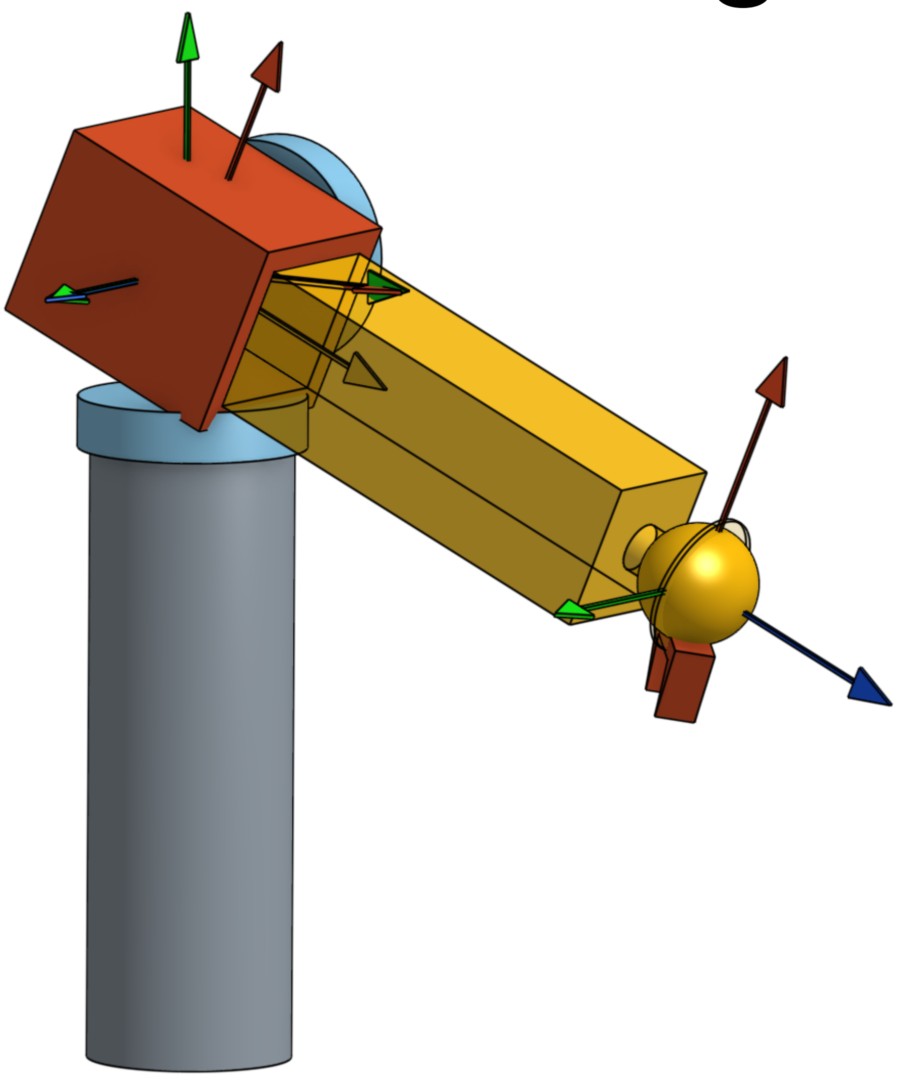
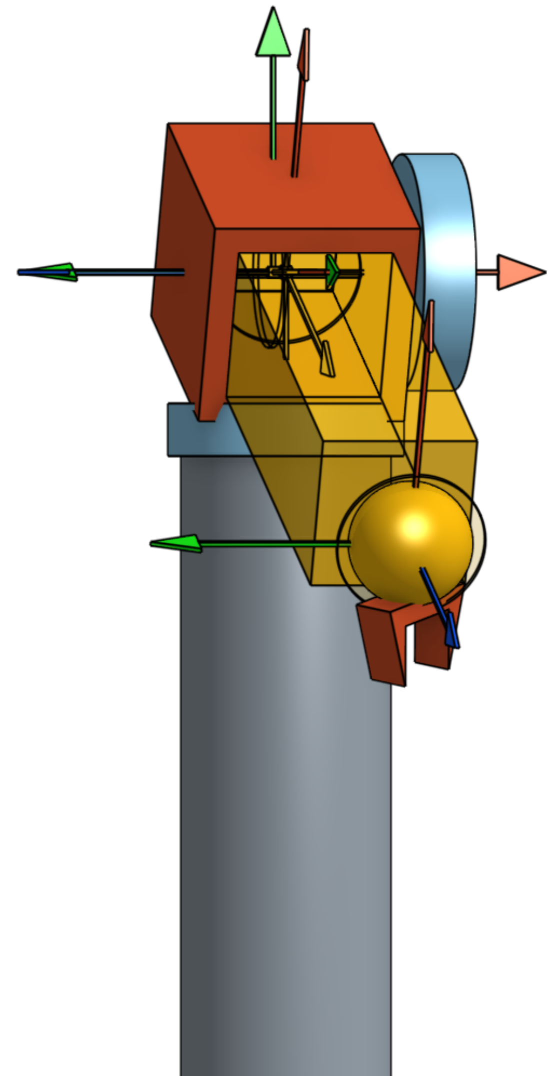
Robotics: Fundamentals

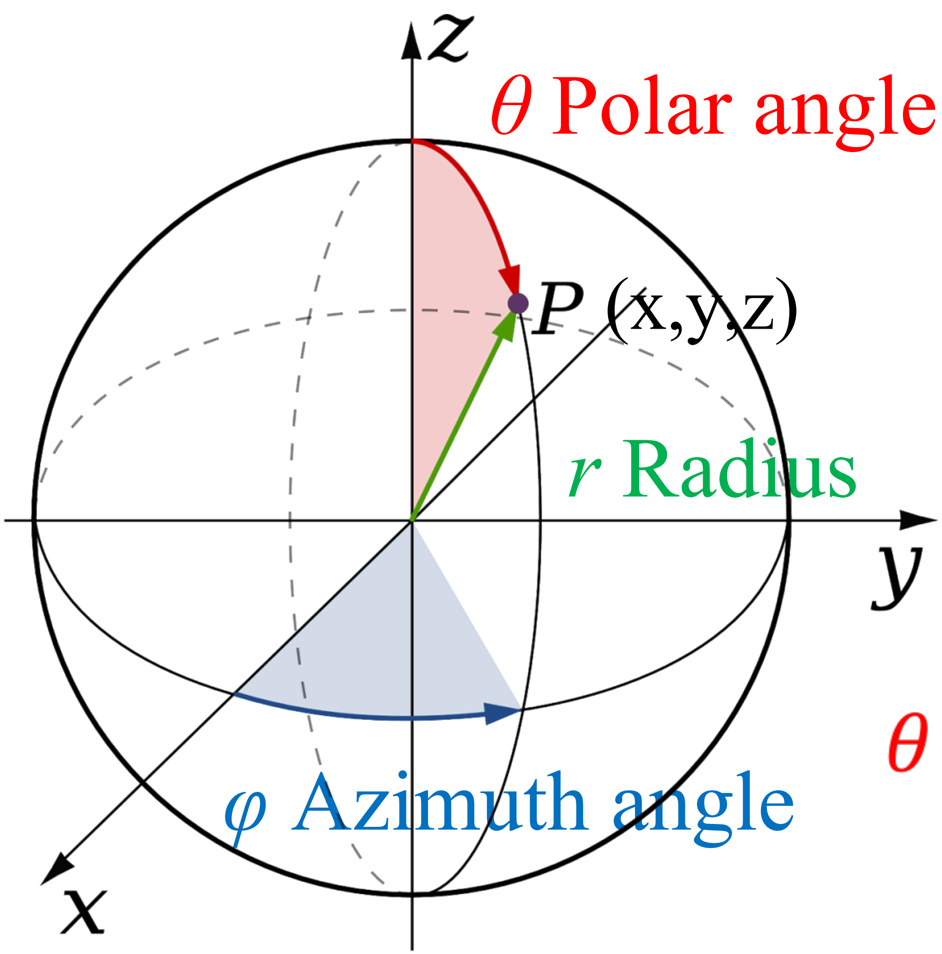
Video 6.4
Mark Yim

Inverse Position
Spherical Configuration (RRP)





Spherical Coordinates

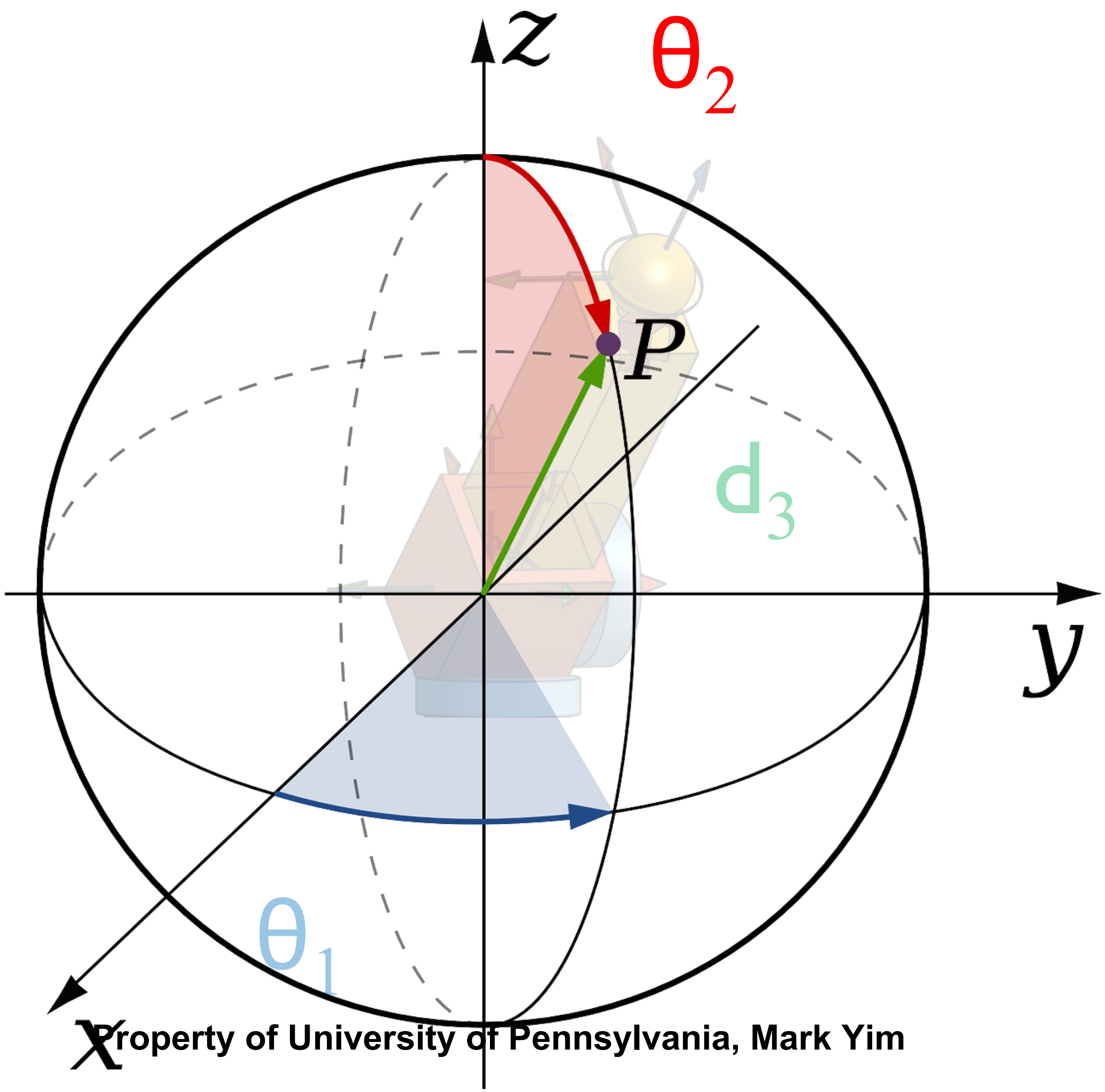


$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = atan2(y, x)$$

$$\theta = atan2\left(\sqrt{x^2 + y^2}, z\right)$$

Inv. Pos. Spherical (RRP)



