Homework 1 in EL2620 Nonlinear Control

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Problem

The problem considered is the analysis of a type of 2-species population model, which is the following one:

$$\frac{dx}{dt} = x(a + bx + cy) \tag{1a}$$

$$\frac{dy}{dt} = y(d + ex + fy) \tag{1b}$$

Where $a, b, c, d, e, f \in \mathbb{R}$ and $x = x(t), y = y(t)$ are functions such that:

$$x(t), y(t) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

They are non negative functions starting at time $t = 0$.
x, y describe the evolution of two species on a grassy island, and depending on the value of the constants x may prey on y, or have other kind of behaviours.

The analysis of such model is broken down into the following five sub-problems:

1. Depending on the signs of the coefficients describe the different types of populations models and label those models as:
   - Predator-Prey(x predator, y prey)
   - Prey-Predator(x prey, y predator)
   - Competitive(x and y inhibits each other)
   - Symbiotic(x and y benefit each other)
2. Consider the case when $(a, b, f, d) = (3, -1, -1, 2)$. Draw the phase portrait, interpret the model and determine the type of each equilibrium for the following cases:
   - $(c, e) = (-2, -1)$
   - $(c, e) = (-2, 1)$
   - $(c, e) = (2, -1)$
   - $(c, e) = (2, 1)$
3. Again, consider the previous questions, but analyse the model for the case $(a, e, b, f, c, d) = (1, 1, 0, 0, -1, -1)$.
4. Show that the x- and y-axes are invariant for all values of the parameters (a, b, c, d, e, f). Why is this a necessary feature of a population model? Assuming $(a, e, b, f, c, d) = (1, 1, 0, 0, -1, -1)$ show that a periodic orbit exists.
5. Generalize the population model to $N > 2$ species.

Solution - Question 1

To understand the problem first consider the case where $y = 0$. In this case (1a) becomes $\dot{x} = ax + bx^2$. It's a differential equations with two zeros, i.e. two equilibrium points, $x_{1,2} = (0, -\frac{a}{b})$. Since $x(t) \geq 0\ \forall t$ and the island has finite size, then the case $b > 0$ makes no sense (if $a > 0$ then we have the species keeps on growing without any limit, because $\dot{x}$ is convex and positive; and $a < 0$ has no physical interpretation in this case). Instead, the case $b < 0$ is the one we are interested in:

- if $a > 0$ we have that $\dot{x}(t) \geq 0\ \forall x \in (0, -\frac{a}{b})$, then $\dot{x}(t) < 0\ \forall x > -\frac{a}{b}$. This means that if the initial population is $c_0 \in (0, -\frac{a}{b})$, then x starts to increase, up to a point where the island has no more food/space available, then the population starts to decrease, and the process repeats. In this case the population x does not depend on any other population to grow (for example it may be a population that eats grass).
- if $a < 0$ we have $\dot{x}(t) < 0\ \forall x \geq 0$, thus the population dies due to starvation. This happens because x preys on another species which is not present on the island ($y = 0$).

Now consider both x, y. Again, because of the previous arguments we have $b, f < 0$. The coefficients c, e describe the effect of y on x and vice-versa.

Because of that we can have 4 types of models, depending on the sign of c, e. The sign of those coefficients is important because they have a positive or negative effect on $\dot{x}, \dot{y}$, which represent the instantaneous population change. The 4 types are the following ones:

1. **c > 0, e > 0**: If both coefficients are positive it means that both x, y benefit from an increase of the other population. Thus a, d should be positive, because neither x or y preys on the other one. This is an example of symbiotic populations.

2. **c > 0, e < 0**: this means that x preys on y since x benefits from an increase of y and y has a drawback from the increase of x. In this case we have a Predator-Prey model (x predator, y prey).

3. **c < 0, e > 0**: it's the opposite case of before, in this case we have a Prey-Predator model (x prey, y predator).

4. **c < 0, e < 0**: in this case both populations suffers from an increase of the other. This is an example of competitive populations.

