Algorithms: Sorting Algorithms

Given a sequence of numbers, you have to arrange them in the ascending or descending order. This is called sorting. Sorting algorithms can be iterative or recursive.

Let's learn about the different sorting algorithms and analyze their space and time complexity. First we are going to start with the iterative Insertion Sort.

INSERTION SORT ALGORITHM AND ANALYSIS

"Insertion sort", the name suggests that there is an insertion happening. Understand it with an example where you want to arrange the playing cards which are placed on a table, face down.

1. Take the first card in your left hand.
2. Take the second card in your right hand, compare it with the left hand card

   If smaller then keep this card in the most left side on the left hand
   If larger then keep this card in the right most side on the left hand

3. Repeat step 2 over again.

At every new picking of the card, your left hand will always have sorted cards. Let us right the algorithm in the context of an array containing integers which we will sort using **Insertion Sort.**

```
Insertion_Sort(A)

{

        for( j = 2 to A.length)
        {
                key = A[ j ];
                //insert A[ j ] into sorted sequence A[1 … j-1]
                i = j – 1;
                while( i > 0 and A[ i ] > key)
                        A[i+1] = A[i];
                        i = i - 1;

                A[i+1] = key;

        }

}
```

Let array = 9, 6, 5, 0, 8, 2, 7, 1, 3 (please follow the algorithm above to sort the list).

Time complexity:

The worst case time complexity for insertion sort can be calculated as below. For worst case to happen, the elements should be already in descending order in the array for which we have to do ascending order sorting using Insertion Sort.

For j	Comparison	Movements	Total Operations
2	1	1	2
3	2	2	4
4	3	3	6
…	…	…	…
N	n-1	n-1	2(n-1)

Therefore, total operations done: $2 + 4 + 6 + \ldots + 2(n-1)$

Or: $2(2-1) + 2(3-1) + 2(4-1) + \ldots + 2(n-1)$

Or: $2(1) + 2(2) + 2(3) + \ldots + 2(n-1)$

Or: $2 (1 + 2 + 3 + \ldots + n-1)$

- ⇨ $2(n(n-1))/2$
- ⇨ **$O(n^2)$**

The best case time complexity for Insertion Sort can be calculated as below. For best case to happen, the array should already have the sorted elements inside it.

For j	Comparison	Movements	Total Operations
2	1	0	1
3	1	0	1
4	1	0	1
…	…	…	…
N	1	0	N - 1

Total operations for best case time complexity: $1 + 1 + 1 + \ldots + N-1$

When the array is already sorted you will just compare the element in you right hand (card analogy) with the rightmost element in the left hand only once for each element in the array.

Therefore, best case time complexity = Ω(n)

<u>Space Complexity of Insertion Sort:</u>

Apart from the input array A[], I only need three variables ‘key’, ‘i’ and ‘j’. So, whatever is the size of the input, I only need 3 extra variables for Insertion sort to work.

Therefore, the space complexity for Insertion Sort is **O(1)** or constant space.

Note: Whenever a sorting algorithm executes in constant space, we call such a sorting algorithm INPLACE algorithm

In Insertion Sort, we saw that the time complexity depends on the number of comparisons involved and the number of movements involved. So how can we decrease this complexity? Let's try some methods.

1. Using Binary Search instead of sequential search in the sorted list

Comparisons using Binary Search	Movements using Binary Search	Total complexity
O(logn)	O(n) (no reduction)	O(nx(n-1) elements)) = $O(n^2)$

Using binary search will not reduce the time complexity of the Insertion Sort algorithm.

2. Using Linked List for insertion

Comparisons using Linked List	Movements using Linked List	Total complexity
O(n) (no reduction)	O(1) (direct insertion)	O(nx(n-1 elements)) = $O(n^2)$

Using linked list will also not reduce the time complexity of the Insertion Sort Algorithm.

Time and Space complexity for Insertion Sort

Time Complexity	**Best Case**	**Worst Case**	**Average Case**
	$\Omega(n)$	**$O(n^2)$**	**$\Theta(n^2)$**

Space Complexity	**O(1)**

Let's move on to the next sorting algorithm which is going to work better than Insertion Sort in terms of time complexity. This algorithm is called Merge sort.

MERGE SORT ALGORITHM AND ANALYSIS

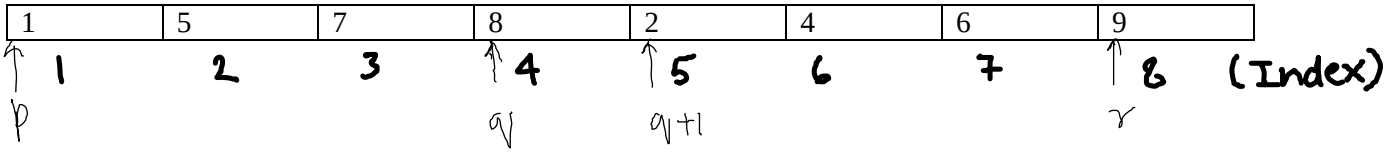
Merge sort falls under the Divide-and-conquer algorithms. It simply means you divide the problem into smaller steps and tackle the problem of sorting using bottom-up approach.

