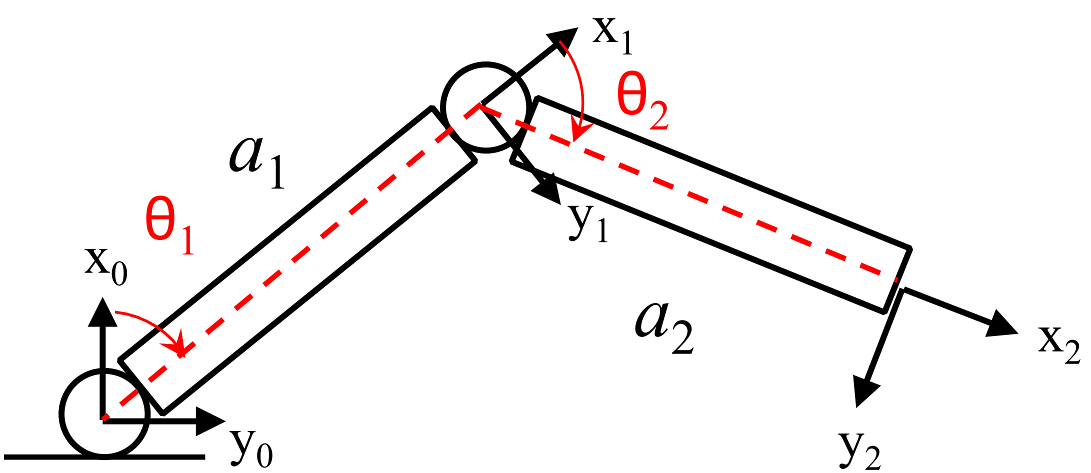
Robotics: Fundamentals

Video 7.5

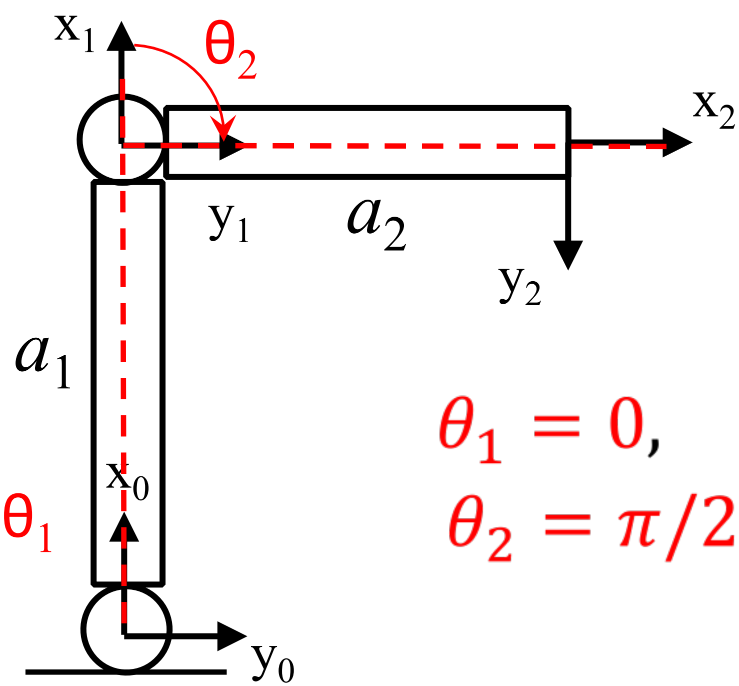
Mark Yim

Singularities

Configurations for which the rank $\mathbf{J}(q)$ is less than its maximum value are called **singularities** or **singular configurations**.



$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_2 \\ a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

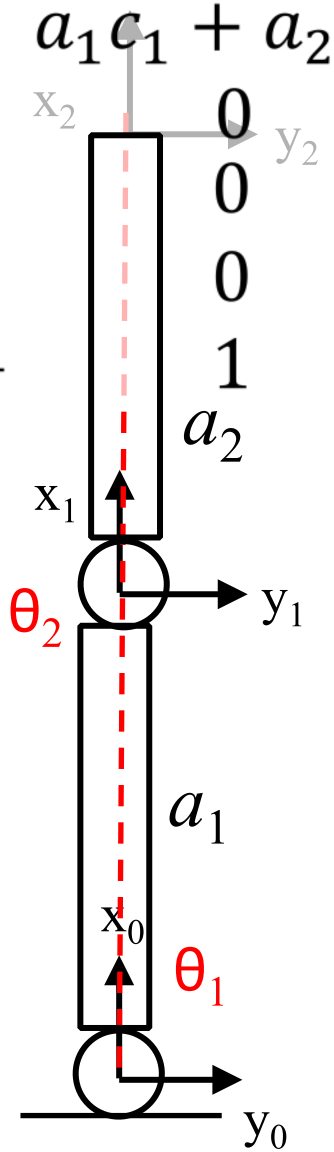


$\theta_1 = 0,\quad s_1 = 0, c_1 = 1$

$\theta_2 = \pi/2,\quad s_2 = 1, c_2 = 0$

$s_{12} = 1, c_{12} = 0$

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix} \end{bmatrix}$$



$$\boxed{\det(\mathbf{J}) = 0}$$

$$\theta_1 = 0, \; s_1 = 0, c_1 = 1$$
$$\theta_2 = 0, \quad s_2 = 0, c_2 = 1$$
$$s_{12} = 0, c_{12} = 1$$

Characteristics at Singular Configurations

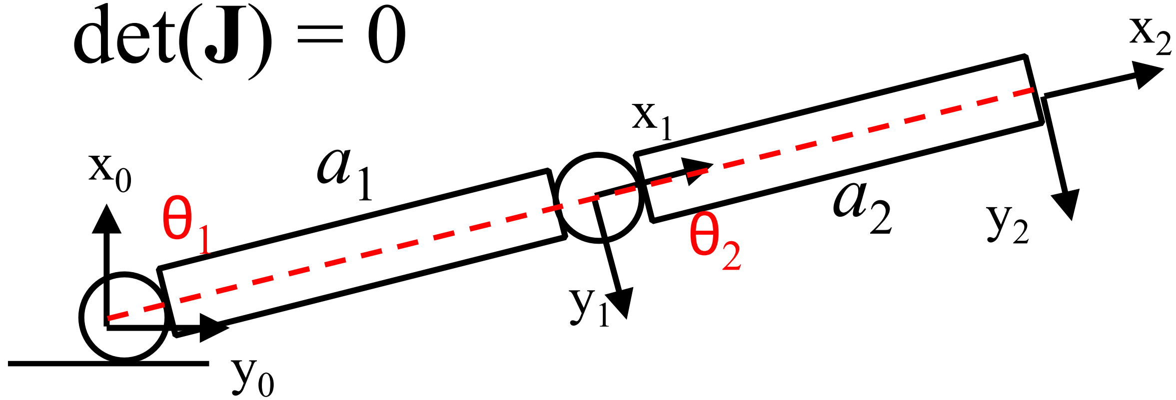
- Directions of motion may be lost
- Infinite joint velocities may be required for finite end-effector velocities
- Theoretically infinite end-effector forces may result from finite joint forces
- Often correspond to points on the boundary of the manipulator workspace.
- There may be no IK solution or there may be infinitely many IK solutions

Using determinant

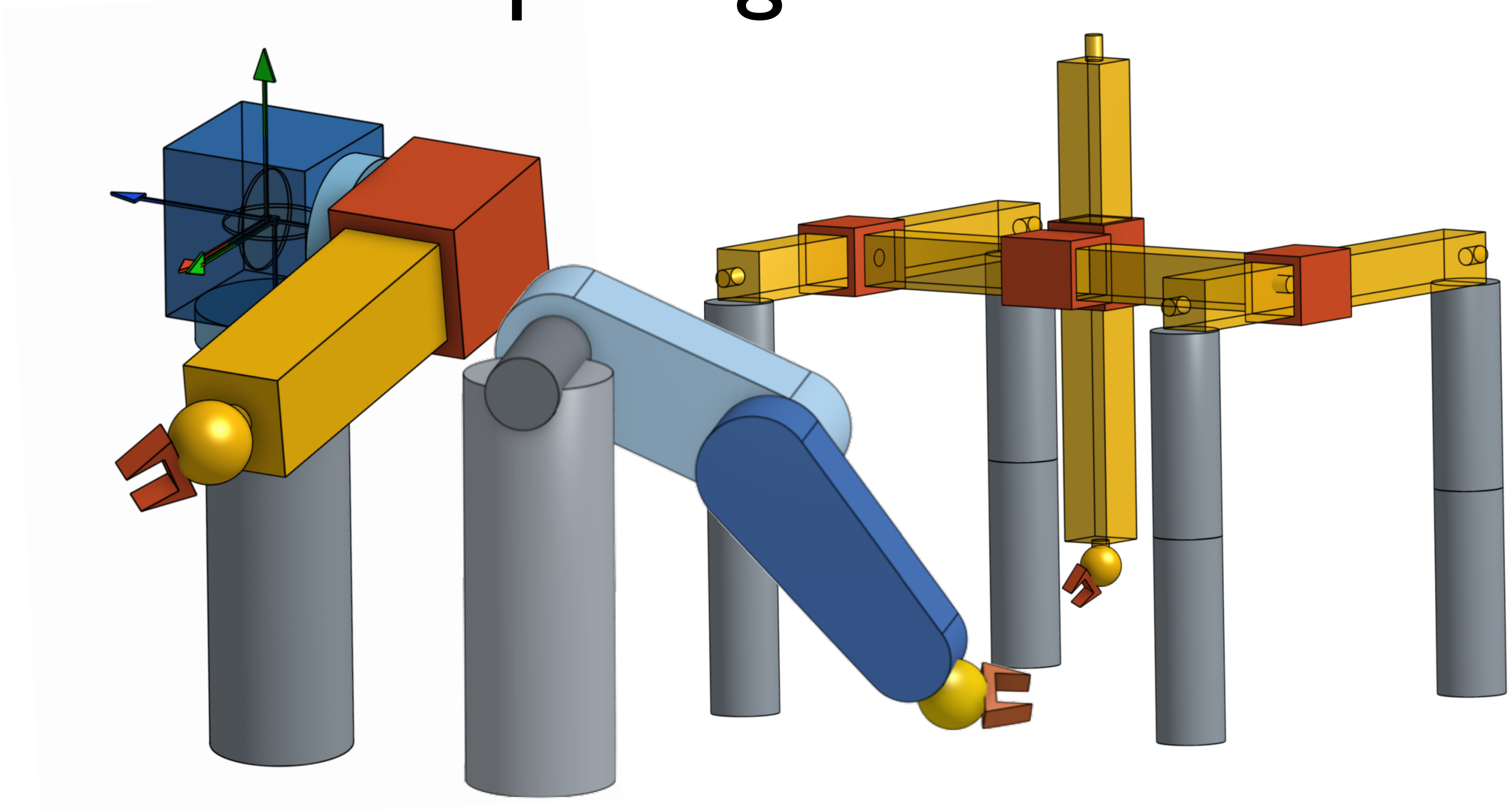
$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_1 & -a_2 s_1 \\ a_1 c_1 + a_2 c_1 & a_2 c_1 \end{bmatrix} \qquad \theta_2 = 0, \pi$$

