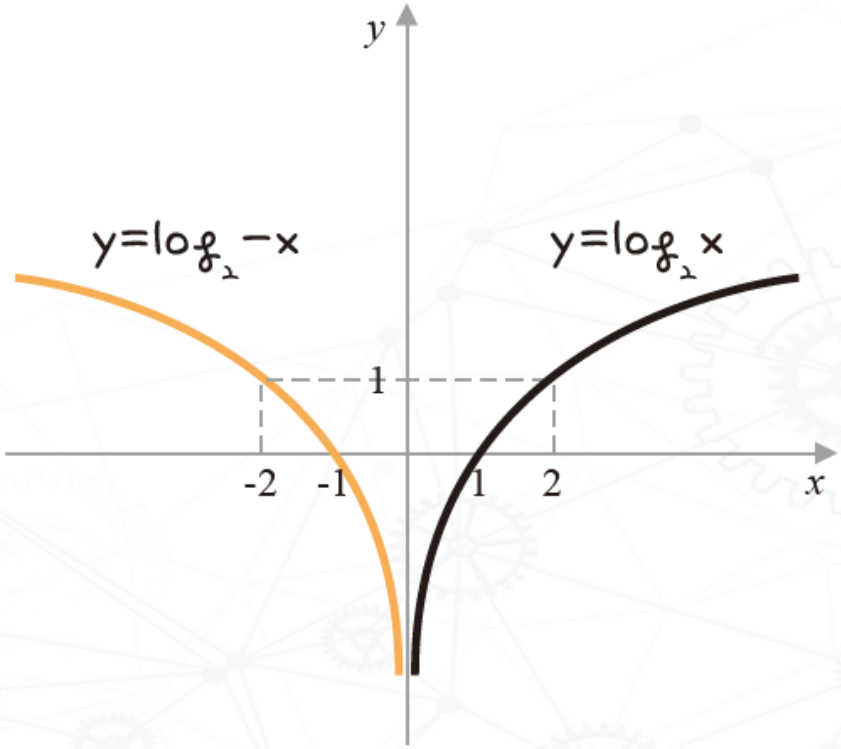
연습 문제

로그함수 $y = \log_2 x$ 그래프를 이용하여 다음 그래프를 그리세요.

