

Control of Linear Vibrations  
Automation and Control Laboratory  
Politecnico di Milano

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Academic Year 2015/2016



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# Chapter 1

## Team introduction

The team is composed by 3 people, all holding a B.Sci. in Engineering not obtained at Politecnico di Milano.

1. *Alessio Russo*: holds a B.Sci. degree in Computer Engineering, enrolled at the M.Sci. degree Automation and Control Engineering at Politecnico di Milano. Because of his high interest in mathematics he prefers to deal with problems using precise models. Currently he's also an ASP student, and his thesis will focus on the implementation of adaptive and robust controllers for the control of unmodelled dynamics of quadrotors, with the use of neural networks and L1 adaptive control techniques.
2. *Gianluca Savaia*:
3. *Alberto Ficicchia*:

## Chapter 2

# Experience introduction

## Chapter 3

# Models

Equations of motion:

$$\begin{aligned} J\ddot{\theta} &= c(t) - c_l(t) - f_m(\dot{\theta}) \\ M\ddot{x} + C\dot{x} + Kx &= F(t) - f_c(\dot{x}) \\ \frac{D}{2}\theta &= x \end{aligned}$$

$f_m$  describes the viscous friction of the motor,  $f_c$  describes the friction of the cart. The gearbox is assumed ideal.

Therefore  $F(t)$  is the transmitted linear force from the motor, thus:

$$F(t)\frac{D}{2} = c_l(t) \Rightarrow F(t) = \frac{2}{D}\left(c(t) - J\ddot{\theta} - f_m(\dot{\theta})\right)$$

In the end we obtain:

$$\left(M + \frac{4}{D^2}J\right)\ddot{x} + C\dot{x} + Kx = \frac{2}{D}c(t) - \frac{2}{D}f_m(\dot{\theta}) - f_c(\dot{x})$$

In case the gearbox is not assumed ideal, we have:

$$J\ddot{\theta} = \begin{cases} c(t) - c_l(t) - f_m(\dot{\theta}) & \text{in contact} \\ c(t) - f_m(\dot{\theta}) & \text{not in contact} \end{cases}$$

And

$$F(t) = \begin{cases} \frac{2}{D}c_l(t) & \text{in contact} \\ 0 & \text{otherwise} \end{cases}$$

### 3.1 Model1 - no BEMF, no disk inertia, no friction cart, no friction motor, no backlash

$$\begin{aligned} M\ddot{x} + C\dot{x} + Kx &= 2\frac{c(t)}{D}, \quad \theta = \frac{2}{D}x \\ \mathcal{L}\{c(t)\} &= 2K_e \frac{1}{2R + 2sL} \mathcal{L}\{v(t)\} \end{aligned}$$

### 3.2 Model2 - no friction cart, no friction motor, no backlash

$$M\ddot{x} + C\dot{x} + Kx = 2\frac{c(t)}{D} - 4\frac{J}{D^2}\ddot{x}, \quad \theta = \frac{2}{D}x$$

$$\mathcal{L}\{c(t)\} = 2K_e \frac{1}{2R + 2sL} (\mathcal{L}\{v(t)\} - 2K_e s \mathcal{L}\{\theta\})$$

### 3.3 Model 3 - no friction motor, backlash

$$M\ddot{x} + C\dot{x} + Kx = 2\frac{c(t)}{D} - 4\frac{J}{D^2}\ddot{x} - f_c(\dot{x}), \quad \theta = \frac{2}{D}x$$

$$\mathcal{L}\{c(t)\} = 2K_e \frac{1}{2R + 2sL} (\mathcal{L}\{v(t)\} - 2K_e s \mathcal{L}\{\theta\})$$

### 3.4 Model 4

$$M\ddot{x} + C\dot{x} + Kx = F(t) - 4\frac{J}{D^2}\ddot{x} - f_c(\dot{x})$$

$$\mathcal{L}\{c(t)\} = 2K_e \frac{1}{2R + 2sL} (\mathcal{L}\{v(t)\} - 2K_e s \mathcal{L}\{\theta\})$$

See introduction for gearbox modelling.

1.  $\mathcal{L}\{\cdot\}$  Laplace transform.
2.  $J$  Disk inertia.
3.  $M$  Cart+load mass
4.  $C$  Spring damping.
5.  $K$  Spring stiffness.
6.  $c(t)$  Torque.
7.  $D$  Disk diameter.
8.  $f_c(t)$  friction applied to the cart.
9.  $f_g(t)$  sliding friction applied to the teeth between the gearbox and the disk.
10.  $f_m$  friction of the motor
11.  $\theta$  angle of the disk.
12.  $v(t)$  tension applied to the motor.
13.  $R, L$  resistance and inductance of the motor
14.  $K_e$  backemf constant.