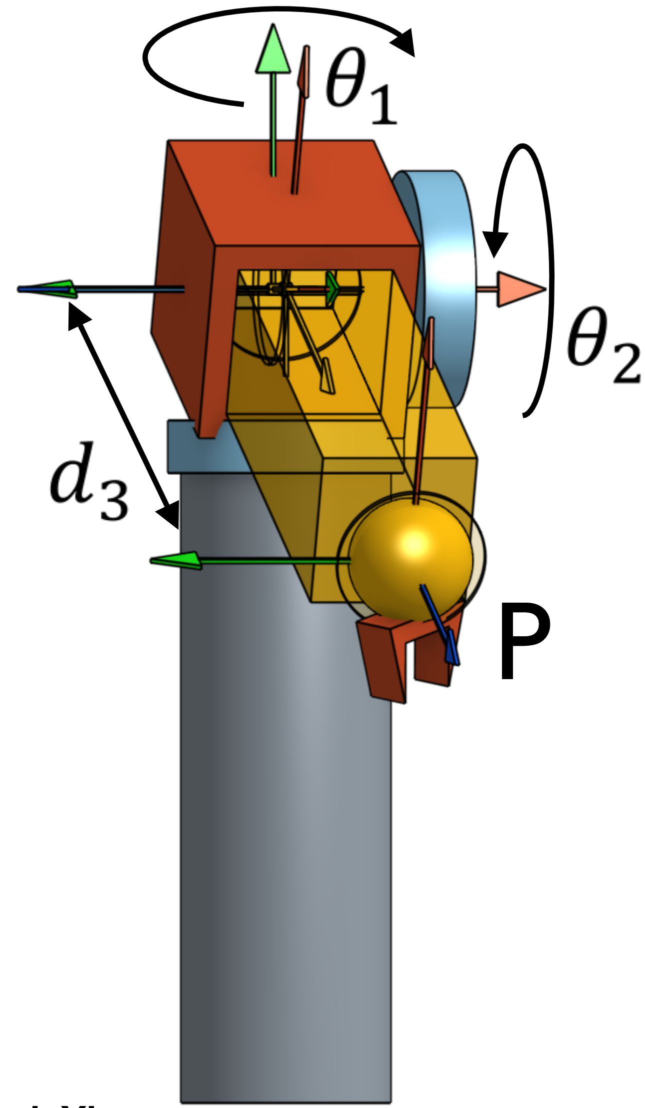
Penn Engineering

Inv. Pos. Spherical (RRP)

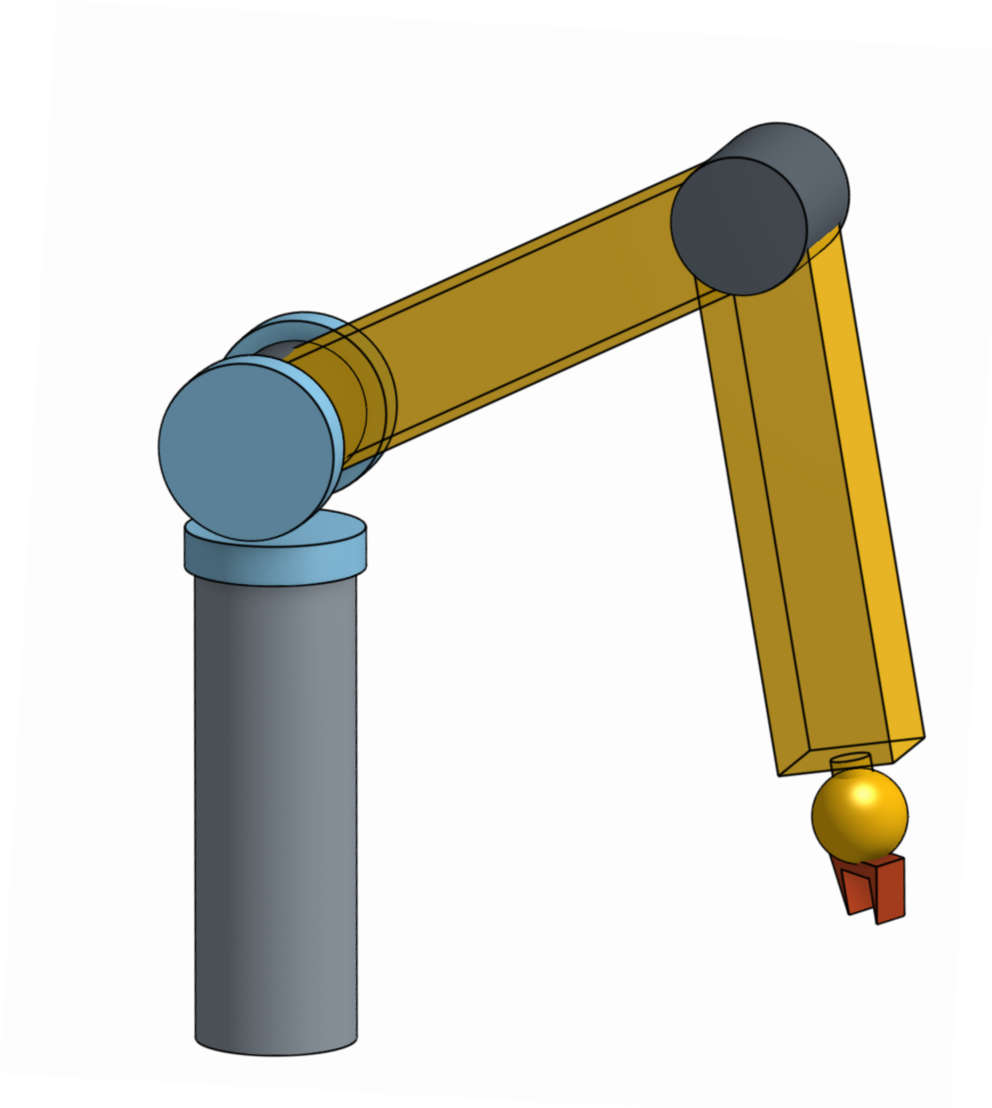
$$\boxed{\theta_1 = atan2(P_y, P_x)}$$

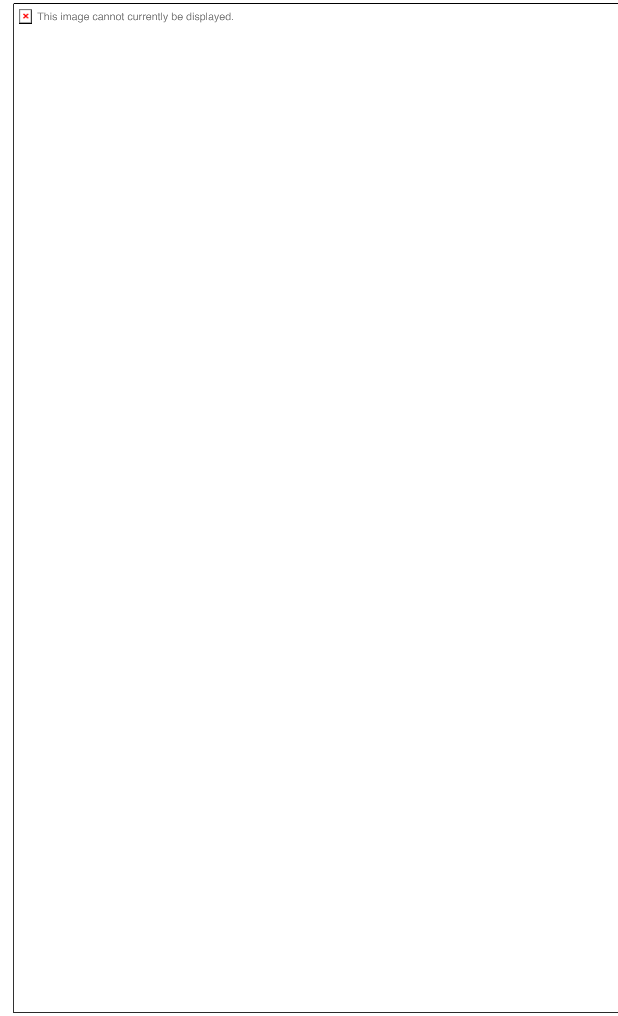
$$\boxed{\theta_2 = atan2\left(\sqrt{P_x^2 + P_y^2}, P_z\right)}$$

$$\boxed{d_3 = \sqrt{P_x^2 + P_y^2 + P_z^2}}$$



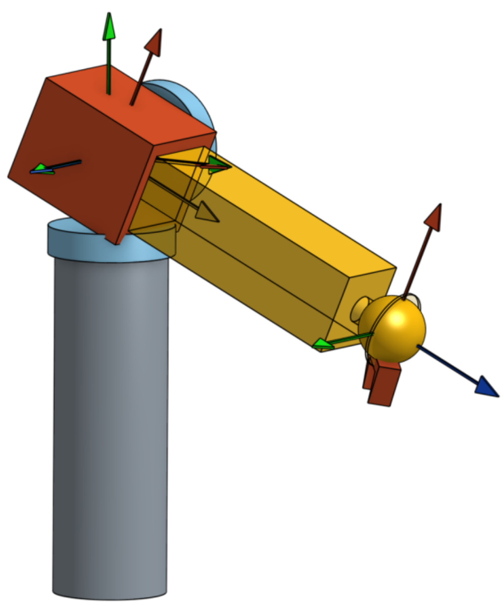
Inv. Pos. Articulated (RRR)

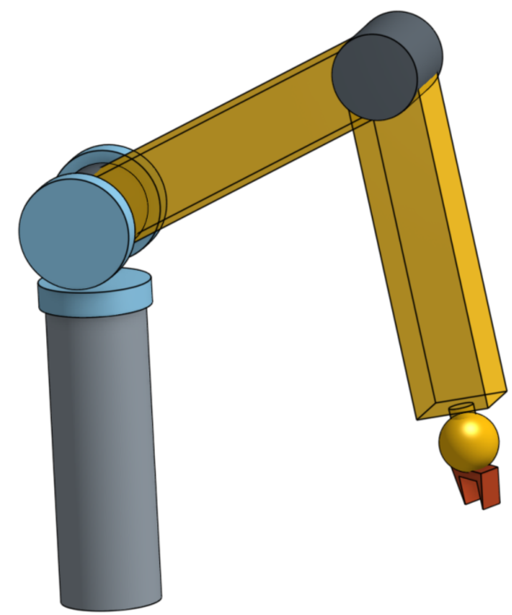




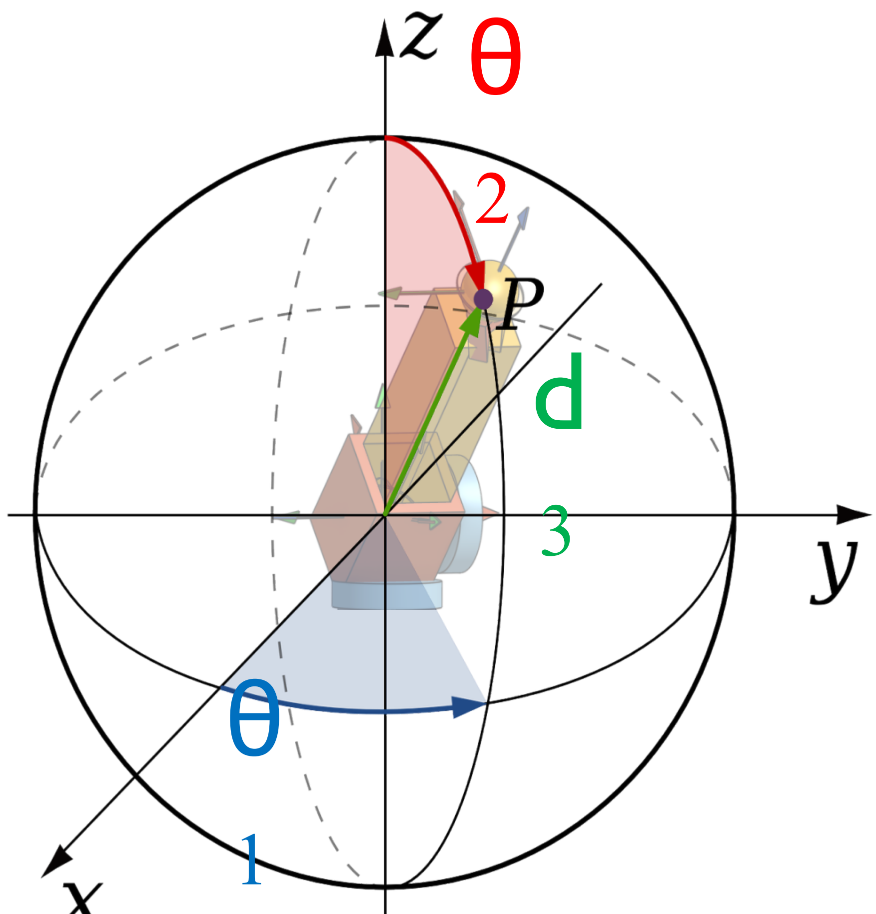
Inv. Pos. Articulated (RRR)

