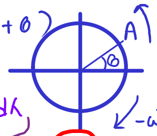
Signals $\rightarrow$ function of independent variables

Systems $\rightarrow$ mapping from one signal to other

System properties:

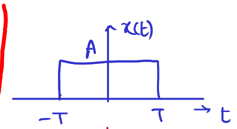
1. Memory: output depends on present only
2. causal: output depends on past ($h(t)=0$ for $t<0$)
3. Stability: Bounded input gives bounded o/p ($\int |h(\lambda)|\,d\lambda < \infty$)
4. Shift Invariance: o/p delay by same amount
5. Linear: $a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$

Phasor: rotating complex number $Ae^{j(\omega t+\theta)}$



$e^{j\omega_0 t} \rightarrow \boxed{LSI}_{h(t)} \rightarrow y(t) = \underbrace{e^{j\omega_0 t}}_{\text{eigen function}} \cdot \underbrace{\int h(\lambda)\, e^{-j\omega_0 \lambda}\, d\lambda}_{H(\omega_0)}$

eigen value $\leftarrow H(\omega_0)$



$\downarrow \mathcal{F}$ from $e^{j\omega T} - e^{-j\omega T}$

Area $\rightarrow$ $2AT \dfrac{\sin(\omega T)}{\omega T}$

$= 2AT\, \mathrm{sinc}(2fT)$

Fourier series: $x(t) = \sum_k c_k e^{j\frac{2\pi}{T}kt}$ $\rightarrow$ Periodic with period T

$c_k = \frac{1}{T}\int x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$

$x(t) \rightarrow \boxed{\ } \rightarrow \{c_k\}$ (analysis)

$\{c_k\} \rightarrow \boxed{\ } \rightarrow x(t)$ (synthesis)

$x(t) \rightarrow \boxed{\text{LSI system}}_{h(t)} \rightarrow y(t) = \sum c_k H(\omega_k) e^{j\frac{2\pi}{T}kt}$

$\hookrightarrow$ Periodic with period T

for periodic output shift invar. is sufficient

$\omega_0 = \frac{2\pi}{T}$ $\quad \omega \rightarrow$

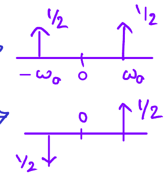
$\cdots -2\omega_0 \quad -\omega_0 \quad 0 \quad \omega_0 \quad 2\omega_0 \cdots$

as $T \rightarrow \infty$, $\omega_0 \rightarrow 0$

$Y(\omega) = X(\omega)\cdot H(\omega) \rightarrow$ linear, memory less

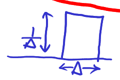
$\cos\omega_0 t \xrightarrow{\mathcal{F}}$

$\sin\omega_0 t \xrightarrow{\mathcal{F}}$



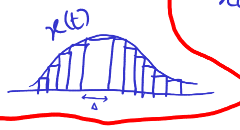
$e^{-at^2} \longrightarrow \sqrt{\frac{\pi}{c}}\, e^{-\omega^2/4c}$

LSI system: **Impulse:** $\lim_{\Delta\to 0}$



$\Rightarrow x(t_0) = \lim_{\Delta\to 0}\int x(t_0)\,\delta(t-t_0)\, dt$

$x(t) = \int x(\lambda)\,\delta(t-\lambda)\, d\lambda =$



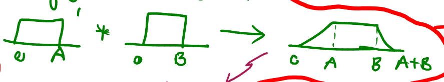
$\delta(t) \rightarrow \boxed{\mathcal{S}} \rightarrow h(t)$

$\delta(t-t_0) \longrightarrow h(t-t_0)$

$x(t) \longrightarrow \int x(\lambda)\cdot h(t-\lambda)\, d\lambda$

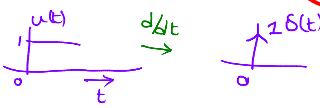
Convolution: graphical method

1. shift 2. mirror



unit step function:

$u(t) = \begin{cases} 1 & ; t \ge 0 \\ 0 & ; t < 0 \end{cases}$



$u(t) \rightarrow \boxed{\mathcal{S}} \rightarrow s(t)$

$\frac{d}{dt}\downarrow \qquad \downarrow d/dt$

$\delta(t) \rightarrow \boxed{\mathcal{S}} \rightarrow h(t)$

$y(t) = x(t) * h(t) = \int x(\lambda)\, h(t-\lambda)\, d\lambda$

1. $x(t) * h(t) = h(t) * x(t)$
2. $x(t) * (h_1 * h_2) = (x * h_1) * h_2$
3. $x * (\alpha h_1 + \beta h_2) = \alpha(x * h_1) + \beta(x * h_2)$

