

LOGICAL DEDUCTION IN AI

INFERENCE BY RESOLUTION REFUTATION



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Predicate Logic

Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.

No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.

All dancers are graceful. Ayesha is a student. Ayesha is a dancer. Therefore some student is graceful.

Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

New Additions in Proposition (First Order Logic)
Variables, Constants, Predicate Symbols and New Connectors: \exists (there exists), \forall (for all)

Wherever Mary goes, so does the Lamb. Mary goes to School. So the Lamb goes to School.

Predicate: goes(x,y) to represent x goes to y

New Connectors: \exists (there exists), \forall (for all)

F1: $\forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$

F2: $\text{goes}(\text{Mary}, \text{School})$ ✓ ground instance

G: $\text{goes}(\text{Lamb}, \text{School})$ ✓

To prove: $(F1 \wedge F2) \rightarrow G$ is always true

VALID

Inferencing in Predicate Logic

Domain: D

Constant Symbols: M, N, O, P, ...

Variable Symbols: x, y, z, ...

Function Symbols: F(x), G(x, y),

H(x, y, z)

Predicate Symbols: p(x), q(x, y),

r(x, y, z),

Connectors: ~, \wedge , \vee , \rightarrow , \exists , \forall

Terms:

Well-formed Formula:

Free and Bound Variables:

Interpretation, Valid, Non-Valid,
Satisfiable, Unsatisfiable

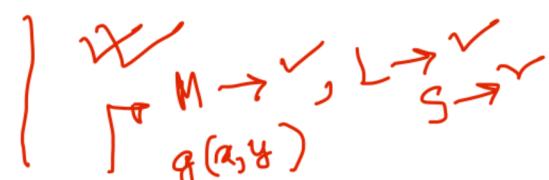
What is an Interpretation? Assign a domain set D, map constants, functions, predicates suitably. The formula will now have a truth value

Example:

F1: $\forall x(g(M, x) \rightarrow g(L, x))$

F2: $g(M, S)$

G: $g(L, S)$



Interpretation 1: D = {Akash, Baby, Home, Play, Ratan, Swim}, etc.,

Interpretation 2: D = Set of Integers, etc.,

How many interpretations can there be?

To prove Validity, means $(F1 \wedge F2) \rightarrow G$ is true under all interpretations

To prove Satisfiability means $(F1 \wedge F2) \rightarrow G$ is true under at least one interpretation

n 2^n Truth Propositional logic

Resolution Refutation for Propositional Logic

To prove validity of $\sim F = ((F_1 \wedge F_2 \wedge \dots \wedge F_n) \rightarrow G)$
 we shall attempt to prove that $\sim F = (F_1 \wedge F_2 \wedge \dots \wedge F_n \wedge \sim G)$ is unsatisfiable
 \rightarrow FALSE under all interpretations

Steps for Proof by Resolution

Refutation: $(\neg a \vee b) \wedge (\neg b \vee c) \wedge (\neg c \vee a)$

1. Convert of Clausal Form / Conjunctive Normal Form (CNF Product of Sums).
2. Generate new clauses using the resolution rule.
3. At the end, either False will be derived if the formula $\sim F$ is unsatisfiable implying F is valid.

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

$$\begin{array}{l} F_1: (a \rightarrow (b \wedge c)) \\ F_2: \sim b, G: \sim a, \sim G: a \end{array} \quad \begin{array}{l} C_1: (\sim a \vee b) \\ C_2: (\sim a \vee c) \\ C_3: \sim b \\ C_4: a \end{array}$$

Let $C_1 = a \vee b$ and $C_2 = \sim a \vee c$ then a new clause $C_3 = b \vee c$ can be derived.

RESOLUTION RULE

(Proof by showing that $((C_1 \wedge C_2) \rightarrow C_3)$ is a valid formula). ✓

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C_1 = a$ and $C_2 = \sim a$ from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates. ✓

Resolution Rule is Sound and Complete

Clauses of Clause Form: $\sim F = (C_1 \wedge C_2 \wedge C_3 \wedge C_4)$
 where: $C_1: (\sim a \vee b)$
 $C_2: (\sim a \vee c)$
 $C_3: \sim b$
 $C_4: a$

To prove that $\sim F$ is False

Applying Resolution Refutation

Let $C_1 = a \vee b$ and $C_2 = \neg a \vee c$
then a new clause $C_3 = b \vee c$ can be derived.
(Proof by showing that $((C_1 \wedge C_2) \rightarrow C_3)$ is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C_1 = a$ and $C_2 = \neg a$ from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

$$F_1: (a \rightarrow (b \wedge c)) = (\neg a \vee b) \wedge (\neg a \vee c)$$

$$F_2: \neg b \quad \checkmark$$

$$G: \neg a \quad \checkmark$$

$$\neg G: a \quad \checkmark$$

$$(F_1 \wedge F_2) \rightarrow G$$

F1 \wedge F2 \wedge G

Clauses of Clause Form: $\neg F = (C_1 \wedge C_2 \wedge C_3 \wedge C_4)$
where: $C_1: (\neg a \vee b) \quad \checkmark$
 $C_2: (\neg a \vee c) \quad \checkmark$
 $C_3: \neg b \quad \checkmark$
 $C_4: a \quad \checkmark$

To prove that $\neg F$ is False

New Clauses Derived
 $C_5: \neg a$ (Using C1 and C3)
 $C_6: \text{False}$ (using C4 and C5)

$C_5: \neg a \rightarrow \text{FALSE}$

Example

Let $C_1 = a \vee b$ and $C_2 = \neg a \vee c$
then a new clause $C_3 = b \vee c$ can be derived.

