Homework 3 in EL2620 Nonlinear Control

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Problem

The problem consists of the analysis of the pitch angle dynamics of a JAS 39 Gripen, a military aircraft. Specifically, Pilot Induced Oscillations are analysed. This kind of oscillations happen when the pilot panics and tries to give maximum command. The control system of the aircraft is based on gain scheduling of speed and altitude in order to compensate for the non linearities of the aircraft. The case considered is for an aircraft of normal load $M = 0.6$, altitude 1km and speed below the speed of sound. The velocity is important since at higher velocities the aircraft is stable, whilst for lower is unstable. The direction in which the aircraft is pointing, with reference to the earth frame, define the pitch angle θ, and the angle between the aircraft and its center of mass velocity is the angle of attack α.
The aircraft has two controllable surfaces to control θ: the elevator (installed on the tail of the aircraft), and the spoiler (installed on the wings of the aircraft). Their angles are denoted by δ_e, δ_s. The dynamics of the rudder servos are denoted by x_e, x_s and the control inputs are u_e the elevator command, and u_s, the spoiler command. We consider a linear model $\dot{x} = Ax + Bu$ where $x = (\alpha, \dot{\theta}, \theta, \delta_e, \delta_s, x_e, x_s)^T, u = (u_e, u_s)^T$. Notice that B is made such that the control can affect only x_e, x_s.
The aircraft model is shown in figure 1. The aircraft is controlled as follows:

$$u(t) = -Lx(t) + (K_f, K_f)^T u^f_{\text{pilot}}(t)$$

where L is derived using linear quadratic control theory and K_f is chosen such that the steady state gain is correct. The internal loop, based on state feedback, stabilizes $\alpha, \dot{\theta}$. The outer loop denotes the pilot, where he control the pitch angle. The pilot signal is also filtered by a low pass filter, with time constant T_f. The analysis of the problem is broken down into the following six sub-problems:

1. First the dynamical modes of the aircraft are analysed, and classified into flight dynamics mode or rudder dynamics mode. The model is then checked to see if it is stable or not.
2. A first nominal design for the state feedback, called *design1*, is used. The L matrix and the eigenvalues of $A - BL$ analysed. Then we discuss why the pilot can be modelled by a PD-Controller with time delay. The system is then simulated, and the rudder angles analysed.
3. Now the Pilot Induced Oscillations (PIO) are analysed. The pilot is panicking and tries to compensate the error $\theta_{\text{ref}} - \theta$ with maximal command signals. The PD-Controller is replaced by a relay. The relay is analysed and the Describing Function Method is used to predict a possible limit cycle.
4. Various designs, *design2* and *design3* are compared. The results are discussed and analysed using the Describing Function Method.
5. A control strategy that outperform the previous design to reduce the PIO amplitude is given. The performance is analysed and compared with the previous design.
6. (Extra) Rate limitations on the hydraulic servos are now included on the model. Rate limitations are discussed and the PIO are analysed based on the new model. Next a non-linear filter [3], that compensate the rate limiter, is analysed.

Solution - Question 1

The eigenvalues of the systems are given by solving the characteristic polynomial:

$$\det(\lambda I - A) = 0$$

The solution, calculated using Matlab, is the following:
$\lambda_i = (-3.65, 1.066, 0.0027, -20, -20, -125, -125)^T$. Two of the poles, $\lambda_2 = 1.0662, \lambda_3 = 0.0027$ have positive real part thus the model is unstable.

This is because the flight dynamics is unstable and we can make the following reasoning: we expect the internal elevator state and the internal spoiler state to have very fast dynamics, since they correspond to the dynamics of the rudder servos, same for the elevator and spoiler angle. In fact $\lambda_4, \lambda_5, \lambda_6, \lambda_7$ have large value, in an absolute sense, and those eigenvalues should corresponds to the rudder dynamics, therefore $\lambda_1, \lambda_2, \lambda_3$ should correspond to the flight dynamics. Moreover it is normal to assume that the rudder dynamics do not depend on the flight dynamics. To prove all of that, observe the matrix A:

$$A = \begin{bmatrix} -1.3936 & 0.9744 & -0.0019 & -0.5349 & -0.0071 & -1.0562 & 0.4891 \\ 5.6870 & -1.1827 & 0.0002 & -25.9398 & 7.9642 & -23.8594 & -6.2531 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20.0000 & 0 & -312.5000 & 0 \\ 0 & 0 & 0 & 0 & -20.0000 & 0 & -312.5000 \\ 0 & 0 & 0 & 0 & 0 & -125.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -125.0000 \end{bmatrix} \tag{1}$$

and notice that $\delta_e, \delta_s, x_e, x_s$ (last four rows) have dynamics that do no depend on $\alpha, \theta, \dot{\theta}$ (first three row/states), which indicates the rudder does not depend on the pitch or angle of attack, as expected. Then:

$$\text{eig}(A) = \text{eig}(A_1) \cup \text{eig}(A_2)$$

where eig(A) denotes the eigenvalues of the matrix A and $A_1 = A(1{:}3{,}1{:}3), A_2 = (4{:}7{,}4{:}7)$. A_1 represents the first 3 states, the flight dynamics, and A_2 represents the states of the rudder dynamics. A simple check gives that $\text{eig}(A_1) = (-3.6452, 1.0662, 0.0027)$, so the flight dynamics are unstable.
This can also be proven by checking the eigenvectors associated to each eigenvalue, and see that each eigenvalue is mainly associated with one state.

Solution - Question 2

The aircraft is controlled by linear state feedback:

$$u(t) = -Lx(t) + (K_f, K_f)^T u_{\text{pilot}}^f(t)$$

where L is derived using linear quadratic control theory and K_f is chosen such that the steady state gain is correct. The L matrix, using the first design, is the following one:

$$L = \begin{pmatrix} 0.159 & 0.067 & 0 & -0.086 & 0.025 & 0 & 0 \\ -0.206 & -0.087 & 0 & 0.112 & -0.034 & 0 & 0 \end{pmatrix}$$

L acts on $\alpha, \dot{\theta}, \delta_e, \delta_s$. The pilot directly affect x_e, x_s, which act on δ_e, δ_s and ultimately on θ. The eigenvalues of $A - BL$ are given by:

$$\det(\lambda I - (A - BL)) = 0$$
$$\lambda_i = (-3.57, -1.3165, -0.0002, -20.0084, -20.0286, -122.6750, -124.9776)^T$$

We see that the flight dynamics now are stabilized. There is an eigenvalue λ_3 close to the origin, which is very slow and represents the pitch dynamics. In fact, its corresponding eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 0.0015 \\ 0.0002 \\ -1 \\ 0.0002 \\ -0.0003 \\ 0 \\ 0 \end{pmatrix}$$

As seen from $\mathbf{v}_3$, λ_3 has a high dependency on the pitch angle, which proves that λ_3 is mainly associated to the pitch angle. The reason of why it is near the origin is the following: first of all it would be strange to see the pitch with a faster dynamics, since aircrafts, because of their size and weight, are not so agile (in fact before it was $\lambda_3 \approx 0.002$). Moreover, the feedback loop was not intended to stabilise that eigenvalue, since the control of the pitch angle is left to the pilot, in fact the feedback mainly acts on λ_2, which is mainly associated to the pitch rate (can be checked by looking at the corresponding eigenvector).