Solution - Question 2

Now consider the following system:

$$\begin{cases} \dot{x}(t) = x(3 - x + cy) \\ \dot{y}(t) = y(2 + ex - y) \end{cases}$$

The equilibrium points are given by setting $\dot{x} = 0, \dot{y} = 0$:

$$\begin{cases} 0 = x(3 - x + cy) \\ 0 = y(2 + ex - y) \end{cases}$$

Let $\mathbf{p}_i = (x_i, y_i)$ be the i-eth equilibria. The first three equilibria are given by $\mathbf{p}_1 = (0,0), \mathbf{p}_2 = (0,2), \mathbf{p}_3 = (3,0)$. The 4-th equilibria is given by:

$$\begin{cases} x = 3 + cy \\ y = 2 + 3e + cey \end{cases} \Rightarrow \begin{cases} x = 3 + c\dfrac{2+3e}{1-ce} \\ y = \dfrac{2+3e}{1-ce} \quad ce \neq 1 \end{cases}$$

Thus $\mathbf{p}_4 = (\frac{3+2c}{1-ce}, \frac{2+3e}{1-ce})$ with $ce \neq 1$.

To study the equilibrium points we can linearise the system around those equilibrium points to study the behaviour of the system, by using the Hartman-Grobman theorem [1, p. 288]. We will therefore linearise the system around the equilibrium points, calculate the eigenvalues, and determine the equilibria based on the eigenvalues. The linearisation is given by $\dot{\tilde{\mathbf{x}}}_i = A\tilde{\mathbf{x}}_i$, with $\tilde{\mathbf{x}}_i = \mathbf{x} - \mathbf{x}_i$ and the Jacobian matrix $A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}_i}$.
The Jacobian matrix in this case is:

$$A = \begin{bmatrix} 3 - 2x + cy & cx \\ ey & 2 + ex - 2y \end{bmatrix}$$

Now we analyze each equilibrium point separately in order to classify them. Since we have a second-order system, we know that we can classify by looking at the eigenvalues of the Jacobian: [2, p. 37]

- If they are **real**, equilibrium is going to be a stable node if both are negative, an unstable node if both are positive, and a saddle point if one is positive and the other is negative.

- If they are **complex**, we have to look at the real part. If it is positive, we have an unstable focus; negative belongs to a stable focus, and zero implies a non-hyperbolic equilibrium.

We start from $\mathbf{p}_1 = (0,0)$. The Jacobian at this point becomes:

$$A_{(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Whose eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. Since both of them are positive, the equilibrium point $\mathbf{p}_1$ is an unstable node.

For the point $\mathbf{p}_2 = (0,2)$ the Jacobian turns out to be:

$$A_{(0,2)} = \begin{bmatrix} 3 + 2c & 0 \\ 2e & -2 \end{bmatrix}$$

And the eigenvalues becomes $\lambda_1 = 3 + 2c$ and $\lambda_2 = -2$. In this case, we have to look at the value of c to classify this point. If $c < -\frac{3}{2}$ this equilibrium is a stable node; and it is a saddle point if $c > -\frac{3}{2}$.

If we have a look to the third point, $\mathbf{p}_3 = (3,0)$, whose Jacobian is:

$$A_{(3,0)} = \begin{bmatrix} -3 & 3c \\ 0 & 3e + 2 \end{bmatrix}$$

And the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2 + 3e$. As in the previous case, the character of this point would depend of the value of e. If $e < -\frac{2}{3}$ this point would be a stable node; and a saddle point otherwise.

Finally, the fourth point $\mathbf{p}_4 = \left(\frac{3+2c}{1-ce}, \frac{2+3e}{1-ce}\right)$ is strongly related to the values of c and e, so we match each tuple (c, e) with its character.

- For $(c, e) = (-2, -1)$, $\mathbf{p}_4 = (1, 1)$, and the eigenvalues are $\lambda_1 = -2.41$ and $\lambda_2 = 0.4142$, being a saddle point.

- For $(c, e) = (-2, 1)$, $\mathbf{p}_4 = (-\frac{1}{3}, \frac{5}{3})$, and the eigenvalues are $\lambda_1 = 0.786$ and $\lambda_2 = -2.11$, being a saddle point.