[insert videos here running next
To each other simultaneously]

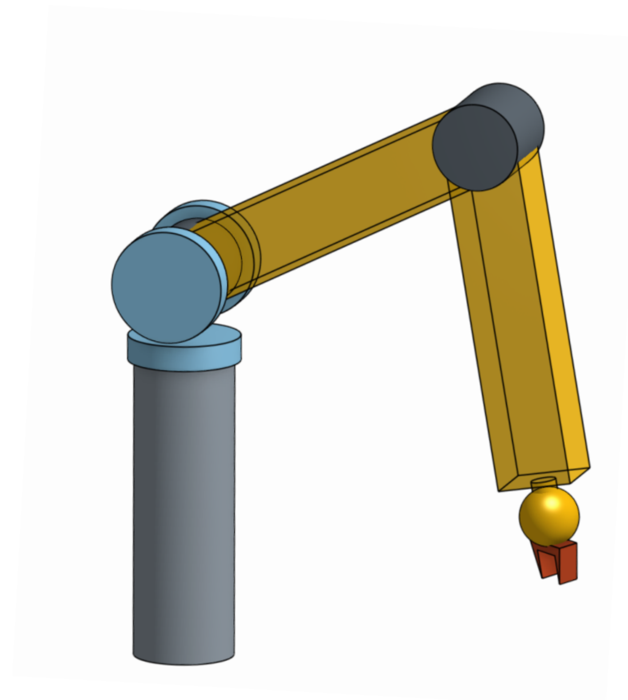




Inv. Pos. Articulated (RRR)

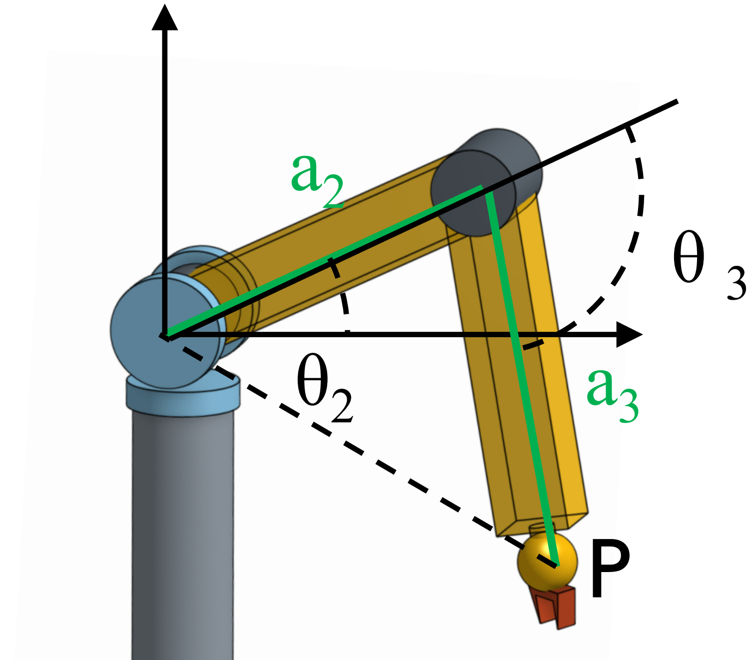


$$\phi = atan2(y, x)$$



$$\boxed{\theta_1 = atan2(y, x)}$$

Inv. Pos. Articulated (RRR)



$$\boxed{\theta_1 = atan2(P_y, P_x)}$$

$$\boxed{\theta_2 = \mathrm{atan2}(P_y, P_x) - \mathrm{atan2}(a_3 s_3,\, a_2 + a_3 c_3)}$$

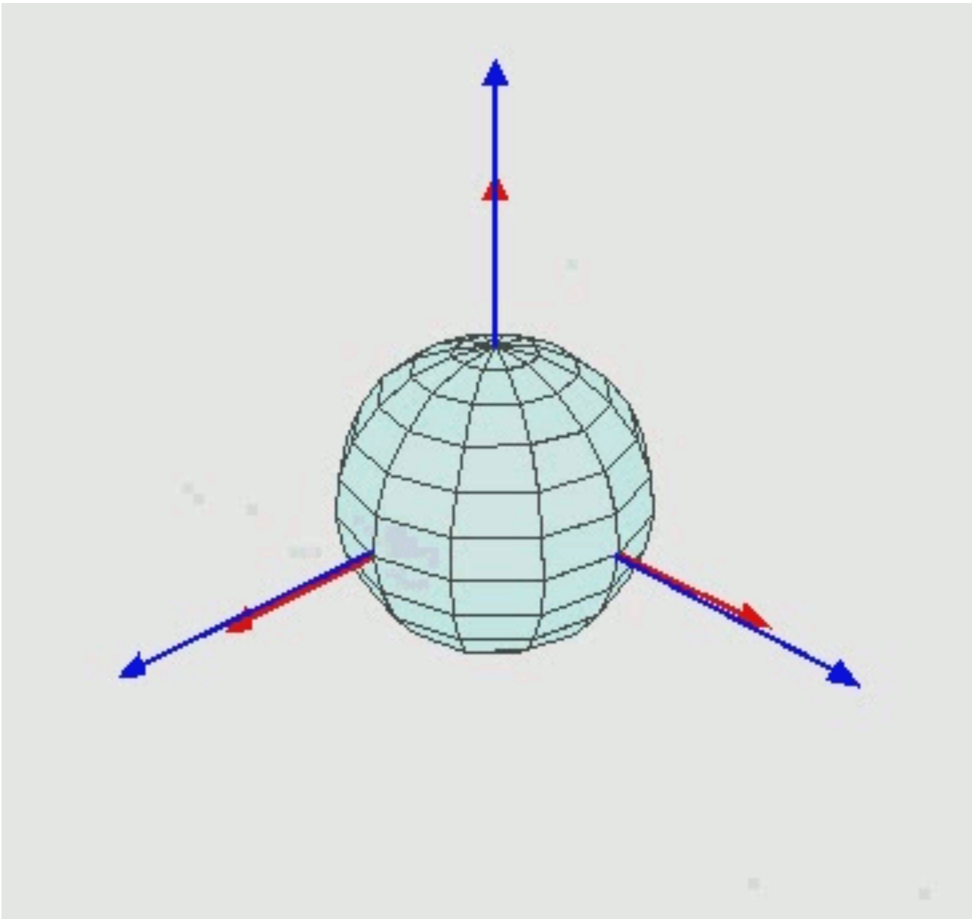
$$\boxed{\theta_3 = \pm\cos^{-1}\left(\frac{P_x^2 + P_y^2 - (a_2^2 + a_3^2)}{2a_2 a_3}\right)}$$

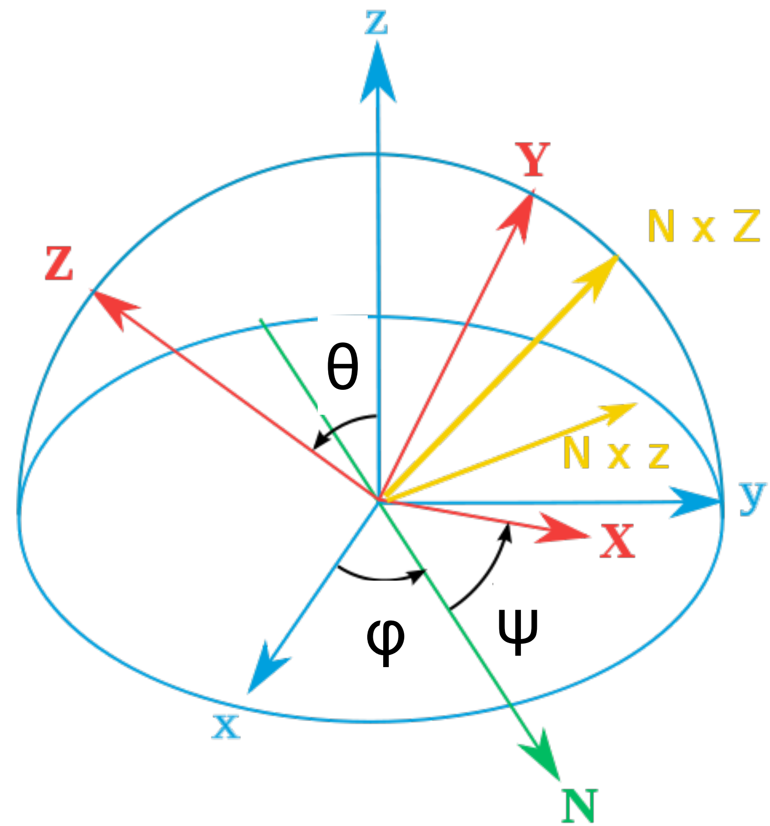
Robotics: Fundamentals

Video 6.5

Mark Yim

Sequential Rotations





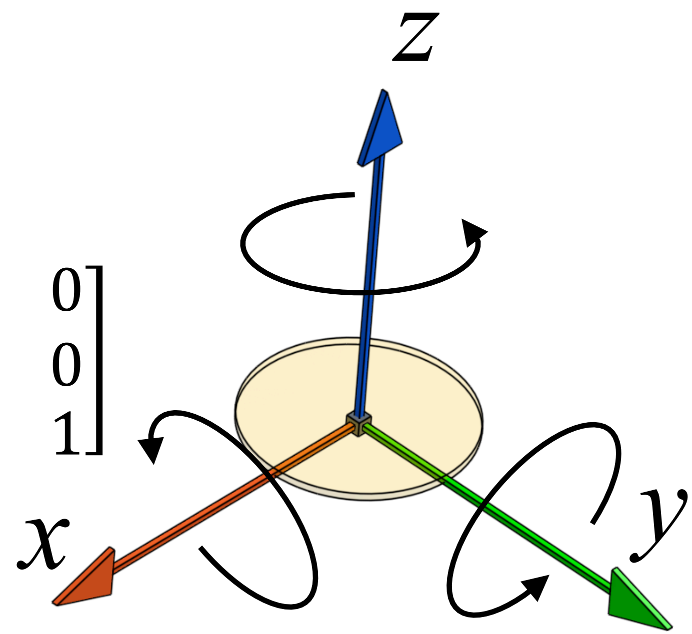
By Euler2.gif: Juansempere derivative work: Xavax - This file was derived from Euler2.gif:, CC BY-SA 3.0,