The pilot may be modelled with a PD-controller with a time delay of $0.3s$ as follows:

$$u_{\text{pilot}} = K_p \frac{1 + T_d s}{1 + \frac{T_d}{N} s} e^{-0.3s} (\theta_{\text{ref}} - \theta)$$

where θ_{ref} is the desired pitch angle, $K_p = 0.2, T_d = 0.5, N = 10$. It's a reasonable model because:

1. It's natural to have some time delay because a human person needs time to react, and there may be some delay due to the servos.

2. The proportional part is also natural since the pilot tries, in normal condition, to give commands proportional to the error.

3. The derivative action acts on the error, so we are looking at the derivative of the error, which is a crude approximation of what will the error look in the future. Thus the pilot tries to control based on what he thinks will happen. if the error changes rapidly the pilot may panic, and that is well represented by the derivative action. (the slope of the error changes continuously, so the derivative action changes sign frequently).

4. The integral action is not included since the pilot does not care about the history of the error, but only about what is the error now.

We now check a simulation of the closed loop system, including the pilot. As reference check figures 2, 3. First, the rudder angles should not be too high in order not to overuse the actuators and avoid saturation effects. Moreover the rudder angles heavily influence the pitch rate, which changes the pitch, and the attack angle. In fact, notice that the pitch rate affects both the attack angle and the pitch in a similar way since $A_{12} \approx 1$ and $A_{32} = 1$. In figure 3 the attack angle at the peak is about 0.4 rad which is about $11.5°$ degrees, below the average critical angle of attack (it's in the range $[15°, 20°]$). Above that critical angle the aircraft would stall [2].

Solution - Question 3

Now we analyse the system when the pilot is in an "emergency situation", and he is replaced by a "relay pilot" that models the fact that the pilot is panicking and he is trying to compensate the error $\theta_{\text{ref}} - \theta$ with maximum command signals. This behaviour may induce sustained oscillations in the system (PIO - Pilot Induced Oscillations). This is usual for system with slow unstable dynamics, such as the aircraft, that cannot respond fast enough to the command input.

To analyse the PIO mode we can make use of the describing function method [1, p. 280] to check for a possible periodic solution. Suppose the error $e(t) = \theta_{\text{ref}} - \theta$ is the input to the non linearity and also $e(t)$ is oscillating with amplitude A and period ω. Let $u(t)$ be the output of the non linearity and suppose the plant is low-pass filter such that $|G(jn\omega)| \ll |G(j\omega)|$ for $n \geq 2$, then we may use the describing function method to model the non linearity, so that it is replaced by a function

$$N(A, \omega) = \frac{b_1(\omega) + ja_1(\omega)}{A}$$

where b_1, a_1 are the first Fourier coefficients of the signal $u(t)$.

This gives as output of the plant an oscillating signal of maximum amplitude:

$$\max|y(t)| \approx |G(j\omega)|\sqrt{a_1^2 + b_1^2} = |G(j\omega)||N(A, \omega)|A$$

The describing function for a relay is:

$$N(A) = \frac{4D}{\pi A}$$

where A is the amplitude of the oscillation and D is the output value, in absolute sense, of the relay system, and in our case $D = 0.2$.
To check for existence of period solution the idea is that we have sustained oscillations with pulsation ω if the open loop-gain is 1 and the phase-lag is $-\pi$ for that pulsation:

$$G(j\omega)N(A) = -1 \Leftrightarrow G(j\omega) = -\frac{1}{N(A)} \qquad (2)$$

Notice that because of that condition, the output in module has maximum amplitude:

$$\max|y(t)| \approx |G(j\omega)||N(A, \omega)|A = A$$

In our case the describing function gives no phase contribution, so we first check for which ω:

$$\arg G(j\omega) = -\pi \qquad (3)$$

To do so we linearised the system and analysed the Bode and Nyquist plots (figures 4, 5).
By checking the plots, the condition (3) is satisfied for $\omega \approx 2.77 \frac{\text{rad}}{\text{sec}}$: the Nyquist curve for that pulsation has imaginary part which is almost 0 and real part -0.402. In fact $|G(j2.77)| = -7.92dB \approx 0.402$.
Then we need to find for which A we may obtain oscillations, by using the condition (2):

$$G(j2.77) \approx -0.402 = -\frac{\pi A}{4D} \Rightarrow A \approx 0.1024$$

The period of the oscillation is given by:

$$\frac{2\pi}{\omega} = \frac{2\pi}{2.77} \approx 2.27 \text{ seconds}$$

By simulating the system we see the aircraft oscillating around the set point 1rad of ± 0.1rad with period ~ 2.34 seconds, which is almost what we obtained with the describing function analysis. So the prediction, using the describing function analysis, gives very good results for this case.

Solution - Question 4

In this question we analyze two different designs. Both of them change L and K_f to a faster design, and the difference between them are the filter time constant T_f, from 0.3 to 0.03 seconds.

The process we are going to follow is the one described in question **3**.

1. We start analyzing *design2*. Bode and Nyquist plot are shown in figures 6 and 7 respectively.

   From this analysis, we get that the condition (3) is satisfied for $\omega \approx 4.41\frac{\text{rad}}{\text{sec}}$. In addition, the Nyquist curve in this point gives real part -0.4212 and imaginary part almost 0. If we apply (2) we obtain the amplitude:

   $$G(j4.41) \approx -0.4212 = -\frac{\pi A}{4D} \Rightarrow A \approx 0.1073$$

   And the period turns out to be 1.428 seconds.

2. Now we focus in *design3*. Both Bode and Nyquist curves can be found in figures 8 and 9.

   Following the same proceeding, the condition (3) is satisfied for this design for $\omega \approx 8.15\frac{\text{rad}}{\text{sec}}$. The Nyquist curve for this point shows, for this ω, a real part of -0.25, and an almost 0 imaginary part. Now, applying condition (2):

   $$G(j8.15) \approx -0.25 = -\frac{\pi A}{4D} \Rightarrow A \approx 0.0637$$

   And the period in this case is 0.771 seconds.

Now we compare all designs under analysis. This comparative is depicted in figure 10. Some conclusions can be obtained from this plots:

1. The concordance between the theoretical and simulated values are very good, being the amplitude 0.106 and 0.071, and the period 1.479 and 0.853 seconds; for *design2* and *design3*, respectively.