- For $(c, e) = (2, -1)$, $\mathbf{p}_4 = (\frac{7}{3}, -\frac{1}{3})$, and the eigenvalues are $\lambda_1 = 0.82$ and $\lambda_2 = -2.82$, being a saddle point.
- For $(c, e) = (2, 1)$, $\mathbf{p}_4 = (-7, -5)$, and the eigenvalues are $\lambda_1 = -2.41$ and $\lambda_2 = 14.42$, being a saddle point.

Having characterized all equilibria, we comment the phase plane for all values of (c, e) given in the formulation.

1. $(c, e) = (-2, -1)$. The phase portrait of the system described is shown in Figure 1. As we show in the first section, this case corresponds to a **competitive** scenario, in which both species can survive either eating grass or the other. So, there are two stable nodes, $(0, 2)$ if y eats x; and $(3, 0)$ otherwise.
2. $(c, e) = (-2, 1)$. The phase portrait of the system described is shown in Figure 2. This case belongs to ***x* prey, *y* predator**. Consequently, the unique stable node is located in $(0, 2)$, since y has eaten all x.
3. $(c, e) = (2, -1)$. The phase portrait of the system described is shown in Figure 3. Now we are in the dual case, in which ***y* is prey and *x* is predator**, and the stable solution corresponds to $(3, 0)$.
4. $(c, e) = (2, 1)$. The phase portrait of the system described is shown in Figure 4. In this case, the society is **symbiotic**, and there is no stable point. The species will grow given the high amount of food available.

Solution - Question 3

In this section, we are going to use the same theoretical background and proceedings described in section 2 to solve this problem. Now the system is:

$$\begin{cases} \dot{x}(t) = x(1 - y) \\ \dot{y}(t) = y(x - 1) \end{cases}$$

We find the equilibrium points by setting $\dot{x} = \dot{y} = 0$, obtaining the equilibrium points $\mathbf{p}_1 = (0, 0)$ and $\mathbf{p}_2 = (1, 1)$. The Jacobian matrix for this system turns out to be:

$$A = \begin{bmatrix} 1 - y & -x \\ y & x - 1 \end{bmatrix}$$

And, particularized for the equilibrium points,

$$A_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; A_{(1,1)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For the first equilibrium, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, which represents a saddle point. Nevertheless, the eigenvalues for the second equilibrium are $\lambda_{1,2} = \pm i$, belonging to a non-hyperbolic equilibrium.

A phase portrait representation for this system could be founded in Figure 5.

Let's comment what is going on in this case. The difference between this system and the one in section 2 is that now x is the unique specie that eats grass, and y preys on x. So, now, y depends on x since there isn't more food for y if x dies.

Therefore, there is a vicious circle in this ecosystem. Since y eats x, the population of x decreases. This causes lack of enough food for y, which implies that the population of y also decreases. But now, knowing that there are few predators, x increases. This generates more food available for y, and increases consequently. And thus, circle is completed.

Solution - Question 4

1. Consider the original system defined by the equations (1a) and (1b).
   First, consider the positive x-axis $W_1 = \{(x, y) \in \mathbb{R}^2 : y = 0, x \geq 0\}$. Then:
   $$\frac{dx}{dt} = ax + bx^2$$
   $$\frac{dy}{dt} = 0$$
   Consider now the initial condition $(x_0, 0) \in W_1$. Since $\dot{y}(t) = 0$ and $y(t) = 0$ the system can vary only along $(x(t), 0) \in W_1 \Rightarrow W_1$ is an invariant set. Same reasoning can be applied to $W_2 = \{(x, y) \in \mathbb{R}^2 : x = 0, y \geq 0\}$ and find that W_2 is an invariant set. It's a necessary feature of the population model because otherwise it would mean that y specimen can transform or evolve into a x specimen (or viceversa).
2. As found in the previous section (3) for the coefficients $a = e = 1, b = f = 0, c = d = -1$ the model represents a population x that eats grass and a population y which preys on x. This model has two equilibria, as previously shown, of which the one in $(x, y) = (1, 1)$ has the eigenvalues of the Jacobian matrix which are purely imaginary (i.e. null real part). Therefore we can not use the Hartman-Grobman theorem to characterize the behaviour of the system around that equilibria. It can be proven that $(1, 1)$ it's a center equilibria, therefore there is a periodic orbit around that point.