By Juansempere - Own work, GFDL,

Euler Angle Rotation

$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

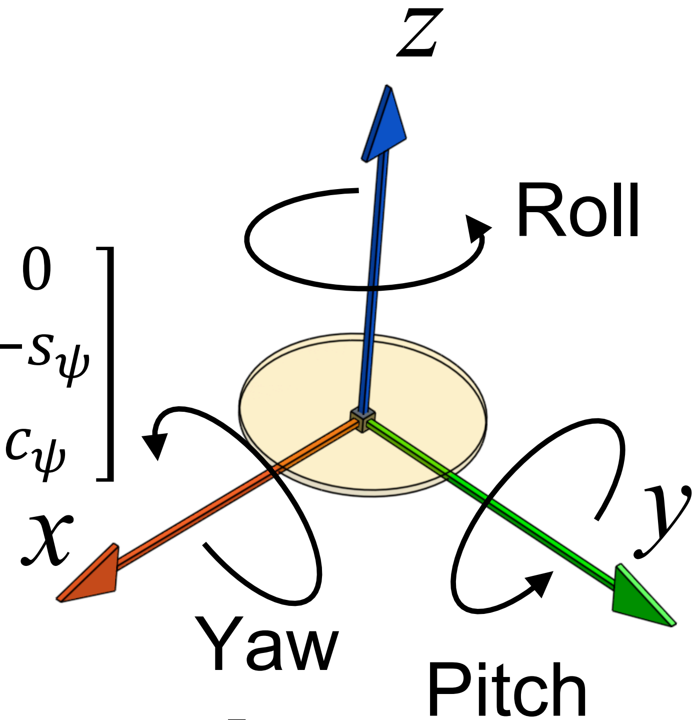


$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Roll Pitch Yaw Rotation

$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{X,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$



$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\psi s_\theta & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

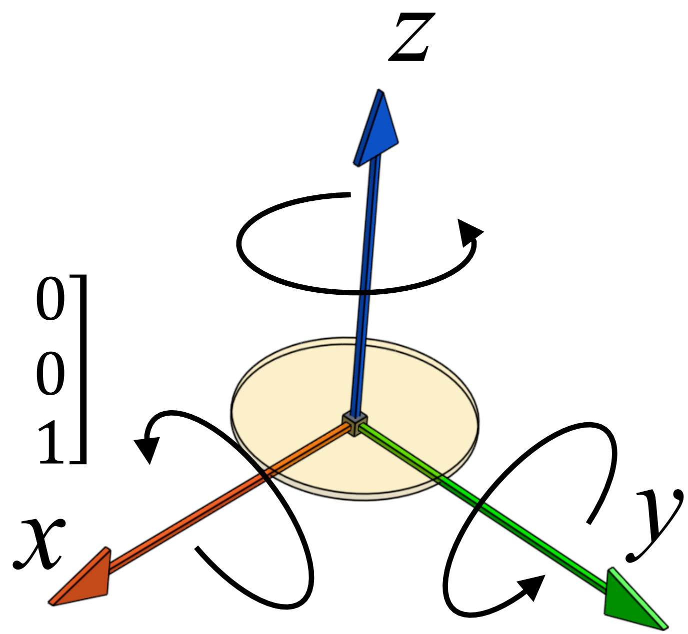
12 possible orderings

Proper Euler angles	Tait-Bryan angles
$X_1Z_2X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$X_1Z_2Y_3 = \begin{bmatrix} c_2 c_3 & -s_2 & c_2 s_3 \\ s_1 s_3 + c_1 c_3 s_2 & c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 \\ c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 \end{bmatrix}$
$X_1Y_2X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$X_1Y_2Z_3 = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 \\ s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1X_2Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$Y_1X_2Z_3 = \begin{bmatrix} c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 \\ c_2 s_3 & c_2 c_3 & -s_2 \\ c_1 s_2 s_3 - c_3 s_1 & c_1 c_3 s_2 + s_1 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1Z_2Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$Y_1Z_2X_3 = \begin{bmatrix} c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 \\ s_2 & c_2 c_3 & -c_2 s_3 \\ -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 \end{bmatrix}$
$Z_1Y_2Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$	$Z_1Y_2X_3 = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix}$
$Z_1X_2Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$	$Z_1X_2Y_3 = \begin{bmatrix} c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 \\ c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 \\ -c_2 s_3 & s_2 & c_2 c_3 \end{bmatrix}$

Euler Angle Rotation

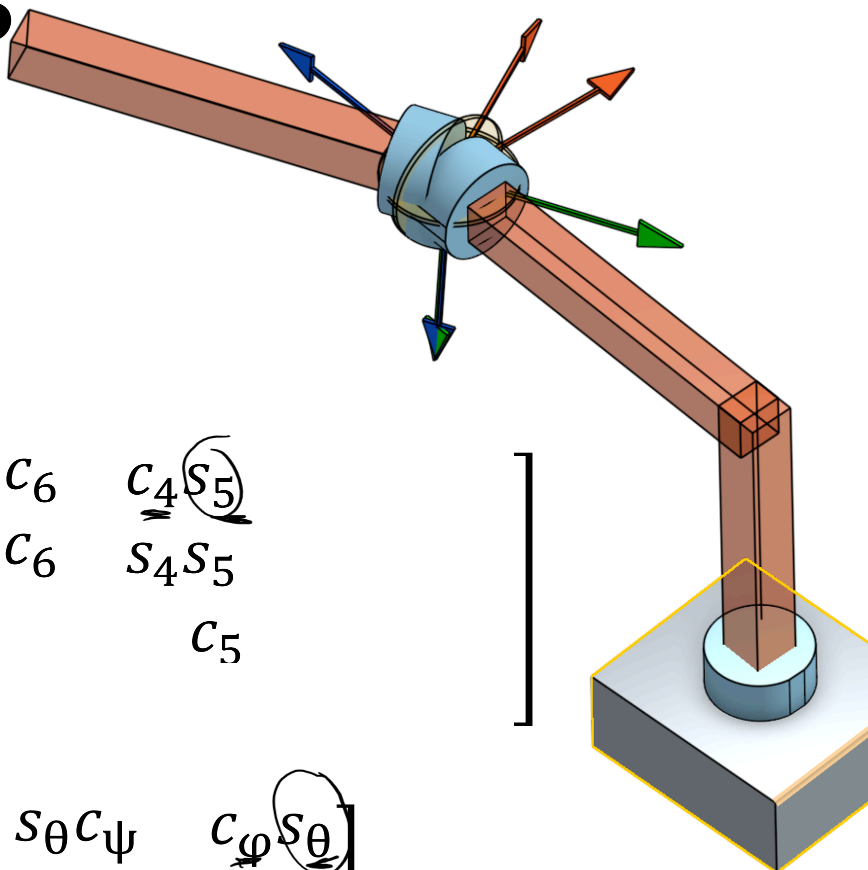
$$R_{01} = R_{Z,\phi} R_{Y,\theta} R_{Z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Euler Angle vs Spherical Wrist



$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

$$= \begin{bmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\theta c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$\theta_4 = \phi, \quad \theta_5 = \theta, \quad \theta_6 = \psi$$

Inverse Orientation (ZYZ)

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 = ?$$
$$\theta_5 = ?$$
$$\theta_6 = ?$$

$$c_5 = r_{33}$$

$$\theta_5 = \pm cos^{-1}(r_{33})$$

Euler Angle Rotation

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow \theta_4 = ?$

$\rightarrow \boxed{\theta_5 = \pm \cos^{-1}(r_{33})}$

$\rightarrow \theta_6 = ?$

$$\underline{atan2(y,x)} = atan2(Cy, Cx), C > 0$$

$$\rightarrow \theta_4 = atan2(r_{23}, r_{13}), \quad \underline{s_5 > 0}$$

$$\theta_4 = atan2(-r_{23}, -r_{13}), \underline{s_5 < 0}$$

Penn Engineering

Euler Angle Rotation

$$T_{36} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_5c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\theta_4 = atan2(\pm r_{23}, \pm r_{13})}$$

$$\boxed{\theta_5 = \pm cos^{-1}(r_{33})}$$

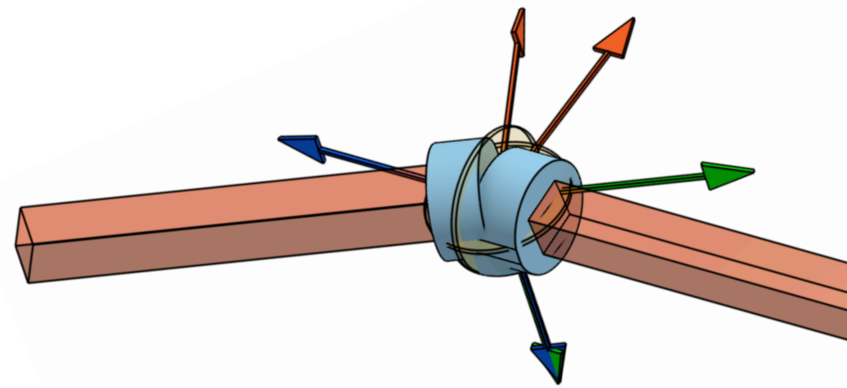
$$\rightarrow \theta_6 = ?$$

$$atan2(y, x) = atan2(Cy, Cx), C > 0$$

$$\theta_6 = atan2(r_{32}, -r_{31}), \quad s_5 > 0 \leftarrow$$

$$\theta_6 = atan2(-r_{32}, r_{31}), \quad s_5 < 0 \leftarrow$$

Spherical Wrist (ZYZ)



$s_5 > 0$ Two solutions $s_5 < 0$

$$\theta_4 = atan2(r_{23}, r_{13})$$

$$\theta_5 = cos^{-1}(r_{33})$$

$$\theta_6 = atan2(r_{32}, -r_{31})$$

$$\theta_4 = atan2(-r_{23}, -r_{13})$$

$$\theta_5 = -cos^{-1}(r_{33})$$

$$\theta_6 = atan2(-r_{32}, r_{31})$$

Robotics: Fundamentals

Video 6.6a
Mark Yim

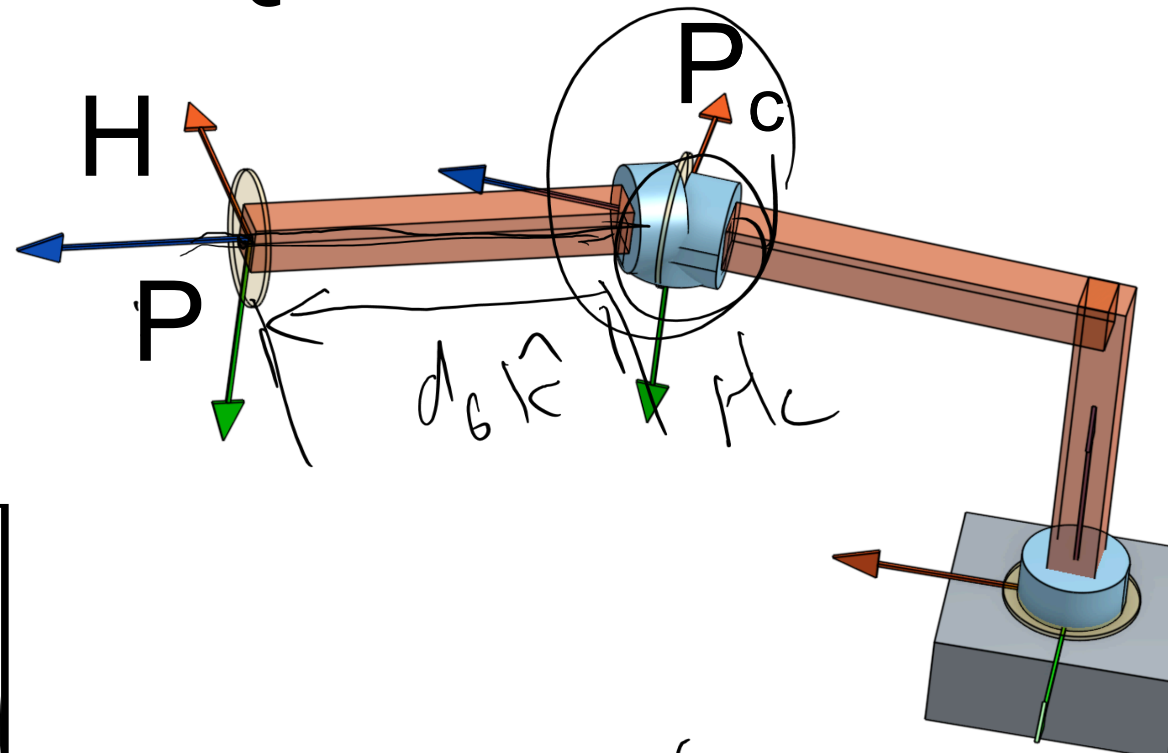
Inverse Pos. + Ori. For P_c

- From Inverse Position we have the joint variables $q_{1\ldots3}$ for $T_{03} = \begin{bmatrix} R_{03} & P_c \\ 0 & 1 \end{bmatrix}$
- From Inverse Orientation we have the joint variables $q_{4\ldots6}$ for a given arbitary R
- Can we put the two together to solve our initial problem for 6DOF?