2. If we compare the values from *design2* with those obtained under *design1*, we can see that the amplitude is about the same in *design2*, whereas the period is smaller. This could be explained by considering the fact that the system now is faster but the pilot command is still slow: so the oscillations have the same amplitude but now happens more frequently.

3. Regarding *design3*, we can see that both amplitude and period has decreased compared to those obtained under *design1*. Now the faster reaction of the pilot are captured by the low pass filter: this reduces the amplitude of the oscillations, because the pilot command is faster, despite making them more frequent (because of the fact that the system is faster).

Solution - Question 5

In this question we suggest a new design in order to improve the performance of the system. This new design is based on a combination between *design1* and the filter time constant set in *design3*.

Under these conditions we analyze the system, as explained in question **3**. The reference plots for the Bode and Nyquist curves are depicted in figures 11 and 12.

Now we apply condition (3). In this case, we obtain $\omega \approx 5.99\frac{\text{rad}}{\text{sec}}$ (period of 1.0489 seconds). The Nyquist point for this pulsation gives a real part of -0.147, and a imaginary part almost 0. So, condition (2) gives us the amplitude of:

$$G(j5.99) \approx -0.147 = -\frac{\pi A}{4D} \Rightarrow A \approx 0.0374$$

Figure 13 shows this suggested design in contrast with the other designs analyzed along this work. We first check the accuracy of the results obtained. The simulated amplitude is 0.043 radians, and the period is 1.128 seconds.

Comparing all design, we can see that our improved system gives the smallest amplitude of the oscillations. Now, this system is slower and also the pilot signal is faster, which is the key point to understand why now this amplitude is small.

If we look at the period, we can see that it is smaller than the one in *design3*, but greater that the obtained under *design1*.
Overall it's a plausible model since the period is still high, so the pilot has not to do many quick movements.

Another way to reduce the PIO amplitude would be to change the filter on the pilot command in order to change the Nyquist locus.

References

[1] Hassan K Khalil. *Nonlinear systems.* Prentice Hall, Upper Saddle river, 3. edition, 2002. ISBN 0-13-067389-7.

[2] Wikipedia *Angle of attack.* `https://en.wikipedia.org/wiki/Angle_of_attack`

[3] Rundquist, L., K. Stahl-Gunnarsson, and J. Enhagen. *Rate Limiters with Phase Compensation in JAS 39 Gripen.* European Control Conference, 1997. ISBN 978-3-9524269-0-6.

[4] Lars Rundqwist and Karin Stihl-Gunnarsson *Phase compensation of rate limiters in unstable arircraft.* 1996 IEEE International Conference on Control Applications , September 1996.

[5] Henning Schmidt et al. *Exercises and Homework for EL2620 Nonlinear Control.* Automatic Control Dept. at KTH. October 2015.

Solution - Extra Question (6)

The model considered so far does not model the rate limitations imposed by the hydraulic servos. This is an important aspect because when a rate limiter is saturated it means that the control signal sent to the actuator is increasing/decreasing more quickly than what the actuator is able to output: this leads to a reduction of the phase margin, because the output of the rate limiter has a phase shift compared to the input, and it increases the possibility of PIO.
This is easily explained by the fact that the phase shift also affects the pilot, thus creating a time delay effect, happening when the pilot starts to command the aircraft with large and rapid inputs, like in the relay model. In industry, software rate limiters are applied to the control commands in order to prevents the hydraulic servos from rate limiting [3, p. 3944], and those software rate limiters are used in conjunction with algorithms or filters to provide phase compensation.
For an input signal $u(t)$ the rate limiting effect happens if:

$$\lim_{\delta\to 0}\left|\frac{u(t)-y(t-\delta)}{\delta}\right| > r \tag{4}$$

where r[unit/s] is the rate limit value. When the rate limiting effect starts to apply, we have $|\dot{y}(t)| = |r|$.
Condition (4) for a rate limiting effect to happen may be simplified to $\frac{r}{a\omega} < 1$ [3, p. 3945], for sinusoidal input $u(t) = a\sin(\omega t), a > 0$, In fact, suppose $y(t) = u(t)$, then the rate limiting effect will start when condition 4 is satisfied:

$$\lim_{\delta\to 0}\left|\frac{u(t)-y(t-\delta)}{\delta}\right| = |\dot{u}(t)| = |a\omega\cos(\omega t)| > r$$
$$\Rightarrow \frac{r}{a\omega} < |\cos(\omega t)| \Rightarrow \frac{r}{a\omega} < 1$$

Therefore the amplitude and pulsation of a signal are very important, they may determine the activation of the rate limiting effect. Notice that if rate limiting started at t_0, then:

$$y(t) = \text{sgn}(\dot{u}(t_0))r(t-t_0) + u(t_0)$$

Rate limiters were introduced as in article [3, p. 3947] in our analysis, and some values of r were tested in order to analyse what happens to the limit cycle. The model changed in the following way (figure 14): rate limiters are now present on both the pilot command and on the sum of the pilot command and feedback command, as in [3, p. 3947]. The system was simulated using the three designs and using a rate limiting value of $r = 0.5$ first (figure 15) and $r = 1$ later (figure 16). Obviously, for $r \to \infty$ the system output will converge to the output of the system that has no rate limiters. For $r = 0.5$ the amplitude of the oscillations for *design2* and *design3* increases, this is due to the fact that $K_f > 1$ and the gain of the input command is increased, thus activating the rate limiting effect. Instead, the output for *design1* for both $r = 0.5$ and $r = 1$ does not change: this is easily explained since $K_f < 1$, thus lowering the gain of the signal. Moreover the internal feedback loop does not influence so much since the rate of that loop is not very high, thus there is no rate limitation effect. For $r \to 0$ obviously we have instability.

A simple method to avoid the rate limiting effect is to either reduce the gain or use an additional low pass filter on the pilot input, in order to affect $u(t)$. Although being a solution, it reduces the aircraft response [3, p. 3944]. Other ideas may be to reduce gain as function of frequency or as function of the input rate, use feedforward or logical expressions [4, p. 20]. Early methods also used the idea of differentiating, use saturation and integrate: this permits to have $\text{sgn}(\dot{u}) = \text{sgn}(\dot{y})$, thus perfect phase compensation. But after rate limiting $y \neq u$, since saturation removed some information [4, p. 20].

In [3] the idea of using a feedback loop to phase compensate the rate limiter is used, and the inspiration came from anti-windup methods. The error $e(t) = y(t) - u(t)$ is fed back into a filter, in order to stabilise the controller. There are three main variants for this method, and they were all tested using as input $u(t) = 5\sin(t)$ for $t \leq 10$, $u(t) = 0$ for $t > 10$. The results can be seen in figure 17, where the rate limiters have $r = 1$.
The three main variants are:

1. *Feedback with integrator*: the error is fed back into an integrator. This provide good phase compensation, especially for low frequency signals, but where rate limiting ceases there may be a bias in the output because of the integration [3, p. 3945]. An example of the bias effect can be seen in figure 17 for $t > 10$.