(1) $y = \log_2(x-2)$

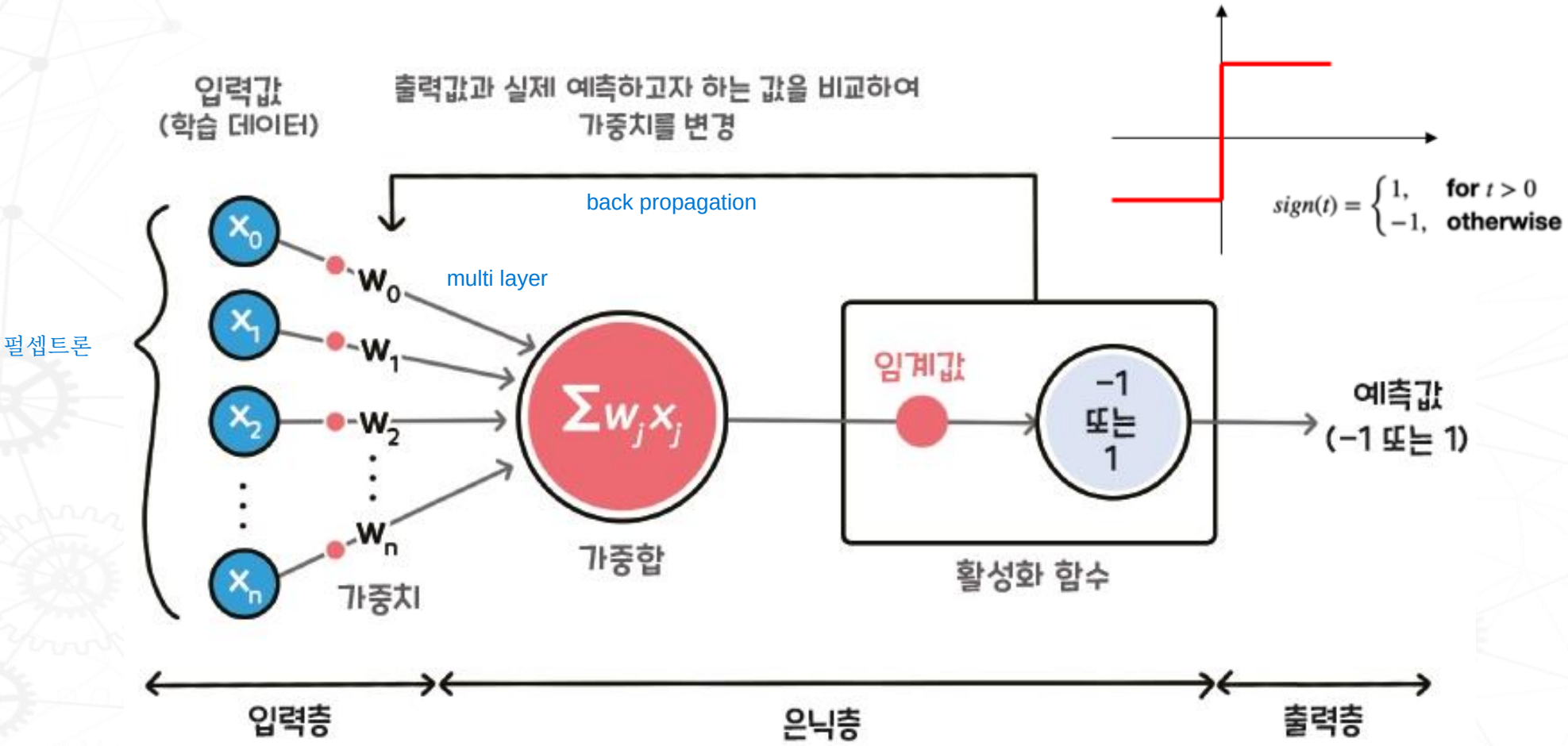
(2) $y = \log_2 -x$





Neural Network Framework

❖ Concepts



Neural Network Framework

MATHEMATIC

❖ Definition

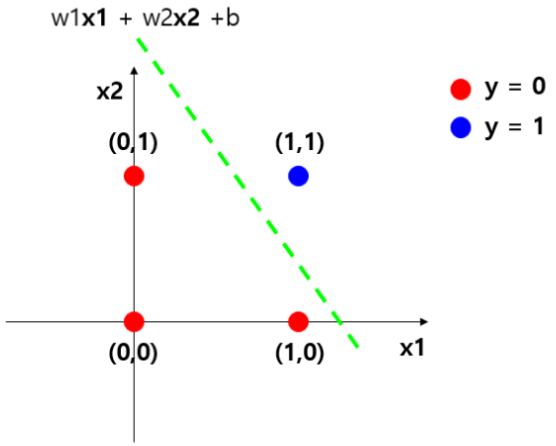
구분	설명
입력층	학습하고자 하는 데이터를 입력받음
은닉층	모든 입력 노드로부터 입력값을 받아 가중합을 계산하고, 이 값을 활성화 함수에 적용하여 출력층에 전달
출력층	최종 결과 출력
가중치	입력 신호가 출력에 미치는 영향을 조절하는 매개변수로, 입력값의 중요도를 결정
편향	가중합에 더하는 상수로, 하나의 뉴런에서 활성화 함수를 거쳐 최종적으로 출력되는 값을 조절