To do so we use a very simple idea: first we find a function $V(x,y)$ such that is constant along solutions of the system. Then, if we find a closed bounded curve along which $V(x,y) = const.$ then this curve is a solution of the system, and since it is closed and bounded, the solution is periodic.
The system is given by:

$$\begin{cases} \dot{x}(t) = x(1-y) \\ \dot{y}(t) = y(x-1) \end{cases}$$

To find $V(x,y)$ we notice that $\frac{dy}{dx}$ is function only of x, y and that can be easily solved:

$$\frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}$$

$$\int \frac{1-y}{y} dy = \int \frac{x-1}{x} dx$$

$$\ln|y| - y = x - \ln|x| + c$$

Since $x, y \geq 0$, then :

$$\ln(y) - y - x + \ln(x) = c$$

where $c \in \mathbb{R}$ is a constant, thus our function $V(x,y)$ is:

$$V(x,y) = \ln(y) - y - x + \ln(x)$$

To prove that V is conserved we calculate $\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$ and show that it is equal to 0:

$$\begin{aligned} \dot{V}(x,y) &= \Big(\frac{1}{x} - 1\Big)(x - xy) + \Big(\frac{1}{y} - 1\Big)(yx - y) \\ &= 1 - x - y + xy + x - yx - 1 + y = 0 \end{aligned}$$

Now to prove that the solutions are periodic around $(1,1)$ we must prove that $V(x,y)$ around that point has level sets that are simple closed bounded curves.
First notice that $V(1,1) = -2$ and that $V(x,y)$ has only 1 stationary point, which is $(1,1)$. Next, the Hessian matrix of $V(x,y)$ is :

$$\mathbf{H}(x,y) = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix}$$

which is negative definite in $(1,1) \Rightarrow V(x,y)$ is concave in $(1,1)$ and has a maxima in that point. Since $V(x,y)$ is a continuous function in $(1,1)$ and concave, in a neighbourhood of $(1,1)$ $V(x,y)$ is a quadratic form, and has the shape of a cone. Then $V(x,y) = V(1,1) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T \mathbf{H}_{(1,1)}(\mathbf{x}-\mathbf{x}_0) + O(||\mathbf{x}-\mathbf{x}_0||^2)$, where $\mathbf{x}_0 = (1,1)$.
Then $V(x,y) \approx -4 + 2x + 2y - x^2 - y^2 = \hat{V}(x,y)$. Since V is continuous, for $|\mathbf{x} - \mathbf{x}_0|$ sufficiently small we can use $\hat{V}(x,y)$, the second order approximation of V to analyse the level sets of $V(x,y)$ around $(1,1)$. To do so we must ensure that we analyse the level sets *near* $(1,1)$: if γ is a level set of V then:

$$|\mathbf{x} - \mathbf{x}_0| < \delta, \mathbf{x} \in \gamma$$

with $\delta > 0$ sufficiently small in order to ensure that the 2-nd order approximation is valid. Since V is concave, and $V(\mathbf{x}_0) = -2$, this condition, for continuity of V, is translated into:

$$\forall \varepsilon > 0 \exists \delta : |\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |V(x,y) + 2| < \varepsilon$$

Then $|V(x,y) - \hat{V}(x,y)| \to 0$ as $\varepsilon \to 0$. Consider the level set $V(x,y) = -2-\varepsilon$ ($-\varepsilon$ because $V(1,1)$ is a local maxima): for $\varepsilon \to 0$ we can look at $\hat{V}(x,y) = -2 - \varepsilon \Rightarrow -2 + 2x + 2y - x^2 - y^2 + \varepsilon = 0$. This is simply a circle centred in $(1,1)$ with radius $\sqrt{\varepsilon}$, in fact we can write it like $\varepsilon - (x-1)^2 - (y-1)^2 = 0$. For $\varepsilon > 0$, and sufficiently small, the level set is a circumference of radius $\sqrt{\varepsilon}$: this is a closed, simple and bounded curve (it can be bounded by a circumference of radius ε) $\Rightarrow V(x,y)$ has level sets (fig. 6) that are simple closed bounded curves for $\mathbf{x}$ sufficiently close to $(1,1)$. The solutions (x,y) on those curves then have a periodic orbit thus $(1,1)$ is a center equilibria.