### 3.5 2 DOF - Model

To derive the equations of motion we can use the Lagrangian approach. Let  $T, V, D$  be the kinetic, potential and dissipated energy. Then:

$$\begin{aligned} T &= \frac{1}{2} \left( M_1 + \frac{4}{D^2} J \right) \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 \\ V &= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 \\ D &= \frac{1}{2} c_1 \dot{x}_1^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2 \end{aligned}$$

Let  $Q$  be the external forces acting on the systems:

$$\begin{aligned} Q_1 &= \frac{2}{D} c(t) - \frac{2}{D} f_m(\dot{\theta}) - f_c(\dot{x}_1) \\ Q_2 &= -f_c(\dot{x}_2) \end{aligned}$$

The equations of motion are given by:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_i} \right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} + \frac{\partial D}{\partial \dot{x}_i} = Q_i$$

$$\begin{aligned} \left( M_1 + \frac{4}{D^2} J \right) \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2) x_1 &= k_2 x_2 + c_2 \dot{x}_2 + \frac{2}{D} (c(t) - f_m(\dot{\theta})) - f_c(\dot{x}_1) \\ M_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 &= k_1 x_1 + c_1 \dot{x}_1 - f_c(\dot{x}_2) \end{aligned}$$

## Chapter 4

# System Identification

The system considered can be easily modelled and identified without the need to use black-box identification to identify the system.

For completeness both *white-box* and *grey-box* identification were used.

First of all the problem of whether to consider a *closed* or *open* loop system is considered. In fact *back-emf* can be seen as a gain acting on the closed loop.

Then, using both *white-box* and *grey-box* identification we identified the main parameters of the system:

1. Resistance and inductance for the motor.
2. Mass, stiffness and damping for the cart and the springs.

Next identification of non-linearities are considered.

### 4.1 Open vs Closed loop identification

In this experiment we had the necessity to choose whether to consider back-emf in the identification process or to completely ignore it.

As a matter of fact, ignoring it would mean to neglect a feedback component. But how much can it affect identification of other parameters?

Consider for example the following 2-nd order system, such as the system considered in the experiment:

$$G(s) = \frac{1}{Ms^2 + cs + k}$$

First consider a feedback loop with a constant gain  $\rho$  on the feedback. Thus the closed loop transfer function is:

$$T(s) = \frac{G(s)}{1 + \rho G(s)} = \frac{1}{Ms^2 + cs + (k + \rho)}$$



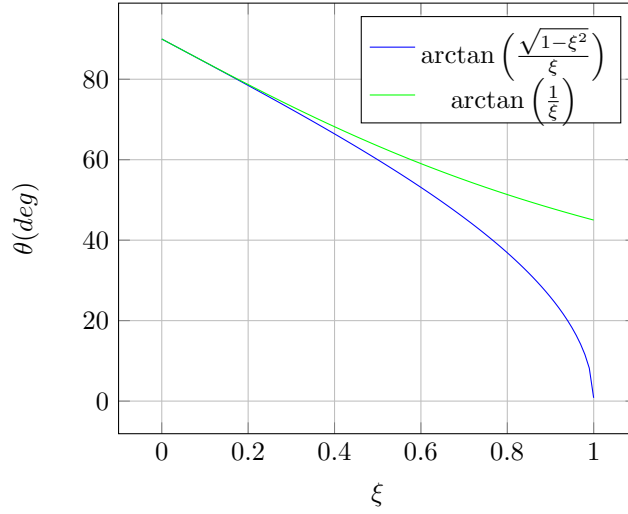


Figure 4.1: Comparison of the approximated value of  $\theta$  with the real one

The effect of  $\rho$  is to change the length of the poles, i.e. their absolute value, since for polynomial with real coefficients the zero-degree coefficient is the product of all roots.

Just compare with  $s^2 + 2\xi\omega_0 s + \omega_0^2$ , it's easy to see that  $\omega_0^2 = \frac{k+\rho}{M}$ .

In our case back-emf acts on the velocity of the cart, so if we have a feedback loop on the position, on the feedback we have  $\gamma s$ , and the closed loop transfer function is:

$$T(s) = \frac{1}{Ms^2 + cs + k + \gamma s}$$

So what is the effect of  $\gamma s$ ? Again, if we compare with  $s^2 + 2\xi\omega_0 s + \omega_0^2$  we have:

$$c + \gamma = 2\xi\omega_0$$

Where  $\xi$  has a strict relationship with the angle formed between the real negative axis and a pole,  $\theta$  :

$$\theta = \arctan\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

So the effect of  $\gamma s$  is to rotate the poles, but to which extent is this effect negligible?