Neural Network Framework

❖ Limitations of perceptron

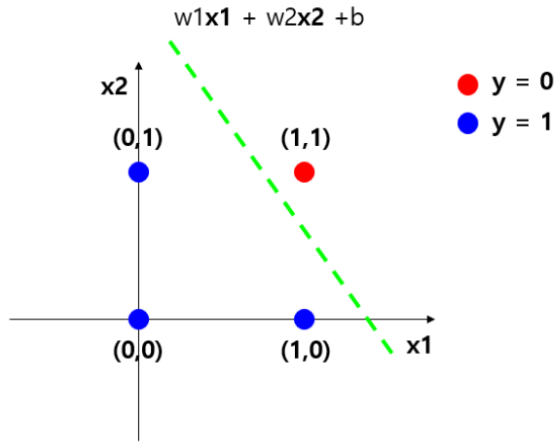
AND

X1	X2	반환값(Y)
0	0	0
0	1	0
1	0	0
1	1	1



NAND

X1	X2	반환값(Y)
0	0	1
0	1	1
1	0	1
1	1	0

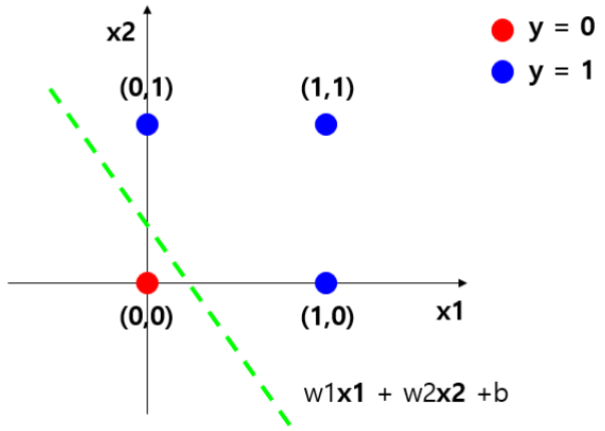


Neural Network Framework

❖ Limitations of perceptron

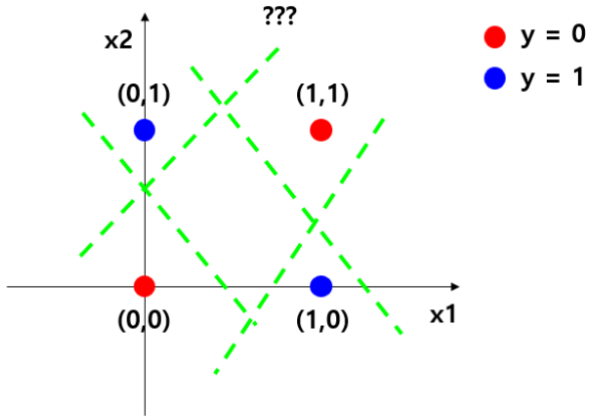
OR