2. *Feedback with low pass filter*: the error is fed back into a low pass filter. In this way when rate limiting ceases there is no bias. When rate limiting starts $e(t) < 0$, thus reducing the input signal. Despite not giving any bias, it gives less phase compensation compared to an integrator [3, p. 3945]. Another drawback is that for high frequencies the circuit almost blocks.

3. *Feedback with low pass filter and bypass*: the idea is to use the feedback with low pass filter circuit but using only the low frequency components of the input signal. Moreover, we don't need the high frequency components to be phase compensated. So, first the signal $u(t)$ is filtered by a low pass filter, obtaining the signal $u_l(t)$. Next, the high frequency components are obtained by doing $u(t) - u_l(t) = u_h(t)$. $u_l(t)$ is limited by the first rate limiter and phase compensated. Then the output signal is added to $u_h(t)$ and fed into a second rate limiter. At this point only the component $u_h(t)$ is limited (and uncompensated). Note that both rate limiter have the same settings and are software rate limiters usually [3, p. 3945]. Compared to the previous filter, this one has better phase compensation, because high frequency components do not interfere with the phase compensation of low frequency components. The parameters for this circuit are tuned based on the bandwidth of the system and of the input signal. Moreover, because of the low pass filter, the output of the compensated rate limiter has lower amplitude compared to a normal rate limiter. Check figure 18 for a scheme of the filter.

Stability properties in closed loop depend on the presence of high frequency disturbances. If not presents, the closed loop system with compensated rate limiters has better stability properties. With disturbances any of the filters can be the better one or the worst one, it depends on the rest of the loop [3, p. 3946]. Moreover, stability properties can always be predicted by using the Describing Function method [3, p. 3946].

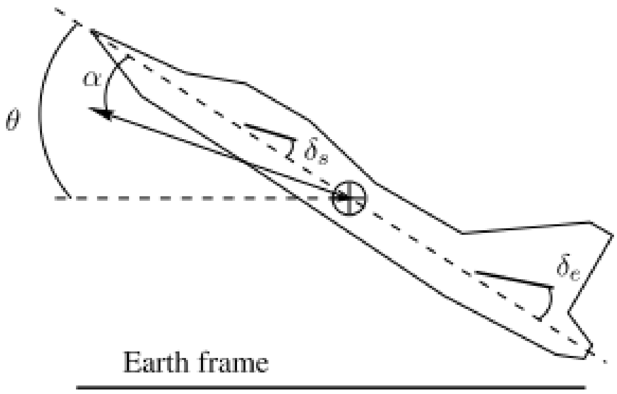

Figure 1: Aircraft model considered, obtained from [5, p. 15].

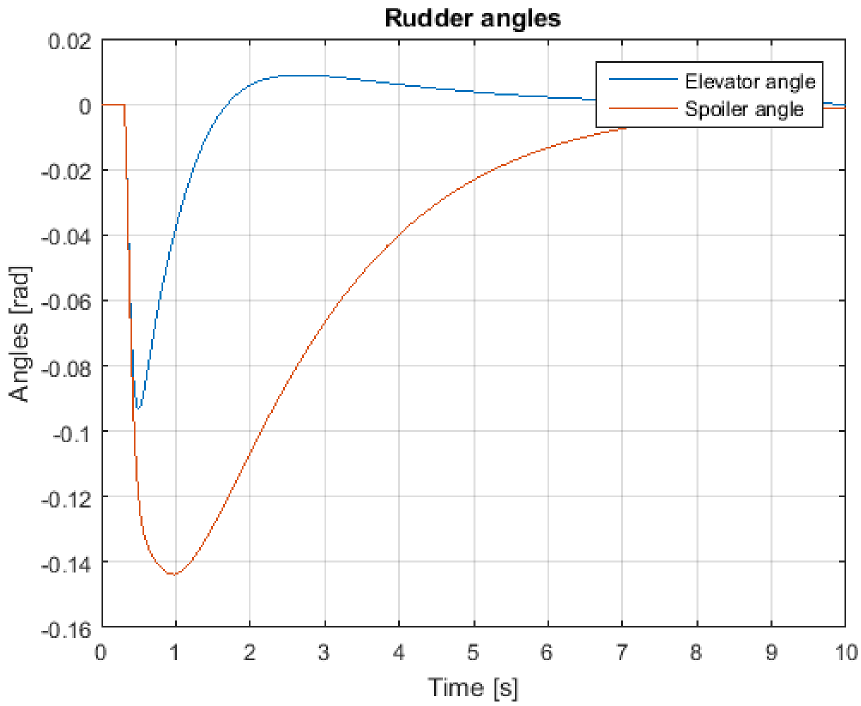

Figure 2: Plot of the rudder angles by simulating the system using the PD-Controller as model for the pilot and *design1* settings.

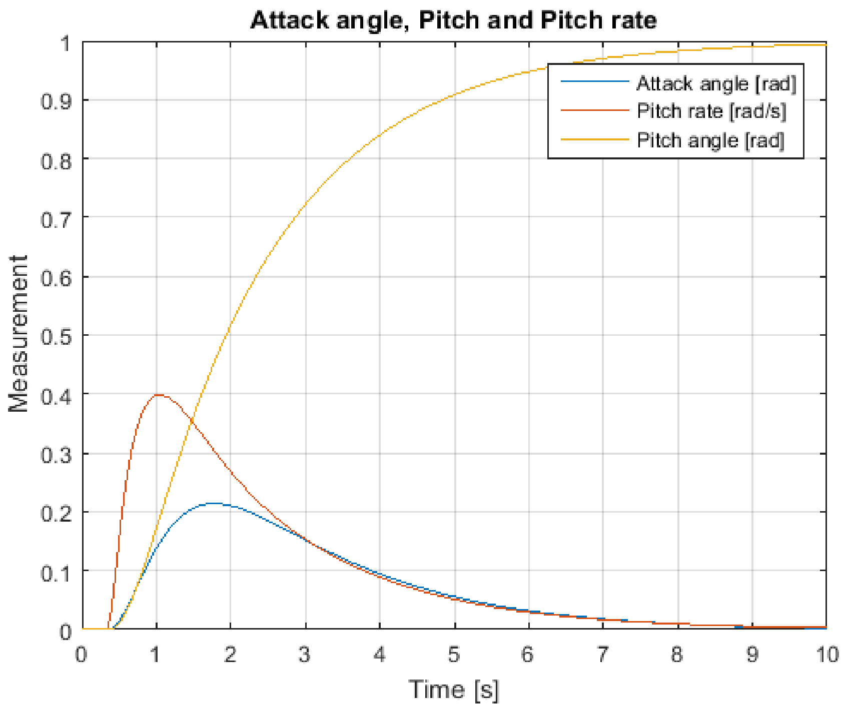

Figure 3: Plot of the angle of attack, pitch and pitch rate by simulating the system using the PD-Controller as model for the pilot and *design1* settings.