Solution - Question 5

Starting from the general system described in (1), the goal for this question is to generalize the system for more than 2 species.

We introduce a change in notation. Now, for each specie i, identified by x_i, the constant that shows if eats grass becomes a_i, and the relationship between this specie with another specie j is denoted by r_{ij}.

If we generalize the system for N species, we obtain for each one (recall $a_i, r_{ij} \in \mathbb{R}, i, j = 1, ..., N$):

$$\dot{x}_i(t) = x_i \left(a_i + \sum_{j=1}^{N} r_{ij} x_j \right)$$

References

[1] Shankar Sastry. *Nonlinear systems: analysis, stability, and control*, volume 10. Springer, New York, N.Y., 1999. ISBN 0-387-98513-1.

[2] Hassan K Khalil. *Nonlinear systems*. Prentice Hall, Upper Saddle river, 3. edition, 2002. ISBN 0-13-067389-7.

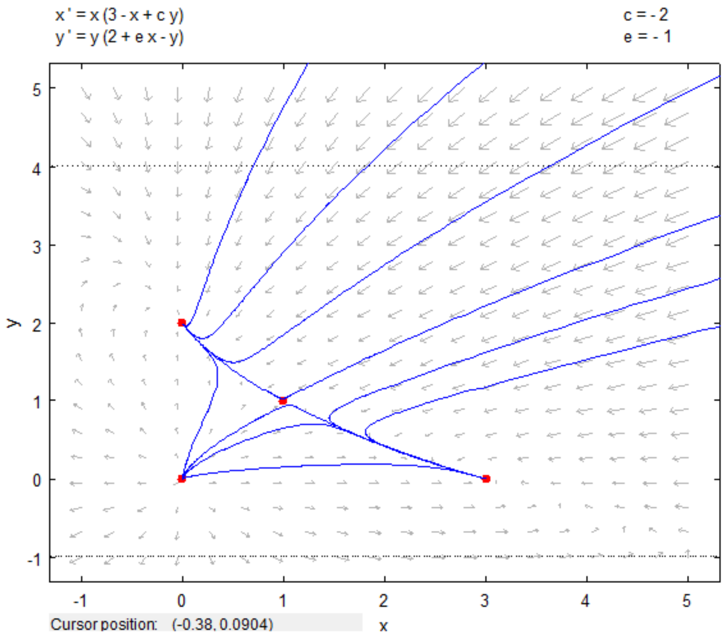

Figure 1: Phase portrait for the competitive system. This has an unstable focus at $(0, 0)$, a saddle point at $(1, 1)$ and two stable nodes at $(0, 2)$ and $(3, 0)$.
All phase portraits were generated using Matlab (`http://www.mathworks.com`) and pplane8 (`http://math.rice.edu/~dfield/`). Equilibriums are marked by red dots, and trajectories by solid blue lines.