X1	X2	반환값(Y)
0	0	0
0	1	1
1	0	1
1	1	1



XOR

X1	X2	반환값(Y)
0	0	0
0	1	1
1	0	1
1	1	0



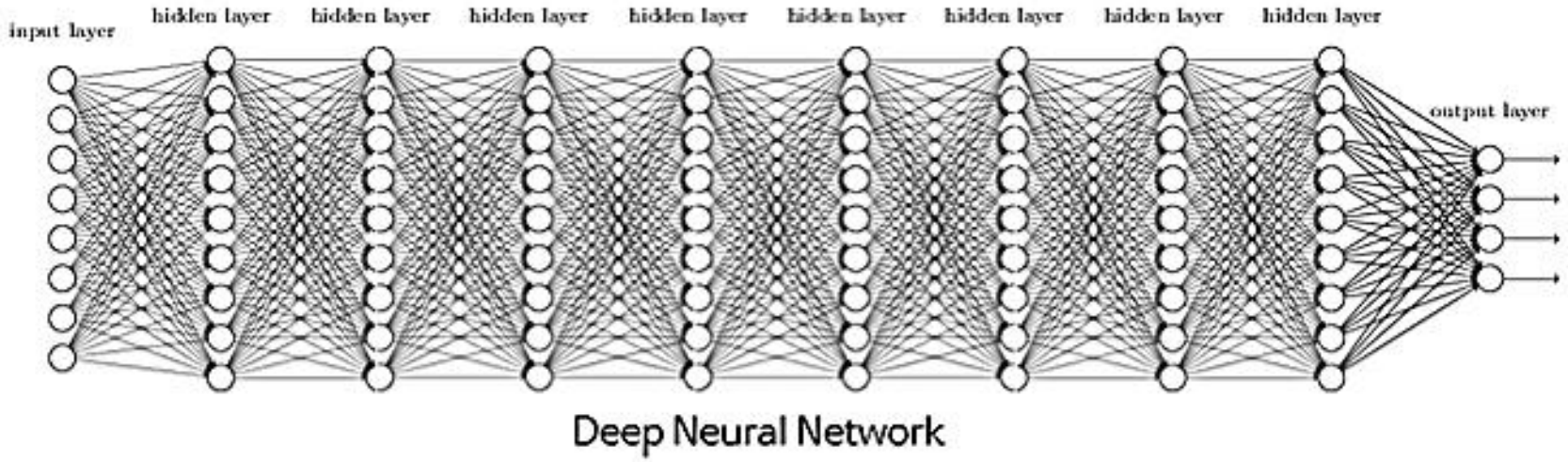
XOR에서 decision boundary를

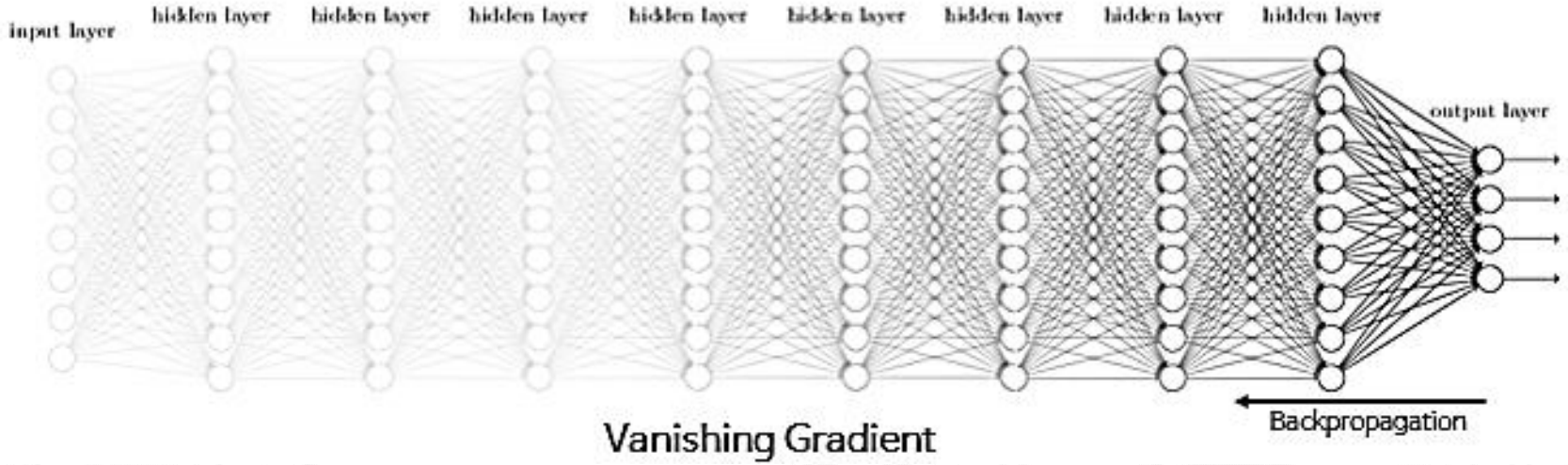
3차원으로 생각하면, (1, 1)을

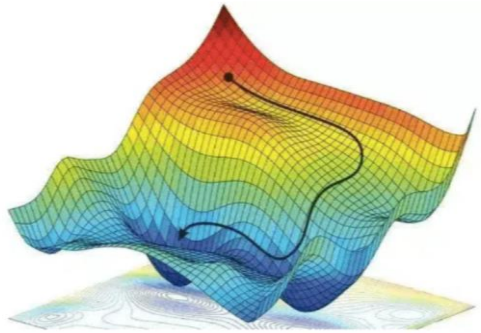
(0, 0) 기준으로 접으면 아래쪽에 분포하게 되고, 나머지 점은 decision boundary 위에 분포하게 되어 참이 된다.