From data we are mainly dealing with values of  $\xi \in (0, 0.5)$ , so we can approximate the value of  $\theta$ :

$$\theta \approx \arctan\left(\frac{1 - \frac{\xi^2}{2}}{\xi}\right) = \arctan\left(\frac{1}{\xi} - \frac{\xi}{2}\right) \approx \arctan\left(\frac{1}{\xi}\right)$$

Notice that in the last step we made use of the fact that  $\frac{1}{\xi} \gg \frac{\xi}{2}$ . Check figure 4.1 to compare the approximation.

Then, how much does  $\theta$  change for a small variation of  $\xi$ ?

$$\frac{d\theta}{d\xi} = -\frac{1}{1+\xi^2} = -1 + \frac{\xi^2}{1+\xi^2}$$

For  $\xi < 0.5$  the change is almost linear, as seen from figure 4.1. Moreover  $\frac{d\theta}{d\xi} \approx -1$  for  $0 < \xi < 0.5$ , so the slope of the curve is almost  $-1$ .

In our case  $\xi = \frac{c+\gamma}{2\omega_0} = \frac{c}{2\omega_0} + \frac{\gamma}{2\omega_0}$ , so the contribution of the backemf is  $\frac{\gamma}{2\omega_0}$ .

From the motor datasheet  $\gamma \ll 1$  and from experiments  $\omega_0$  is always greater than  $10 \frac{rad}{sec}$ , therefore the contribution is small, less than 1 and since the contribution to  $\theta$  is linear with proportion  $\sim -1$  also the change in  $\theta$  is less than 1 degree, therefore backemf can be ignored and open-loop identification can be applied.

## 4.2 White box identification

### 4.2.1 Detached system: cart and springs identification

To accurately identify the mass of the cart and the stiffness/damping of the spring, the motor was detached from the cart, in order to reduce influence of friction due to the pinion and rack.

So we obtain a system like the one considered in figure 4.2.

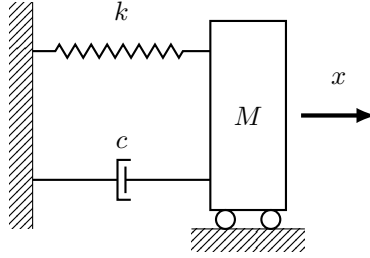


Figure 4.2: Cart detached from the motor diagram.

The differential equation governing this system is given by:

$$M\ddot{x} + c_i\dot{x} + k_ix = f(t)$$

where  $M$  [kg] is the total mass of the system,  $c_i$  [Ns m<sup>-1</sup>] comprehends the damping of the  $i$ -eth spring and the viscous damping of the sliding guide. Finally  $k_i$  [N m<sup>-1</sup>] is the stiffness of the  $i$ -eth spring, and  $f(t)$  represents external forces acting on the system (such as non-linear friction components).

### Experiment description

For each spring we conducted 2 experiments, one without any load and one with a load of 0.986 [kg], each repeated 3 times. To accurately identify the mass of the cart and the stiffness/damping of the spring, the motor was detached from the cart, in order to reduce influence of friction due to the pinion and rack.

For each experiment the cart was released from an initial condition  $x(0) = x_0 \neq 0$  and 0 velocity, such that the force that the spring was exerting on the cart was sufficient enough to make negligible the very small component of the static friction acting on the cart.

Notice that the initial condition differs for each spring since the stiffness is very different for each spring.

If we neglect the external forces acting on the cart, which are negligible since they are small non-linear components, then the system considered is:

$$\begin{cases} M\ddot{x} + c_i\dot{x} + k_ix = 0 \\ x(0) \in [1, 3]\text{cm} \\ \dot{x}(0) = 0 \end{cases} \quad (4.1)$$

Then data regarding the position of the cart is collected, and from that data the pulsation, damping ratio, mass and stiffness are retrieved.