Before we go into the algorithm and analysis of it, let's understand the heart of the algorithm, which is a procedure called Merging. For clarity, here is the algorithm

```
MERGE(A, p, q, r)
{
        n1 = q-p+1
        n2 = r – q
        Let L[1…n1+1] and R[1 … n2 + 1] be new arrays
        for( i=1 to n1)
                L[i] = A[p + i-1]
        for( j=1 to n2)
                R[j] = A[q + j]
        L[n1 + 1] = ∞
        R[n2 + 1] = ∞
        for(k = p to r)
                if(L[i] <= R[j]
                        A[k] = L[i]
                        i=i+1
                else A[k] = R[j]
                        j=j + 1
}
```

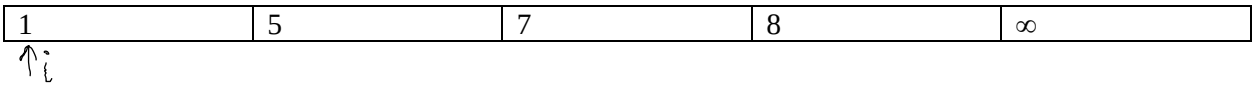
Working of MERGE:

We have a list of elements where left half and right half are sorted.

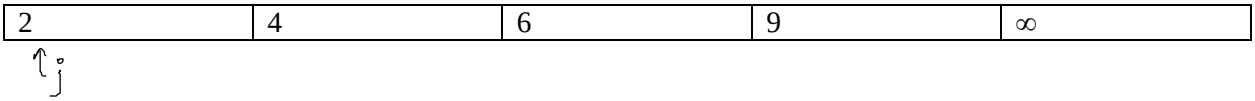


We divide the list above into two lists L and R containing the first half sorted elements and the second half sorted elements.

L = An array of size q-p+1 where last element is a very big number denoted by ∞ here.



R = An array of size r-q where last element is a very big number denoted by ∞ here.



Now, we compare the elements of L and R and arrange them in the ascending order in the final array. The final array looks like below.

A =

1	2	4	5	6	7	8	9

For merge sort to work we need to copy the elements from the input which takes ‘n’ extra space other than the input. And to compare the two sorted lists it takes one comparison per input, therefore it will take ‘n’ comparisons.

Time Complexity = O(n)

Space Complexity = O(n)

If the size of list L and R is ‘n’ and ‘m’ respectively, the time and space complexity will be O(n+m). The lists L and R should have sorted elements in them.

Note: We add ∞ in the end of the two lists for proper comparison. If L = 10,20,30,40, ∞ and R = 1,2,3,4, ∞ then A = 1,2,3,4.. initially but to enter the elements of L we need some element to compare it to, so the fifth element in R is compared with the elements of L to add them to the sorted list A. Then A will become 1, 2, 3, 4, 10, 20, 30, 40

Now that we have seen how the MERGE method is working. Let’s use it to create Merge Sort algorithm.

Working of MERGE SORT

```
Merge_sort(A, p, r)
{
        if p < r
                q = floor[(p+r)/2]
                Merge_sort(A, p, q)
                Merge_sort(A, q+1, r)
                MERGE(A,p,q,r)
}
```