$$\boldsymbol{T}_{06}(q_1 \cdots q_6) = \boldsymbol{H}$$

Offset from P_c wrist center

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} \\ \boldsymbol{0} & 1 \end{bmatrix}$$



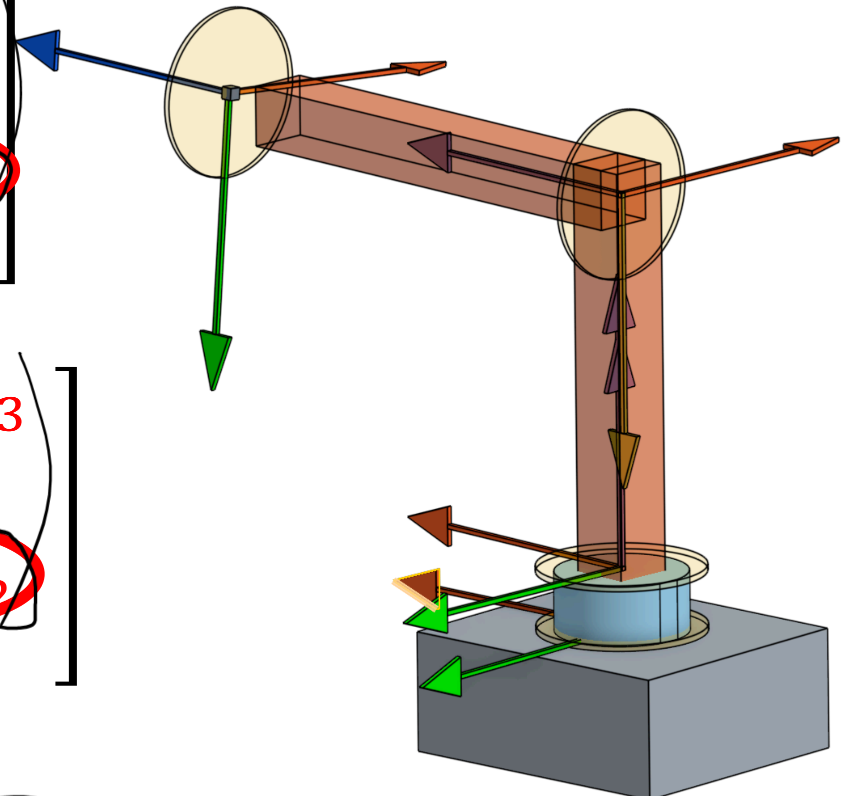
$$\boldsymbol{P_c} = \boldsymbol{P} - d_6\hat{\boldsymbol{k}}$$

$$\begin{bmatrix} P_{cx} \\ P_{cy} \\ P_{cz} \end{bmatrix} = \begin{bmatrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{bmatrix}$$

$$\boldsymbol{H_c} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{P} - d_6\hat{\boldsymbol{k}} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{R}] & \begin{matrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{matrix} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix}$$

Offset from P_c wrist center

$$H_c = \begin{bmatrix} [\boldsymbol{R}] & \begin{matrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{matrix} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix}$$



$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

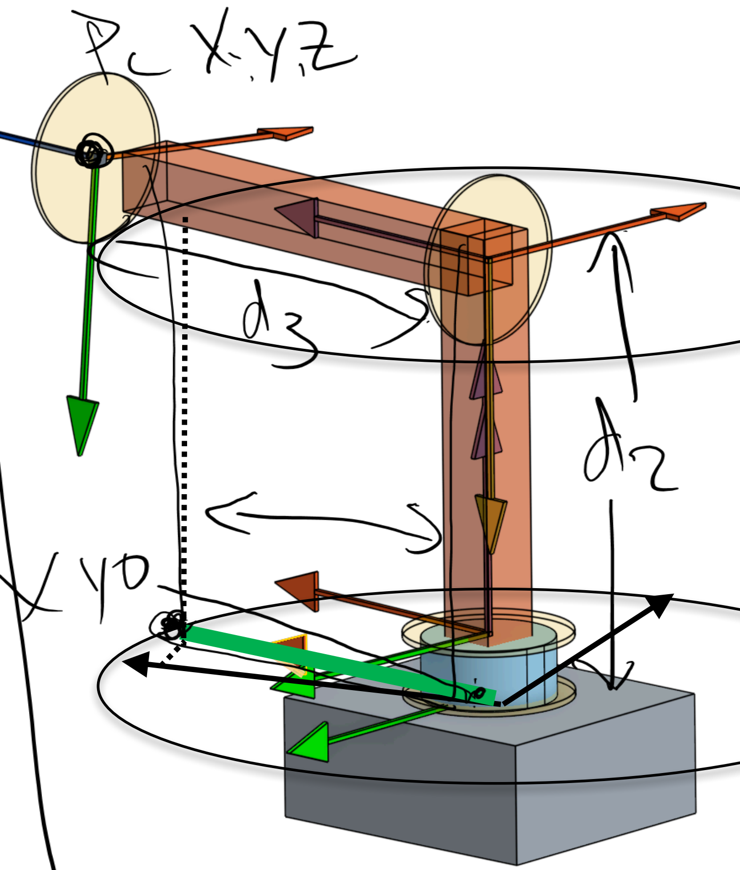
$$\boxed{d_2 = P_z - d_6 r_{33} - d_1}$$

Offset from P_c wrist center

$$H_c = \begin{bmatrix} & [\boldsymbol{R}] & & P_x - d_6 r_{13} \\ & & & P_y - d_6 r_{23} \\ & & & P_z - d_6 r_{33} \\ [0 & 0 & 0] & & & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_3 = \sqrt{x^2 + y^2}$$

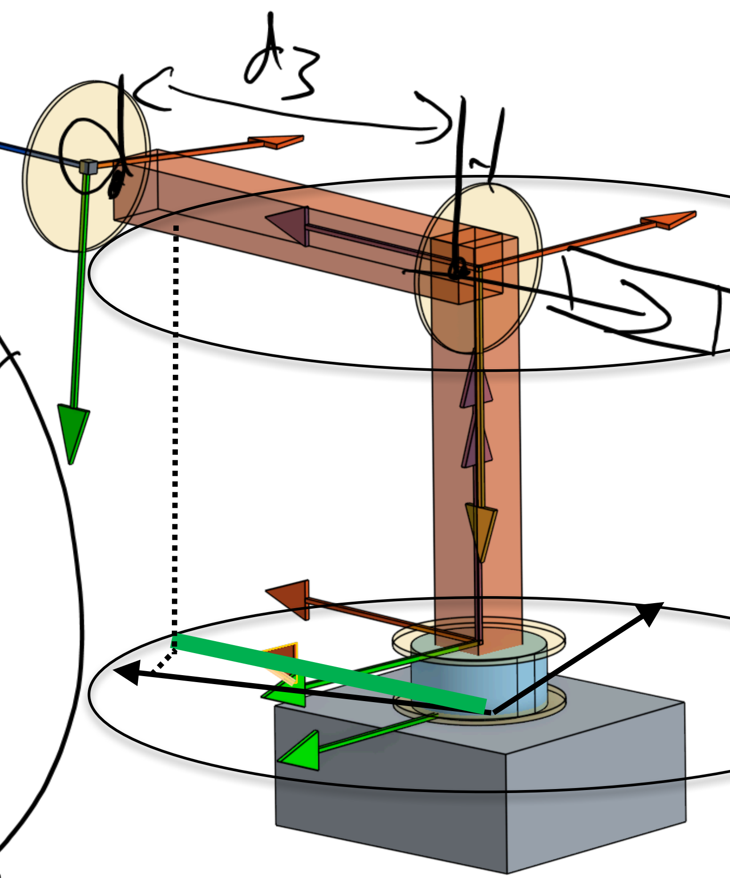


$$\boxed{d_3 = \sqrt{\left(P_y - d_6 r_{23}\right)^2 + \left(P_x - d_6 r_{13}\right)^2}}$$

Offset from P_c wrist center

$$H_c = \begin{bmatrix} [\boldsymbol{R}] & \begin{matrix} P_x - d_6 r_{13} \\ P_y - d_6 r_{23} \\ P_z - d_6 r_{33} \end{matrix} \\ [0 \quad 0 \quad 0] & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



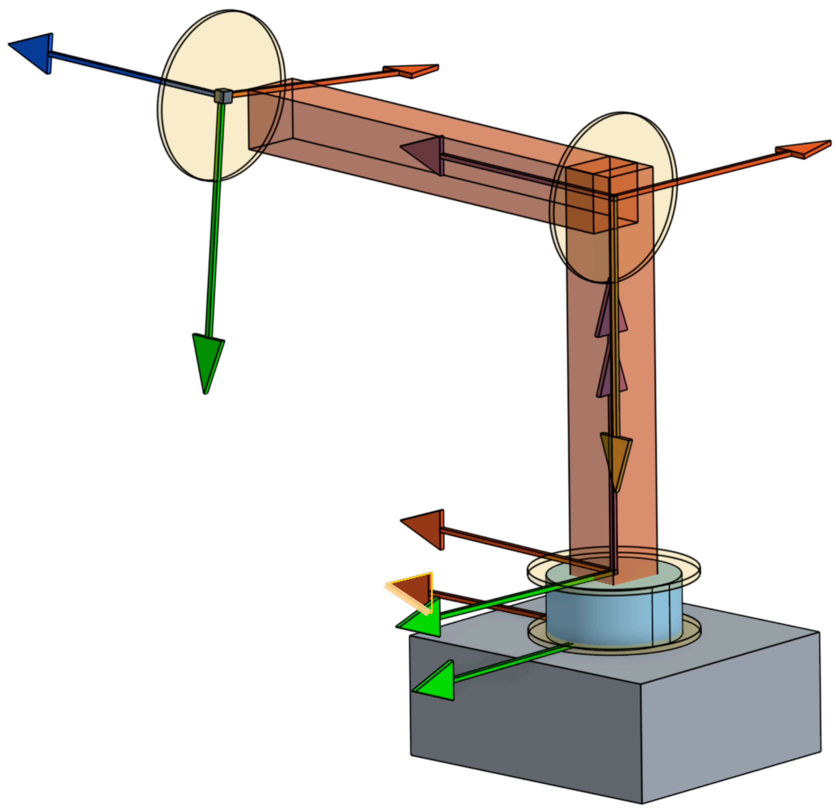
$$\theta_1 = atan2(Csin, \mathrm{C}cos), C > 0$$

$$\boxed{\theta_1 = atan2(-(P_x - d_6 r_{13}), P_y - d_6 r_{23}), d_3 > 0}$$

$$\boxed{\theta_1 = atan2(P_x - d_6 r_{13}, -P_y + d_6 r_{23}), d_3 < 0}$$

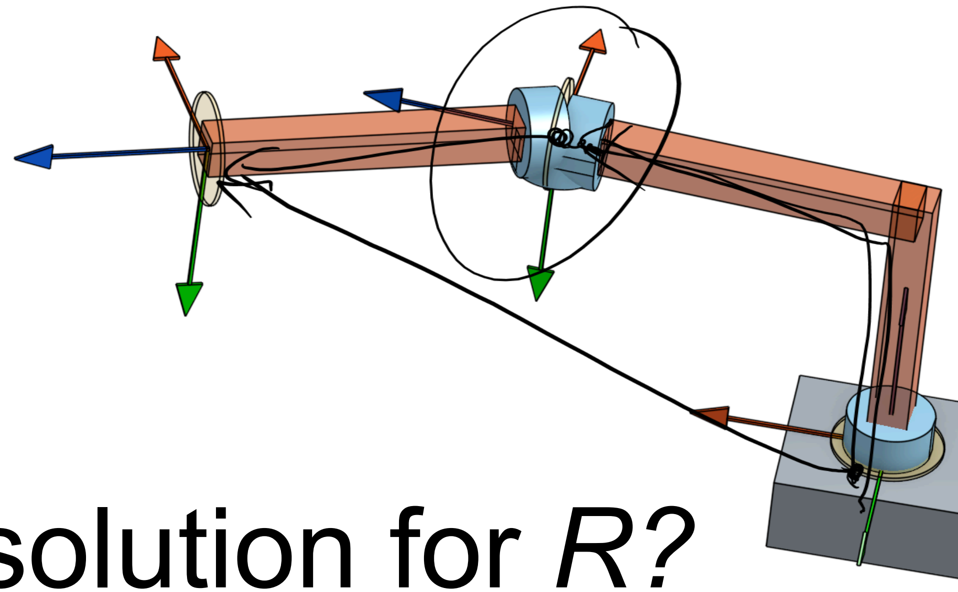
2 Solutions with negative d_3

- [insert video showing sliding back and forth (going negative)]