### Experiment analysis

Using 4.1 the response in time can be obtained by using the Laplace transform. Let  $X(s)$  be the Laplace transform of  $x(t)$ , then:

$$mX(s)(s^2 - x(0)s) + cX(s)(s - x(0)) + kX(s) = 0$$

and:

$$X(s) = x(0) \frac{(ms + c)}{ms^2 + cs + k}$$

If we solve in  $X(s)$  and then apply the inverse Laplace transform, we obtain the response in time:

$$x(t) = e^{-\xi\omega_0 t} (A \cos(\omega t) + B \sin(\omega t))$$

where  $\xi = \frac{c}{2\sqrt{Mk}}$ ,  $\omega_0 = \sqrt{\frac{k}{M}}$ ,  $\omega = \omega_0 \sqrt{1 - \xi^2}$ , and  $A, B$  depend on  $x(0), \xi$ .

Since the pulsation is the same for both sinusoidal components we have:

$$x(t) = Ce^{-\xi\omega_0 t} \sin(\omega t + \phi)$$

Where  $C = \sqrt{A^2 + B^2}$ ,  $\phi = \arctan(A/B)$ .

Knowing those equations we are able to extract data from the response in the following way:

- To measure  $\omega$  we can just extract the period  $T$ : the difference in time between the first and second peak is taken, and that difference is the period. Then  $\omega$  is just  $\frac{2\pi}{T}$ . We consider only the first and second peak because at the beginning non-linearities such as static friction are negligible.
- To measure  $\xi$  also the first and second peak are considered. Let  $t_0, t_1$  be the times at which there is the first and second peak. Notice that  $t_0 = 0, t_1 = T$ , and  $x(T) = Ae^{-\xi\omega_0 T}$ . Then, consider:

$$\log\left(\frac{x(0)}{x(T)}\right) = \log(e^{\xi\omega_0 T}) = \xi\omega_0 T = \frac{\xi}{\sqrt{1 - \xi^2}} 2\pi$$

Then

$$\xi = \frac{a}{\sqrt{a^2 + 1}}, \quad a = \frac{1}{2\pi} \log\left(\frac{x(0)}{x(T)}\right)$$

Once  $M, k$  are known we can calculate the damping from  $c = 2\xi\sqrt{Mk}$ . Observe that for  $a \sim 0 \Rightarrow \xi \sim a$ .

Since damping

- To identify each spring and the mass of the cart we made use of the fact that we have two type of experiments for each spring: one without any load, and one with a load of 0.986 kg. We obtain a system of linear equations:

$$\begin{cases} \frac{k_i}{m_c + m_l} = \omega_l^2 \\ \frac{k_i}{m_c} = \omega_{nl}^2 \end{cases}$$

Where  $m_c$  is the mass of the cart,  $m_l$  the mass of the load,  $\omega_l$  the pulsation of the system with the load,  $\omega_{nl}$  the pulsation of the system without the load. It's a system with two unknowns ( $k_i, m_c$ ) and two equations, so we can solve it. We can rewrite it in matrix form:

$$\begin{bmatrix} 1 & -\omega_l^2 \\ 1 & -\omega_{nl}^2 \end{bmatrix} \begin{bmatrix} k_i \\ m_c \end{bmatrix} = \begin{bmatrix} \omega_l^2 m_l \\ 0 \end{bmatrix}$$

and solve for  $(k_i, m_c)$ .

### Experiment results

Since there are 3 springs let's denote the set of springs as  $K = \{k_l, k_m, k_h\}$  where  $l$  stands for low,  $m$  for medium and  $h$  for high. In a similar manner we define the various pulsation: for example  $\omega_{m-nl}$  is the pulsation for the system with spring  $k_m$  and no load.

**Pulsation** In the table below are shown the various mean of the pulsation and their relative variance:

$(\omega_{avg} [\text{rad s}^{-1}], \omega_{std} [\text{rad s}^{-1}])$	$k_h$	$k_m$	$k_l$
<b>with load</b>	(21.2989, 0 )	(14.2800 , 0.0671)	(10.6495 , 0 )
<b>with no load</b>	(34.9066 , 0 )	(23.7101, 0.1792)	(17.6991, 0.1005)

Table 4.1: Pulsation of the cart detached from the motor. Various configuration are shown (with a load of 0.986 [kg] and no load) for the various springs.

It's interesting to note that even if we considered to average all the periods by considering the various peaks of the signal, and not only the first two peaks, we would have obtained the same results. This is an hint of the fact that the principal non-linearity, i.e. static friction, is negligible.

**Cart mass and springs stiffness** By using the mean pulsation the resultant average mass of the cart  $m_c$  is 0.5685 [kg] with standard deviation 0.0141 [kg]. Results also for the springs are shown in table 4.2.1.