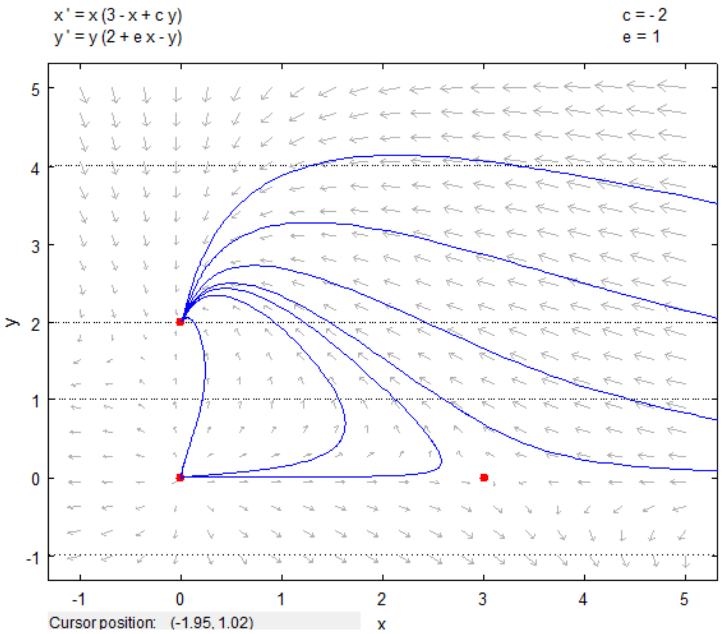

Figure 2: Phase portrait for the "y-predator" system. This has an unstable focus at $(0, 0)$, a saddle point at $(3, 0)$ and a stable node at $(0, 2)$. The second saddle point at $(-\frac{1}{3}, \frac{5}{3})$ is not considered since that point does not make physical sense (population must be ≥ 0).

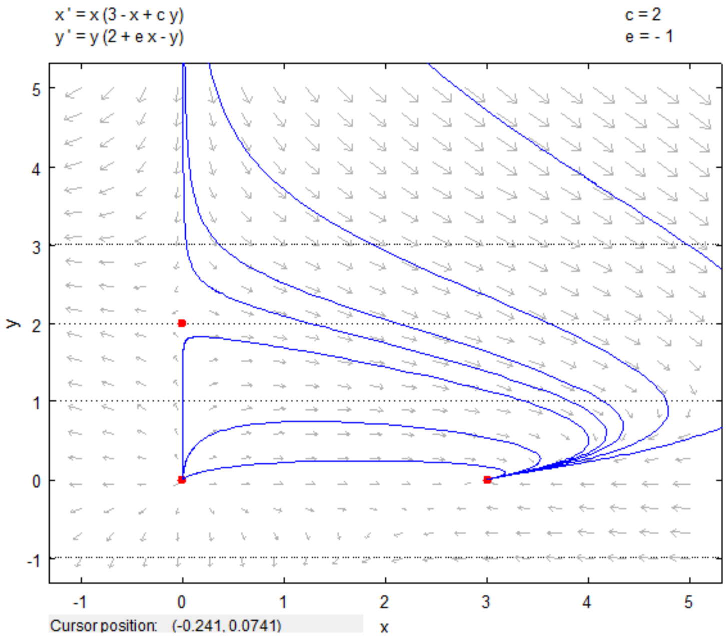

Figure 3: Phase portrait for the "x-predator" system. This has an unstable focus at $(0, 0)$, a saddle point at $(0, 2)$ and a stable node at $(3, 0)$. The second saddle point at $(\frac{7}{3}, -\frac{1}{3})$ is not considered since that point does not make physical sense (population must be ≥ 0).

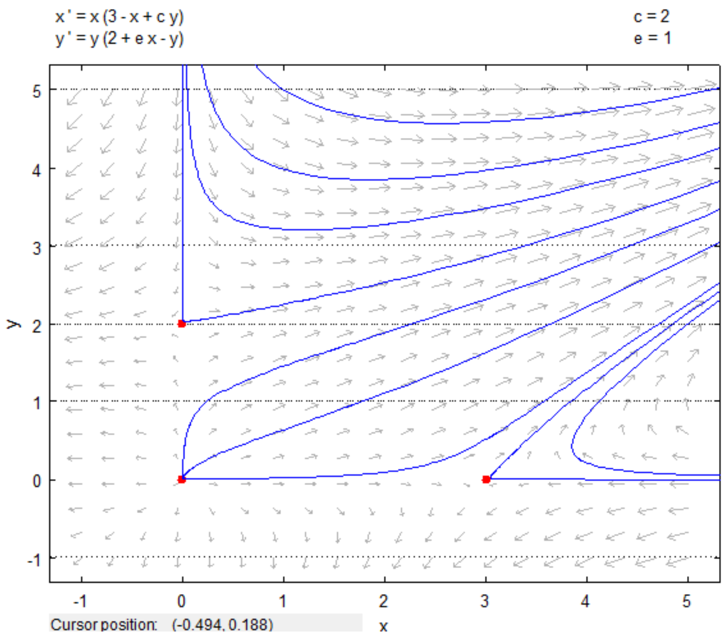

Figure 4: Phase portrait for the symbiotic system. This has an unstable focus at $(0, 0)$ and two saddle points at $(0, 2)$ and $(3, 0)$. The third saddle point at $(-7, -5)$ is not considered since that point does not make physical sense (population must be ≥ 0).

