Homework 2 in EL2620 Nonlinear Control

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Problem

The problem consists on the analysis of back-lash. We have to consider the drive line in a crane, whose angular-control diagram is formed by a P controller, an electric motor and the back-lash. This model is depicted in figure 1.

Now we focus on the back-lash. It can be modeled by a relation between the angles input θ_{in} and output θ_{out}, or by the angular velocities. In this case, the model turns out to be:

$$\dot{\theta}_{\text{out}} = \begin{cases} \dot{\theta}_{\text{in}}, & |\,\theta_{\text{in}} - \theta_{\text{out}}\,| = \Delta \wedge \dot{\theta}_{\text{in}}(\theta_{\text{in}} - \theta_{\text{out}}) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The analysis of the back-lash model and its influence is broken down into the following five subproblems:

1. Starting from the angular-velocity back-lash model in open loop, draw θ_{in}, θ_{out}, $\dot{\theta}_{\text{in}}$ and $\dot{\theta}_{\text{out}}$ when the input is ($\theta_{\text{in}}(0) = 0$, $\theta_{\text{out}}(0) = -\Delta$):

$$\dot{\theta}_{\text{in}}(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

2. For the same model, prove that its gain is equal to 1, is passive and can be bounded by a sector $[k_1, k_2] = [0, 1]$.

3. Now we suppose that we have the angular-velocity model in a feedback loop, as represented in figure 2. Under this circumstances, compute:

   (a) Considering the gain derived in **2**, what are the constraints that must be imposed to $G(s)$ to have a BIBO stable feedback system, by using Small Gain Theorem?

   (b) What will be the constraints to $G(s)$ if the passivity theorem is used for having a BIBO stable closed-loop system?

   (c) What will be the constraints to $G(s)$ if the Circle Criterion is used for having a BIBO stable closed-loop system?

4. In the closed-loop system used in **3**, and showed in figure 2, now we suppose that the transfer function is:

$$G(s) = \frac{K}{s(1 + sT)} \tag{3}$$

   In this case, BIBO stability cannot be concluded by using the Small Gain Theorem or the Passivity Theorem. Explain why. For which $K > 0$ the Circle Criterion ensures BIBO stability, given $T = 1$?

5. Simulate the crane system in Simulink.

   (a) Compare the simulated model with the block diagrams presented in this work. What is the disturbance introduced in the simulated version?

   (b) Now set $K = 0.25$. Is the closed-loop system BIBO stable from d_{in} to ($\dot{\theta}_{\text{in}}$, $\dot{\theta}_{\text{out}}$)? Why cannot be BIBO stable to (θ_{in}, θ_{out})?

   (c) Finally consider other values for K. Compare with what was found in **4**. What happens if there is no back-lash?

Solution - Question 1

A sketch of the signals θ_{in}, θ_{out}, $\dot{\theta}_{\text{in}}$ and $\dot{\theta}_{\text{out}}$ when the input is the one described in equation (2) is presented in figure 3. We analyse the effect of the back-lash by looking at the signals θ_{out} and $\dot{\theta}_{\text{out}}$.

Because of the initial condition $\theta_{\text{out}}(0) = -\Delta$, in $t = 0$ there is contact. In addition, the input starts moving in the positive sense, which translates this movement directly from the input to the output.

Once the input switches its sense, we have to wait 2Δ, which is the time employed to move from the contact position to the other one when the velocity is 1. Furthermore, if $2\Delta > 1$, there will not be any movement, since the contact will not be reached before the movement stops.

Considering that this condition is not fulfilled, and thus there is movement in this part, the final position will be $1 - \Delta - (1 - 2\Delta) = \Delta$, which corresponds to the initial condition minus the time that there is movement.

Solution - Question 2

1. The gain of a system S, with input $u(t)$ and output $y(t)$, by definition [2, p. 209] is given by:

$$\gamma(S) = \sup_{u \in L_2} \frac{\|y(t)\|_2}{\|u(t)\|_2}$$

In our case $y(t) = \dot{\theta}_{\text{out}} = f(u(t))$, and $u(t) = \dot{\theta}_{\text{in}}$. $f(u(t))$ is described by equation (1), which means that $y = u$ when in contact, otherwise $y = 0$. Thus:

$$\|y\|_2^2 = \int_0^\infty f(u(t))^2 dt \leq \int_0^\infty u(t)^2 dt = \|u\|_2^2$$