Inverse Orientation

$$H_c = \begin{bmatrix} [R] & [P - d_6\hat{k}] \\ [0] & 1 \end{bmatrix}$$

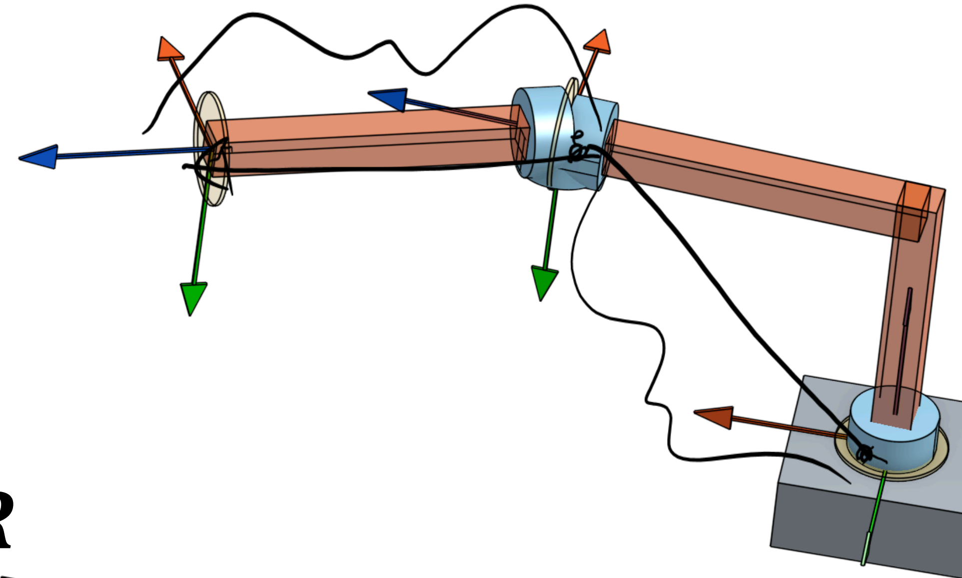


Spherical Wrist solution for *R*?

$$\begin{cases} \theta_4 = atan2(\pm r_{23}, \pm r_{13}) \\ \theta_5 = \pm\cos^{-1}(r_{33}) \\ \theta_6 = atan2(\mp r_{32}, \pm r_{31}) \end{cases}$$

Inverse Orientation

$$H_c = \begin{bmatrix} [R] & [P - d_6\hat{k}] \\ [0] & 1 \end{bmatrix}$$



$$T_{06} = \begin{bmatrix} [R_{03}][R_{36}] & P_c \\ [0] & 1 \end{bmatrix}$$

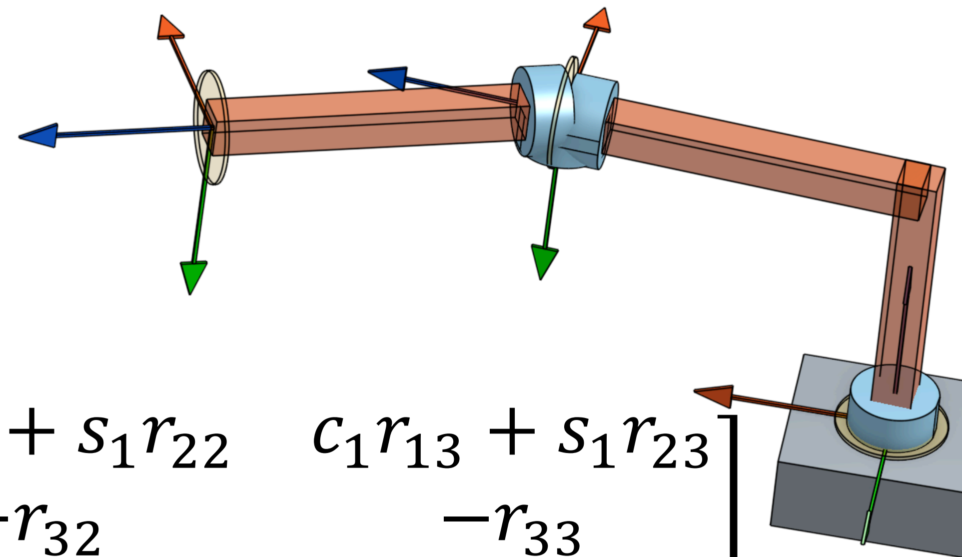
$$R_{06} = R_{03}R_{36} = R$$

$$R_{36} = (R_{03})^{-1}R$$

$$R_{36} = (R_{03})^{T}R$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Orientation



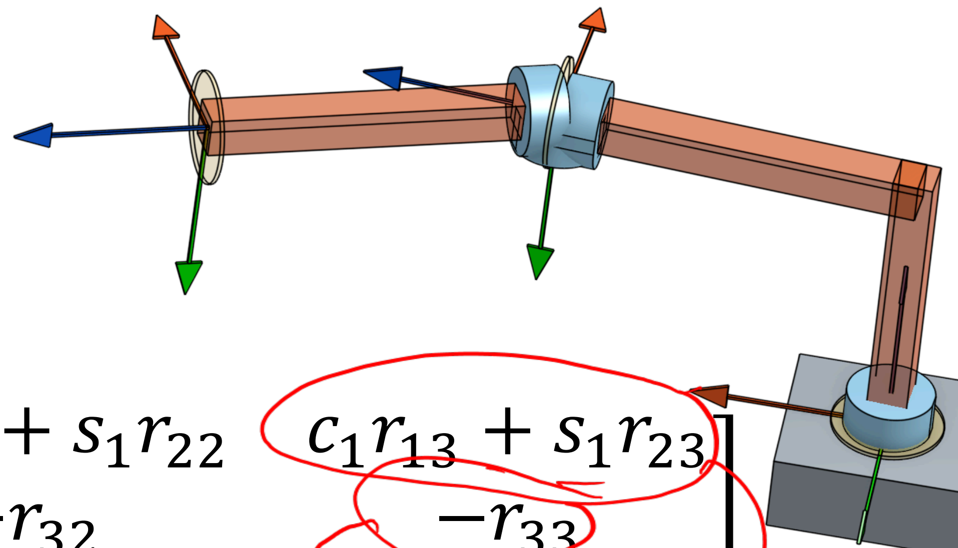
$$R_{36} = \begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} \\ -r_{31} & -r_{32} & -r_{33} \\ c_1 r_{21} - s_1 r_{11} & c_1 r_{22} - s_1 r_{12} & c_1 r_{23} - s_1 r_{13} \end{bmatrix}$$

$$R_{36} = (R_{03})^T R$$

$$T_{03} = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Orientation

$$\theta_4 = atan2(r_{23}, r_{13})$$
$$\theta_5 = \pm\cos^{-1}(r_{33})$$
$$\theta_6 = atan2(-r_{32}, r_{31})$$



$$R_{36} = \begin{bmatrix} c_1 r_{11} + s_1 r_{21} & c_1 r_{12} + s_1 r_{22} & c_1 r_{13} + s_1 r_{23} \\ -r_{31} & -r_{32} & -r_{33} \\ c_1 r_{21} - s_1 r_{11} & c_1 r_{22} - s_1 r_{12} & c_1 r_{23} - s_1 r_{13} \end{bmatrix}$$

$$\boxed{\theta_4 = atan2(-r_{33}, c_1 r_{13} + s_1 r_{23})}$$
$$\boxed{\theta_5 = \pm\cos^{-1}(c_1 r_{23} - s_1 r_{13})}$$
$$\boxed{\theta_6 = atan2(-\pm(c_1 r_{22} - s_1 r_{12}), \pm(c_1 r_{21} - s_1 r_{11}))}$$

IK for Cylindrical RPP

2 soln

$$\boxed{\theta_1 = atan2\left(\mp(P_x - d_6 r_{13}), \pm(P_y - d_6 r_{13})\right)}$$

$$\boxed{d_2 = P_z - d_6 r_{33} - d_1}$$

$$\boxed{d_3 = \pm\sqrt{(P_y - d_6 r_{23})^2 + (P_x - d_6 r_{13})^2}}$$

POS

2 soln

$$\boxed{\theta_4 = \ atan2(\mp r_{33}, \pm c_1 r_{13} \pm s_1 r_{23})}$$

$$\boxed{\theta_5 = \pm \cos^{-1}(c_1 r_{23} - s_1 r_{13})}$$

$$\boxed{\theta_6 = atan2(- \pm(c_1 r_{22} - s_1 r_{12}), \pm(c_1 r_{21} - s_1 r_{11}))}$$

ORI

Robotics: Fundamentals

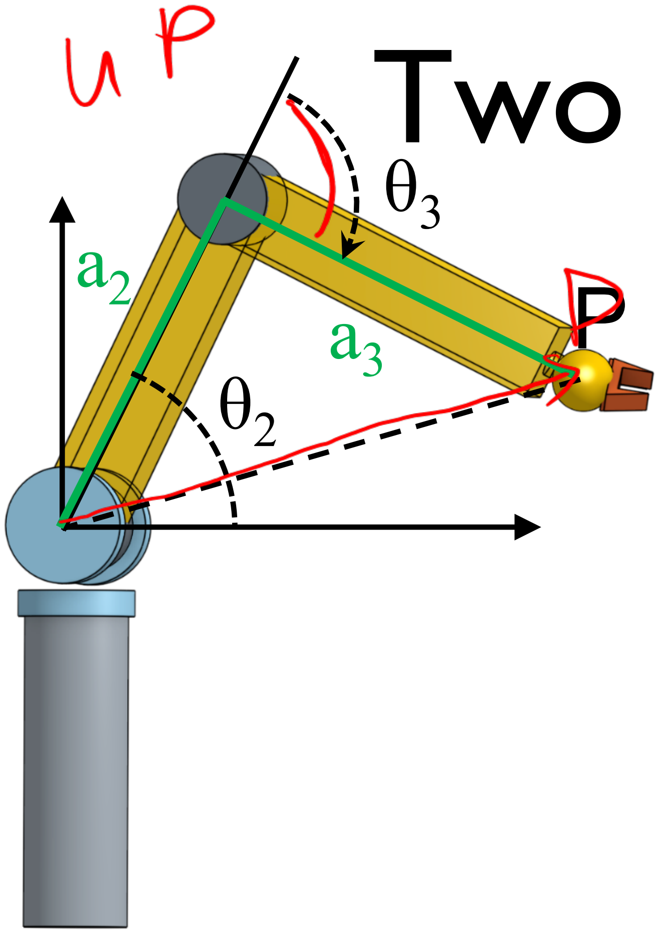
Video 6.6b

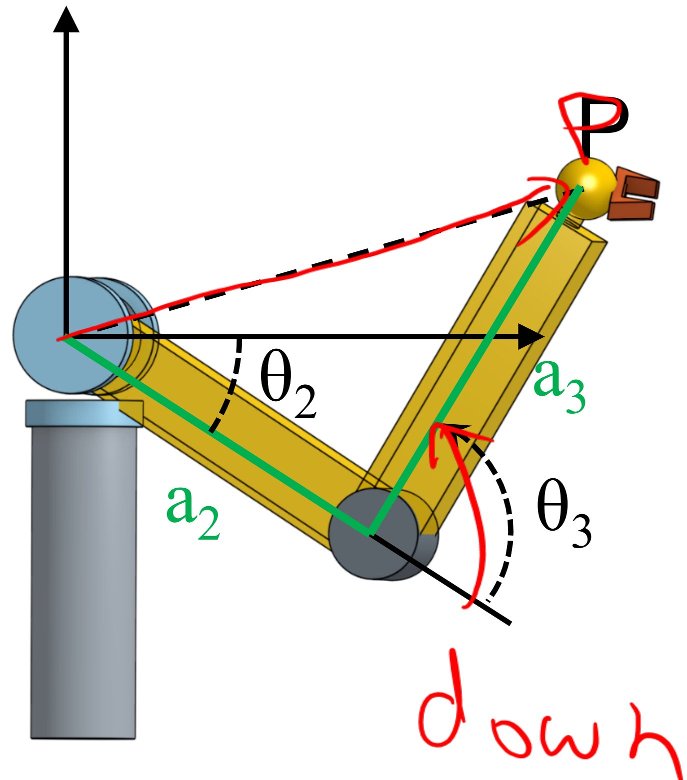
Mark Yim

Special cases

- Multiple solutions
  - How many
- Gimbal lock
- Representation singularity
- Joint singularity
- Dextrous workspace