(Proof by showing that $((C_1 \wedge C_2) \rightarrow C_3)$ is a valid formula).

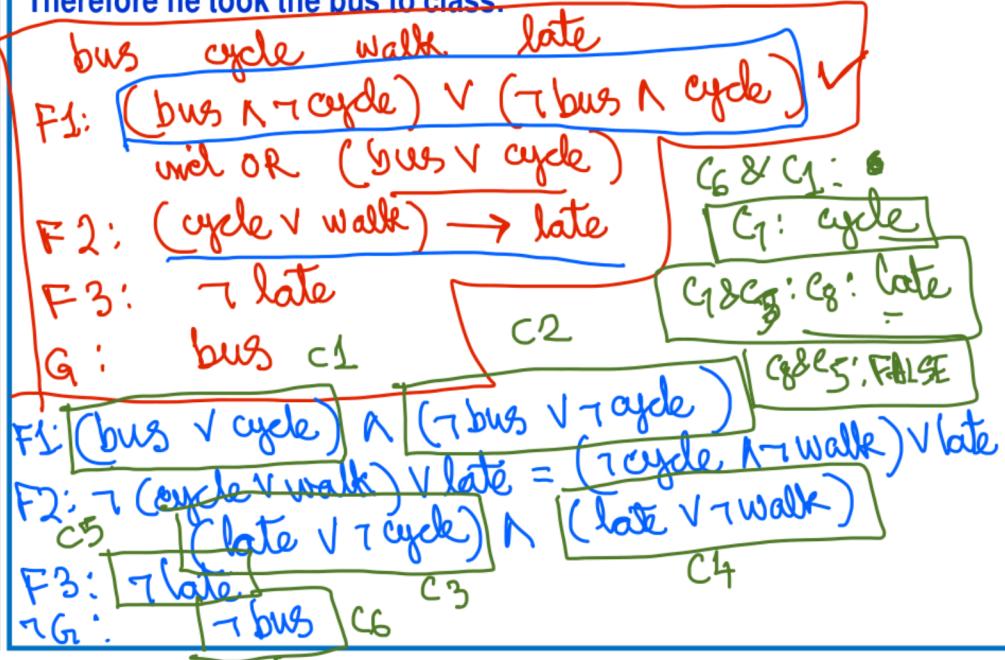
To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C_1 = a$ and $C_2 = \neg a$ from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

Rajesh either took the bus or came by cycle to class. If he came by cycle or walked to class he arrived late. Rajesh did not arrive late. Therefore he took the bus to class.



Resolution Refutation for Predicate Logic

Given a formula F which we wish to check for validity, we first check if there are any free variables. We then quantify all free variables universally.

Create $F' = \neg F$ and check for unsatisfiability of F'

STEPS:

Conversion to Clausal (CNF) Form:

- Handling of Variables and Quantifiers, Ground Instances
- Concept of Unification
- Principle of Most General Unifier (mgu)
- Repeated application of Resolution Rule using mgu

$F_1: \forall x(\text{goes}(\text{Mary}, x) \rightarrow \text{goes}(\text{Lamb}, x))$ constants

$F_2: \text{goes}(\text{Mary}, \text{School})$

variables

$G: \text{goes}(\text{Lamb}, \text{School})$

predicates functions

To prove: $(F_1 \wedge F_2) \rightarrow G$ is valid

CONVERSION TO CLAUSAL FORM IN PREDICATE LOGIC

- Remove implications and other Boolean symbols converting to equivalent forms using \neg , \vee , \wedge
- Move negates (\neg) inwards as close as possible
- Standardize (Rename) variables to make them unambiguous
- Remove Existential Quantifiers by an appropriate new function / constant symbol taking into account the variables dependent on the quantifier
(Skolemization)
- Drop Universal Quantifiers
- Distribute \vee over \wedge and convert to CNF

$F_1 \wedge F_2 \wedge \neg G$ $\forall x \{ \neg \text{goes}(\text{Mary}, x) \vee \neg \text{goes}(\text{Lamb}, x) \}$

C1: $\neg \text{goes}(\text{Mary}, x) \vee \neg \text{goes}(\text{Lamb}, x)$
C2: $\neg \text{goes}(\text{Mary}, \text{School})$ C3: $\neg \text{goes}(\text{Lamb}, \text{School})$
C4: $\neg \text{goes}(\text{Lamb}, \text{School}) \rightarrow \text{FALSE}$