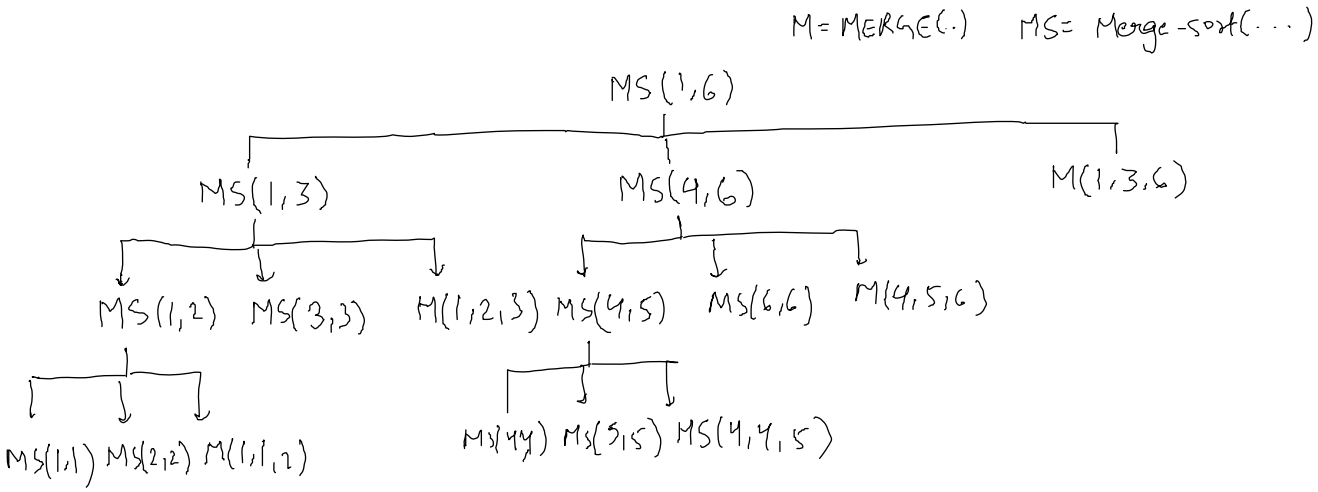
As said earlier, Mege Sort falls under the category of Divide and Conquer algorithms. In this sorting technique we divide the array to be sorted into single element array (single element array is already sorted) and move up from there to give us the final sorted array.

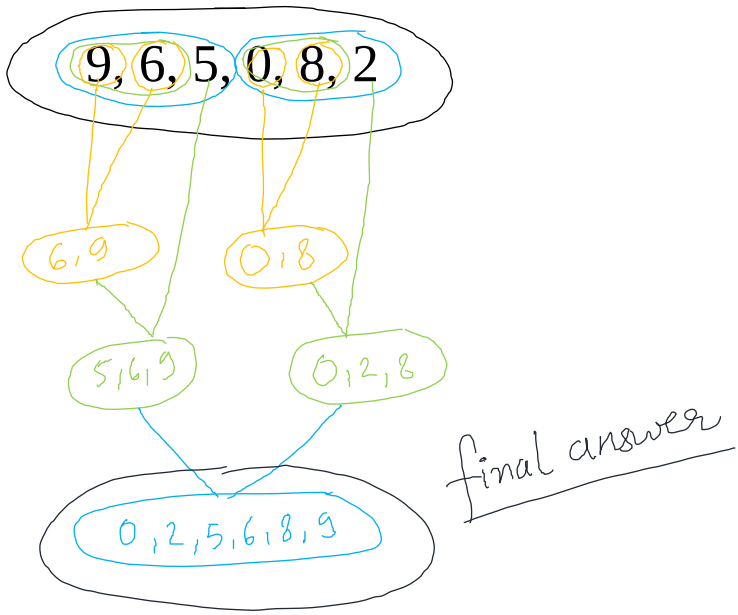
This is a recursive algorithm where each function call is going to call three functions:

Merge_sort(A, p, q)
Merge_sort(A, q+1, r)
MERGE(A,p,q,r) Let’s see how it is doing the work using an example.

Sort the array A = [9, 6, 5, 0, 8, 2] using merge sort.

Recursion tree for Merge Sort.





Space Complexity Analysis:

We know that the extra space required for the Merge Operation is O(n).

We need extra space in the stack for function calling. To analyze this, we need

1. Total number of function calls made
2. The order in which they are made
3. The height of the stack that is needed for this.

Total number of function calls made for above example: 16 so do we really need a stack of size 16 for this? Let's see.

That is indeed not the case: The height of the stack is equal to the height of the recursion tree made by the merge sort algorithm.

Note: All the function calls in a level will be carried out in the stack memory of same level. Level 3 calls will be made on the third memory cell of the stack. (Interesting).

The height of the recursion tree for an input size of n = ceil(logn) + 1. For every level we need a cell in the stack which let's say occupies 'k' space units.

Therefore, for stack we need k(logn + 1) space.

- ⇨ **Space complexity for stack: O(k(logn+1)) or O(logn)**
- ⇨ **For merge procedure space required is of O(n)**

Total space complexity = O(n + logn) or O(n) for merge sort.

Now let us analyze the time complexity of the merge sort algorithm

Time Complexity Analysis:

Let us say we are sorting an array of size n.

Let's say that the time taken by merge sort on array of size n = T(n)

```
Merge_sort(A, p, r)          → T(n)
{
        if p < r
                q = floor[(p+r)/2]
                Merge_sort(A, p, q)      → T(n/2)
                Merge_sort(A, q+1, r)    → T(n/2)
                MERGE(A,p,q,r)           → O(n)
}
```

$\therefore T(n) = 2T(n/2) + O(n)$

Using Master's Theorem

$a = 2 \quad b = 2 \quad k = 1 \quad p = 0$

$a = b^k \; \& \; p > -1$

$\therefore T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$

$T(n) = \Theta(n^{\log_2 2} \log^{0+1} n)$

$T(n) = \Theta(n \log n)$

Time and Space complexity for Merge Sort

Time Complexity	Best Case	Worst Case	Average Case
	Ω(nlogn)	**O(nlogn)**	**Θ(nlogn)**

Space Complexity	O(n)

Let's see some questions on Merge Sort.