Properties of Fourier Transform:

1. $x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$
2. $X(t) \xrightarrow{\mathcal{F}} x(-f) = \frac{1}{2\pi} x(-\omega)$
3. $e^{j\omega_0 t} x(t) \longrightarrow X(\omega-\omega_0)$
4. $\frac{d}{dt} x(t) \longrightarrow j\omega X(\omega)$
5. $t\, x(t) \longrightarrow j\frac{d}{d\omega} X(\omega)$
6. $x(at) \longrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
7. $\int x(t)\,\bar{y}(t)\, dt = \int X(f)\,\overline{Y(f)}\, df$ (Parseval theorem)

Dirichlet conditions: (Sufficient conditions)

1. Absolute integrable
2. Bounded Variation
3. Finite no. of discontinuties over finite interval

Discrete impulse: $\delta[n] = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{else} \end{cases}$

unit step: $u[n] = \begin{cases} 1 & ; n \ge 0 \\ 0 & ; n < 0 \end{cases}$

$x[n] = \sum_k x[k]\,\delta[n-k]$; $x[n] \longrightarrow \boxed{LSI}_{h[n]} \rightarrow y[n]$

$y[n] = \sum_k x[k]\, h[n-k]$

Nyquist thm: To reconstruct the signal (Band limited)

$f_s > 2f_B$ if there is a freq. component at f_B

$- f_s = 2f_B$ if no frequency component at f_B

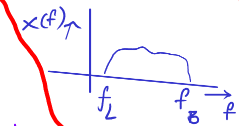
Cross correlation:

$R_{xy}(\tau) = \int x(t+\tau)\,\bar{y}(t)\, dt$

$= \int X(f)\cdot\bar{Y}(f)\cdot e^{j2\pi f\tau}\, df$

If $x(t), y(t)$ are periodic with period T

$R_{xy}(\tau) = \frac{1}{T}\int_{(T)} x(t+\tau)\,\overline{y(t)}\, dt \rightarrow$ Periodic with period T



$f_L = 0 \rightarrow$ low pass signal

$f_L > 0 \rightarrow$ Band pass signal

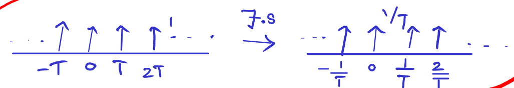
Reconstruction (interpolation): pass through low pass filter with cutoff $f_s/2$

Auto correlation:

$R_{xx}(\tau) = \int X(f)\,\bar{X}(f)\, e^{j2\pi f\tau}\, df$

$R_{xx}(\tau) \xrightarrow{\mathcal{F}} |X(f)|^2$ energy spectral density

Sampled signal: $X_s(f) = \sum_n c_n X(f - nf_s)$



Discrete signals: $x[n] = x(nT_s)$

$x[2n] \xrightarrow{\mathcal{Z}} \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right]$

$x[-n] \longrightarrow X(z^{-1})$

$a^n x[n] \longrightarrow X\left(\frac{z}{a}\right)$

Discrete Time Fourier Transform (DTFT)

$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$ — normalized frequency

$X(\Omega+2\pi) = X(\Omega) \quad \forall\, \Omega$

IDFT

$x[m] = \frac{1}{2\pi}\int_{(2\pi)} X(\Omega)\cdot e^{j\Omega m}\, d\Omega$

$t\,x(t) \longrightarrow -\frac{d}{ds} X(s)$

$n\,h[n] \longrightarrow -z\frac{d}{dz} H(z)$

Delay: $H(s) = e^{-st_0} \longrightarrow$ irrational

$H(z) = z^{-D} \longrightarrow$ rational

Differentiation:

$H(s) = s \longrightarrow$ rational

$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\Omega\, e^{j\Omega n}\, d\Omega$

