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Test 1 – CS41001 (ThOC)

1. $VIS = \{ (M, x, p) \mid M$ on input x visits state p during computation $\}$

$p \in Q$
$x \in \Sigma^*$

→ VIS is undecidable. It is possible to define a turing machine that reduces Halting Problem to VIS.

$$HP \leq_M VIS$$

Proof.

$$(N, V) \overset{\sigma}{\longmapsto} (M, x, p)$$

N halts on V $\longleftrightarrow$ M on input x visits state p

Behavior of M on input x,
- erase x from the turing machine tape
- write v on the tap and simulate N on v
- If N halts on V, move to state p

If N halts on v, $L(M) = \Sigma^*$
If N does not halt on v, $L(M) = \emptyset$

∴ When N halts on v, M visits the state p for every such v.

4. (a) PCP is decidable over unary alphabet 1.

We can describe a turing machine M that decides unary PCP.

Given unary PCP instance:

$$A = \{w_1, w_2 \dots w_k\}, \quad B = \{x_1, x_2 \dots x_k\}$$
$$= \{1^{a_1}, 1^{a_2} \dots 1^{a_k}\}, \quad = \{1^{b_1}, 1^{b_2} \dots 1^{b_k}\}$$
$$(a_i, b_i \geq 0 \ \forall \ 1 \leq i \leq k)$$

Behavior of M on input (A, B),

→ check if $a_i = b_i$ for some i, If so, accept

→ Check if there exist i, j such that $a_i > b_i$ and $a_j < b_j$. If so, accept

→ If none of the above holds, reject.

∴ M halts in all inputs (A, B) and either accepts or rejects.

∴ M is a total turing machine and PCP over 1 is decidable.

PAGE NO
DATE

$PRO = \{ G \mid G$ is CFG and ambiguous $\}$

3.b) G is ambiguous → undecidable

Reducing from PCP. PCP $\leq_m$

Proof
Let $A = w_1, w_2, \ldots w_k$, $B = x_1, x_2, x_3 \ldots x_k$ over Σ

PCP instance $= (A, B)$

Let $(a_1, a_2 \ldots a_k)$ be symbols not in Σ.

Let G be CFG given by,

$$S = X \mid Y$$
$$X = w_i X a_i \mid w_i a_i \quad \text{for } 1 \leq i \leq k$$
$$Y = x_i Y a_i \mid x_i a_i \quad \text{for } 1 \leq i \leq k$$

where $S, X, Y \notin \Sigma \cup \{a_1, \ldots a_k\}$

Suppose G is ambiguous, and let

$z \in (\Sigma \cup \{a_1 \ldots a_k\})^*$
$\alpha, \beta, \gamma_1, \gamma_2 \in (\Sigma \cup \{a_1 \ldots a_k, S, X, Y\})^*$ and
$C \in \{S, X, Y\}$ such that

- $S \overset{*}{\Rightarrow} \alpha C \beta \Rightarrow \alpha \gamma_1 \beta \overset{*}{\Rightarrow} z$
- $S \overset{*}{\Rightarrow} \alpha C \beta \Rightarrow \alpha \gamma_2 \beta \overset{*}{\Rightarrow} z$
- $\gamma_1 \neq \gamma_2$

Since there is atmost 1 derivation $A \Rightarrow \gamma \ldots \Rightarrow z$
and at most one derivation $B \Rightarrow \ldots \Rightarrow z$,
hence $\alpha C \beta = S$

The string obtained by removing all a_i's from z is a solution to PCP instance
$\therefore$ G is ambiguous $\Rightarrow$ PCP instance is solvable.

If PCP instance has a solution, let solution be
$(i_1, i_2 \ldots i_j \in [1,k])$

$w_{i_1} w_{i_2} \ldots w_{i_j} = x_{i_1} x_{i_2} \ldots x_{i_j}$

Concatenating $a_{i_j}, a_{i_{j-1}}, a_{i_2} a_{i_1}$ to both sides,

$w_{i_1} w_{i_2} \ldots w_{i_j} a_{i_j} a_{i_{j-1}} \ldots a_{i_3}, a_{i_2}, a_{i_1}$

$= x_{i_1} x_{i_2} \ldots x_{i_j} a_{i_j} a_{i_{j-1}} \ldots a_{i_2} a_{i_1}$

We can see that they can be derived from S

$S \rightarrow A \rightarrow w_{i_1} w_{i_2} \ldots w_{i_j} a_{i_j} \ldots a_{i_2} a_{i_1}$

$S \rightarrow B \rightarrow x_{i_1} x_{i_2} \ldots x_{i_j} a_{i_j} \ldots a_{i_2} a_{i_1}$

$\therefore$ PCP instance is solvable $\Rightarrow$ G is ambiguous

PAGE NO.
DATE

$\therefore$ PCP instance is solvable $\Leftrightarrow$ G is ambiguous

$\therefore$ Hence a reduction is possible from PCP to PRO.

$\therefore$ PCP is undecidable $\Rightarrow$ G is ambiguous is also undecidable over Σ

$\frac{15}{30}$ 3. a) $\mathcal{L}(G) = \mathcal{L}(G)\mathcal{L}(G)$ is undecidable

Using reduction from halting problem complement.

$$\neg HP \leq_m \{G \mid \mathcal{L}(G) = \mathcal{L}(G)\mathcal{L}(G) \text{ and } G \text{ is CFG}\}$$

Proof. $(M, w) \rightarrow \neg HP$ instance

Let G be grammar for $L = \neg VALCOMP(M, w)$
(studied in class)

If M does not halt on w, then $VALCOMP(M, w) = \emptyset$ and so $L = \mathcal{L}(G) = \Delta^*$ and $\Delta^* = \Delta^*\Delta^*$

If M halts on w, then $VALCOMP(M, x) \neq \emptyset$

Let $\alpha = \# C_0 \# C_1 \# C_2 \ldots \# C_N \#$ be a valid computation history of M on w.

then let

$a = \# c_0$

$b = \# c_1 \# c_2 \ldots \# c_N \#$

So a = Invalid computation history
(does not end with #)

b = Invalid computation history
(head is not the leftmost cell / left end marker in first configuration c_1).

So $a, b \in L = \neg VALCOMP(M, x)$

But $\alpha = ab \in VALCOMP(M, w)$

$\therefore \alpha \notin L$

$\therefore \mathcal{L}(G)$ does not satisfy $\mathcal{L}(G) = \mathcal{L}(G)\mathcal{L}(G)$

$\therefore$ undecidable

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2. Undecidable

$S = \{ \langle M \rangle \mid M$ is a TM that accepts w^R whenever it accepts w
$\mathcal{L}(M) = \mathcal{L}(M)^R \}$

Let $A = \{ \langle M, w \rangle \mid M$ is a TM that accepts $w \}$

~~[illegible]~~

Behaviour of A on input $\langle M, w \rangle$,

- check if $\langle M, w \rangle$ is a valid encoding, else reject
- construct a TM M_2 from M and w:

$$M_2 = \begin{cases} \text{accepts } x, & x \in L(00^*11^*) \\ \text{rejects} & \text{otherwise} \end{cases}$$

- Run on $\langle M_2 \rangle$
- if M_2 accepts, accept
- else reject.

So if $y \in L(00^*11^*)$

$y^R \notin L(00^*11^*)$

So TM A recognises $L(00^*11^*)$, then A cannot recognise $(L(00^*11^*))^R$

$\therefore$ A TM cannot have both $L(M)$ and $L(M)^R$.

$\therefore$ <u>S is undecidable</u>