Q1. Given "logn" sorted list each of size "n/logn". What is the total time required to merge them into one list?

Total number of sorted lists = logn

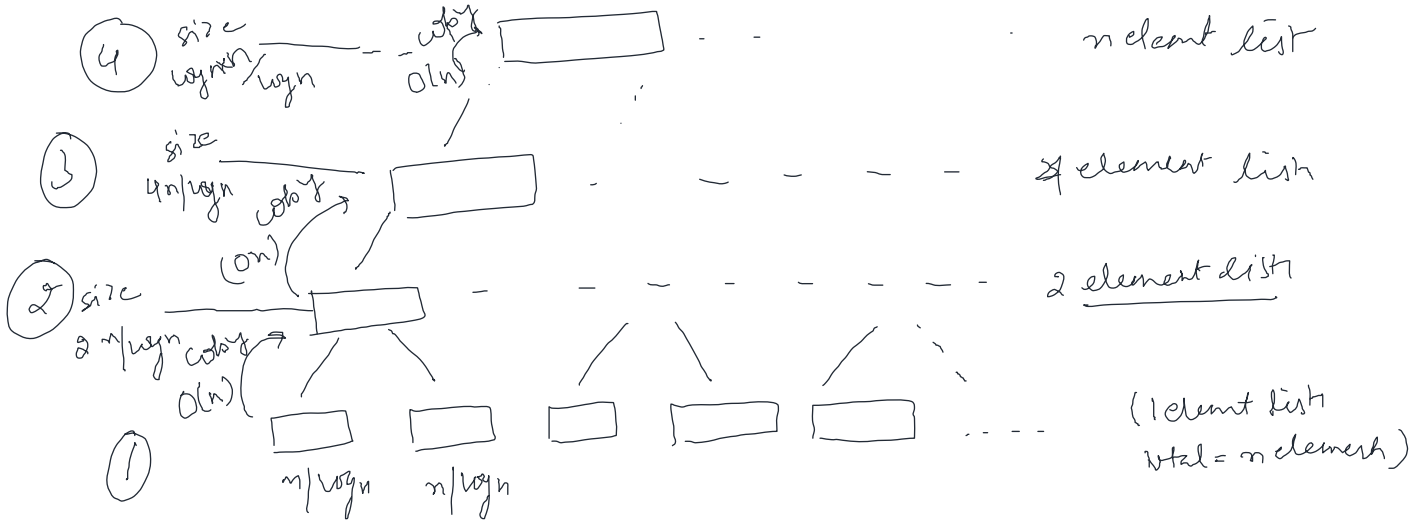
Each list is of size: n/logn

Total number of elements : logn x n/logn = n

For copying the elements in level of the recursion tree =O(n)

This copying has to be done for the height of the tree= loglogn

Total time complexity : O(nloglogn)



total complexity = O(n) X no. of climbings I have to do to reach last level of tree

from (1) to (2) climbings = log 2 (or 1 climb)

" (2) to (3) " = log 4 (or 2 climbs)

from 1 to last (top of tree) climbings = log (log n) (or log n climbings)

∴ complexity = O(n log log n)

Q2. "n" strings each of length "n" are given then what is the time taken to sort them?

If two strings are of length n, then the time taken to compare them will be O(n)

At the bottom level we are given n string each of length n.

For one comparison time taken = O(n)

For n comparisons time taken = $O(n^2)$

For copying in the next level time taken = n strings of size n = $O(n^2)$

Total time taken for comparison + copying in **one level** = $2.O(n^2)$ or $O(n^2)$

This has to be repeated until we reach the top level. So how many levels we are going to traverse for that?

To go from bottom to (bottom -1) level we take one step or log2 steps or ***log(size of array at bottom-1 level)***

From bottom to (bottom -2) level we take two steps or log4 steps or ***log(size of array at bottom -2 level)***

We know the size of array at the top level has to be 'n'.

Therefore to reach the top level, we have to take log(size of top array) or logn steps.

Total time complexity : (number of steps taken) X (time complexity for copying and comparison on each level)

- ⇨ **Total complexity = $O(n^2 logn)$**

POINTS ON MERGE SORT

1. Merge sort uses divide and conquer paradigm
2. The time complexity for two lists of size m and n to merge is O(m+n)
3. Questions can be asked like "What is the order of the array (given in quesion) in the second pass of two way merge sort?"

LET'S MOVE ON TO OUR NEXT SORTING ALGORITHM