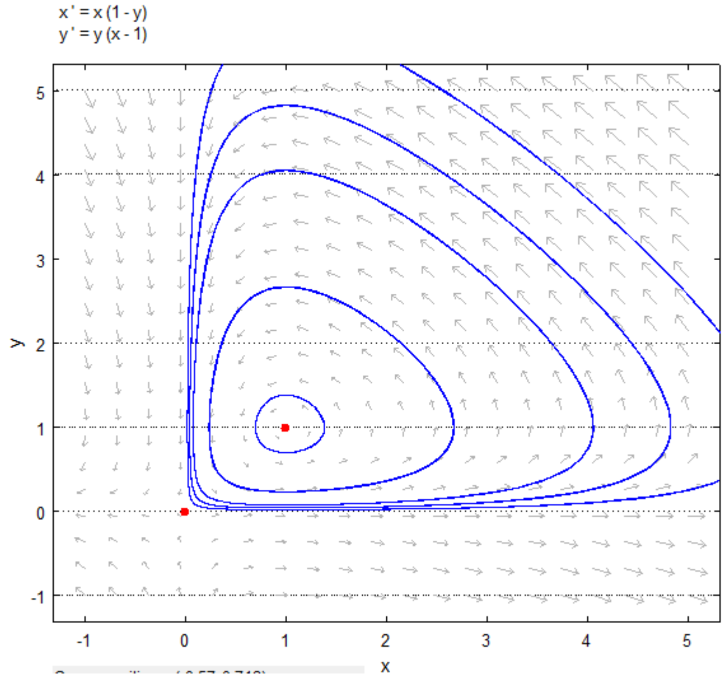

Figure 5: Phase portrait for the system in exercise 3. This has an saddle point at $(0, 0)$ and a center point in $(1, 1)$, as shown in section (4).

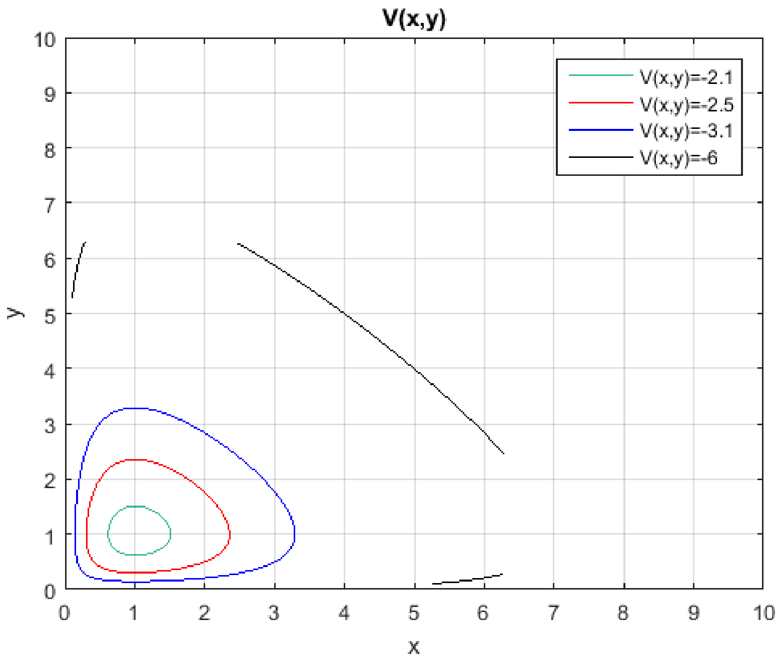

Figure 6: Level sets of $V(x, y)$, described in section (4). Notice that the level set $V(x, y) = -c$, for c sufficiently big, is not a closed curve. For $c = -2.5$ we have a simple, closed, bounded curve. The curve $\gamma = \{(x, y) \in \mathbb{R}, \varepsilon > 0 : V(x, y) = -2 - \varepsilon\}$ for $\varepsilon \to 0$ becomes a circumference centred in $(1, 1)$. This behaviour can be seen for $V(x, y) = -2.1$, as shown in the figure.