LINEAR ALGEBRA (MA20105)

Problems Sheet-6:

Notation: $\mathbb{F} = (\mathbb{R}, \mathbb{C})$ will always denote a field and $\mathbb{F}^n := \mathbb{F} \times \cdots \times \mathbb{F}$ (n times).

$\mathbb{R}$ will denote the field of real numbers

$\mathbb{C}$ will denote the field of complex numbers

Problems related to Linear functionals and dual space:

1. In $\mathbb{R}^3$, let $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (0, 1, -2)$, $\alpha_3 = (-1, -1, 0)$.

   (i) f is a linear functional on $\mathbb{R}^3$ such that

   $$f(\alpha_1) = 1, \ f(\alpha_2) = -1, \ f(\alpha_3) = 3,$$

   and if $\alpha = (a, b, c)$, find $f(\alpha)$.

   (ii) Describe explicitly a linear functional f on $\mathbb{R}^3$ such that

   $$f(\alpha_1) = f(\alpha_2) = 0$$

   but $f(\alpha_3) \neq 0$.

   (iii) Let f be any linear functional such that

   $$f(\alpha_1) = f(\alpha_2) = 0$$

   and $f(\alpha_3) \neq 0$. If $\alpha = (2, 3, -1)$, show that $f(\alpha) \neq 0$.

2. Let $\mathcal{B} = \{\alpha_1, \alpha_1, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of $\mathcal{B}$.

3. If A and B are $n \times n$ matrices over the field F, show that $trace(AB) = trace(BA)$. Show that similar matrices have the same trace.

4. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ which have degree 2 or less:

   $$p(x) = c_0 + c_1 x + c_2 x^2.$$

   Check that the following are linear functionals i.e. a linear map from $V \to \mathbb{R}$

   $$f_1(p) = \int_0^1 p(x)dx; \ f_2(p) = \int_0^2 p(x)dx; \ f_3(p) = \int_0^{-1} p(x)dx.$$

5. If A and B are $n \times n$ complex matrices, show that $AB - BA = I$ is impossible.