Tutorial 3
Recursive Function Theory

Quick Recap of Notation: $f_i(\cdot)$ denotes the function computed by the TM with encoding $i \in \mathbb{N}$. For a TM $\mathcal{M}$, $\langle \mathcal{M} \rangle$ denotes its encoding in $\mathbb{N}$.

1. Prove that there exists $x_0 \in \mathbb{N}$ such that for all y,

$$f_{x_0}(y) = \begin{cases} y^2 & \text{if } y \text{ is even} \\ f_{x_0+1}(y) & \text{otherwise} \end{cases}$$

Solution: There exists a partial recursive function g in two variables such that

$$g(x, y) = \begin{cases} y^2 & \text{if } y \text{ is even} \\ f_{x+1}(y) & \text{otherwise} \end{cases}$$

Let $\mathcal{M}$ be a TM that does the following on input x, y: check if y is even; if so write y^2 on the tape and halt; otherwise simulate the TM with index $x + 1$ on input y. Clearly $\mathcal{M}$ computes $g(x, y)$.

By Kleene's recursion theorem, there exists $x_0 \in \mathbb{N}$ such that

$$f_{x_0}(y) = g(x_0, y) = \begin{cases} y^2 & \text{if } y \text{ is even} \\ f_{x_0+1}(y) & \text{otherwise} \end{cases}$$

for all y.

2. Define any fixed point for the total recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ defined as follows: for $x \in \mathbb{N}$, the TM with description $\sigma(x)$ computes the function

$$f_{\sigma(x)}(y) = \begin{cases} 1 & \text{if } y = 0 \\ f_x(y+1) & \text{otherwise} \end{cases}$$

Describe a fixed point for σ.

Solution: Let $\mathcal{M}$ be a TM that on input $y \in \mathbb{N}$ outputs 1 if $y = 0$ and outputs a constant $a \in \mathbb{N}$ otherwise. Then $\hat{x} = \langle \mathcal{M} \rangle$ is a fixed point for σ, as justified below.

For $y = 0$, we have

$$f_{\hat{x}}(y) = 1 = f_{\sigma(\hat{x})}(y)$$

and otherwise, we have

$$f_{\hat{x}}(y) = a = f_{\hat{x}}(y+1) = f_{\sigma(\hat{x})}(y).$$

3. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any total recursive function. Prove that σ has infinitely many fixed points i.e., there are infinitely many $w \in \mathbb{N}$ such that $f_w(y) = f_{\sigma(w)}(y)$ for all y.

Hint: Recursion theorem ensures there is atleast one fixed point for 'any' total recursive function. If the set of fixed points is finite, does it contradict recursion theorem?

Solution: Suppose there exists a total recursive function σ with finitely many fixed points. Let the set of fixed points be denoted $\mathcal{F}$. Let g be a partial recursive function such that the indices of all TMs computing g are outside $\mathcal{F}$. That is for all TMs $\mathcal{M}$ computing g, $\langle \mathcal{M} \rangle \notin \mathcal{F}$. In other words, for all TMs $\mathcal{M}$ computing g, $f_{\langle \mathcal{M} \rangle} \neq f_w$ for every $w \in \mathcal{F}$.

Let u be an index of some TM computing g. Now, define a function $\tau : \mathbb{N} \to \mathbb{N}$ implicitly so that

$$\tau(x) = \begin{cases} u & \text{if } x \in \mathcal{F} \\ \sigma(x) & \text{otherwise} \end{cases}$$

Observe that τ is total recursive:

- For any $x \in \mathbb{N}$, check whether $x \in \mathcal{F}$. This can be done in finite time since $\mathcal{F}$ is a finite set.
- If $x \in \mathcal{F}$, then set $\tau(x) = u$; otherwise compute $\sigma(x)$ (which is total recursive) and assign the resulting value to $\tau(x)$.

We now argue that τ has no fixed point. If $x \in \mathcal{F}$, then $\tau(x) = u$ and since $f_u \neq f_w$ for every $w \in \mathcal{F}$, we have (in particular) $f_{\tau(x)} \neq f_x$. Now suppose $x \notin \mathcal{F}$. Then $f_{\tau(x)} = f_{\sigma(x)} \neq f_x$. Combining the two, we have $f_{\tau(x)} \neq f_x$ for every $x \in \mathbb{N}$ thus implying that τ has no fixed points. This contradicts the recursion theorem. Hence, any total recursive function must have inifinitely many fixed points.

4. Let $\mathcal{M}_x$ denote the Turing machine with index $x \in \mathbb{N}$. Here's a statement of the recursion theorem, specialised to language recognisers:

   "Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any total recursive function. Then there exists $x_0 \in \mathbb{N}$ such that $L(x_0) = L(\sigma(x_0))$."

   Use it to provide an alternate proof of Rice's theorem (part I).

   Solution: Let P be any non-trivial property of *r.e.* sets. Then there exist encodings of TMs u, v such that $P(L(u)) = \top$ and $P(L(v)) = \bot$. Assume, for the sake of contradiction, that P is decidable. Define a function $\sigma : \mathbb{N} \to \mathbb{N}$ as follows:

   $$\sigma(x) = \begin{cases} u & \text{if } P(L(x)) = \bot \\ v & \text{otherwise} \end{cases}$$

   By our assumption, σ is a total recursive function. The recursion theorem implies that σ has a fixed point x_0 with $L(x_0) = L(\sigma(x_0))$. Now, if $P(L(x_0)) = \top$, we have

   $$\top = P(L(x_0)) = P(L(\sigma(x_0))) = P(L(v)) = \bot,$$

   thus contradicting our assumption that P is decidable. Therefore, P is undecidable.

   Food for thought: Does a generalisation of Rice's theorem hold for partial recursive functions? That is, can you show that any non-trivial property of the set of partial recursive functions is undecidable?