Assignment 1

CS41001: Theory of Computation — Autumn, 2021
Deadline: 2 October 2021, 23:59 — Total Marks: 100

Solve all problems. Stick to notation used in the classes. Write solutions on white paper, scan and then upload a single pdf file. Make sure that the file size does not exceed 20 MB. Any format other than pdf is not acceptable.

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1. Does a generalisation of Rice's theorem hold for partial recursive functions? That is, can you show that any non-trivial property of the set of partial recursive functions is undecidable? $\boxed{10}$

2. Let L be the set of Turing machines $\mathcal{M}$ with input alphabet $\{0,1\}$ such that $\mathcal{M}$ writes the symbol 1 at some point on its tape. Show that L is undecidable. $\boxed{10}$

3. Let $\mathscr{P}$ be a non-trivial property of *r.e.* sets over Σ. Let

$$S_{\mathscr{P}} = \{A \subseteq \Sigma^* \mid A \text{ is } r.e. \text{ and } \mathscr{P}(A) = \top\}$$

   and

$$T_{\mathscr{P}} = \{\mathcal{M} \mid \mathcal{M} \text{ is a TM and } \mathscr{P}(L(\mathcal{M})) = \top\}.$$

   Note that $T_{\mathscr{P}}$ contains descriptions of precisely those Turing machines that accept sets in $S_{\mathscr{P}}$.

   Show that $T_{\mathscr{P}}$ is *r.e.* if and only if the following conditions hold.

   (a) $\mathscr{P}$ is monotone.

   (b) If A is an infinite language in $S_{\mathscr{P}}$, then there is a finite subset of A in $\S_{\mathscr{P}}$.

   (c) The set of finite languages in $S_{\mathscr{P}}$ is enumerable.

   You may break up the proof into several parts, as hinted below.

   - $\neg$(a) implies $T_{\mathscr{P}}$ not *r.e.*
   - $\neg$(b) implies $T_{\mathscr{P}}$ not *r.e.*
   - $T_{\mathscr{P}}$ is *r.e.* implies (c).
   - (a),(b) & (c) imply $T_{\mathscr{P}}$ is *r.e.* .

   $\boxed{20}$

4. Consider a variant of PCP where all strings (in the two sets) are of length exactly 5. Is this variant decidable? Justify. $\boxed{5}$

5. Prove that $\{G : G \text{ is a CFG and } L(G) = L(G)^{\mathbf{R}}\}$ is undecidable. $\boxed{10}$

6. If $\mathcal{M}$ is a TM making atleast 3 moves, then for any x, $\mathsf{VALCOMPS}_{\mathcal{M}} = \cup_{x\in\Sigma^*}\mathsf{VALCOMPS}_{\mathcal{M},x}$ is a CFL if and only if $\mathcal{L}(\mathcal{M})$ is finite. $\boxed{10}$

   Hint: Pumping lemma for CFLs.

7. Show that there exists an *r.e.* list of Turing machines such that every machine on the list accepts a recursive set and every recursive set is represented by some machine on the list. (Note that the machines need not be total.) $\boxed{15}$

8. Let $Th^+(\mathbb{N})$ be the set of true sentences in the language of first-order number theory without the multiplication operator (and the corresponding identity 1). Show that $Th^+(\mathbb{N})$ is decidable. (That is, the problem of determining whether a given sentence involving only addition operator is true is recursive.) $\boxed{20}$