$(k_h [\text{N m}^{-1}], m_c [\text{kg}])$	$(k_m [\text{N m}^{-1}], m_c [\text{kg}])$	$(k_l [\text{N m}^{-1}], m_c [\text{kg}])$
(712.5990 , 0.5848)	(315.5074 , 0.5612 )	(175.2819, 0.5595 )

Table 4.2: Identified springs and cart mass

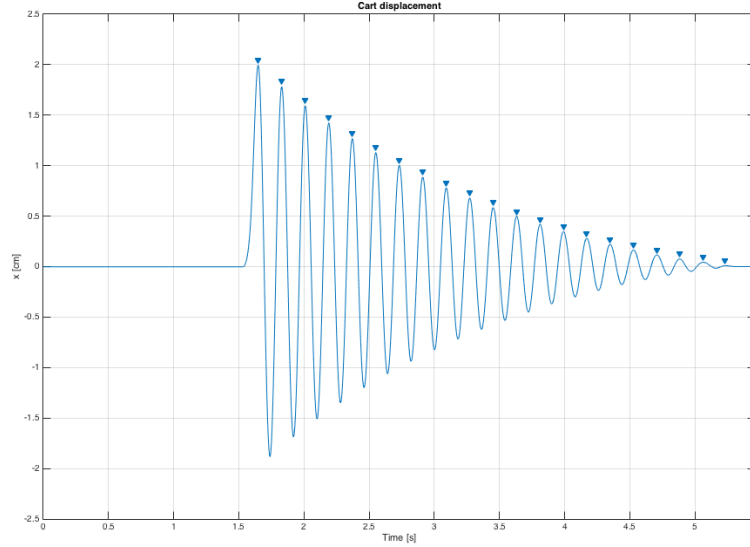


Figure 4.3: Displacement of the cart with spring  $k_h$  and load 0.986 [kg].

**Damping and damping ratio** The mean values for the damping ratio, including their standard deviation, are shown in table 4.2.1 for the various springs, with and without the load.

$(\xi_{avg}, \xi_{std})$	$k_h$	$k_m$	$k_l$
<b>with load</b>	(0.0128, 0.0007)	(0.0238, 0.0018)	(0.0346, 0.0036)
<b>with no load</b>	(0.0179, 0.0025)	(0.0301, 0.0013)	(0.0379, 0.0040)

Table 4.3: Damping ratio. Various configuration are shown (with a load of 0.986 [kg] and no load) for the various springs.

From the values shown in table 4.2.1 it seems that the damping  $C$  is function of the mass, in fact we don't obtain the same damping if we consider the damping ratio with no load or with load. For example consider  $k_h$ : with a load we obtain  $C = 0.0128 \cdot 2 \cdot \sqrt{k_h M} = 0.8520$  [N s m<sup>-1</sup>], without load:  $C = 0.0179 \cdot 2 \cdot \sqrt{k_h m_c} = 0.7206$  [N s m<sup>-1</sup>]. This is most likely an effect due to friction, and the various damping values are shown in table 4.2.1.

$C$ [N s m <sup>-1</sup> ]	$k_h$	$k_m$	$k_l$
<b>with load</b>	0.8520	1.0542	1.1423
<b>with no load</b>	0.7206	0.8063	0.7567

Table 4.4: Damping values. Various configuration are shown (with a load of 0.986 [kg] and no load) for the various springs.

We can therefore linearly characterize the damping value as function of the

mass centered in  $m_c$ , for each spring:

$$C(m) = C_{nl} + \frac{C_l - C_{nl}}{m_l}(m - m_c) = C_{nl} + \alpha(m - m_c)$$

The different values of  $\alpha$ , the difference quotient, are shown in table 4.2.1

	$k_h$	$k_m$	$k_l$
$\frac{C_l - C_{nl}}{m_l} [\text{N s m}^{-1} \text{ kg}^{-1}]$	0.1334	0.2514	0.3911

Table 4.5: Damping difference quotient. Due to friction damping changes for different weights, we can therefore characterize the damping in a linear way with the formula:  $C(m) = C_{nl} + \frac{C_l - C_{nl}}{m_l}(m - m_c) = C_{nl} + \alpha(m - m_c)$ . Values of the difference quotient are shown for the different springs.

#### 4.2.2 Motor identification

#### 4.2.3 Overall system identification

### 4.3 Gray box identification



## 4.4 Non-linearities identification

## Chapter 5

# Control of 1 Degree of Freedom

## Chapter 6

# Control of 2 Degree of Freedom

## Chapter 7

# Control of 3 Degree of Freedom

## Chapter 8

# Conclusions

## Chapter 9

# Appendix