여기서 이러한 개념을 hidden layer라 한다.

Vanishing Gradient Problem



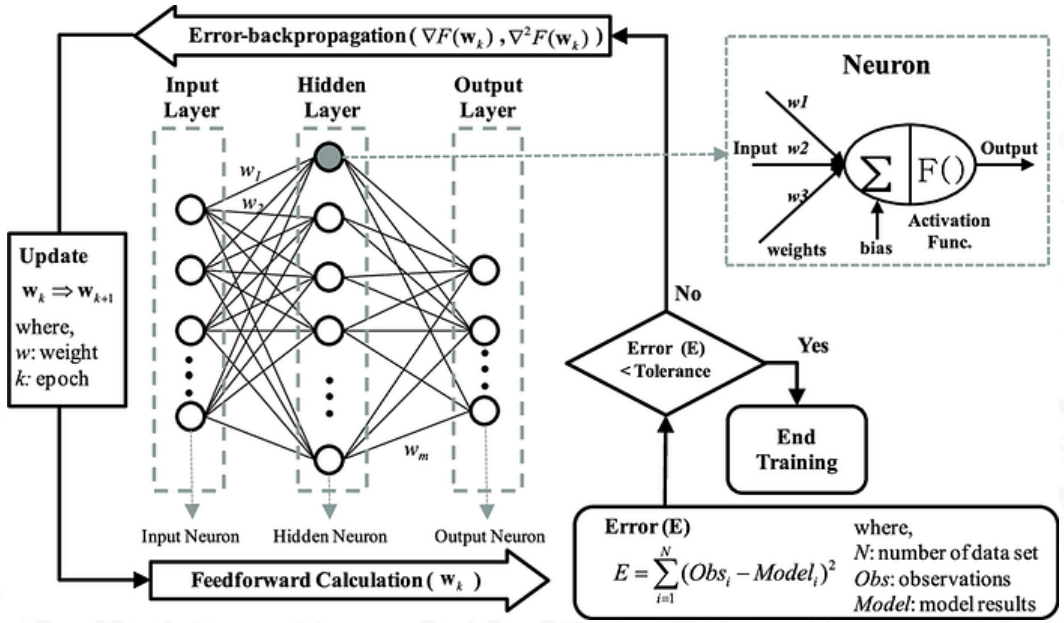




<그림 1> 인공 신경망의 비용 함수(Cost Function) 그래프

Vanishing Gradient Problem

- Why activation function?
  - If the value of the neuron you are trying to convey is available from -∞ to ∞ we cannot determine whether a signal needs to be transmitted to the neuron
  - Turn data into non-linear, helping to deepen the network
  - Linear systems can be implemented as a single hidden layer even as the network deepens

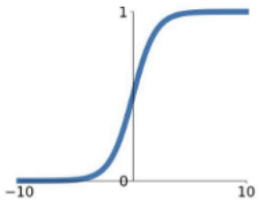


Vanishing Gradient Problem

- Kinds of Activation function
  - Activation helps determine whether neurons need to be activated or not
  - Function: Mechanisms by which neurons process and transmit information through neural networks

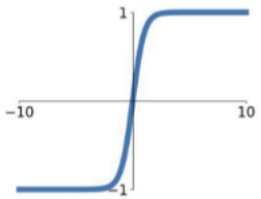
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



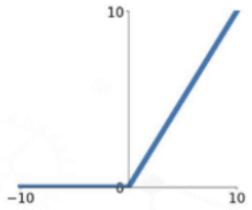
tanh

$$\tanh(x)$$



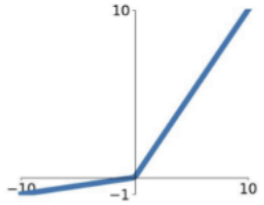
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$

