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Assignment - 2

1. Gram - Schmidt Process

Given $A = (\beta_1^t, \beta_2^t, \beta_3^t)$

$$= \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix} \\ \;\; v_1 \quad v_2 \quad v_3 \end{array}$$

Let the orthonormal basis be (q_1, q_2, q_3) in $\mathbb{R}^3$

(a) First check if QR decomposition is possible.

* Check if β_1^t, β_2^t & β_3^t is independent.

$$c_1\beta_1 + c_2\beta_2 + c_3\beta_3 = 0 \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1, c_2, c_3 \in \mathbb{R}$

Row echelon form for A

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & -1 & 4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$ $\qquad$ $R_2 \sim \frac{1}{3}R_2, R_3 \sim \frac{1}{2}R_3$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \text{Pivots} = 1, -1, 1$$

$R_2 \leftrightarrow R_3$

$\therefore$ $c_1 + c_2 = 0$

$-c_2 + 2c_3 = 0$

$c_3 = 0$

$\Rightarrow$ $c_3 = 0,\ c_2 = 0,\ c_1 = 0.$

$\therefore$ As $c_1 = 0 = c_2 = c_3 = 0$

$\therefore$ A is a basis containing $(\beta_1^t, \beta_2^t \,\&\, \beta_3^t)$

$\therefore$ QR decomposition is possible

(b) Let Q be (q_1, q_2, q_3) in $\mathbb{R}^3$

Using Gram-Schmidt process,

Let $v_1 = \beta_1^t,\ v_2 = \beta_2^t,\ v_3 = \beta_3^t$

$$\therefore q_1 = \frac{v_1}{\|v_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{1^2+0^2+1^2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Now $q_2' = v_2 - (q_1^t v_2) q_1$

$$\Rightarrow q_2' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} [1\ 0\ 1] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{1}{2}\right) \left(\underset{1\times 3}{[1\ 0\ 1]} \underset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{1}{2}\right) [1 + 0 + -1] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{1}{2}\right)(0)\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore q_2 = \frac{q_2'}{\|q_2'\|} \quad \text{(Normalising } q_2'\text{)}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now $q_3' = v_3 - (q_2^t v_3) q_2 - (q_1^t v_3) q_1$

$$\therefore q_3' = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$- \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow q_3' = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \frac{1}{\sqrt{2}}(0+0-4) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$- \frac{1}{\sqrt{2}}(0+0+4) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{-4}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)(4)\left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \left(\frac{\cancel{4}^{\,2}}{\cancel{2}}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore q_3 = \frac{q_3'}{\|q_3'\|} = \frac{\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}}{\sqrt{0^2+3^2+0^2}}$ (Normalising)

$= \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \boxed{\text{Orthogonal Matrix } (Q) = (q_1, q_2, q_3) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}}$

Checking if Q has orthonormal basis in $\mathbb{R}^3$

$\|q_1\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1$

$\|q_2\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1$

$\|q_3\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{0 + 1 + 0} = 1$

$q_1^t q_2 = \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \left[\frac{1}{2} + 0 - \frac{1}{2}\right] = 0$

$$q_1^t q_3 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [0+0+0] = \underline{0}$$

$$q_2^t q_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \left[\frac{1}{2} + 0 + \left(\frac{-1}{2}\right)\right]$$

$$= \underline{0}$$

$$q_2^t q_3 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= [0+0+0] = 0$$

$$q_3^t q_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = [0+0+0] = 0$$

$$q_3^t q_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = [0+0+0] = 0$$

$$\therefore \; q_i^t q_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } 1 \le i, j \le 3 \;\&\; i \ne j \end{cases}$$

$\therefore$ $\{q_1, q_2, q_3\}$ are orthonormal vectors.

$\therefore$ Q is orthogonal matrix with orthonormal vectors.

Now for calculating R in $(QR = A)$

$$R = \begin{bmatrix} q_1^t v_1 & q_1^t v_2 & q_1^t v_3 \\ 0 & q_2^t v_2 & q_2^t v_3 \\ 0 & 0 & q_3^t v_3 \end{bmatrix}$$

Upper triangular Matrix.

$$= \begin{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \\ 0 & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \\ 0 & 0 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} & 0 + 0 + \frac{4}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & 0 + 0 - \frac{4}{\sqrt{2}} \\ 0 & 0 & 0 + 3 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & -4/\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \boxed{R = \begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}}$$

To verify $A = QR$ $\qquad \boxed{A = QR}$

$$\Rightarrow \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}}$$

$$QR = \begin{bmatrix} 1 & 1 & 2-2+0 \\ 0 & 0 & 0+0+3 \\ 1 & -1 & 2+2+0 \end{bmatrix}$$

$$QR = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix}$$

$$\therefore \boxed{QR = A}$$

Hence verified.

Date
Page

2. (a) Let E be projection of (x_1, x_2) on subspace W.
$\therefore b^t = (x_1, x_2)$
then E can be written as,

$E = ka$ where a is a vector in W.
$a^t = (3, 4)$

$$\therefore k = \frac{a^t b}{a^t a} = \frac{\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \frac{3x_1 + 4x_2}{9+16} = \frac{3x_1 + 4x_2}{25}$$

$$\therefore k = \frac{3x_1 + 4x_2}{25}$$

$$\therefore E = ka = \left(\frac{3x_1 + 4x_2}{25}\right)\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\boxed{\therefore E(x_1, x_2) = \left(\frac{9x_1 + 12x_2}{25}, \frac{12x_1 + 16x_2}{25}\right)}$$

(b) Projection matrix $= \dfrac{aa^t}{a^t a} = \dfrac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix}}{\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$

$$= \frac{\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}}{25} = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$\therefore$ Matrix of E in the standard ordered basis is

$$\Rightarrow \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

$$\Rightarrow \frac{9}{25}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{12}{25}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{12}{25}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ \frac{16}{25}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) Let $V = W^{\perp}$ where V is a subspace of R^2 spanned by vector (x,y)

$$W^t V = 0 \Rightarrow [3\ 4]\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x + 4y = 0$$

$$\Rightarrow x = -\frac{4y}{3}$$

Let $(x,y) = (-4,3)$

So $(-4,3)$ is orthogonal to W.

$$\therefore \boxed{W^{\perp} = L\{(-4,3)\}}$$

$$\boxed{W^{\perp} = \{(x,y) \in R^2;\ x = \frac{-4y}{3}\}}$$

(d) For $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we need orthogonal basis u, v
(orthonormal)

such that $E(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $E(v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$E(u) = 1 \cdot u + 0 \cdot v = u$

Orthonormal basis: $u = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$, $v = \pm \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}}$$

$\|u\| = 1$
$\|v\| = 1$
$u^t v = 0$