Two solutions

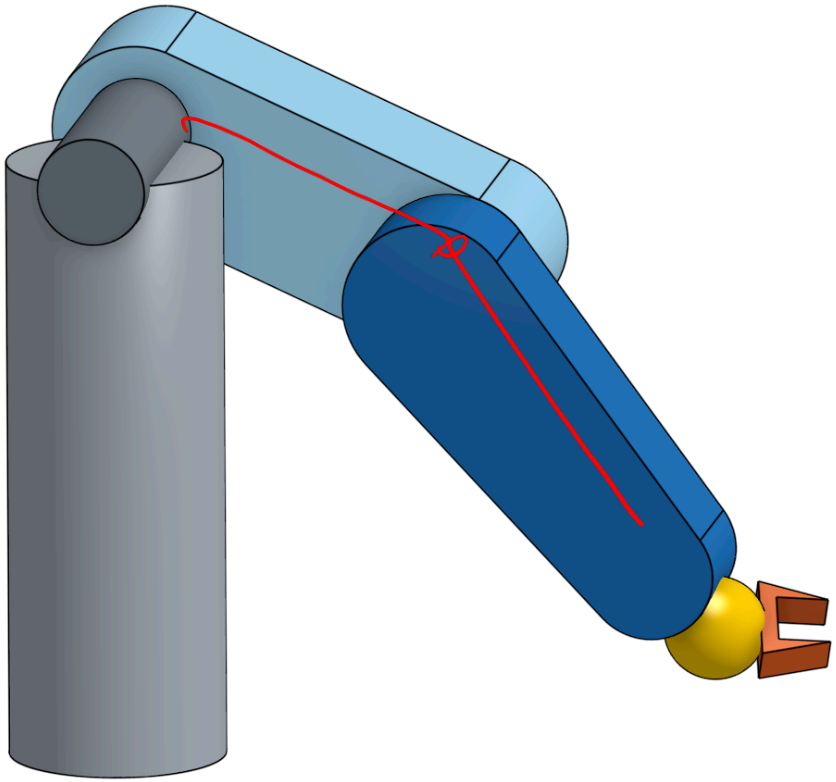


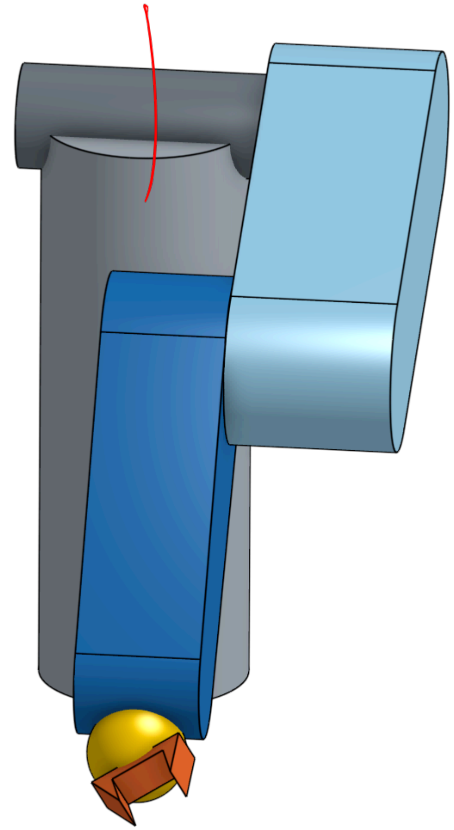


Positive θ_3

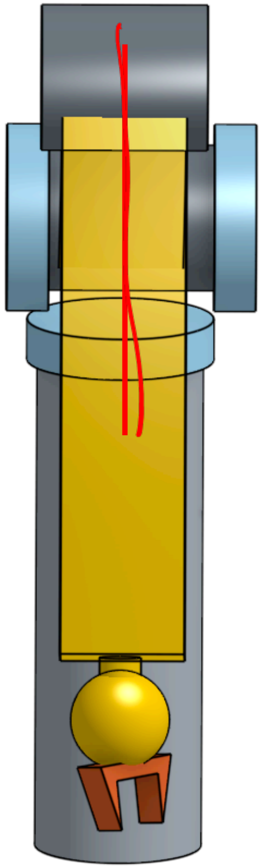
Negative θ_3

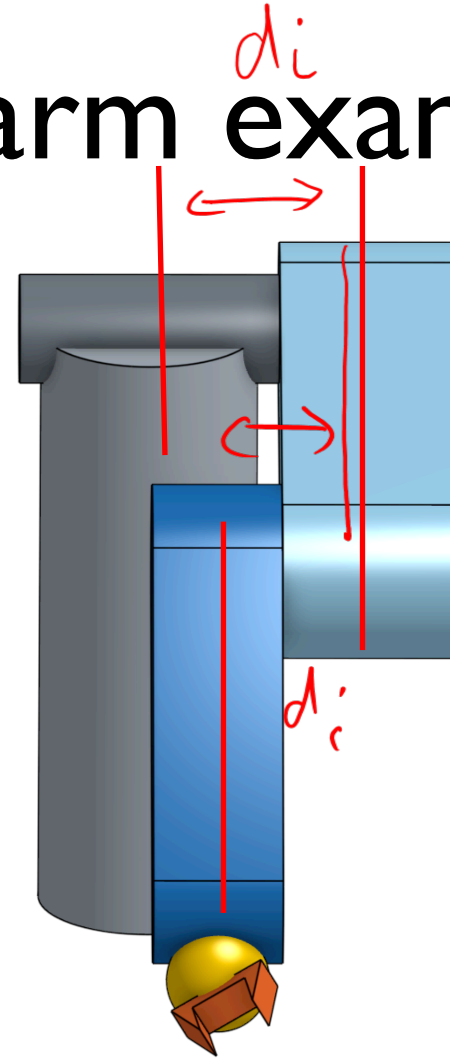
Articulated arm example



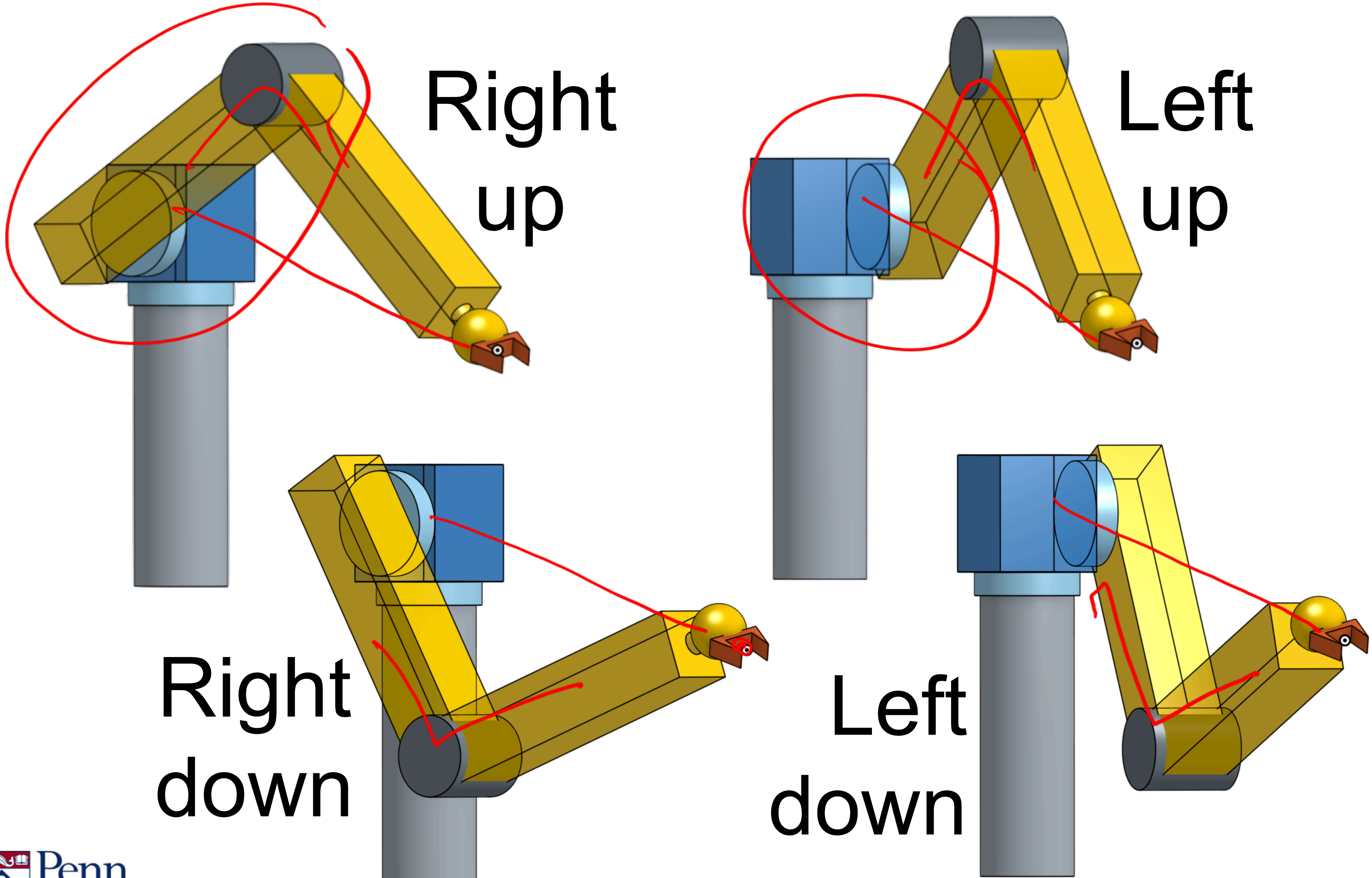


Articulated arm example





Four Solutions to 6R robot



Wrist multiple solution

$$\boxed{\theta_4 = atan2(\mp r_{33}, \pm c_1 r_{13} \pm s_1 r_{23})}$$

$$\boxed{\theta_5 = \pm cos^{-1}(c_1 r_{23} - s_1 r_{13})}$$

$$\boxed{\theta_6 = atan2(-\pm(c_1 r_{22} - s_1 r_{12}), \pm(c_1 r_{21} - s_1 r_{11}))}$$