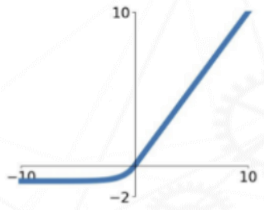


Maxout

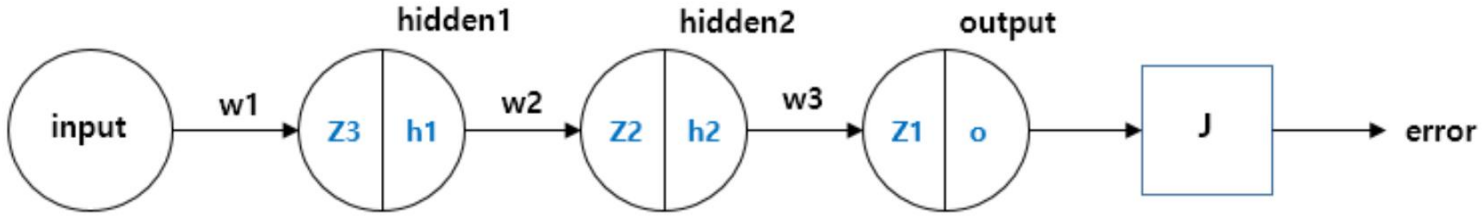
$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

ELU

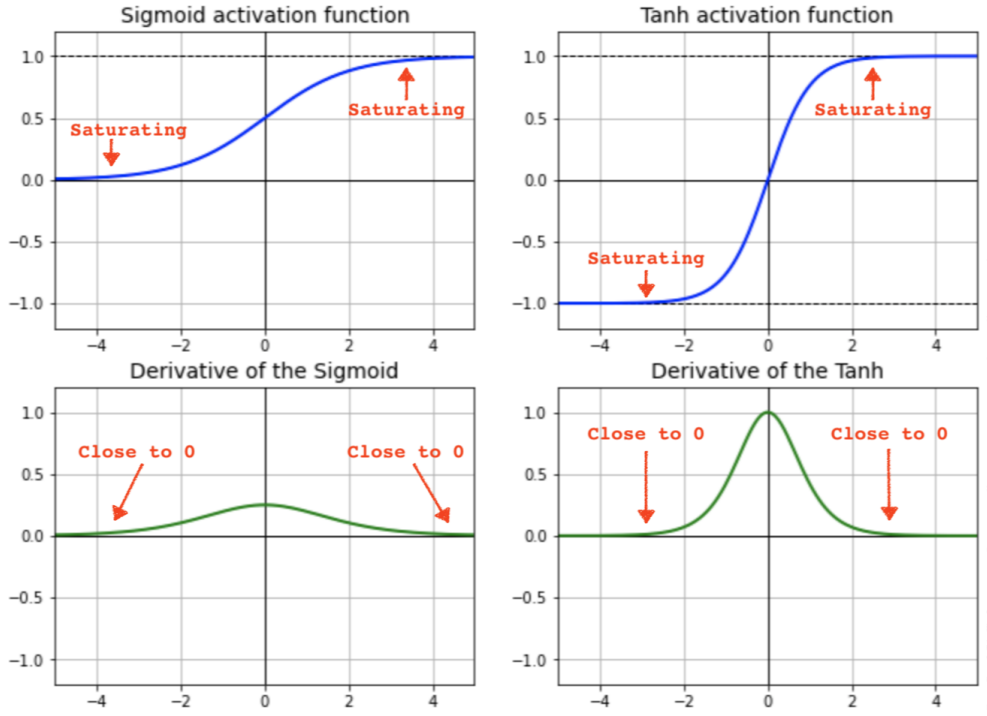
$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



