Basic Principle

$$u \approx U = \sum_k a_k \phi_k$$

$u \rightarrow$ exact, $U \rightarrow$ approximate.

$a_k =$ Degrees of Freedom (dof) (Galerkin, Collocation, Continuous Galerkin (CG), Discontinuous Galerkin (DG), Penalty, Staggered Grid, Tau)

$\phi_k =$ Orthogonal Polynomial (Fourier, Legender, Chebyshev)

Why, Where

WHY

1. Higher accuracy, fast convergence
2. Low # of dof for given accuracy
3. Low phase / dispersion error
4. Low dissipation error
5. Easy implementation of Boundary Condition (BC)
6. Efficient Parallalisation

WHERE

1. Detailed physics is needed (e.g. Turbulence)
2. Long time integration (dispersion and dissipation)

Model Problem

$$u_t + cu_x = 0, \qquad x \in [0, 2\pi], \qquad t>0$$

Def:

Inner Product:

$$(u, v) = \int_0^{2\pi} uv^* dx$$

Fourier Galerkin (FG)

$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx} \approx U = \sum_{k=-N/2}^{N/2} a_k e^{ikx}$$

$$\Rightarrow \quad \sum_{k=-N/2}^{N/2} (\dot{a}_k + ikca_k)e^{ikx} = R$$

$$\textit{Project}$$

$$\Rightarrow \quad \left(\sum_{k=-N/2}^{N/2} (\dot{a}_k + ikca_k)e^{ikx}, e^{imx} \right) = (R, e^{imx})$$

$$\Rightarrow \quad (\dot{a}_m + imca_m)2\pi = (R, e^{imx}) = 0, \quad m = -N/2, .., N/2 \quad (\textit{Galerkin})$$

$$\Rightarrow \quad \dot{a}_k + ikca_k = 0 \ \ \& \ \ a_k(0) = \hat{u}_k(0)$$

RK4 / Leap Frog

Accuracy

Example: $u_0(x) = sin(\pi(cos(x)))$

N	FD-2	FD-4	FG
8	1.11	9.62×10^{-1}	9.87×10^{-2}
16	6.13×10^{-1}	2.36×10^{-1}	2.55×10^{-4}
32	1.99×10^{-1}	2.67×10^{-2}	1.05×10^{-11}
64	5.42×10^{-2}	1.85×10^{-2}	6.22×10^{-13}

Galerkin

Basis

$$\phi = \sum_{k=-N/2}^{N/2} c_k e^{ikx} \in \mathcal{P}_p^{N/2}$$

$\mathcal{P}_p^{N/2}$: Fourier Polynomial of Degree N/2

Galerkin

$$(U_t + cU_x, \phi) = 0; \quad \forall \;\; \phi \in \mathcal{P}_p^{N/2}$$

- Non-Linear Problems are tricky!

Variable Coefficient PDE

$$u_t + uu_x = \nu u_{xx}$$

- Will mix modes.
- Aliasing Error appears.
- Use Pseudospectral Method (Derivative in spectral space & multiplication in real space)
- For de-aliasing use $\frac{3}{2}$ padding instead of $\frac{2}{3}$ truncation.
- In fully developed turbulence, high frequencies; pump energy into low frequencies. Hence aliasing error changes physics.
- For smooth looking solution, use of cosmetic filtering is permitted.

Navier-Stokes Equation

Burgers Equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)$$

- Initial Condition : $u(x,0) = \sin(x),\ \nu = 0 = f$
- $\nu \neq 0 \Rightarrow$ No Gibbs Oscillation
- $f \neq 0 \Rightarrow$ Intermittency

Simulation Method : Pseudo-Spectral

$$u(x,t)$$

$$\downarrow FFT$$

$$\widehat{u_k}(k,t) \xrightarrow{Multiply} ik * \widehat{u_k}(k,t) \xrightarrow{IFFT} \frac{du(x,t)}{dx} \rightarrow$$

$$\xrightarrow{Multiply} u(x,t) * \frac{du(x,t)}{dx} \xrightarrow{RK4} u(x,t+1)$$

Burgers Equation with $\nu = 0$

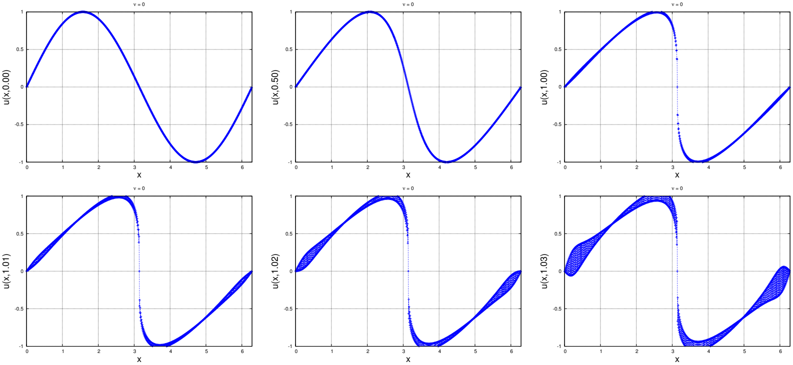


Figure: Compare with: Fig 1, PRE, 84, 016301, Fig 1, PNAS, 97, 12413

- Can be simulated better with WENO schemes!
- "Tygers" are the artificial effect of Galerkin truncation. Hence will not appear in any FD based scheme!