$$\det(\mathbf{J}) = 0$$



Decomposing 6DOF arms



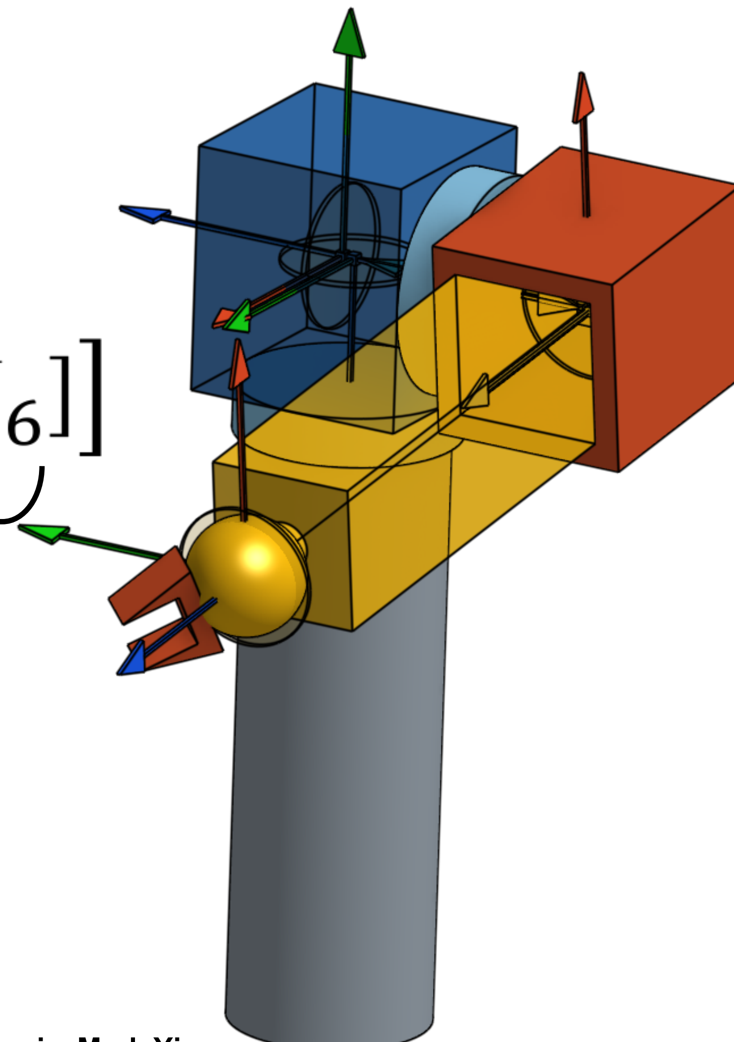
Arm singularities Wrist singularities

Decomposing 6DOF arms

$\mathbf{J}(q)$ is 6x6 and is singular
if and only if
$\det(\mathbf{J}) = 0$

$$\mathbf{J} = \Big[\underbrace{[\mathbf{J}_1][\mathbf{J}_2][\mathbf{J}_3]}_{}\underbrace{[\mathbf{J}_4][\mathbf{J}_5][\mathbf{J}_6]}_{}\Big]$$

$$\mathbf{J} = [\mathbf{J}_P|\mathbf{J}_O]$$



Decomposing 6DOF arms

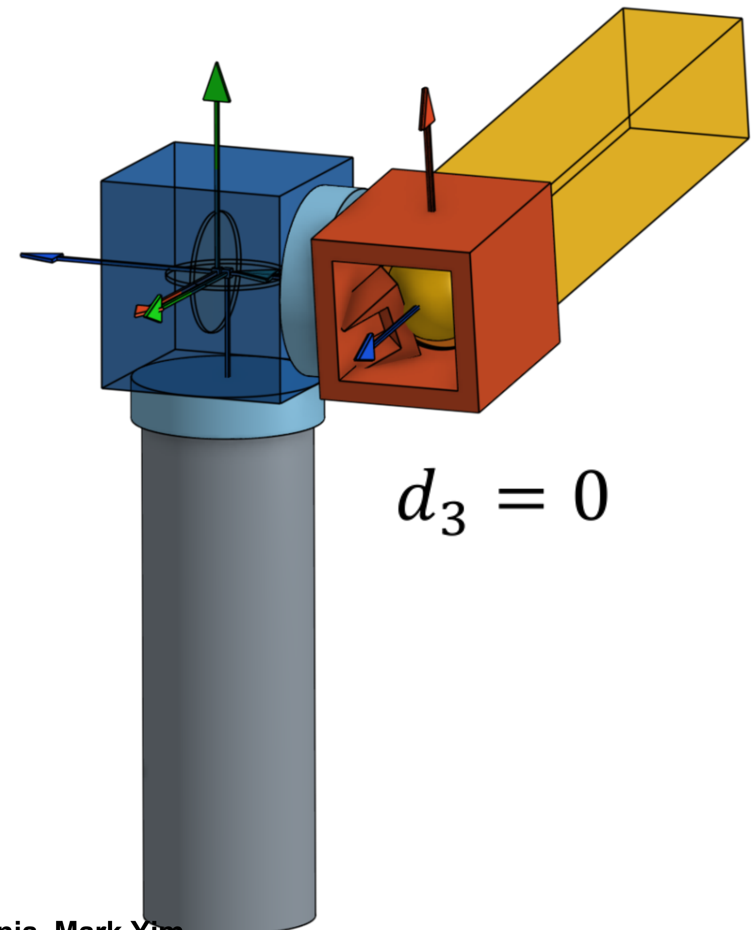
$$\mathbf{J}_O = \begin{bmatrix} \hat{\mathbf{z}}_3 \times (\boldsymbol{P}_6 - \boldsymbol{P}_3) & \hat{\mathbf{z}}_4 \times (\boldsymbol{P}_6 - \boldsymbol{P}_4) & \hat{\mathbf{z}}_5 \times (\boldsymbol{P}_6 - \boldsymbol{P}_5) \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

$$P_3 = P_4 = P_5 = P_6$$

Choose $P_6 = P_c$

$$\mathbf{J}_O = \begin{bmatrix} [0] & [0] & [0] \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

$$\mathbf{J} = [\mathbf{J}_P | \boxed{\mathbf{J}_O}]$$

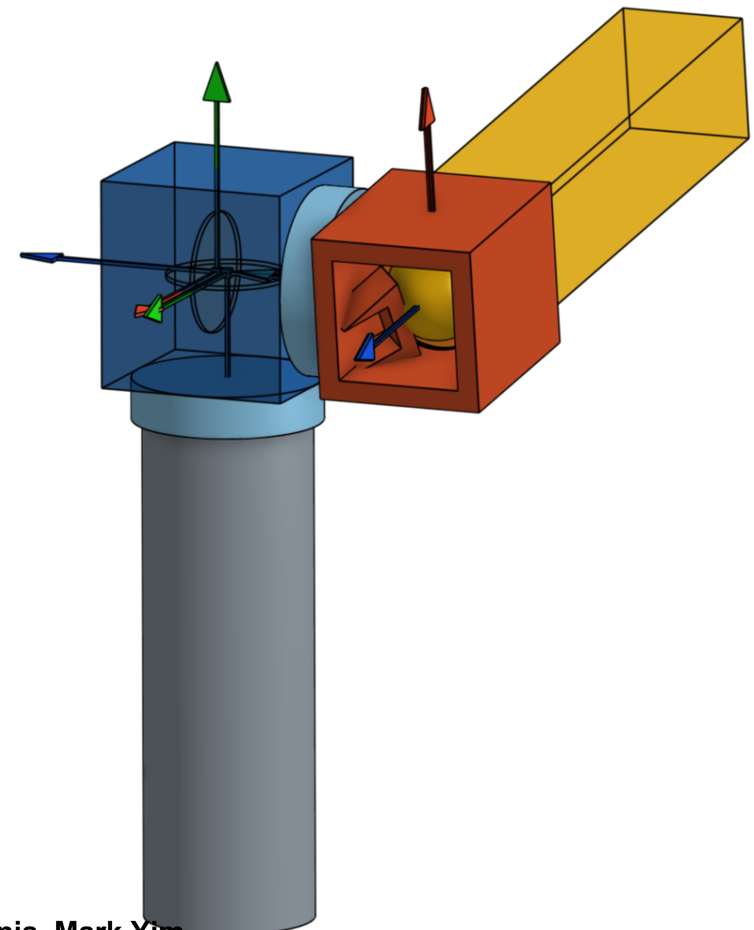


$d_3 = 0$

Decomposing 6DOF arms

$$\mathbf{J}_O = \begin{bmatrix} \hat{\mathbf{z}}_3 \times (\boldsymbol{P}_6 - \boldsymbol{P}_3) & \hat{\mathbf{z}}_4 \times (\boldsymbol{P}_6 - \boldsymbol{P}_4) & \hat{\mathbf{z}}_5 \times (\boldsymbol{P}_6 - \boldsymbol{P}_5) \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$

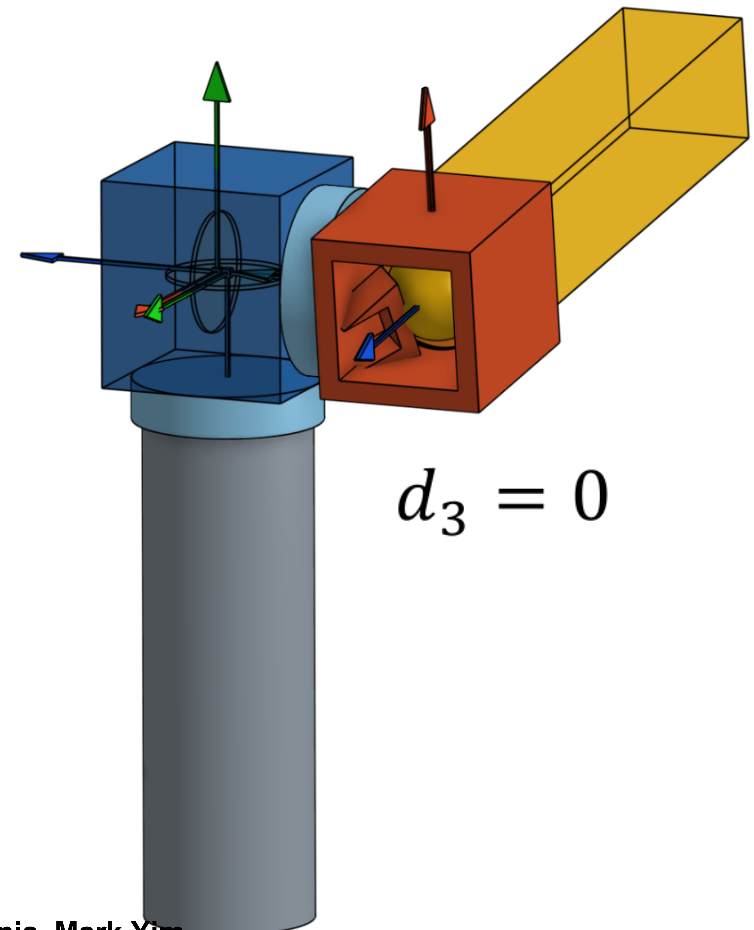
$$\mathbf{J}_O = \begin{bmatrix} [0] & [0] & [0] \\ \hat{\mathbf{z}}_3 & \hat{\mathbf{z}}_4 & \hat{\mathbf{z}}_5 \end{bmatrix}$$



$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}$$

$$\det(\mathbf{J}) = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{0} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix}$$
$$= \det(\mathbf{J}_{11}) \det(\mathbf{J}_{22})$$

Decomposing 6DOF arms



$d_3 = 0$

$\mathbf{J}_{22}$ is orientation part of wrist joints

$$\mathbf{J}_{22} = [\hat{\mathbf{z}}_3 \quad \hat{\mathbf{z}}_4 \quad \hat{\mathbf{z}}_5]$$

$$= \det(\mathbf{J}_{11}) \det(\mathbf{J}_{22})$$

Wrist Singularities

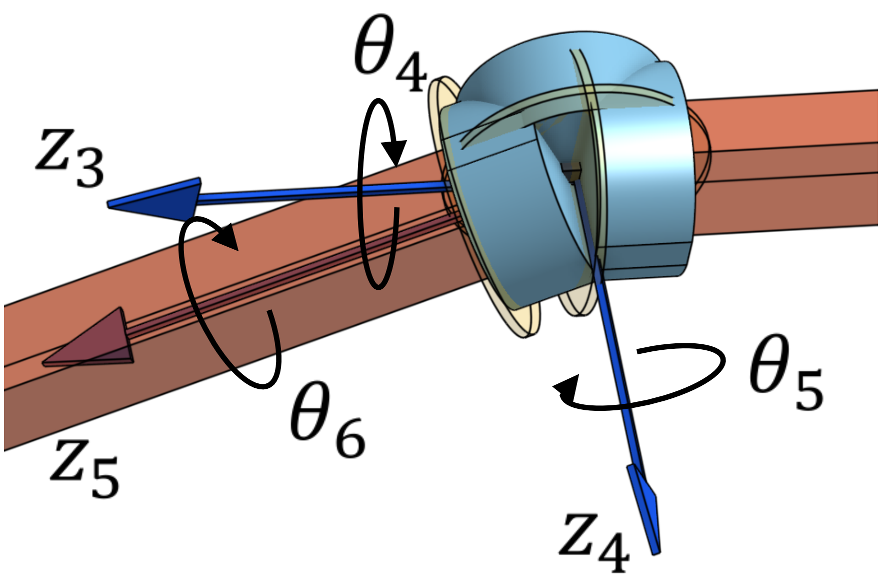
Singularities occur in the wrist:

- if and only if joint axis are collinear (0 or π radians)
- unavoidable if moving through that point.