$H(z) \longrightarrow$ irrational

$e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a}$

$t^M e^{-at} u(t) \longrightarrow \frac{M!}{(s+a)^{M+1}}$

Laplace transform

$e^{st} \longrightarrow \boxed{h(t)}_{LSI} \longrightarrow y(t) = H(s)\,e^{st}$

$s = \sigma + j\omega$

$H(s) = \int_{-\infty}^{\infty} h(\lambda)\, e^{-s\lambda}\, d\lambda$

$Y(s) = H(s)\cdot X(s)$

$s = j\omega \rightarrow$ Fourier transform

$x(t) = \frac{1}{2\pi j}\int_C X(s)\, e^{st}\, ds$

Z- transform

$z^n \longrightarrow \boxed{h[n]}_{LSI} \rightarrow H(z)\,z^n$

$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$

$Y(z) = X(z)\cdot H(z)$

$z = e^{-j\Omega n} \longrightarrow$ DTFT

$x[n] = \frac{1}{2\pi j}\oint_C X(z)\, z^{m-1}\, dz$

C: Circle centered at $z=0$ and in ROC

$H(s), H(z) \rightarrow$ rational

$\Rightarrow H(s) = \frac{N(s)}{D(s)}$

System $H(s)$

stable – $j\omega$ axis $\in$ ROC

causal – All poles to left of $j\omega$ axis

System $H(z)$

stable – unit circle ($z=1$) $\in$ ROC

Causal – ($z \rightarrow \infty$) in ROC

$\Rightarrow$ ROC is exterior of outermost pole

$\frac{1}{1-az^{-1}}$, ROC $|z| > a \longrightarrow a^n u[n]$

ROC $|z| < a \longrightarrow -a^n u[-n-1]$

$\frac{1}{(1-az^{-1})^2}$; $|z| > a \longrightarrow (n+1)a^n u[n]$

; $|z| < a \longrightarrow -(n+1)a^n u[-n-1]$

$z^{-K} \longrightarrow \delta[n-K]$

$x[n]$ a sequence i.e mapping from $\mathbb{Z} \to \mathbb{C}$

Discrete time systems.

$x[n] \to \boxed{\text{syst}} \to y[n]$

$\delta[n] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{else} \end{cases}$

- causal, stable, rational system all poles lie inside unit circle

implementable

Forced response → contributed by input

Natural response → contributed by system alone

Resonant response → contributed by common poles of system and input

Finite Impulse Response (FIR) filter: stable, linear phase.

Design: → Truncate impulse response of ideal filter by keeping symmetry upto desired length.

- Impulse response completely characterizes LSI system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Steps in convolution:

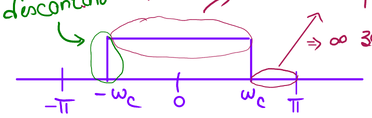
1. stationary platform $x[k]$
2. moving train (stepwise) $h[n-k]$

Ideal filters infinitely non causal (cannot be made causal by delay)

Irrational, unstable

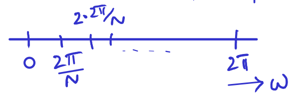
Realistic specs:

1. Pass band tolerance
2. stop band tolerance
3. Transition band width



DFT: Discrete Fourier Transform.

Discretize $X(\Omega) \to N$ points

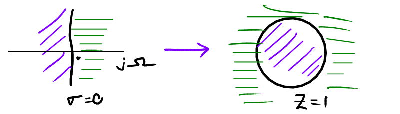


Mapping from s to z domain:

- Rationality, stability must be preserved.

⇒ $Re(s) < 0 \quad \leftrightarrow \quad |z| < 1 \quad z = re^{j\theta}$

⇒ $\sigma < 0 \quad \Leftrightarrow \quad r < 1$



$$\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} = \tilde{x}[n] \to \text{Dual of Sampling thm.}$$

- Sampling freq. axis at interval of $\frac{2\pi}{N}$

⇒ Repeats after $\frac{2\pi}{\text{freq. spacing}} = N$

$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$ → Periodic with period N

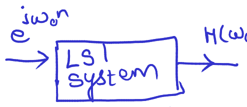
If $\tilde{x}[n] = x[n]$ in support of $x[n]$, then there is no time domain aliasing

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} \leftarrow \text{IDFT}$$

LSI system is BIBO stable if

$$\sum_{k\in\mathbb{Z}} |h[k]| < \infty$$

Finite Impulse Response (FIR) → unconditionally stable



$$H(\omega_0) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega_0 n}$$

$z = \frac{1+s}{1-s}$ $\quad \Omega: -\infty \to \infty \quad \omega: -\pi \to \pi$

$$s = \frac{1-z^{-1}}{1+z^{-1}} = j\tan\left(\frac{\omega}{2}\right); \quad z = e^{j\omega}$$