. When in contact we have the equality, so:

$$\gamma(BL) = \sup_{u \in L_2} \frac{\|u(t)\|_2}{\|u(t)\|_2} = 1$$

2. A system S is said to be passive [2, p. 227], if the input $u(t)$ and the output $y(t)$ satisfy:

$$\langle y, u \rangle_T = \int_0^T y^T(t) u(t) dt \geq 0 \quad \forall T > 0$$

Which means the phase of the system is between $[-\frac{\pi}{2}, \frac{\pi}{2}]$, since if we interpret u, y as vectors in the Euclidean space, from the euclidean scalar product we obtain:

$$\cos(\phi) = \frac{\langle y, u \rangle_T}{|y|_T |u|_T}$$

When in contact the system has phase 0, since $y = u$, and when not in contact $y \perp u$, which means that the output is orthogonal to the input, i.e. $|\phi| = \frac{\pi}{2}$. We can prove this using the previous formula. If in contact:

$$\langle y, u \rangle_T = \int_0^T y^T(t) y(t) dt = \|y\|_2^2 \geq 0 \quad \forall T > 0$$

If not in contact:

$$\langle y, u \rangle_T = \int_0^T y^T(t) \cdot 0 dt = 0 \quad \forall T$$

Thus the system is passive

3. A system $f(u)$ is said to be bound by a sector $[k_1, k_2], k_1, k_2 \in \mathbb{R}$ [2, p. 264] if $\forall u \neq 0$ and with $f(0) = 0$ we have:

$$k_1 \leq \frac{f(u)}{u} \leq k_2$$

In the case of a back-lash system we have $\dot{\theta}_{\text{out}} = f(u(t)) = f(\dot{\theta}_{\text{in}})$ which is equal to $\dot{\theta}_{\text{in}}$ if in contact, otherwise $\dot{\theta}_{\text{out}} = 0$, thus the condition $f(0) = 0$ is satisfied. When in contact we have:

$$k_1 \leq 1 \leq k_2$$

When not in contact:

$$k_1 \leq 0 \leq k_2$$

We observe that $k_1 = 0, k_2 = 1$ satisfy both the cases, thus the system can be bound by a sector $[0, 1]$.

Solution - Question 3

Consider the system in figure 2.

1. The gain of the system G is given by: [2, p. 209]

$$\gamma(G) = \sup_{\omega \in (0, \infty)} |G(i\omega)|$$

Suppose G to be BIBO stable, then the Small Gain Theorem [2, p. 217] states that the feedback system is BIBO stable if:

$$\gamma(G)\gamma(\text{BL}) < 1$$

Since $\gamma(\text{BL}) = 1$, then we need G to have the following constraint in order to have a BIBO stable feedback:

$$\gamma(G) < 1$$

2. The Passivity Theorem [2, p. 245] states that a closed loop system, with two subsystems, is BIBO stable if at least one of the system is strictly passive and the other one is passive. In this way we have the total phase of the system to be less than π. In the case considered, the back-lash system is passive, thus we need G to be strictly passive in order to have a BIBO stable closed loop system.

3. In case of the Circle Criterion [2, p. 265], we should assume G to have no poles in the right half plane and find a sector of the nonlinearity, which is $[k_1 = 0, k_2 = 1]$ as found in section **2**. Then, to have BIBO stability for

the closed loop system we need $G(i\omega)$ to not encircle nor intersect the circle defined by the points

$$\left[-\frac{1}{k_1}, -\frac{1}{k_2}\right]$$

So, in this case the Nyquist curve of $G(s)$ should not intersect the line $\text{Re} = -1$.

Solution - Question 4

If BIBO stability could not be ensured, one of the possible causes is that the new linear system is not stable.

As it is stated in [3, p. 99], a linear and time-invariant system is stable if the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = C < \infty$$

Since we are given the Laplace-domain system, we have to compute the inverse transform:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} =$$
$$= \mathcal{L}^{-1}\left\{\frac{K}{s(1+sT)}\right\} = K\left(1 - e^{-t/T}\right) u(t)$$

Where $u(t)$ denotes the Heaviside step function. Now we apply the theorem:

$$\int_{-\infty}^{\infty} |g(\tau)| d\tau = \int_{0}^{\infty} \left|K\left(1 - e^{-t/T}\right)\right| d\tau =$$
$$= \int_0^{\infty} K d\tau + \int_0^{\infty} K e^{-t/T} d\tau = K(T + \lim_{\tau\to\infty} \tau) \to \infty$$

So, the system $G(s)$ is not stable. This implies, for the Small Gain Theorem and Passivity Theorem stated in question **3**:

1. The Small Gain Theorem requires that both systems must be stable. Since one of them is not stable, we cannot conclude BIBO stability using this theorem.