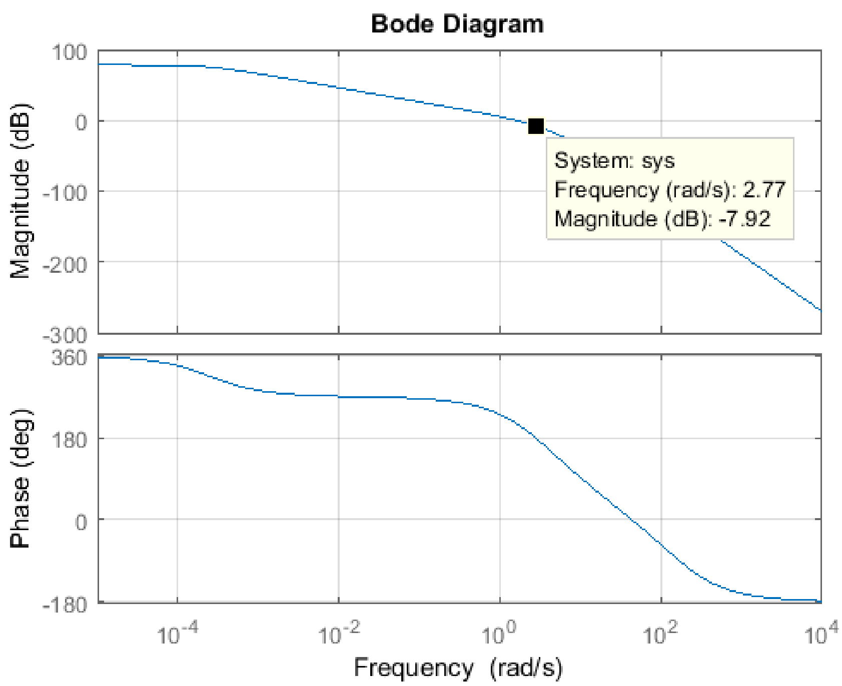

Figure 4: Bode plot of the linearised open loop using the relay model for the pilot and *design1* settings.

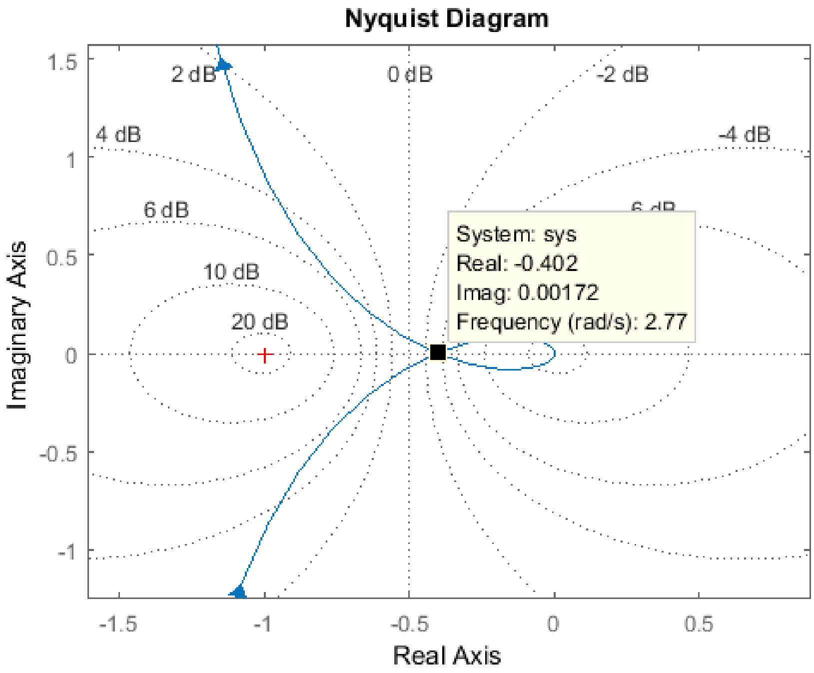

Figure 5: Nyquist plot of the linearised open loop using the relay model for the pilot and *design1* settings.

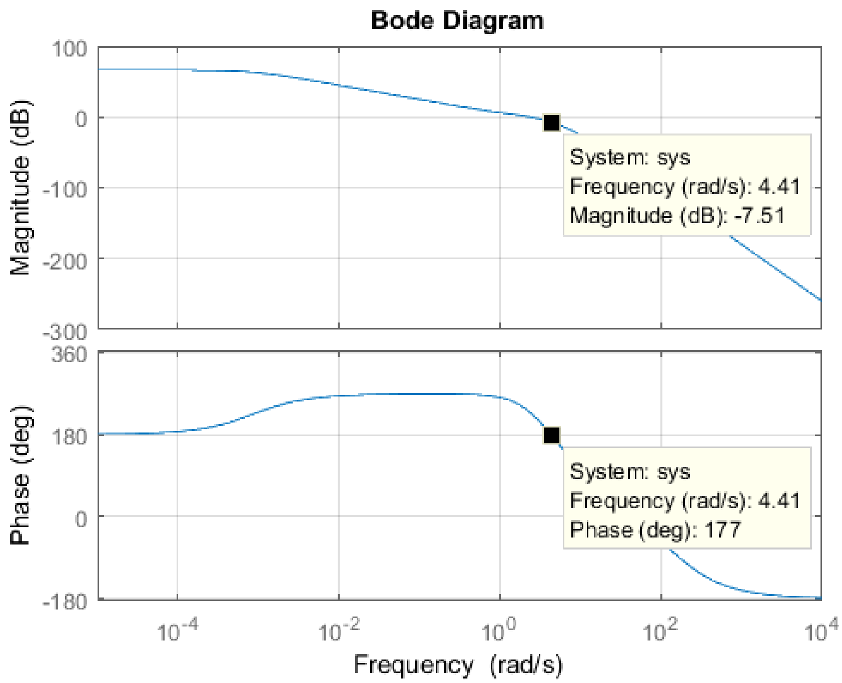

Figure 6: Bode plot of the linearised open loop using the relay model for the pilot and *design2* settings.

