LINEAR ALGEBRA (MA20105)

Problems Sheet-5:

Notation: $\mathbb{F} = (\mathbb{R}, \mathbb{C})$ will always denote a field and $\mathbb{F}^n := \mathbb{F} \times \cdots \times \mathbb{F}$ (n times).

$\mathbb{R}$ will denote the field of real numbers

$\mathbb{C}$ will denote the field of complex numbers

Problems related to the representation of limear map by matrices:

1. Let T be a linear map on $\mathbb{C}^2$ defined by $T(x_1, x_2) = (x_1, 0)$. Let $\mathcal{B}$ be the standard ordered basis of $\mathbb{C}^2$ and let $\mathcal{B}' = (\alpha_1, \alpha_2)$ be the ordered basis defined by $\alpha_1 = (1, i), \alpha_2 = (-i, 2)$.

   (i) What is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

   (ii) What is the matrix relative to the pair $\mathcal{B}', \mathcal{B}$?

   (iii) What is the matrix of T in the ordered basis $\mathcal{B}'$?

   (iv) What is the matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$?

2. Let V be a two dimensional vector space over the field $\mathbb{F}$, and let $\mathcal{B}$ be an ordered basis for V. If T is a linear map on V and $[T]_{\mathcal{B}} =$
$$\begin{bmatrix} a & b \\ c & d, \end{bmatrix}$$
   prove that $T^2 - (a+d)T + (ad - bc)I = 0$.

3. Let T be a linear map on $\mathbb{R}^2$ defined by
$$T(x_1, x_2) = (-x_2, x_1).$$
   (i) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

   (ii) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

   (iii) Prove that for every real number c the linear map $T - cI$ is invertible.

   (iv) Prove that $\mathcal{B}$ is any ordered basis for $\mathbb{R}^2$ and $[T]_{\mathcal{B}} = A$, then $A_{12}A_{21} \neq 0$.

4. Let T be a linear map on $\mathbb{R}^3$ defined by
$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$
   Prove that T is invertible and give a rule for T^{-1} like the one which defines T.

5. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:
$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}, \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$