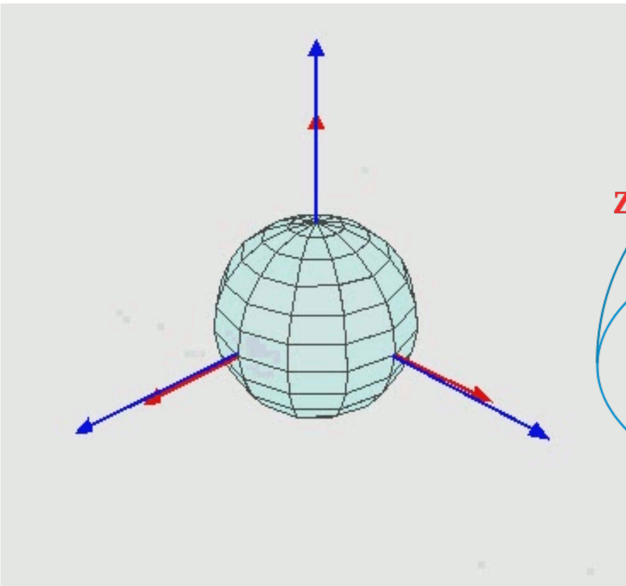
$\mathbf{J}_{22}$ is orientation part of wrist joints

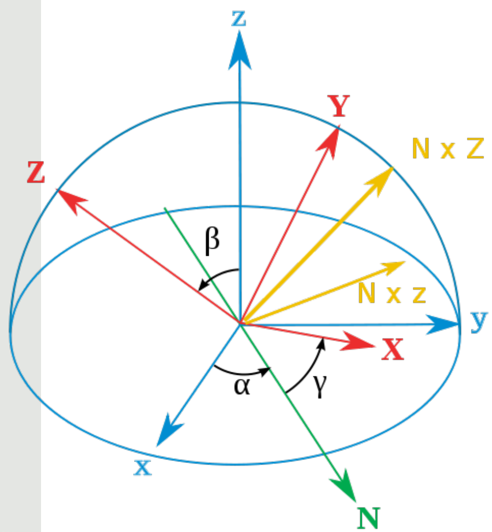
$$\mathbf{J}_{22} = [\hat{\mathbf{z}}_3 \quad \hat{\mathbf{z}}_4 \quad \hat{\mathbf{z}}_5]$$

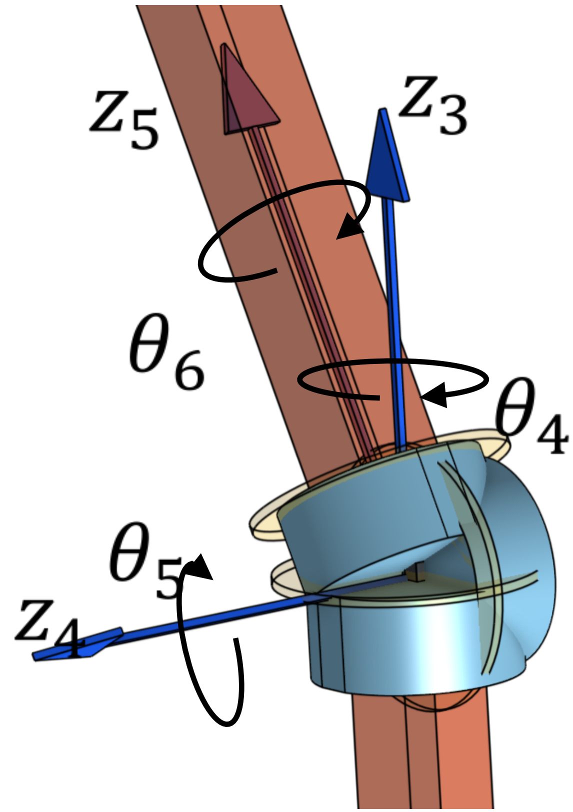
$$\theta_5 = 0 \ or \ \pi$$



Representational Singularities







By Euler2.gif: Juansempererederivative work: Xavax - This file was derived from Euler2.gif:, CC BY-SA 3.0,

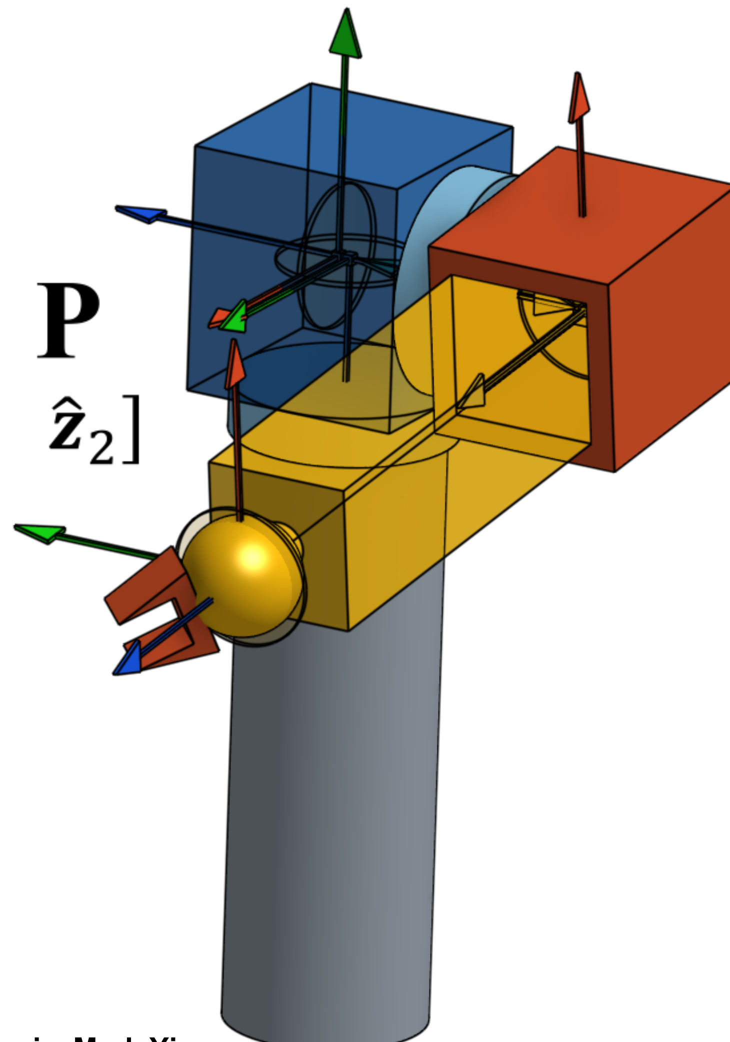
By Juansempere - Own work, GFDL,

Arm Singularities

$\mathbf{J}_{11}$ is the position jacobian of links 1-3

$$\mathbf{J}_{11} = \begin{array}{ccc} \textbf{R} & \textbf{R} & \textbf{P} \\ [\hat{\mathbf{z}}_0 \times (\boldsymbol{P}_6 - \boldsymbol{P}_0) & \hat{\mathbf{z}}_1 \times (\boldsymbol{P}_6 - \boldsymbol{P}_1) & \hat{\mathbf{z}}_2] \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 d_3 s_2 - d_2 s_1 \\ c_1 d_2 + d_3 s_1 s_2 \\ c_2 d_3 \end{bmatrix}$$

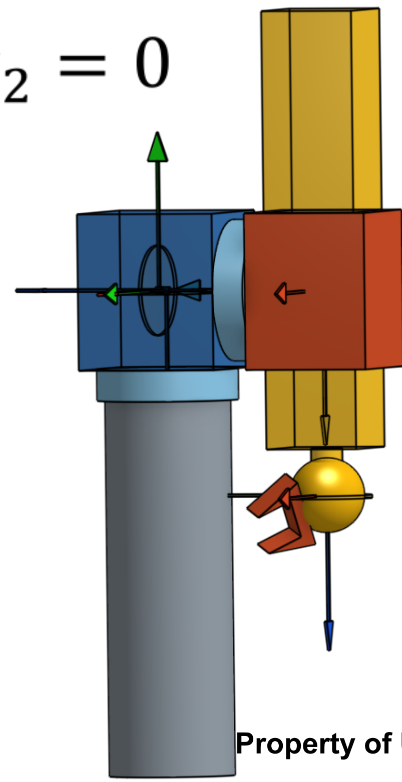


Arm Singularities

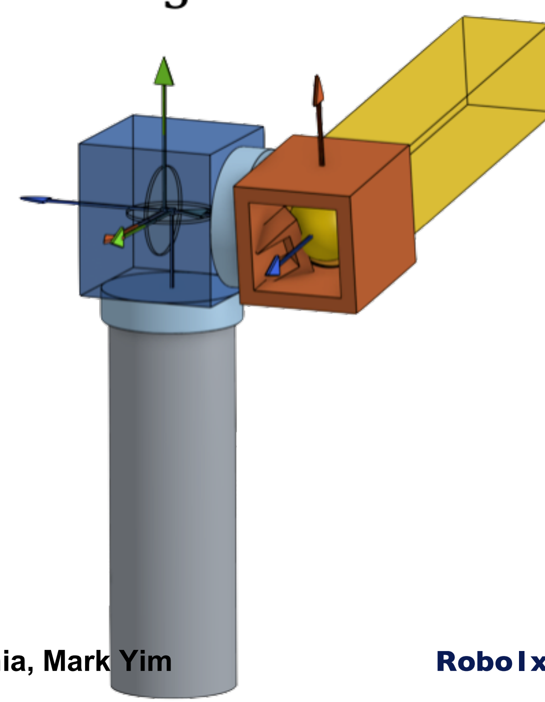
$$\mathbf{J}_{11} = \begin{bmatrix} -s_1 s_2 d_3 - c_1 d_2 & c_1 c_2 d_3 & c_1 s_2 \\ c_1 s_2 d_3 - s_1 d_2 & s_1 c_2 d_3 & s_1 s_2 \\ 0 & -s_2 d_3 & c_2 \end{bmatrix}$$