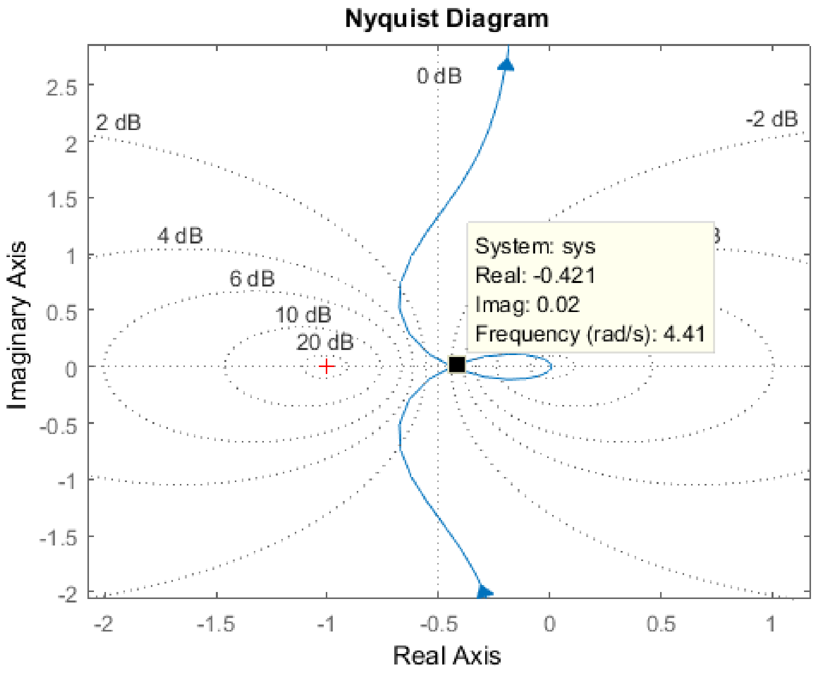

Figure 7: Nyquist plot of the linearised open loop using the relay model for the pilot and *design2* settings.

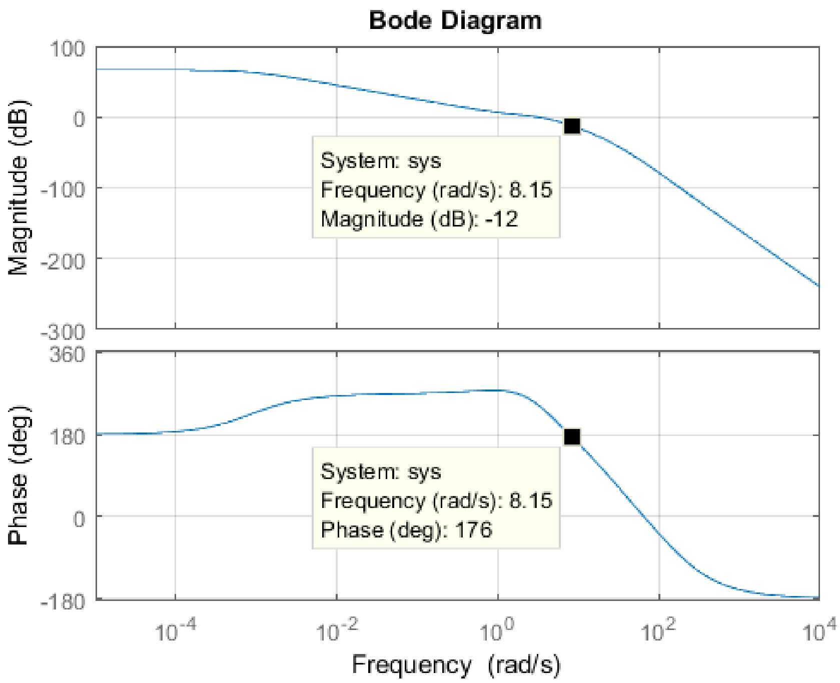

Figure 8: Bode plot of the linearised open loop using the relay model for the pilot and *design3* settings.

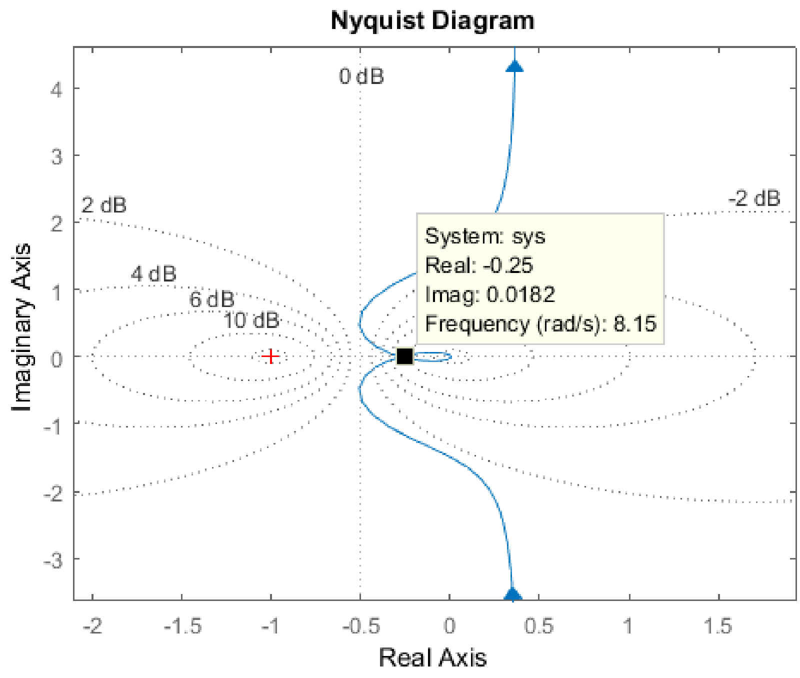

Figure 9: Nyquist plot of the linearised open loop using the relay model for the pilot and *design3* settings.

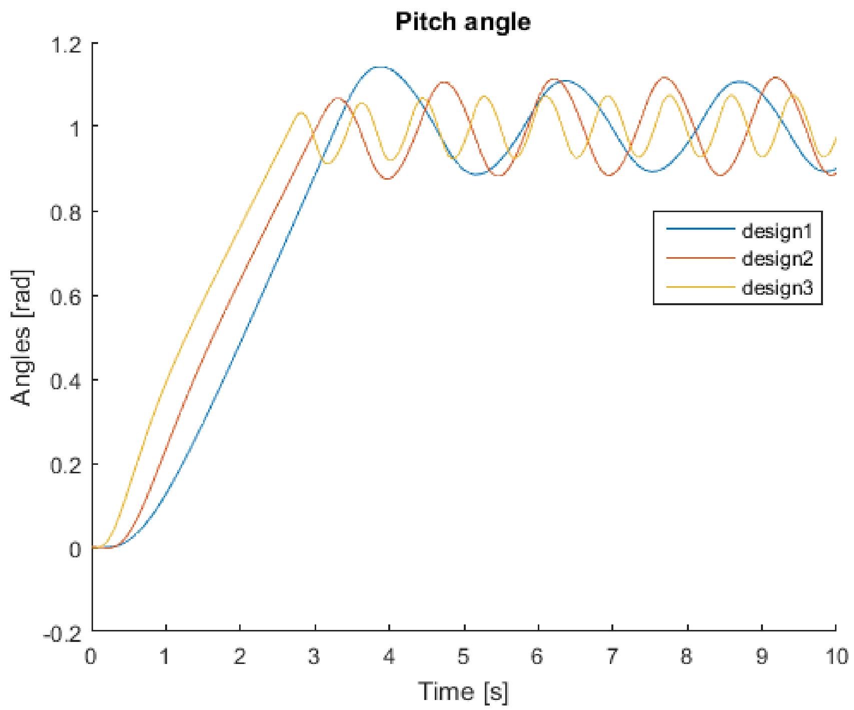

Figure 10: Plots of the pitch angle by simulating the system using the relay model for the pilot and *design1*, *design2* and *design3* settings.

