Tutorial 4

Incompleteness Theorem, Type-0 Grammars

1. Write down formulas in the language of first order number theory for the following.
   (a) The relations $\neq, \geq, \leq, <, >$.
   Solution:
   $$x \neq y := \neg(x = y)$$
   $$x \geq y := \exists z \ (x = y + z)$$
   $$x \leq y := \exists z \ (y = x + z)$$
   $$x > y := \exists z \ (z \neq 0 \wedge x = y + z)$$
   $$x < y := \exists z \ (z \neq 0 \wedge y = x + z)$$
   (b) "There are infinitely many primes".
   Solution:
   $$\forall n \exists p \ (p \geq n \wedge \mathsf{PRIME}(p))$$
   (c) "There are infinitely many twin primes." Twin primes are pairs of primes that differ by 2.
   Solution:
   $$\forall n \exists p \ (p \geq n \wedge \mathsf{PRIME}(p) \wedge \mathsf{PRIME}(p + 2))$$
   (d) "For $n \geq 3$, the equation $x^n + y^n = z^n$ has no postive integer solutions." This is the famous Fermat's last theorem.
   Solution:
   $$\forall n \ (n \geq 3) \longrightarrow \neg (\exists x \exists y \exists z \ x^n + y^n = z^n)$$

2. Prove that there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ that is not provably total in Peano arithmetic.

   Solution: f is total if and only if $\forall x \exists y f(x) = y$. Also, the soundness of PA proof system ensures that if f is provably total, then f is total.

   Assume that all total functions are provably total. Let $\mathcal{M}$ be a TM enumerating all functions f with proofs for $\forall x \exists y f(x) = y$. Let $f^{(1)}, f^{(2)}, \ldots$ be the enumeration. Define a function $g : \mathbb{N} \to \mathbb{N}$ as follows:
   $$g(z) = f^{(z)}(z) + 1 \text{ for all } z.$$
   Since $f^{(z)}$ is total, so is g. Our assumption now implies that there is some $u \in \mathbb{N}$ such that $g = f^{(u)}$. But
   $$g(u) = f^{(u)}(u) + 1 \neq f^{(u)}(u),$$
   thus contradicting our assumption. Therefore, there must exist some total function that is not provably total.

3. (a) Recall the definition of context-sensitive grammars (CSGs): a grammar $G = (N, \Sigma, P, S)$ is context-sensitive if all its productions are the form $\alpha \to \beta$ where $\alpha, \beta \in (N \cup \Sigma)^*$, $\alpha \neq \epsilon$ and $|\alpha| \leq |\beta|$. Show that every context-sensitive grammar is recursive. (In other words, show that every CSG is accepted by a total TM.)

Solution: Here's an algorithm that on input a CSG $G = (N, \Sigma, P, S)$ and string $x \in \Sigma*$, decides whether $x \in L(G)$. Construct a graph H whose vertices are strings from $(N \cup \Sigma)^*$ of length $\leq |x|$. Include edge from α to β if $\alpha \xrightarrow[G]{1} \beta$. Now check whether there is a path from S to x in H. Such a path would correspond to a derivation of x in G. This algorithm halts on any input. Therefore, $L(G)$ is recognised by some total TM and hence recursive.

(b) Show that there does not exist an *r.e.* list of total Turing machines such that every recursive set is accepted by some machine on the list.

Solution: Let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an enumeration of total TMs such that for every recursive sets A, there is some $\mathcal{M}_j$ such that $L(\mathcal{M}_j) = A$. Let $\mathcal{K}$ be a TM that does the following on input n:

- construct $\mathcal{M}_n$
- if $\mathcal{M}_n$ accepts n, reject
- accept otherwise

Since $\mathcal{M}_n$ is total, so is $\mathcal{K}$. Then $B = L(\mathcal{K}) = \{n | n \notin L(\mathcal{M}_n)\}$ is recursive. But, $B \neq L(\mathcal{M}_n)$ for any n contradicting our assumption that every recursive set is accepted by some machine on the list.

(c) Conclude that there is a recursive set that is not context sensitive.

Solution: Fix an encoding for context-sensitve grammars using $\{0, 1\}$. For example, if $G = (N, \Sigma, P, S)$ is a CSG, we could let $|, \to, \{, \}, (,)$ be denoted by $10, 100, ..., 10^6$, respectively; 10^{6+i} denoting i-th symbol of Σ; $10^{6+|\Sigma|+j}$ denoting the j-th symbol of N; using which we can express the entire description (N, Σ, P, S) using $\{0, 1\}$. Let $\mathcal{M}_j$ be the TM deciding the language defined by CSG with encoding j. From part (a), $\mathcal{M}_j$ is total. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be an enumeration of total TMs accepting context-sensitive languages. From part (b), it follows that there is some recursive set B such that $B \neq L(\mathcal{M}_j)$ for any j.

4. Describe unrestricted grammars for

(a) $\{ww \mid w \in \{0, 1\}^*\}$

Solution:

$$S \to RZ \mid \epsilon$$
$$R \to 0RA \mid 1RB \mid 0A \mid 1B$$

$$AZ \to C_0Z \qquad BZ \to C_1Z$$

$$AC_0 \to C_0A \qquad BC_0 \to C_0B$$
$$AC_1 \to C_1A \qquad BC_1 \to C_1B$$

$$0C_0 \to 00 \qquad 1C_0 \to 10$$
$$0C_1 \to 01 \qquad 1C_1 \to 11$$
$$0Z \to 0 \qquad 1Z \to 1$$