$$\det(\mathbf{J}_{\cdot}) = -s_2 d_3^2$$

$-s_2 = 0$

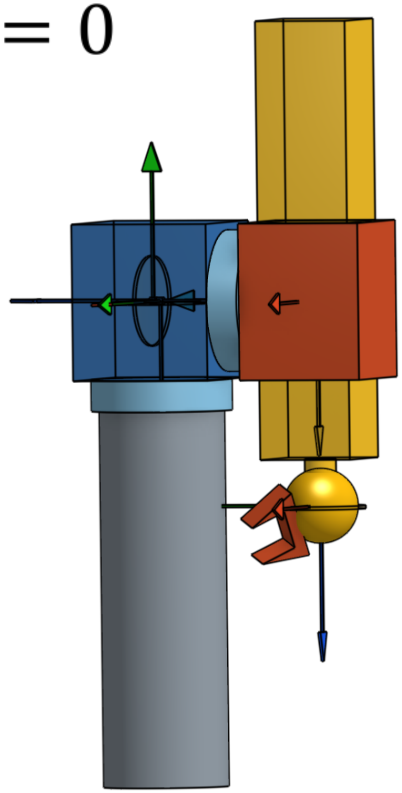


$d_3^2 = 0$

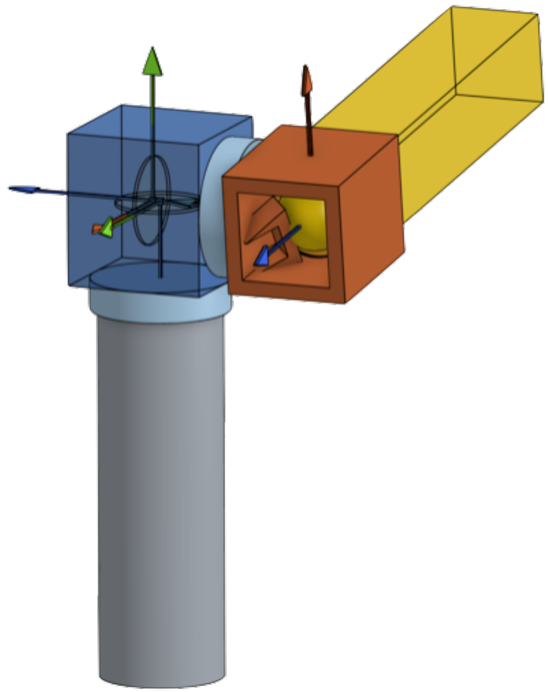


Stanford arm Singularities

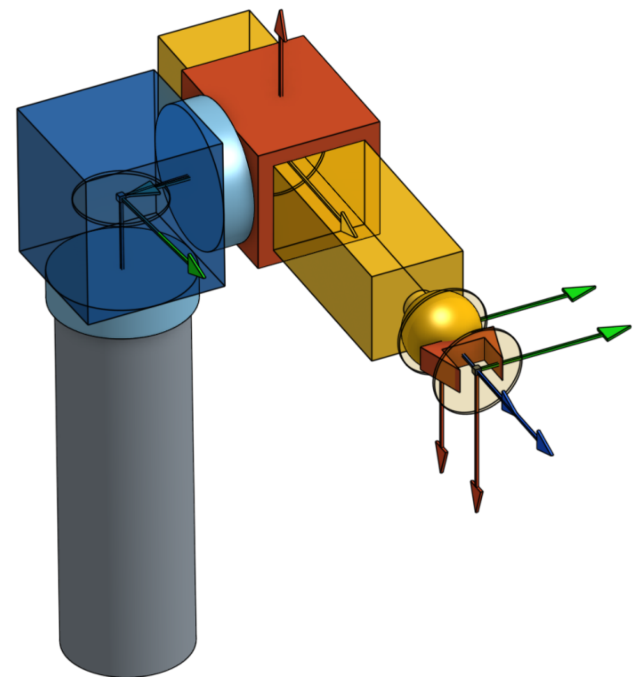
$-s_2 = 0$



$d_3^2 = 0$



$\theta_5 = 0$



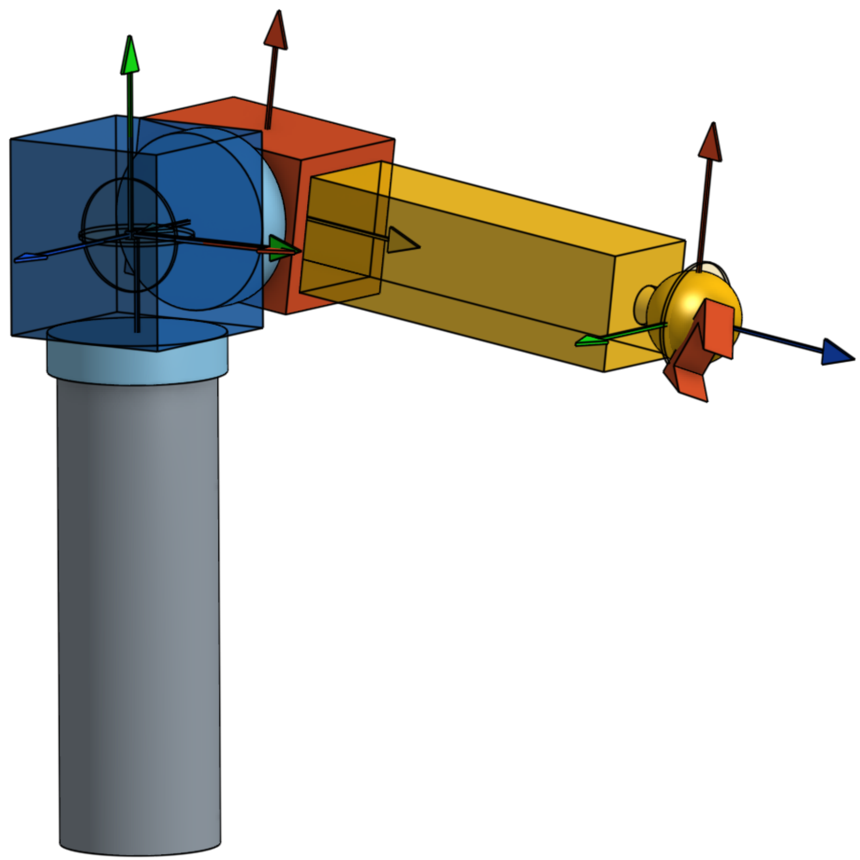
Robotics: Fundamentals

Video 7.6

Mark Yim

Manipulability

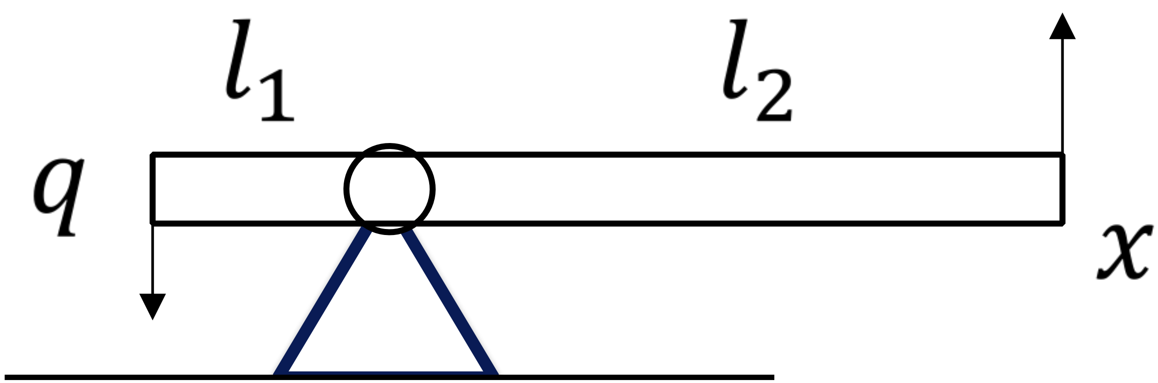
$$\dot{x} = \mathbf{J}\dot{q}$$



Manipulability

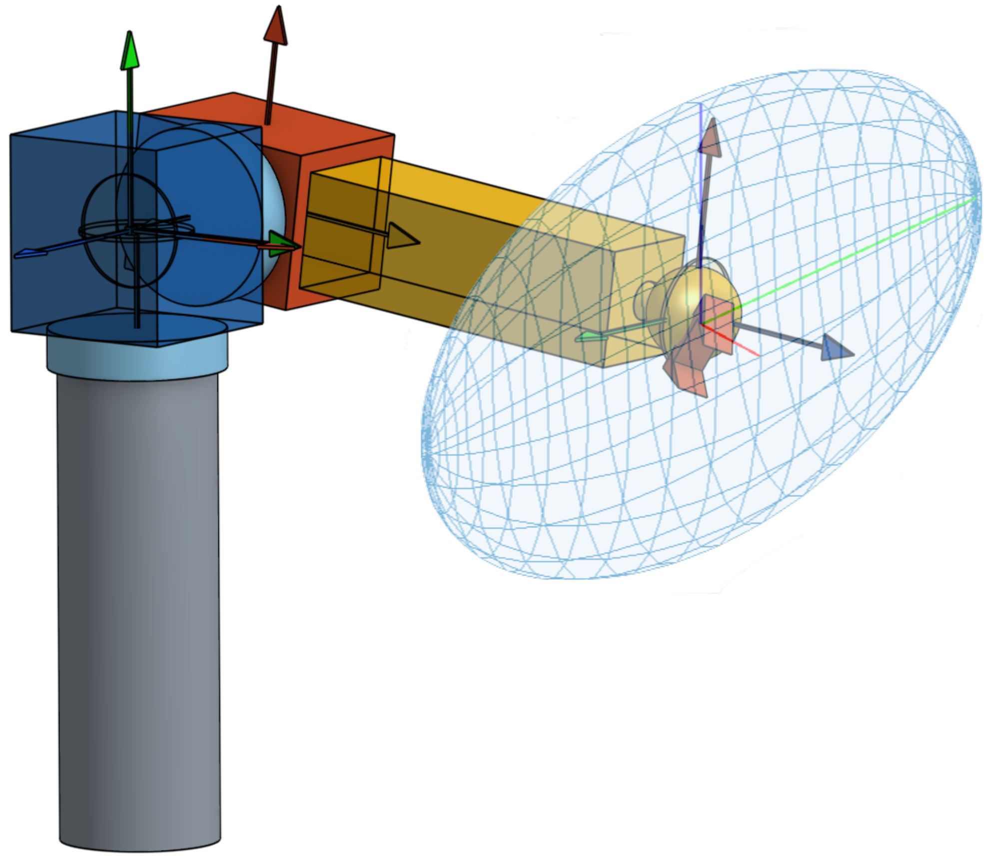
$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$

$$\dot{x} = \frac{l_2}{l_1}\dot{q}$$



Manipulability

$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$



Manipulability

For unit input $||\dot{\boldsymbol{q}}||$=1 and minimum norm solution

$$\|\dot{q}\| = \dot{q}^T \dot{q}$$

$$\xi^T \left(\mathbf{J}\mathbf{J}^T\right)^{-1} \xi \leq 1$$

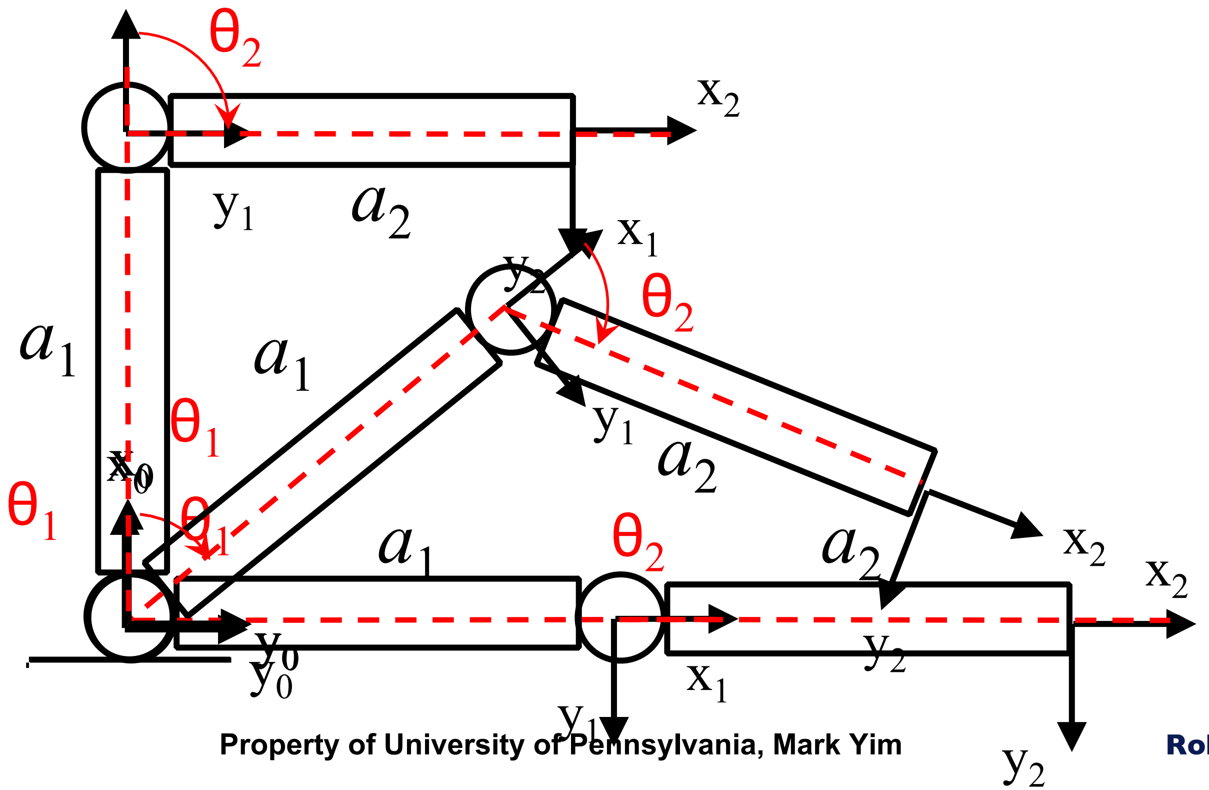
$$\det\left(\mathbf{J}\mathbf{J}^T\right) = \lambda_1^2 \lambda_2^2 \ldots \lambda_n^2$$