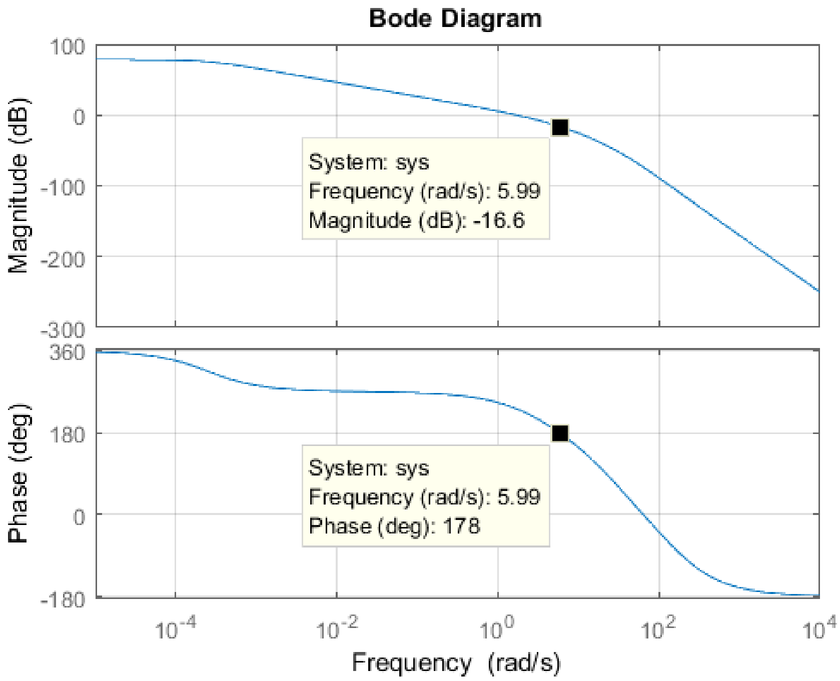

Figure 11: Bode plot of the linearised open loop using the relay model for the pilot and suggested design.

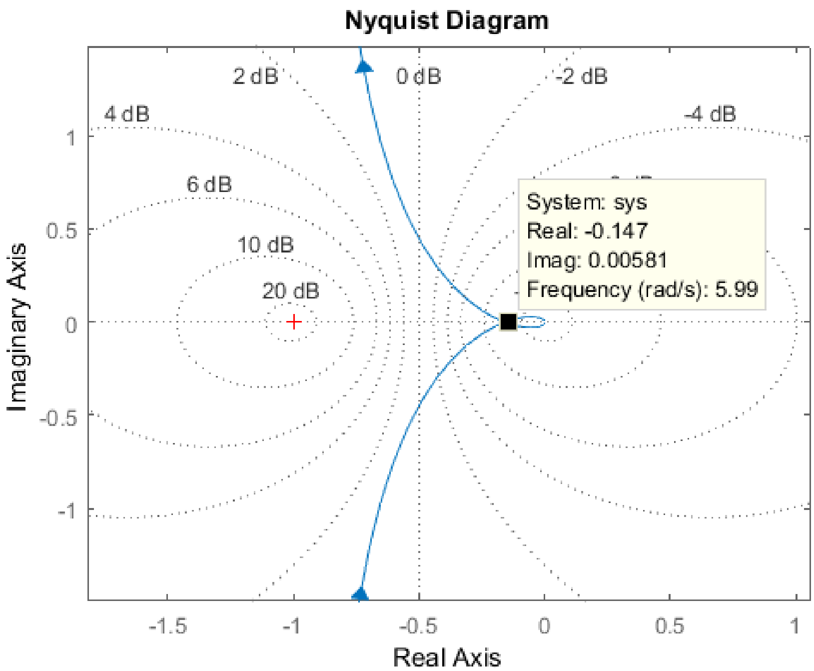

Figure 12: Nyquist plot of the linearised open loop using the relay model for the pilot suggested design.

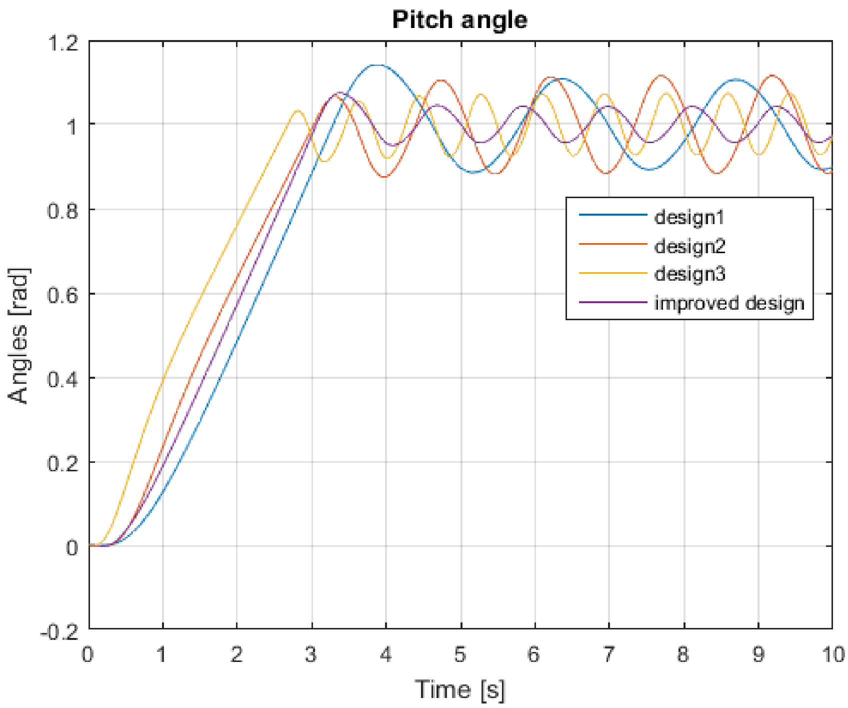

Figure 13: Plots of the pitch angle by simulating the system using the relay model for the pilot and *design1*, *design2* and *design3* settings, comparing them to the improved design suggested in question **5**.