$\Omega = \tan\left(\frac{\omega}{2}\right)$ → monotonically increasing

$y[n] = h[n] * x[n]$; $\quad x[n],\ n = 0,1,\dots,M-1$; $\quad h[n],\ n = 0,1,\dots,L-1$

↳ $M+L-1$

→ We need $N \geq M+L-1$ to reconstruct $y[n]$ using DFT

→ If $N \leq M+L-1$ then $y[n]$ is the circular convolution of $x[n], h[n]$

Discrete Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$x[n] * h[n] \xrightarrow{\mathcal{F}} X(\omega) H(\omega)$$

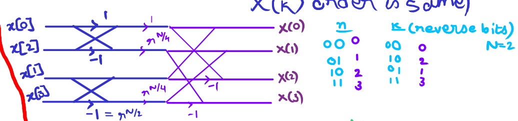
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \to \text{energy}$$

Filter design strategy:

1. Discrete time specs.
2. Go to analog filter specs ($\Omega = \tan\left(\frac{\omega}{2}\right)$)
3. convert non Lowpass filter to LPF equivalent $(s \to s_L)$
4. Design analog LPF
5. Transform back from $s_L \to s$
6. Replace $s \leftarrow \frac{1-z^{-1}}{1+z^{-1}}$

Fast Fourier Transform: let $r = \frac{2\pi}{N}$ ← N^{th} root of unity

Decimation in time: ($x[n]$ order change, $X(k)$ order is same)



$N^2 \to$ multiplications $= \frac{N}{2} + \frac{N}{4} + \dots + 1$

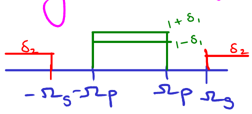
$N(N-1) \to$ Additions $= \frac{N}{2}\cdot \log N \times 2 = N \log N$

Rational Z transform → find partial fraction expansion

↓

find the inverse Z transform

Analog filter design



- Ideal filters have irrational impulse response.

- For rational causal system ROC is $|z| > \beta$, where β is the max. magnitude of finite pole.

Passband	Stopband	Filter type
M	M	Butterworth
E	M	Chebyshev
M	E	Inv. cheby.
E	E	Jacob / elliptical

Chebyshev and Inv. cheby.: same order

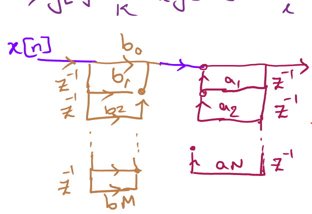
M – Monotone

E – Equiripple

For given set of specs order of filter Non increasing

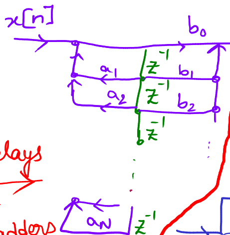
Direct form I: $H(z) = \frac{\sum_\ell b_\ell z^{-\ell}}{1 - \sum_k a_k z^{-k}} = H_{poles}(z) \cdot H_{zeros}(z)$ → Direct form II:

⇒ $y[n] = \sum_k a_k\, y[n-k] + \sum_\ell b_\ell\, x[n-\ell]$



uses less delays

Same number of multipliers and adders



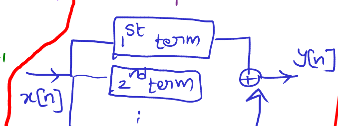
Decimation in freq: $x[n]$ → same order

$X[k]$ → reverse bit order

Dec. in freq. have same computational complexity as decimation in time

Parallel decomposition

↳ use partial fractions

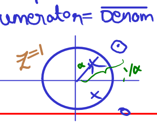


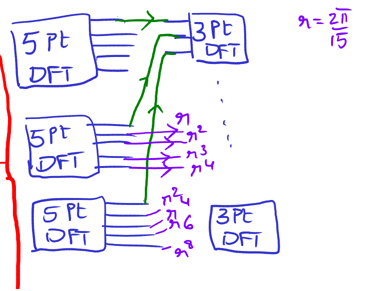
FFT for arbitrary N: ex: $N = 15 = 3 \times 5$

All pass filter

$|H(e^{j\omega})| = 1$

Numerator = Denom





$x[n] \to X[k]$ expansion in time aliasing

$\{x[0], 0, x[1], \dots x[n], 0\} \to \{X[k], X[k]\}$; $\{x[n], 0 \dots\}$ → addition of samples in freq. domain

Padding

as N ↑ more samples of $X(\omega)$