Conversion to Clausal Form

1. Remove implications and other Boolean symbols converting to equivalent forms using \sim , \vee , \wedge
2. Move negates (\sim) inwards as close as possible ✓ ✓
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function /constant symbol taking into account the variables dependent on the quantifier (Skolemization) ✓
5. Drop Universal Quantifiers ✓
6. Distribute \vee over \wedge and convert to CNF ✓

$$\begin{aligned}
 & \forall x (\forall y (\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow (\exists z (\text{likes}(z, x)))) \\
 & \quad \forall x \{ \forall y (\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow \\
 & \quad \quad \exists z (\text{likes}(z, x)) \} \\
 & \quad \forall x \{ \neg (\forall y (\text{student}(y) \rightarrow \text{likes}(x, y))) \vee \\
 & \quad \quad \exists z (\text{likes}(z, x)) \} \\
 & \quad \forall x \{ \neg (\forall y (\neg \text{student}(y) \vee \text{likes}(x, y))) \vee \\
 & \quad \quad \exists z (\text{likes}(z, x)) \} \\
 & \quad \forall x \{ \exists y ((\text{student}(y) \wedge \neg \text{likes}(x, y))) \vee \\
 & \quad \quad \exists z (\text{likes}(z, x)) \} \\
 & \quad \forall x \{ \exists y ((\text{student}(F(x)) \wedge \neg \text{likes}(x, F(x))) \vee \\
 & \quad \quad \exists z (\text{likes}(G(x), z)) \}
 \end{aligned}$$

Substitution, Unification, Resolution

Consider clauses:

- C1: $\neg \text{studies}(x, y) \vee \text{passes}(x, y)$ ✓
- C2: $\text{studies}(\text{Madan}, z)$
- C3: $\neg \text{passes}(\text{Chetan}, \text{Physics})$ ✓
- C4: $\neg \text{passes}(w, \text{Mechanics})$ ✓

What new clauses can we derive by the resolution principle?

Ground Clause and a more general clause

Concept of substitution / unification and the Most General Unifier (mgu)

Resolution Rule for Predicate Calculus: Repeated Application of Resolution using mgu

$$\begin{aligned}\exists x \forall y p(x, y) &\Rightarrow \exists y p(A, y) \Rightarrow p(A, y) \\ \forall x \exists y p(x, y) &\Rightarrow \forall x p(x, f(x)) \\ \text{SKOLEM FN} &\Rightarrow p(x, f(x))\end{aligned}$$

$$\begin{array}{c} C_1 \& C_2 : C' \\ \hline C_1 \& C_2 : C'' \end{array} \frac{\text{passes}(\text{Madan}, z)}{\text{passes}(\text{Madan}, \text{Physics})}$$

C' is more general than C''

$$\begin{array}{c} C_1 \& C_4 : y/\text{Mechanics} \\ \hline C_1 \& C_4 : C''' \end{array} \quad \begin{array}{c} \neg \text{studies}(z, \text{Mechanics}) \\ \hline \text{FALSE} \end{array}$$

C''' is the most general unifier

Examples

F1: $\forall x(\text{contractor}(x) \rightarrow \neg\text{dependable}(x))$ $c(x)$
F2: $\exists x(\text{engineer}(x) \wedge \text{contractor}(x))$ $d(x)$
G: $\exists x(\text{engineer}(x) \wedge \neg\text{dependable}(x))$ $e(x)$

$$\begin{array}{l}
 \text{F1: } \cancel{\forall x \{ \neg e(x) \vee \neg d(x) \}} \quad c_1 \\
 \text{F2: } \frac{e(A) \wedge \cancel{e(A)}}{c_2} \quad c_3 \\
 \text{F3: } \frac{\cancel{\exists x (e(x) \wedge \neg d(x))}}{c_4} \\
 \quad \quad \quad \frac{\forall x (\neg e(x) \vee d(x))}{c_5} \\
 \text{F4: } \cancel{\neg c(A)}; c_6 \\
 \text{FALSE}
 \end{array}$$

F1: $\forall x(\text{dancer}(x) \rightarrow \text{graceful}(x))$
F2: student(Ayesha), F3: dancer(Ayesha)
G: $\exists x(\text{student}(x) \wedge \text{graceful}(x))$

$F1: \forall x \{ Td(x) \vee g(x) \} : C1$
 $F2: S(A) : C2$
 $F3: d(A) : C3$
 $\neg G: \neg \exists x \{ S(x) \wedge g(x) \}$
 $\neg G: \neg \{ \neg S(x) \vee \neg g(x) \} : C4$

0. FIND $\neg A$:
 $\neg \neg G$
 1. Convert To
 classical Form
 2. Resolution Rule
 UNSATISFIABILITY

x/A
 $\neg g(A)$
 x/A
 $\neg d(A)$
 FALSE

Thank you