$$\boxed{\theta_4 = atan2(-r_{33}, c_1 r_{13} + s_1 r_{23})} - \pi$$

$$\boxed{\theta_5 = + cos^{-1}(c_1 r_{23} - s_1 r_{13})} * (-1)$$

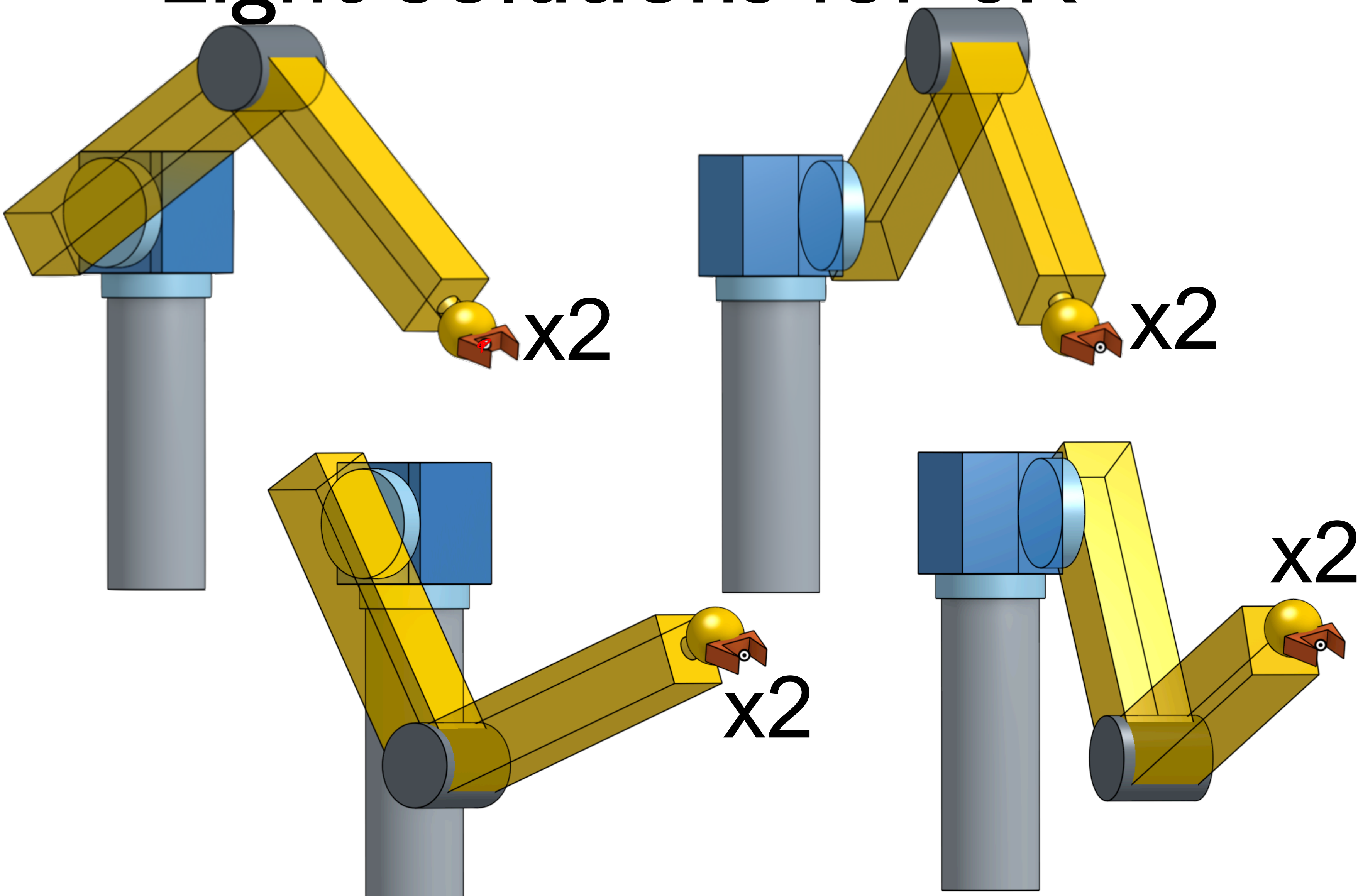
$$\boxed{\theta_6 = atan2(-c_1 r_{22} + s_1 r_{12}, c_1 r_{21} - s_1 r_{11})} - \pi$$

1st solution 2nd solution

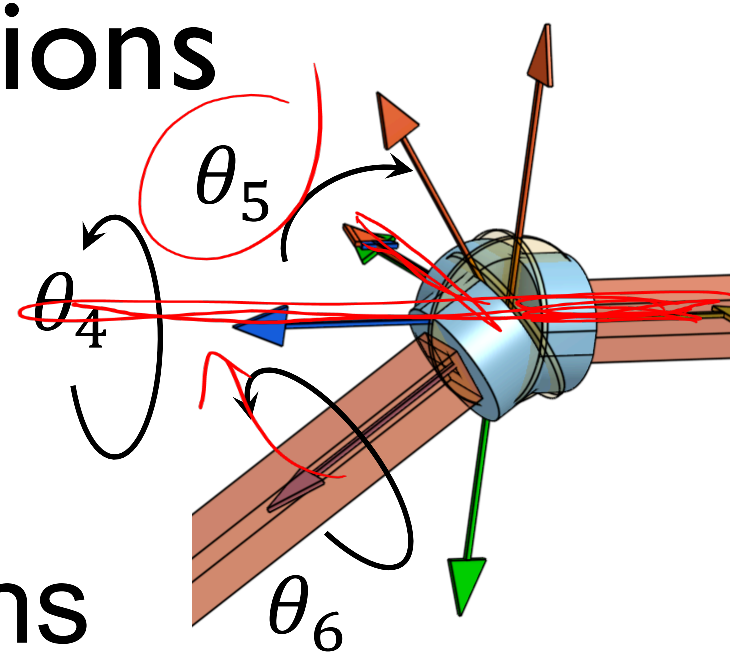
Wrist multiple solution

[insert “wrist solns.mov”]

Eight Solutions for 6R



Infinite solutions



$$\theta_5 = 0,$$
$$\psi = \theta_4 + \theta_6$$

$0 = 1 - 1$

2 ~ 2

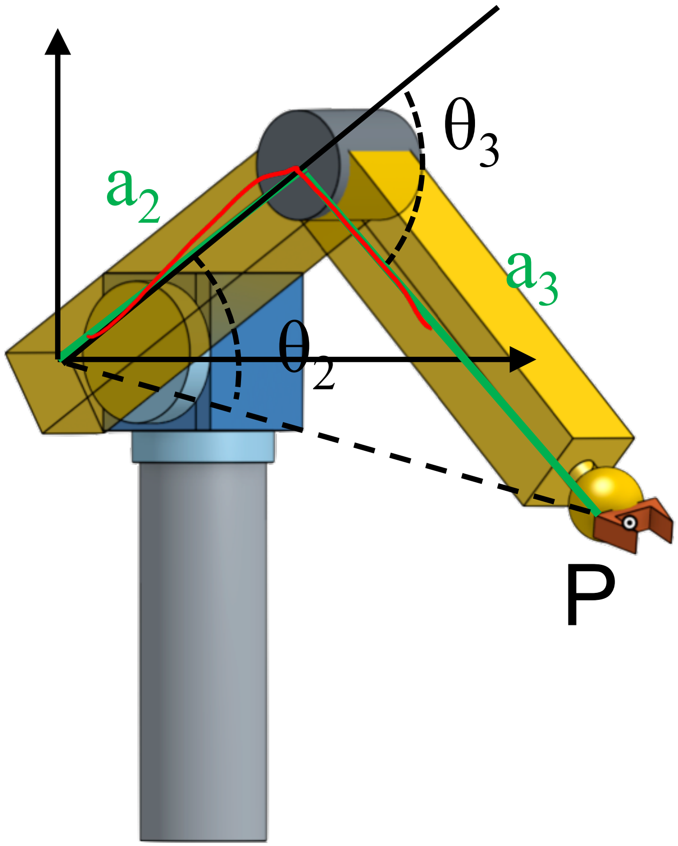
Two Solutions

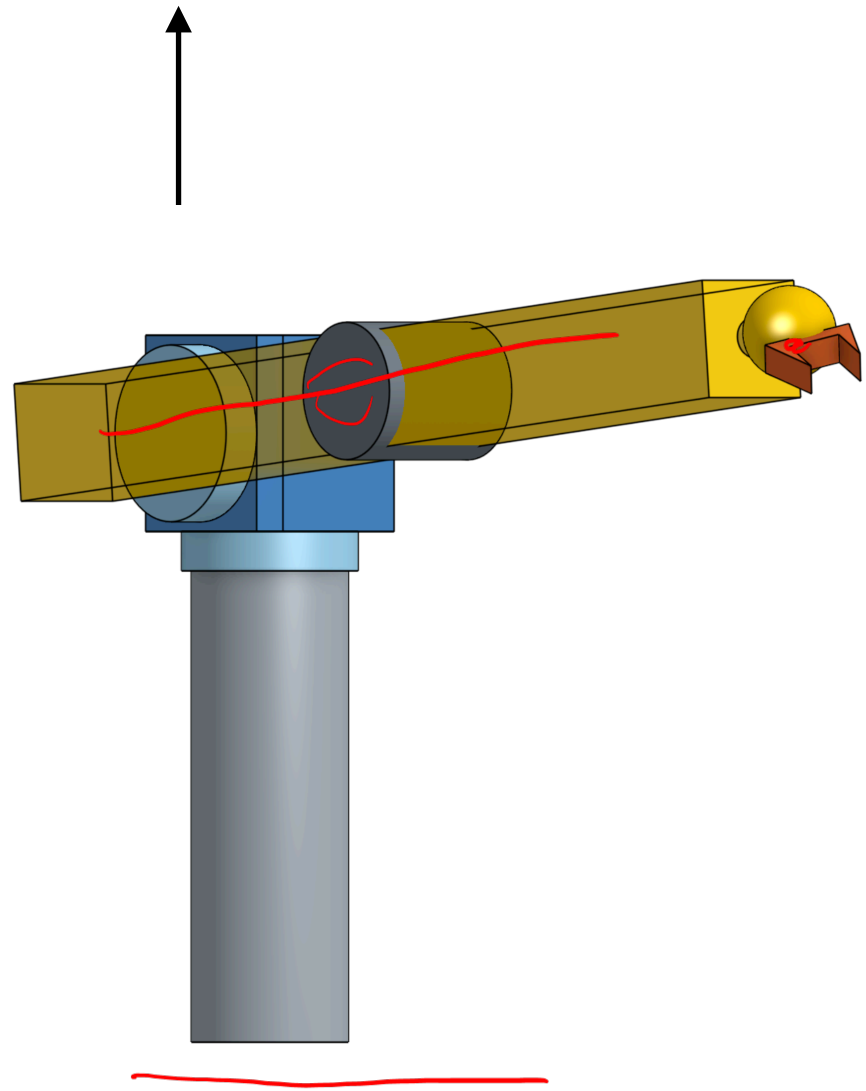
$\boxed{\theta_4 = atan2(r_{23}, r_{13})}$	$\boxed{\theta_4 = atan2(-r_{23}, -r_{13})}$
$\boxed{\theta_5 = cos^{-1}(r_{33})}$	$\boxed{\theta_5 = -cos^{-1}(r_{33})}$
$\boxed{\theta_6 = atan2(-r_{32}, r_{31})}$	$\boxed{\theta_6 = atan2(r_{32}, -r_{31})}$
$s_5 > 0$	$s_5 < 0$

Wrist Flip

- [insert video "gimballock.mov"]
  - Rotate 3 link system?
    - Show each dof
  - Show moving through vertical singularity

Elbow Solutions





Positive θ_3

Single Solution

- [insert video “3LPlanar WS1.mov”

Single Solution

- Insert video “3LPlanarWS limit.mov”
  - [insert video

Dextrous Workspace

- Include “3LPlanar WS2.mov” maybe only first 40 secs or so.

Singular Positions

[insert 3LPlanar xy singularities.mov"

Singular Positions

- Insert "3LPlanar rot singularity.mov"

Singular Positions

- Insert 3Lplanar singularjumping.mov