2. For the Passivity Theorem, we required that $G(s)$ must be strictly passive. Nevertheless, we can only apply the definition of passivity if and only if the system is stable. Since this requirement is not fulfilled, $G(s)$ cannot be passive nor strictly passive. Under these circumstances, Passivity theorem does not give us any conclusion.

Now we are going to analyse what happens if we consider the Circle Criterion. As it was showed in question **3**, the requisite imposed to $G(s)$ is to have no poles in the right half plane. Considering $T = 1$, the system in equation (3) turns out to be:

$$G = \frac{K}{s(1+s)}$$

Whose poles are placed in $s = \{0, -1\}$. In addition, the other condition that leads to BIBO stability is to not intersect the line $\text{Re} = -1$. This condition, for $G(s)$, is equivalent to making sure that:

$$\text{Re } G(j\omega) = \text{Re } \frac{K}{-\omega^2 + j\omega} = -\frac{K\omega^2}{\omega^4 + \omega^2} > -1$$

For $\omega = 0$ the function is not defined, thus first we check the condition $\forall\omega \neq 0$, then we check the condition for $\omega \to 0$:

$$\frac{K}{\omega^2 + 1} < 1 \quad \forall\omega \neq 0$$
$$K - 1 < \omega^2 \quad \forall\omega \neq 0$$

Which holds $\forall\omega \neq 0$ as long as $K - 1 \leq 0 \Rightarrow K \in (0, 1]$, since $K > 0$. Now we check the limit for $\omega \to 0$:

$$\lim_{\omega\to 0} -\frac{K\omega^2}{\omega^4 + \omega^2} = \lim_{\omega\to 0} -\frac{K}{\omega^2 + 1} = -K > -1$$

So, the closed-loop system is BIBO stable for $K \in (0, 1)$, according to the Circle Criterion.

Solution - Question 5

1. The simulation of the crane control system in Simulink is equivalent to the feedback loop presented in figure 2, with some considerations that are going to be discussed in this question.

   Fist of all, parameter $\Delta = 0.5$ in the simulation. This parameter is set in the back-lash model, through the field "Deadband width". Here it is defined the whole margin for the back-lash, that is, from $-\Delta$ to Δ. Since the parameter is equal to 1, we obtain the value for Δ exposed in the beginning. Furthermore, the maximum difference value between θ_{in} and θ_{out} seen in the simulations is equally 0.5.

2. The disturbance that is added in the simulation is now described. There are two step modules which generates an unique pulse, which starts at $t = 1$ and ends at $t = 2$, with amplitude 1.

   Afterwards, this signal goes into an integrator, being this integrated signal the disturbance added. Due to this module, the area below this signal is not bounded, which violates the first assumption of what BIBO stability represents. Since the input signal is not bounded, nothing can be ensured about the boundedness of the output signal.

Let's look at figure 4. As we can see, for the bounded and unbounded disturbances, we obtain similar outputs. For the bounded disturbance, we can conclude that the system is BIBO stable to $(\dot{\theta}_{\text{in}}, \dot{\theta}_{\text{out}})$. The system cannot be BIBO stable to $(\theta_{\text{in}}, \theta_{\text{out}})$ because this signals do not converge to zero.

3. As it was proven in **4**, the whole system is only guaranteed to be BIBO stable if $0 < K < 1$. Nevertheless, regarding the Circle Criterion, this is a sufficient condition, which implies that higher values for K could also result in stable systems.

   In fact, if we simulate for $K = 2$ the system is stable yet. Nevertheless, for $K = 4$, the system is unstable. Both situations are compared in figure 5.

   Finally we are going to consider the case when the back-lash is neglected, described in figure 6. Under these circumstances, $\theta_{\text{in}} = \theta_{\text{out}}$ and $\dot{\theta}_{\text{in}} = \dot{\theta}_{\text{out}}$, due to the fact that the nonlinearity have been removed. So, the parameter K can be increased, when a higher K means a faster response.

References

[1] Henning Schmidt et al. *Exercises and Homework for EL2620 Nonlinear Control.* Automatic Control Dept. at KTH. October 2015.

[2] Hassan K Khalil. *Nonlinear systems.* Prentice Hall, Upper Saddle river, 3. edition, 2002. ISBN 0-13-067389-7.

[3] Alan V Oppenheim. *Signals and systems.* Prentice Hall, 2. edition, 1996. ISBN 0-13-814757-4 .