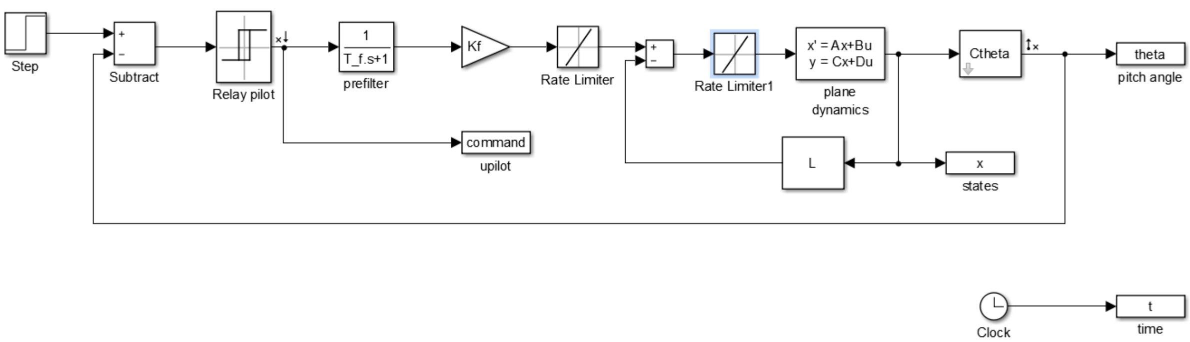

Figure 14: Simulink model of the aircraft including the rate limiters applied on both the pilot command and on the sum of the pilot command and feedback command.

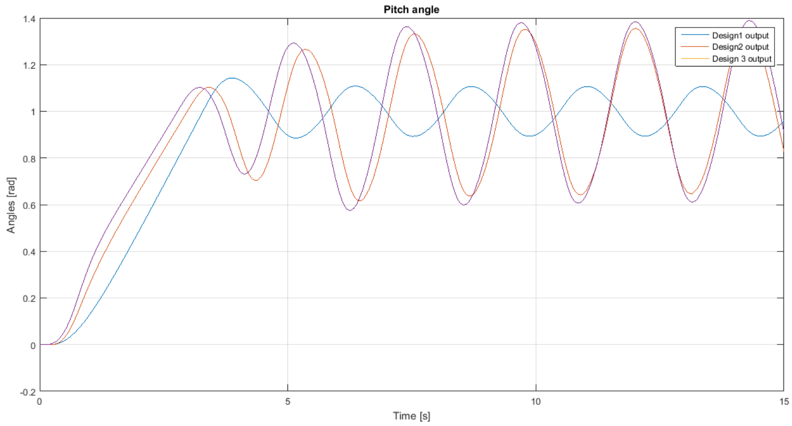

Figure 15: Pitch angle plot for the various design using a rate limit $r = 0.5$.

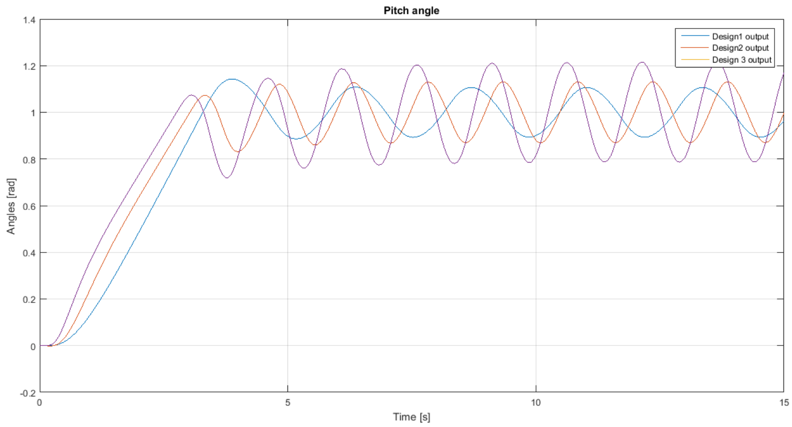

Figure 16: Pitch angle plot for the various design using a rate limit $r = 1$.

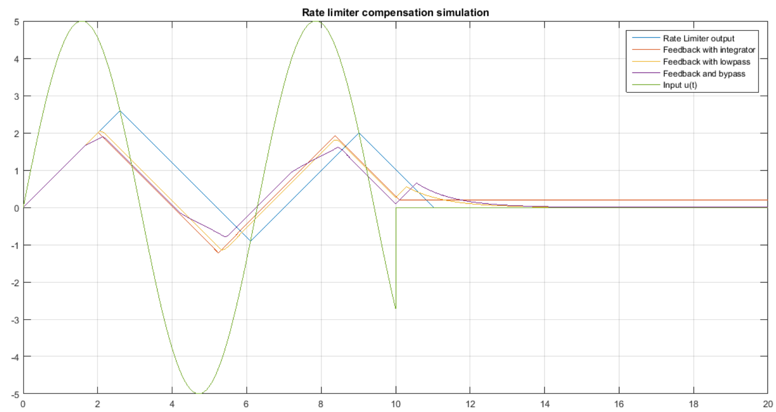

Figure 17: Simulation of the various rate limiter filters for an input signal $u(t) = 5\sin(t)$ for $t \leq 10$, $u(t) = 0$ for $t > 10$. The rate limiters have rate limit value $r = 1$.

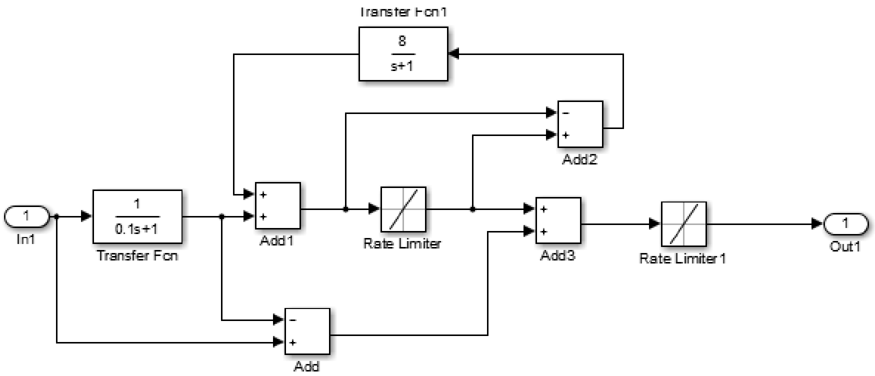

Figure 18: Rate limiter filter with feedback and bypass [3].