- Axis of ellipsoid are eigenvalues
- Volume $= |K\lambda_1\lambda_2 \ldots \lambda_n| = |K \det(\mathbf{J})|$
- Manipulability := $\boxed{\mu = |\det(\mathbf{J})|}$

Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

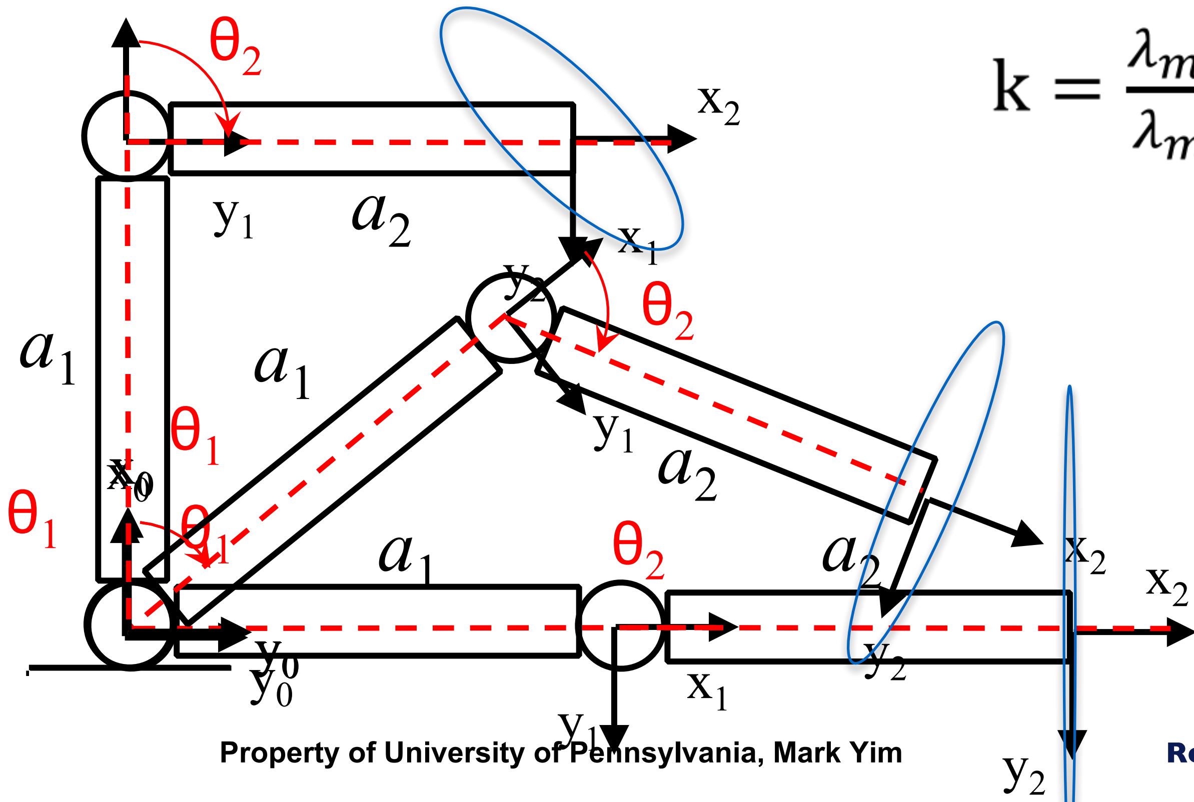
$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$



Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

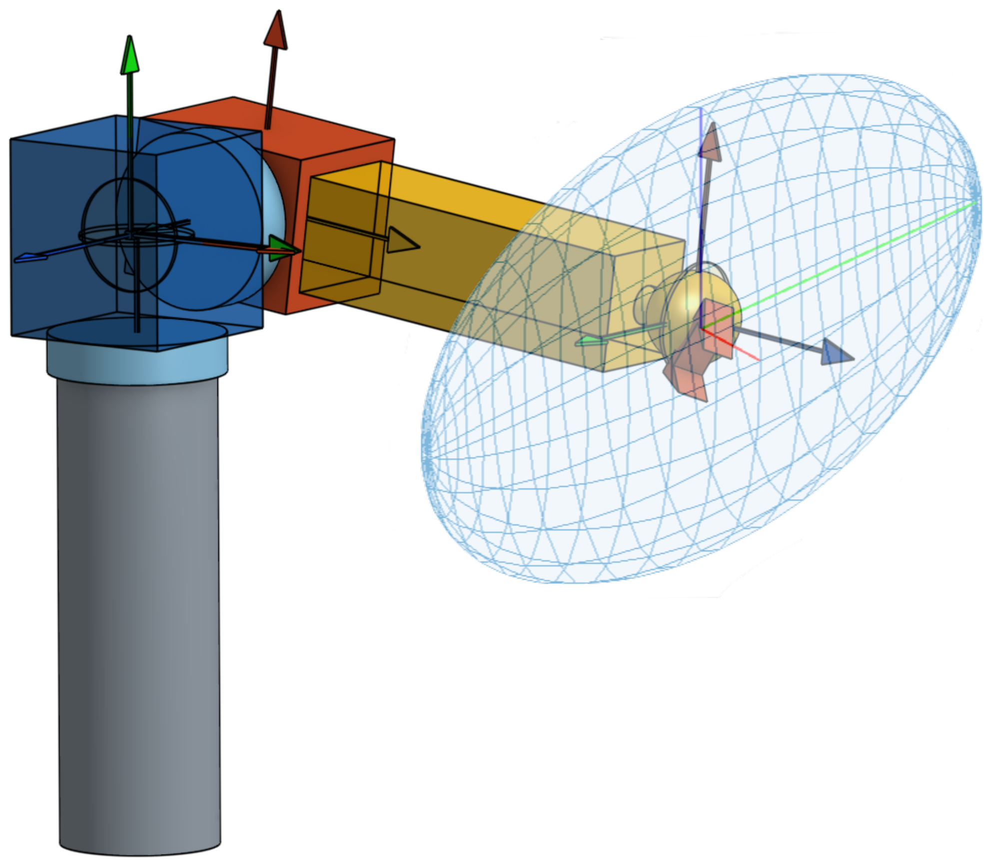
$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$



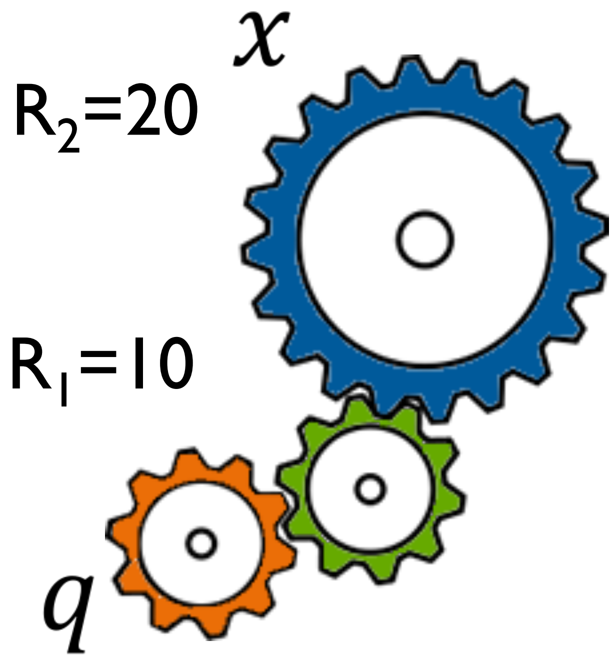
$$k = \frac{\lambda_{max}}{\lambda_{min}}$$

Manipulability

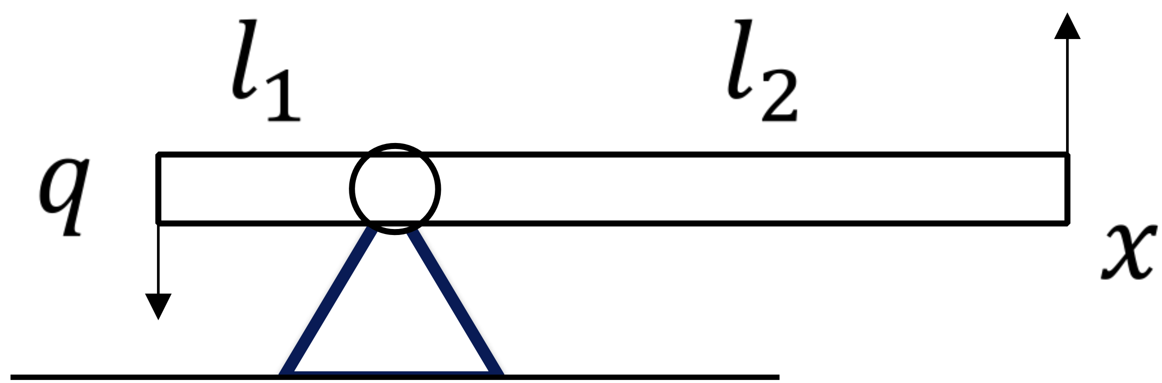
For unit norm input $\|\dot{q}\|=1$ $\quad \dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$



Jacobian Transpose



$$\frac{F_{in}}{F_{out}} = \frac{V_{out}}{V_{in}}$$



Principle of Virtual Work

$$\boldsymbol{F} \cdot \delta \boldsymbol{x} = \boldsymbol{\tau} \cdot \delta \boldsymbol{q}$$

$$\boldsymbol{F}^T \delta \boldsymbol{x} = \boldsymbol{\tau}^T \delta \boldsymbol{q}$$

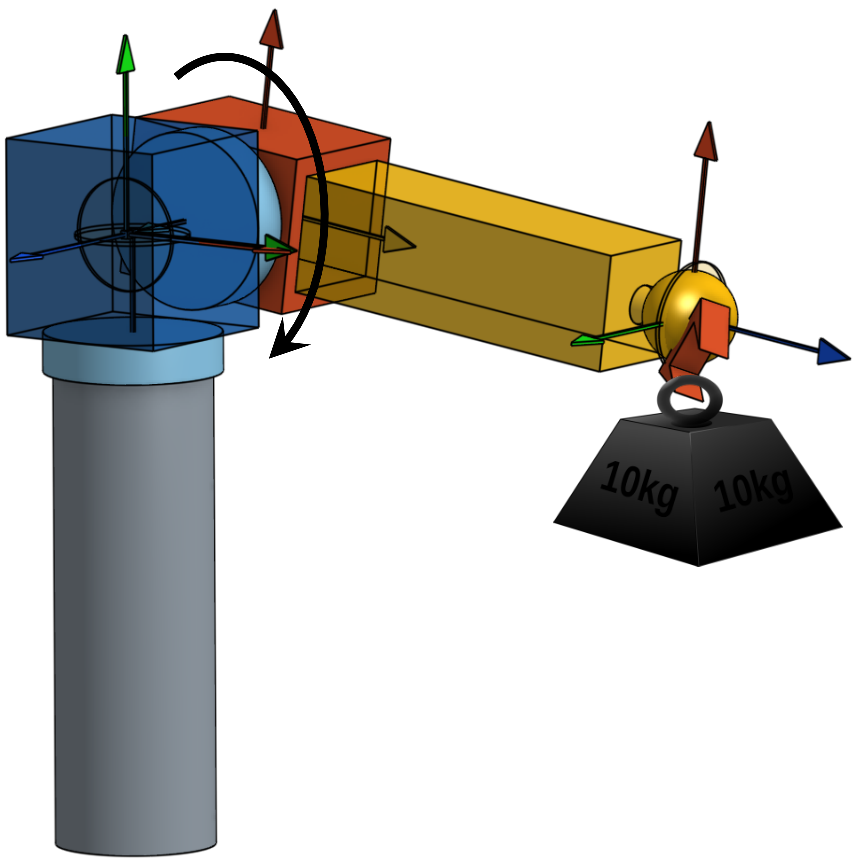
$$\boldsymbol{F}^T \mathbf{J} \delta \boldsymbol{q} = \boldsymbol{\tau}^T \delta \boldsymbol{q}$$

$$\boldsymbol{F}^T \mathbf{J} = \boldsymbol{\tau}^T$$

$$\boxed{\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}}$$

Static Forces

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ F_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix}$$