Burgers Equation with $\nu = 0$

Parameters: $N = 2^{14}$, $L = 2\pi$, $dt = 10^{-6}$, $\nu = 0$

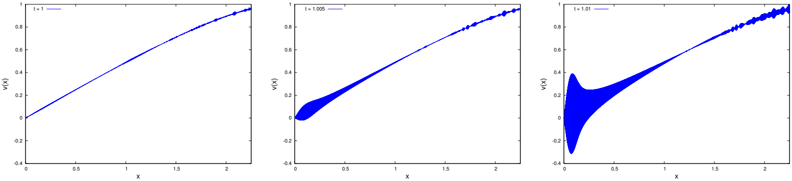


Figure: The occurrance of "tyger" after the generation of shock at time (a) $t = t^\star = \frac{1}{\max|\frac{du_x}{dx}|_{t=0}} = 1$, (b) $t = 1.005$ and at (c) $t = 1.01$.

Burgers Equation with $\nu \neq 0$

Compare with: Fig 11, Pramana, 73, 1, 2009

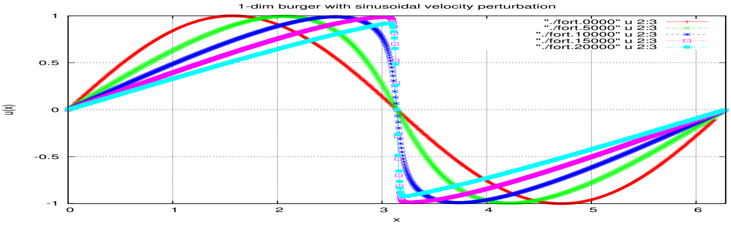


Figure: Parameters: $N = 2^{10}$, $L = 2\pi$, $dt = 10^{-3}$, $\nu = 10^{-2}$

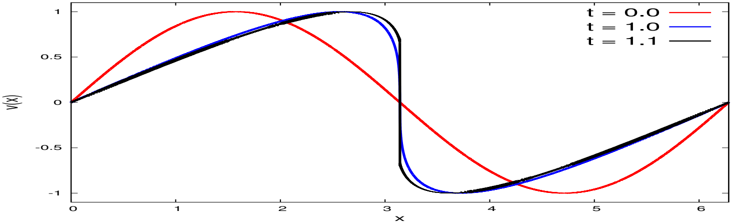


Figure: Parameters: $N = 2^{14}$, $L = 2\pi$, $dt = 10^{-6}$, $\nu = 10^{-6}$

Burgers Equation with $\nu \neq 0$

- Energy Spectra should follow a power law with slope $\frac{1}{k^2}$

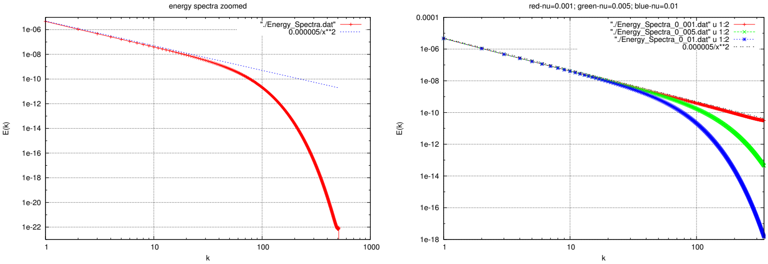


Figure: a. with zero viscosity, b. with different viscosity, Fig 12, Pramana, 73, 1, 2009

- Viscosity acts on higher modes of wavenumber i.e. in small scales.

Homework - 1 [Structure Factor]

Order-P Velocity Correlation Function

$$S_p(\ell) = \langle [v(x+\ell) - v(x)]^p \rangle$$

where $v(x)$ is the velocity at the point x and the angular brackets denote and average over the statistical steady state of the turbulent fluid.
For ℓ in inertial range, $S_p(\ell) \sim \ell^{\xi_p}$
for small Δx,

$$S_p(\Delta x, t) \sim C_p|\Delta x|^p + C'_p|\Delta x|$$

For $p<1$, the first term dominates and for $p>1$ the second term.