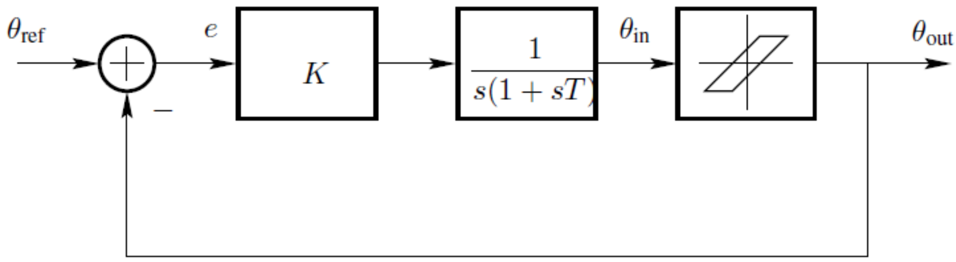

Figure 1: Block diagram which describes the control of the angular position in a crane, obtained from [1, p. 13]. The first block represents the P controller ($K > 0$), the second one the electric motor ($T > 0$), and the third one the back-lash, the nonlinearity under study in this work.

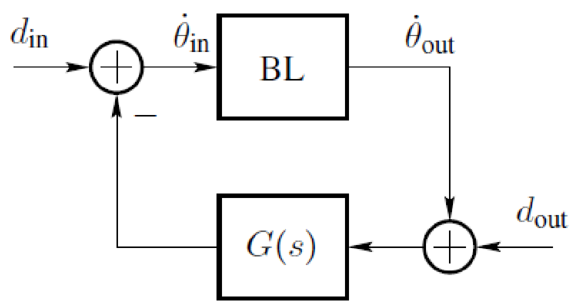

Figure 2: Block diagram which describes a feedback loop in which the angular-velocity back-lash model and an arbitrary linear system $G(s)$ are involved, obtained from [1, p. 14]. Signals d_{in} and d_{out} represents disturbances.

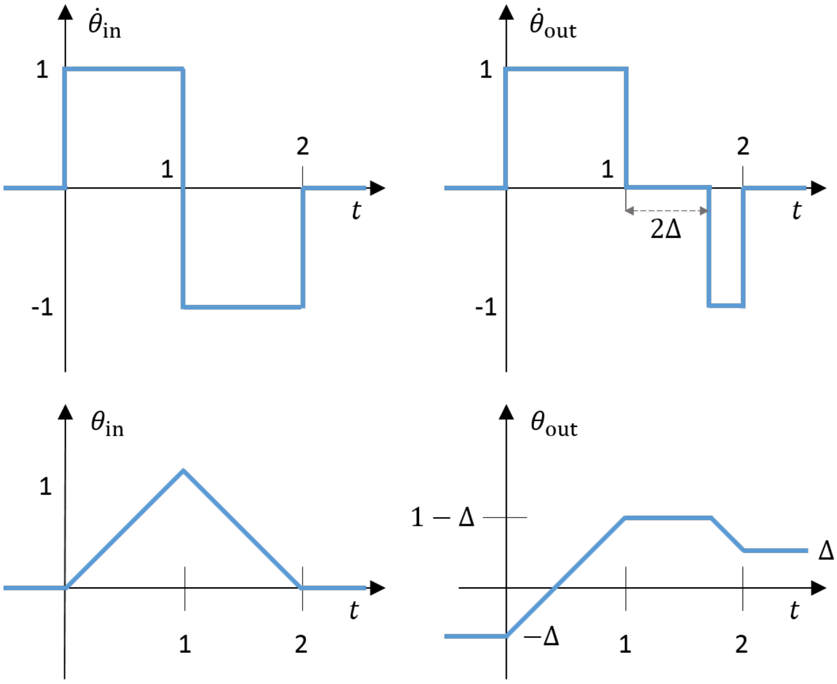

Figure 3: Set of sketches of θ_{in}, θ_{out}, $\dot{\theta}_{\text{in}}$ and $\dot{\theta}_{\text{out}}$ for the angular-velocity back-lash open-loop model, when the input is the one described in equation (2). Furthermore, the initial conditions are $\theta_{\text{in}}(0) = 0$ and $\theta_{\text{out}}(0) = -\Delta$.