$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

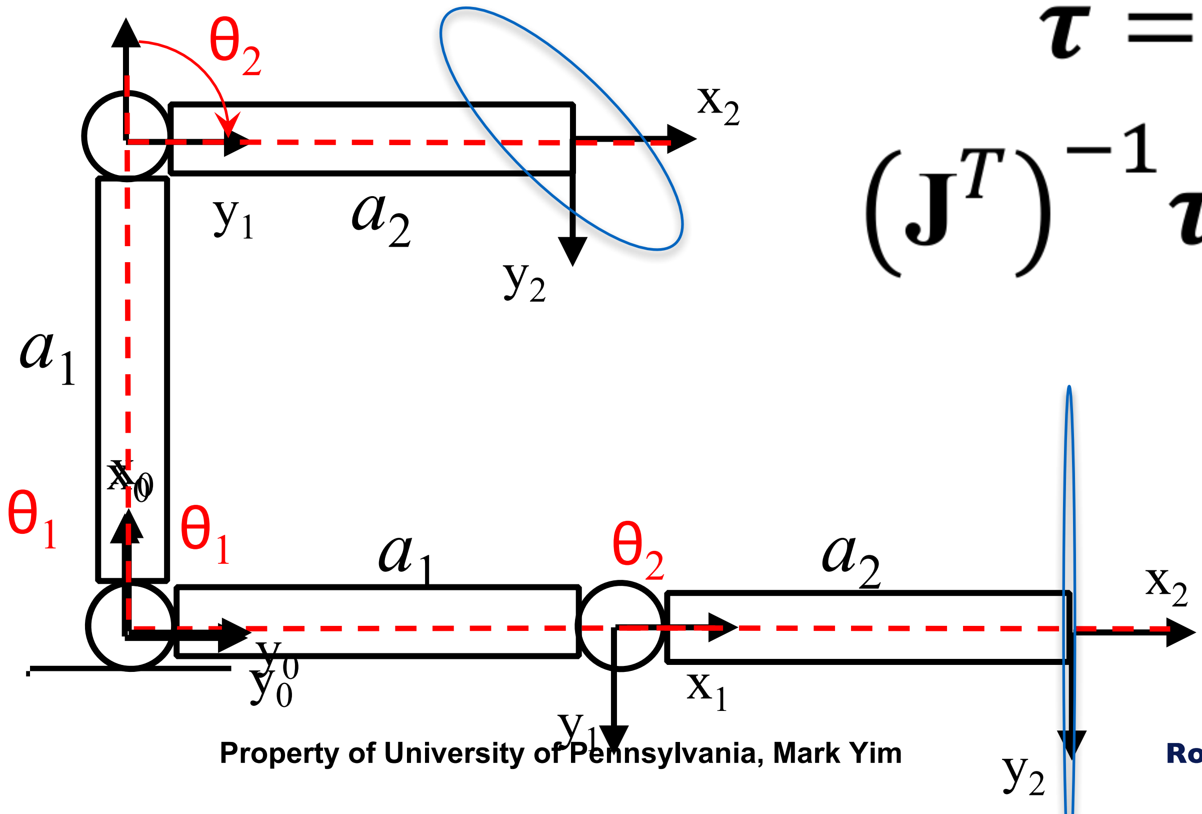
Manipulability

$$\mathbf{J}(\boldsymbol{q}) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mu = |\det(\mathbf{J})| = a_1 a_2 \sin(\theta_2)$$

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

$$\left(\mathbf{J}^T\right)^{-1} \boldsymbol{\tau} = \mathbf{F}$$

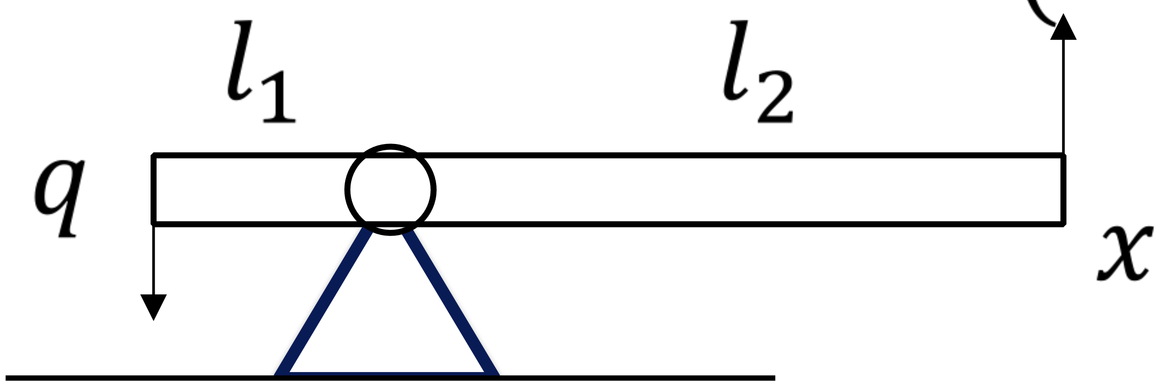


Manipulability

$$\dot{x} = \frac{l_2}{l_1}\dot{q} \qquad \mathbf{F} = \frac{l_1}{l_2}\boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \mathbf{J}^T\mathbf{F}$$

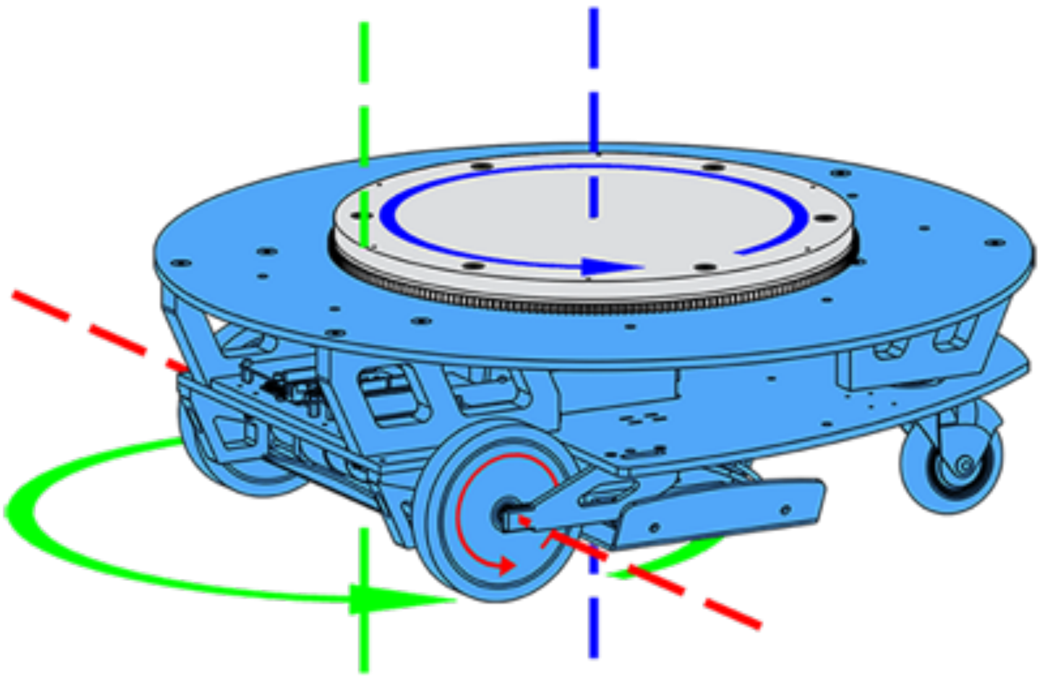
$$\left(\mathbf{J}^T\right)^{-1}\boldsymbol{\tau} = \mathbf{F}$$



Robotics: Fundamentals

Video 7.8
Mark Yim

Jacobian for Mobile Robots

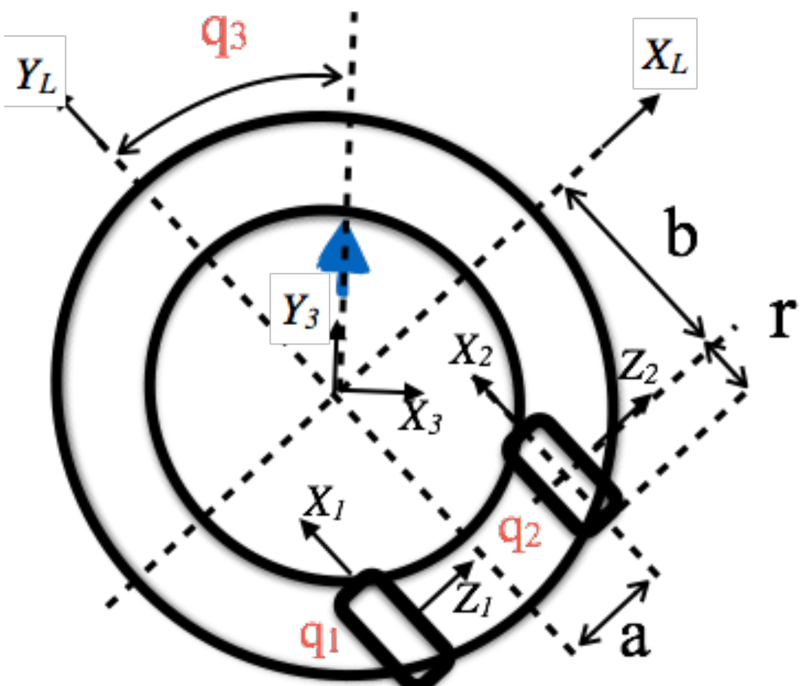


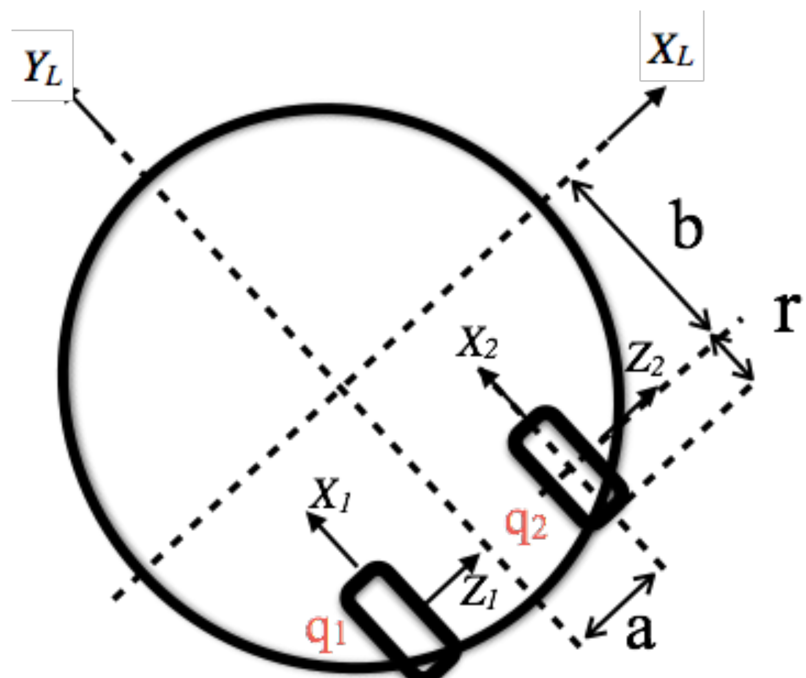
$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Jacobian for Mobile Robots