Reference: "Burgulence", Uriel Frisch, Jeremie Bec, arXiv: nlin/0012033v2 (2001)

Homework - 1 [Structure Factor]

Order-P Velocity Correlation Function

Parameters: $N = 2^{14}$, $L = 2\pi$, $dt = 10^{-6}$, $\nu = 10^{-6}$

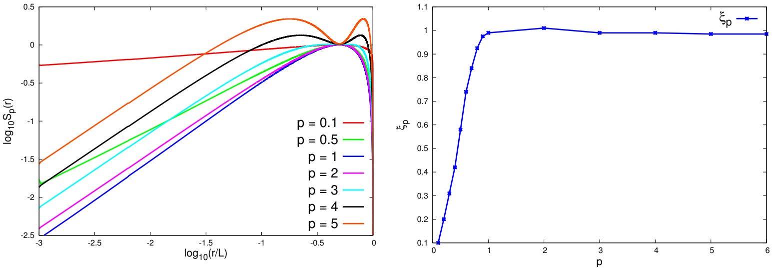
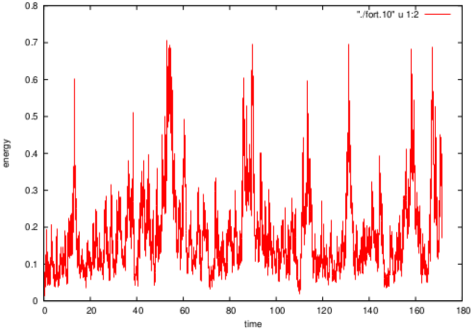


Figure: Compare with: Fig 1(b), 2(a), PRL, 94, 194501

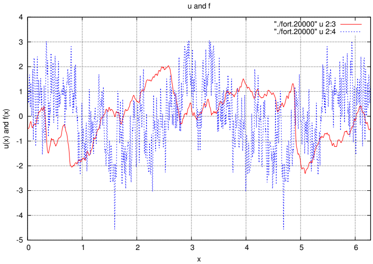
Homework - 2 [Burgers Equation with $f \neq 0$]

• Forcing :: Gaussian random noise with zero mean and Fourier-space Spectrum $\sim \frac{1}{k}$
• One should observe intermittency.



Homework - 2 [Burgers Equation with $f \neq 0$]

The order-p velocity structure function looks like,
(Parameters: $N = 2^{10}$, $L = 2\pi$, $\nu = 10^{-2}$)



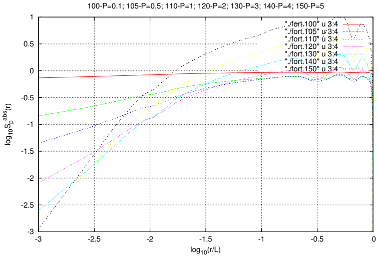


Figure: Compare with: Fig 1(a), (c), PRL, 94, 194501

A nuclear fusion meme by Shawn Zamperini.

Incompressible Navier-Stokes Equation

Basic Equations

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{\omega} = \nu \nabla^2 \vec{\omega}$$
$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

In Two dimensions

$$\vec{\omega} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (u_x, u_y, 0) = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \vec{z}$$

Simulation Method

Pseudocode : Pseudo-Spectral Method

$$\omega(x,y) \xrightarrow{FFT} \hat{\omega}(k_x,k_y) \text{ [at time = t]}$$

$$\Downarrow$$

$$\hat{\psi} = \frac{\hat{\omega}}{k_x^2+k_y^2}$$

$$\hat{u}_x = ik_y\hat{\psi},\ \hat{u}_y = -ik_x\hat{\psi}$$

$$ik_x\hat{\omega},\ ik_y\hat{\omega}$$

$$\Downarrow \text{IFFT}$$

$$u_x, u_y$$

$$\frac{\partial\omega}{\partial x} = \mathcal{IF}[ik_x\hat{\omega}]$$

$$\frac{\partial\omega}{\partial y} = \mathcal{IF}[ik_y\hat{\omega}]$$

$$\Downarrow$$

$$u_x \cdot \frac{\partial\omega}{\partial x},\ u_y \cdot \frac{\partial\omega}{\partial y}$$

$$\Downarrow \text{FFT}$$

Simulation Method cont.