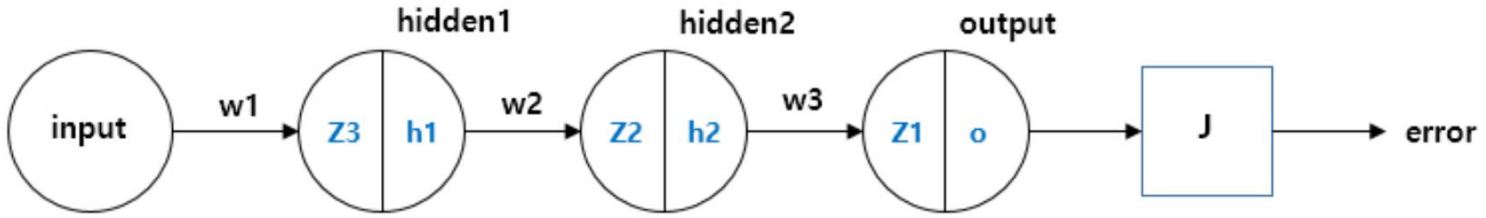
Vanishing Gradient Problem

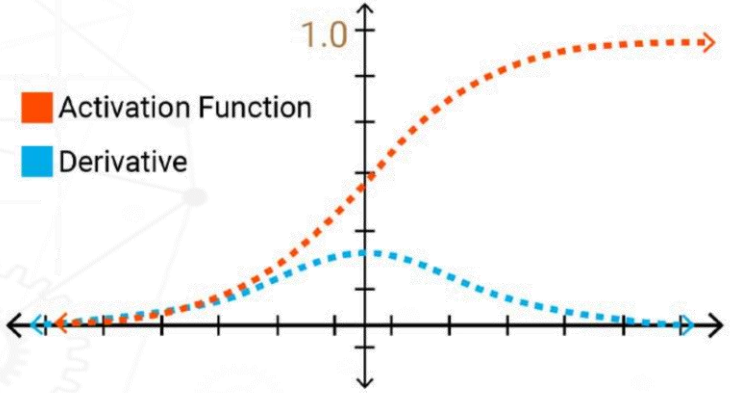


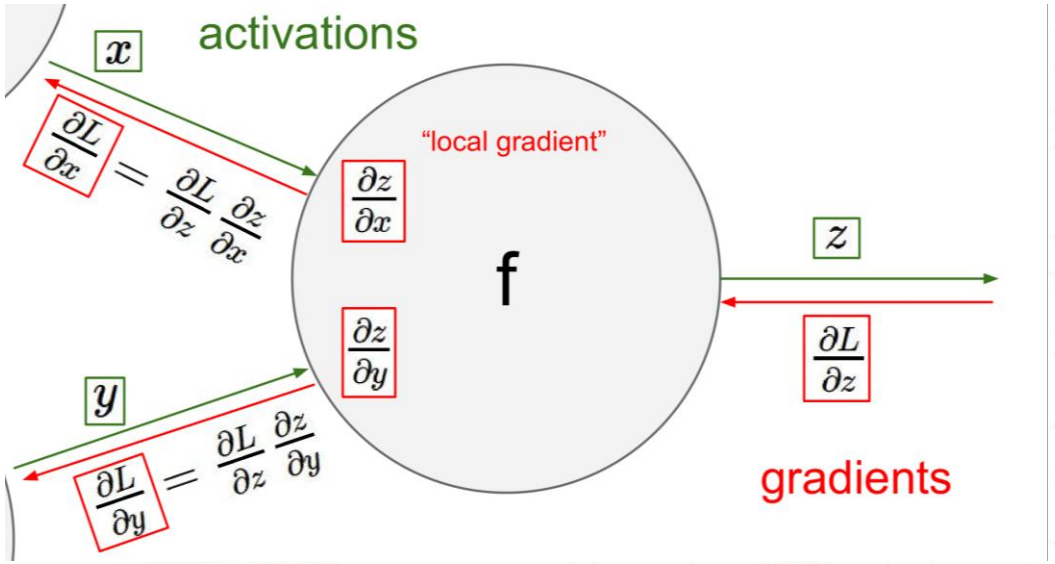
$$z = \sum(\omega x) + b$$



Vanishing Gradient Problem







Vanishing Gradient Problem