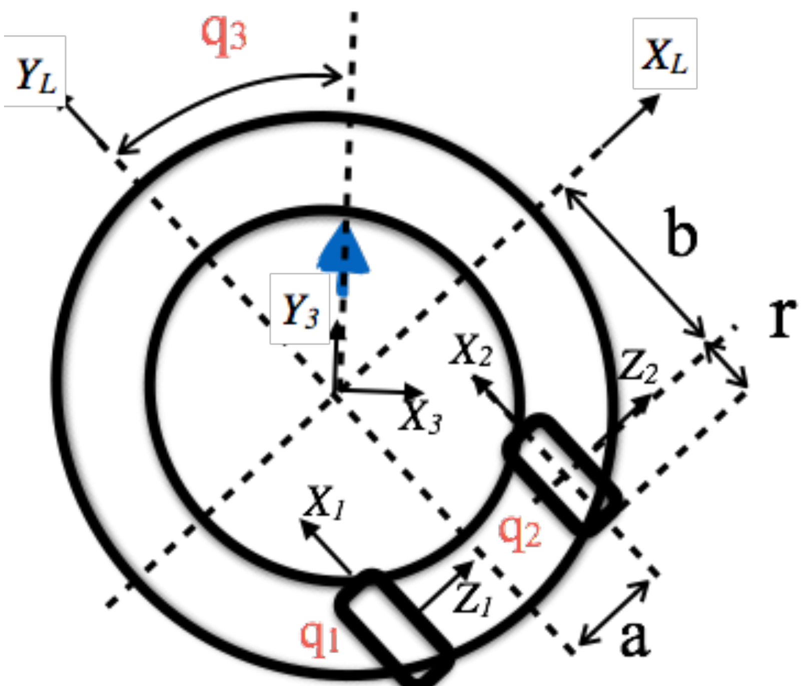
$$\mathbf{J} = [\mathbf{J}_D | \mathbf{J}_3]$$

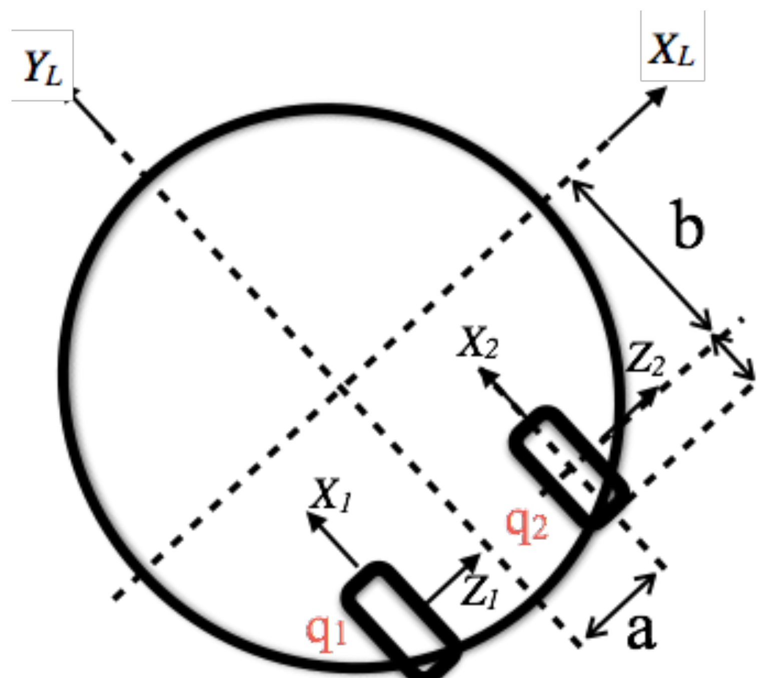




Jacobian for Mobile Robots

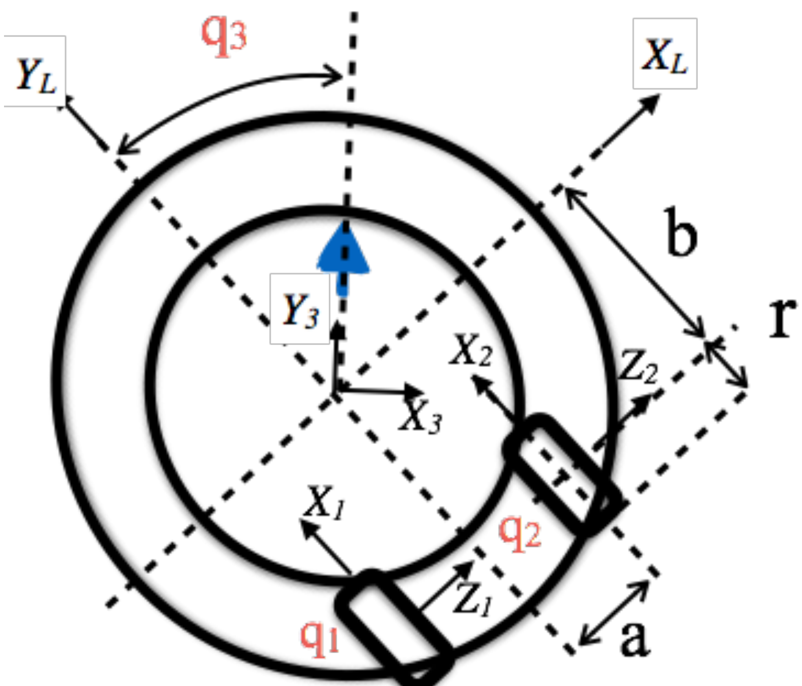
$$\mathbf{J}_D = R_{3L}\mathbf{J}_D^L \qquad \mathbf{J} = [\mathbf{J}_D|\mathbf{J}_3]$$

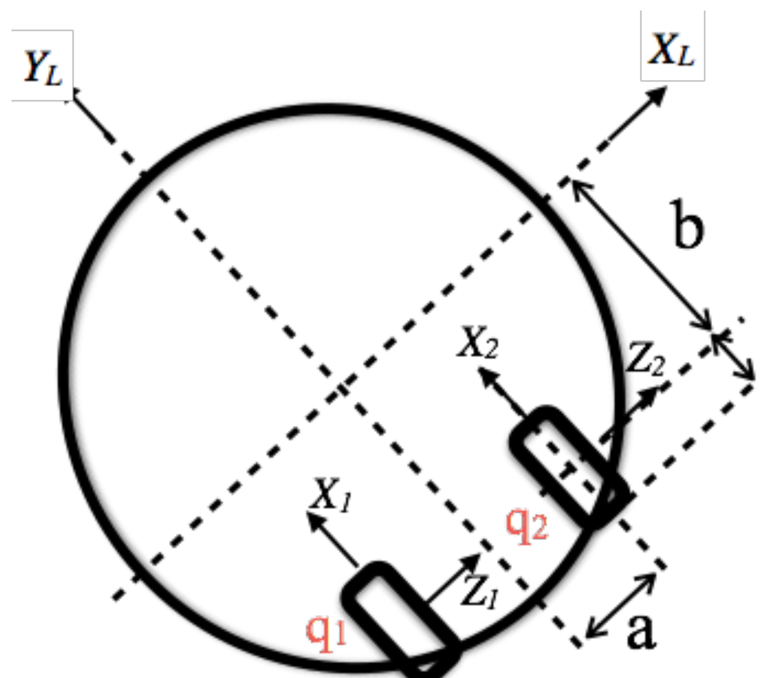




Jacobian for Mobile Robots

$$\mathbf{J}_D = R_{3L}\mathbf{J}_D^L \qquad \mathbf{J}_i = R_{3L}\mathbf{J}_i^L$$





Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \qquad \mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix}$$

$$R_{3L} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

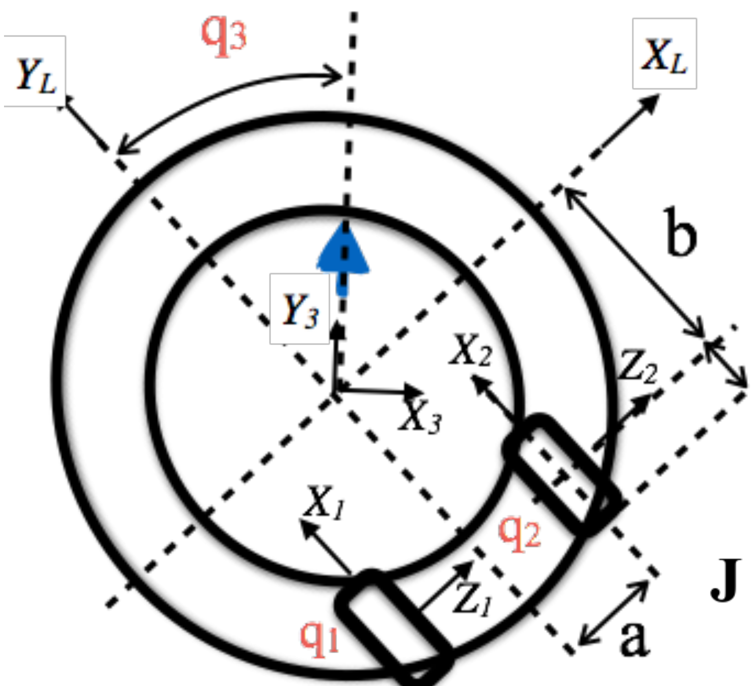
$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \qquad \mathbf{J}_1 = \begin{bmatrix} -\frac{rbc_3}{2a} - \frac{rs_3}{2} \\ -\frac{rbs_3}{2a} + \frac{rc_3}{2} \\ -\frac{r}{2a} \end{bmatrix}$$

Jacobian for Mobile Robots

$$\mathbf{J}_1^L = \begin{bmatrix} -rb/2a \\ r/2 \\ -r/2a \end{bmatrix} \quad \mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \quad \mathbf{J}_3^L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{J}_2^L = \begin{bmatrix} rb/2a \\ r/2 \\ r/2a \end{bmatrix} \qquad \mathbf{J}_2 = \begin{bmatrix} \dfrac{rbc_3}{2a} - \dfrac{rs_3}{2} \\ \dfrac{rbs_3}{2a} + \dfrac{rc_3}{2} \\ \dfrac{r}{2a} \end{bmatrix}$$

Jacobian for Mobile Robots



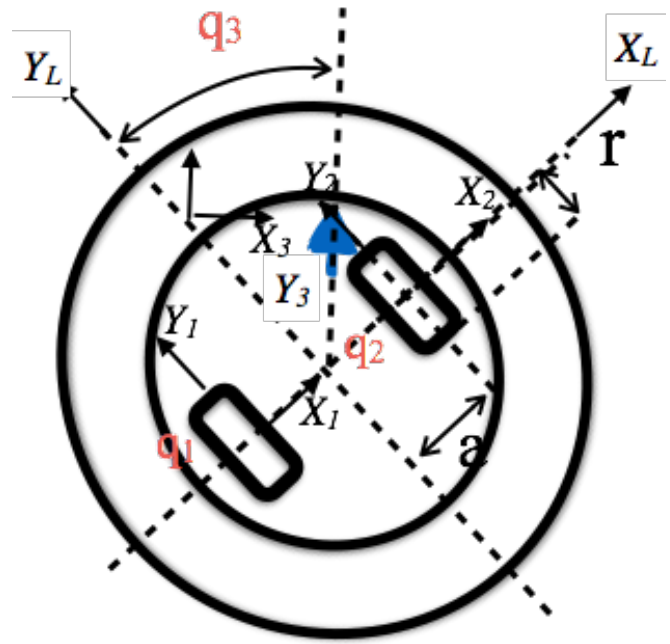
$$\mathbf{J} = [\mathbf{J_1} \quad \mathbf{J_2} \quad \mathbf{J_3}]$$