Pseudocode : Pseudo-Spectral Method

$$\Downarrow \text{FFT}$$

$$NLkx = \widehat{u_x \cdot \frac{\partial \omega}{\partial x}},\ NLky = \widehat{u_y \cdot \frac{\partial \omega}{\partial y}}$$

$$\Downarrow \text{De-Aliazed } (2/3 \text{ rule})$$

$$NLk = NLkx + NLky + \nu \cdot (k_x^2 + k_y^2)\hat{\omega}$$

$$\Downarrow$$

$$\frac{\partial \hat{\omega}}{\partial t} = -NLk$$

$$\Downarrow \text{RK}(4)$$

$$\hat{\omega}(k_x, k_y)\ [\text{at time} = \text{t+dt}]$$

A quick visual

Initial Condition

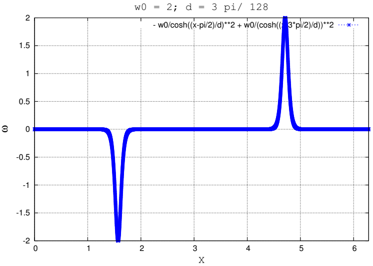
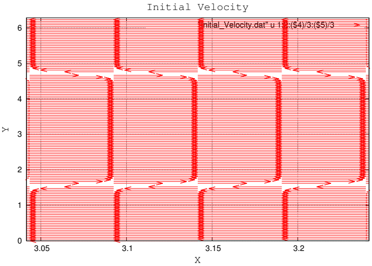


Figure: Vorticity



Velocity

- $\nu = 0.0001$

Movie in the weekend?

KH Instability

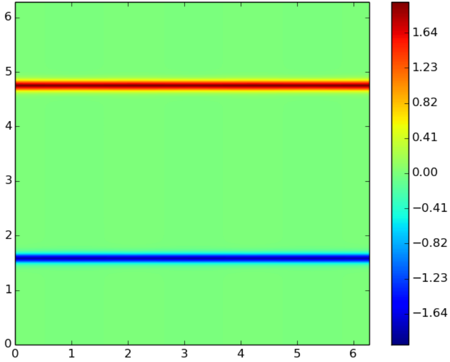
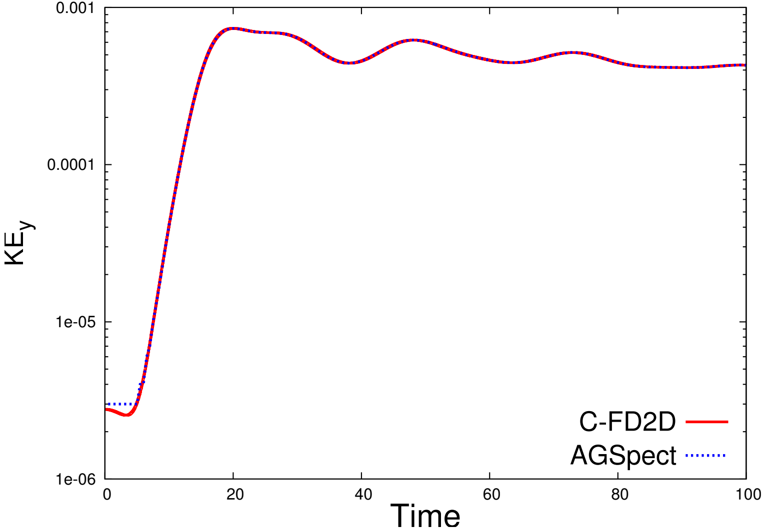


Figure: KH Instability for Incompressible Flow

KH Instability and its nonlinear saturation

Growth Rate Comparison; k = 3



Homework - 3

Analytical Growth Rate for a Broken Jet

P G Drazin, Journal of Fluid Mechanics, **10**, 579 (1961): *Eqn: 43*

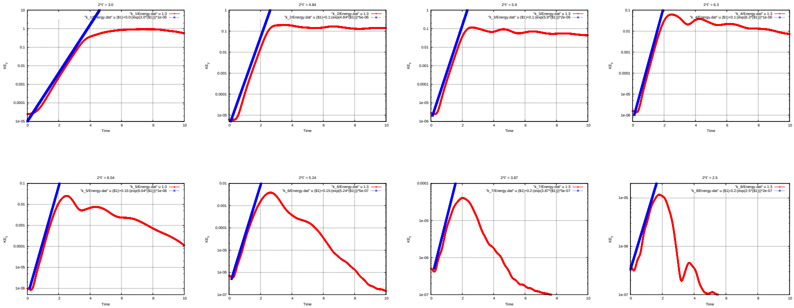
$$\gamma = \frac{k_x U_0}{3}\left[\sqrt{3} - 2\frac{k_x}{R_E} - 2\left\{\left(\frac{k_x}{R_E}\right)^2 + 2\sqrt{3}\frac{k_x}{R_E}\right\}^{\frac{1}{2}}\right]$$

$$R_E = \frac{U_0 d}{\nu}$$

$d =$ Shearing Length

Homework - 3

Numerical Growth Rate - How do you fit these curves?



- $L_x = L_y = 2\pi;\ \omega_0 = 25;\ d = \frac{3\pi}{128};\ \nu = 0.0015$

Homework - 3

Compare the Growth Rates!