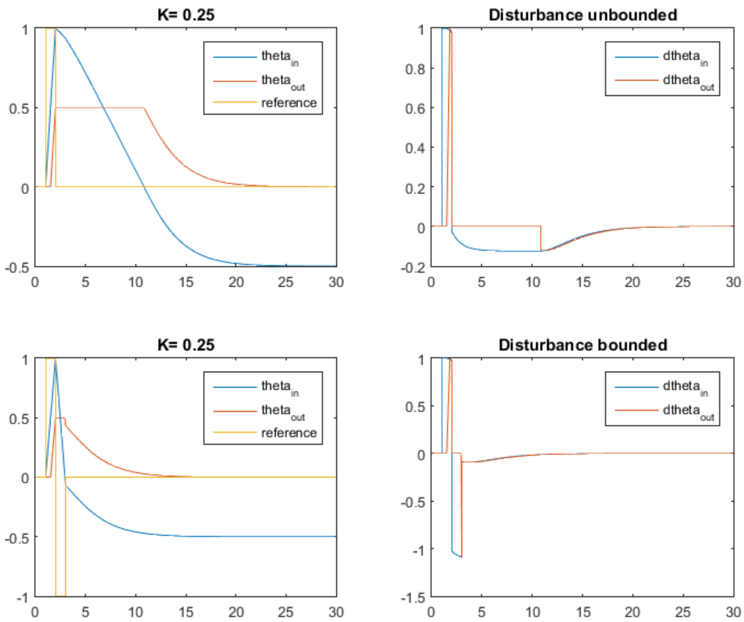

Figure 4: Simulated response of the system, when $K = 0.25$ and for 30 seconds. The upper graphs represent the system when the disturbance is unbounded, whereas the lower graphs belongs to the bounded case.

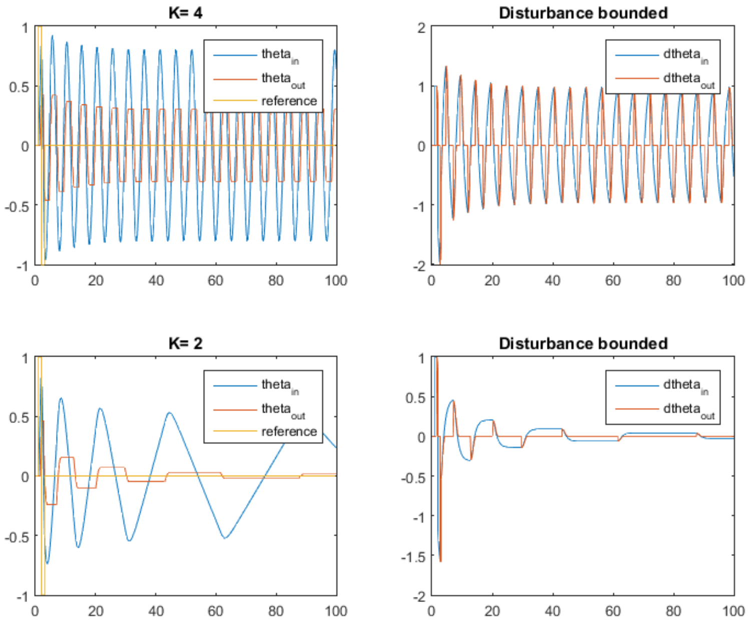

Figure 5: Simulated response of the system, when the disturbance is bounded and for 100 seconds. The upper graphs represent the system when $K = 4$, whereas the lower graphs belongs to the case $K = 2$.

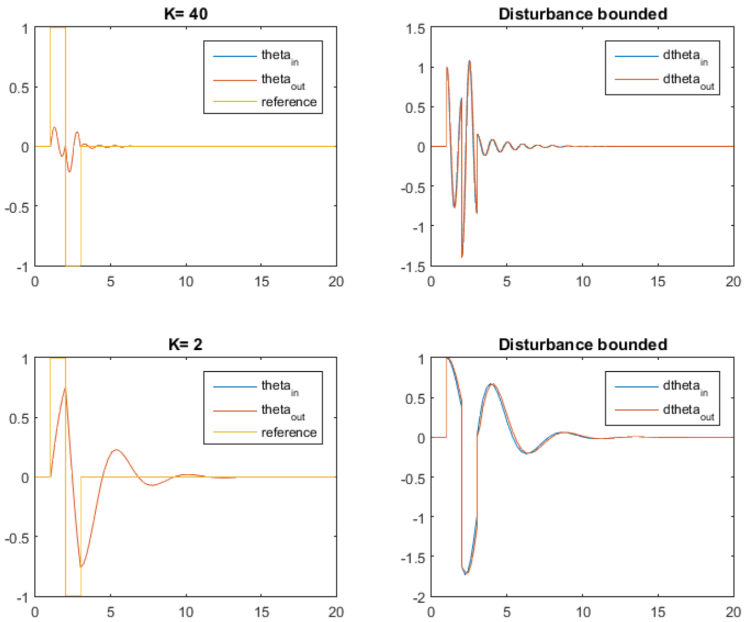

Figure 6: Simulated response of the system, when the disturbance is bounded, the back-lash have been neglected and for 20 seconds. The upper graphs represent the system when $K = 40$, whereas the lower graphs belongs to the case $K = 2$.