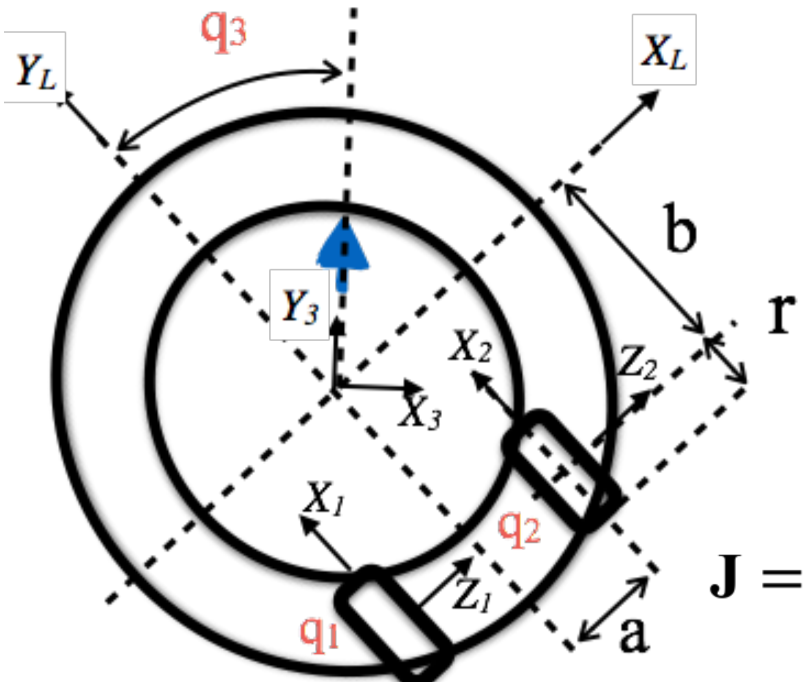
$$\mathbf{J} = \begin{bmatrix} -\frac{rbc_3}{2a} - \frac{rs_3}{2} & \frac{rbc_3}{2a} - \frac{rs_3}{2} & 0 \\ -\frac{rbs_3}{2a} + \frac{rc_3}{2} & \frac{rbs_3}{2a} + \frac{rc_3}{2} & 0 \\ -\frac{r}{2a} & \frac{r}{2a} & 1 \end{bmatrix}$$

$$\mathbf{J} = \frac{1}{2a} \begin{bmatrix} -rbc_3 - ras_3 & rbc_3 - ras_3 & 0 \\ -rbs_3 + rac_3 & rbs_3 + rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$b = 0?$$



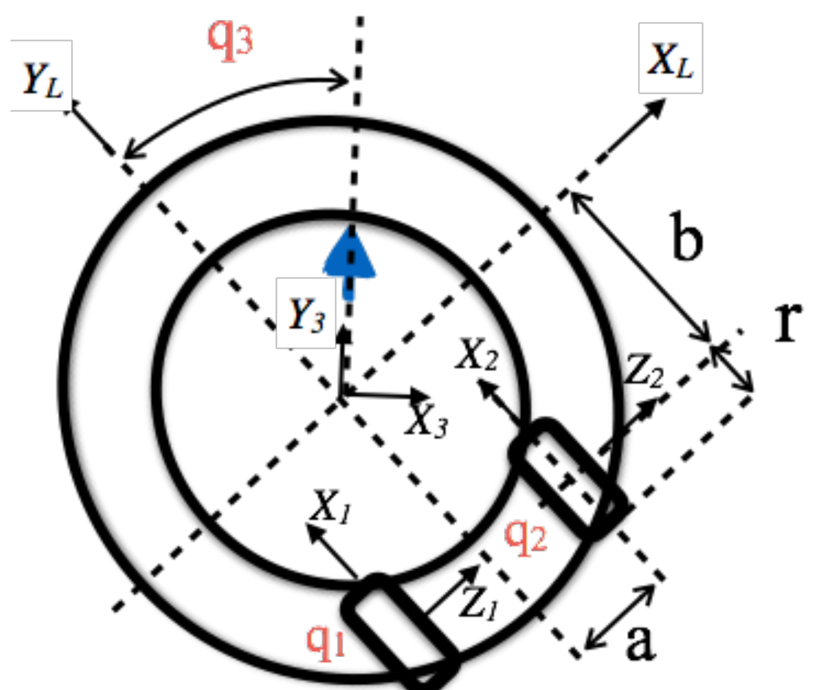


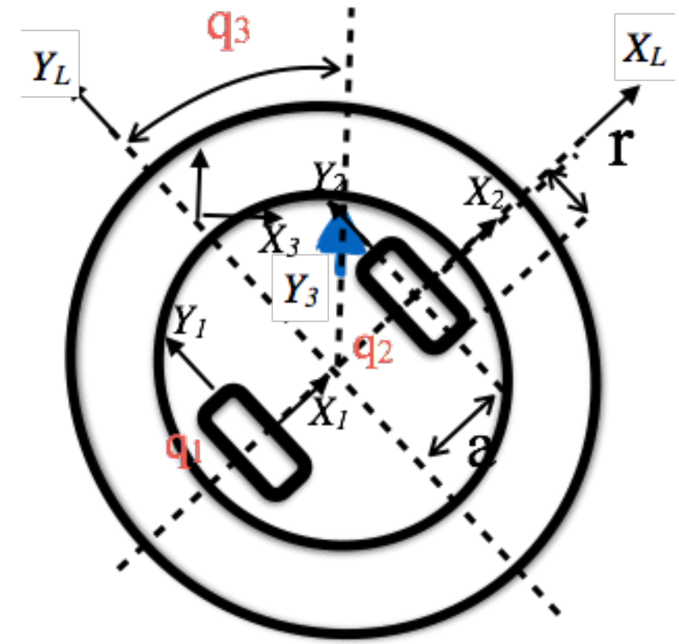
$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -rbc_3 - ras_3 & rbc_3 - ras_3 & 0 \\ -rbs_3 + rac_3 & rbs_3 + rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Jacobian for Mobile Robots

$$b = 0?$$

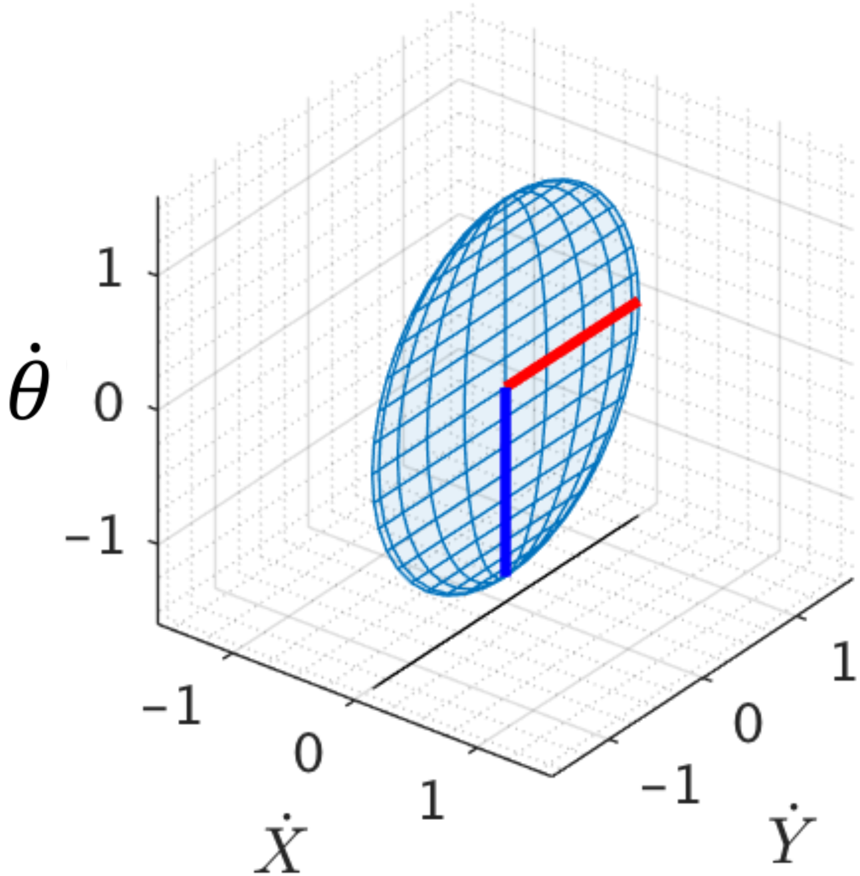
Singularity!

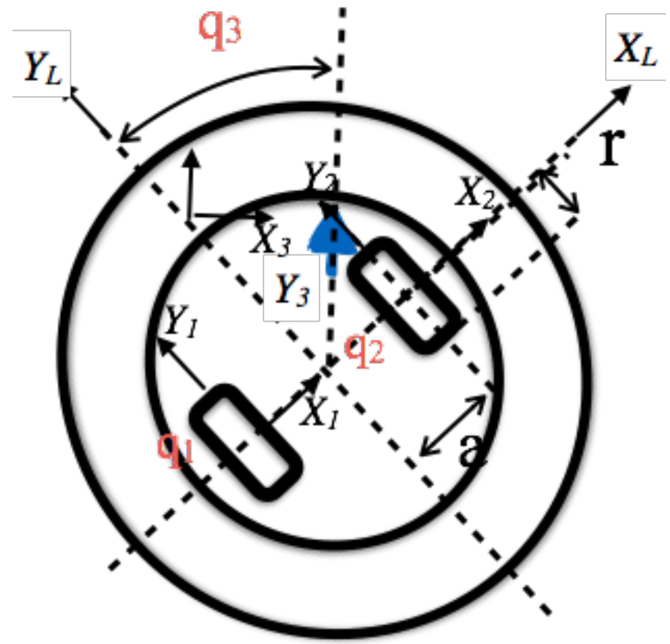




$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -ras_3 & -ras_3 & 0 \\ rac_3 & rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

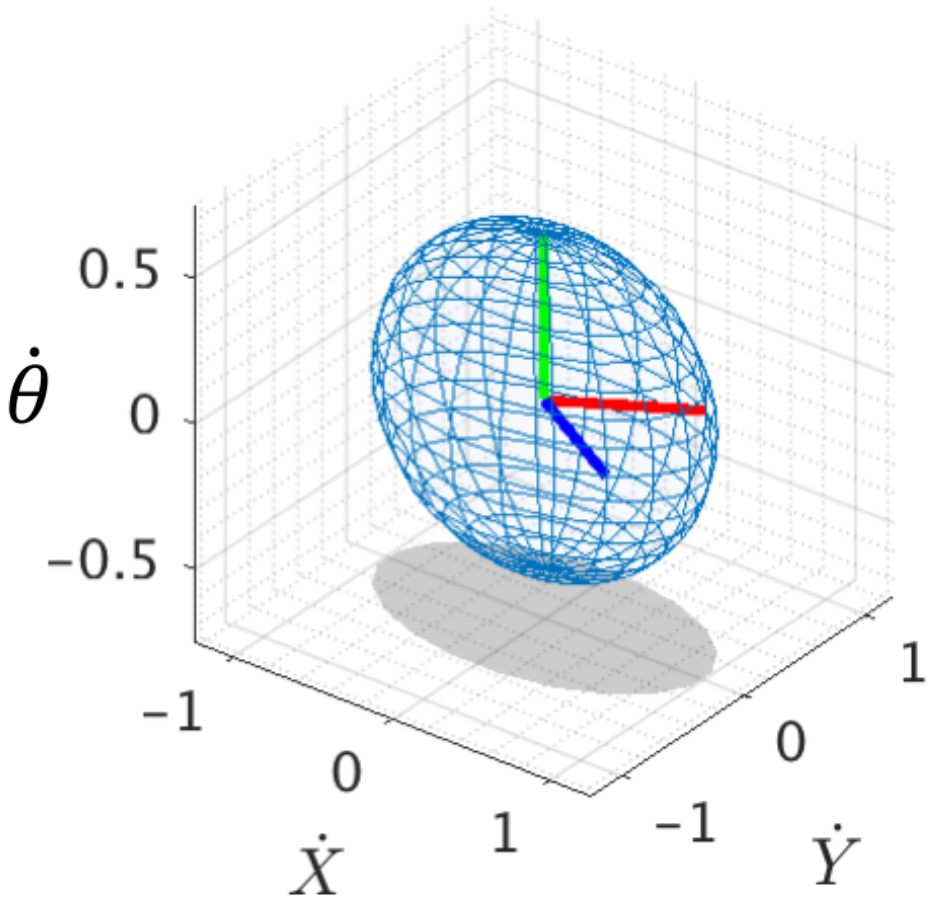
Mobility Ellipsoid

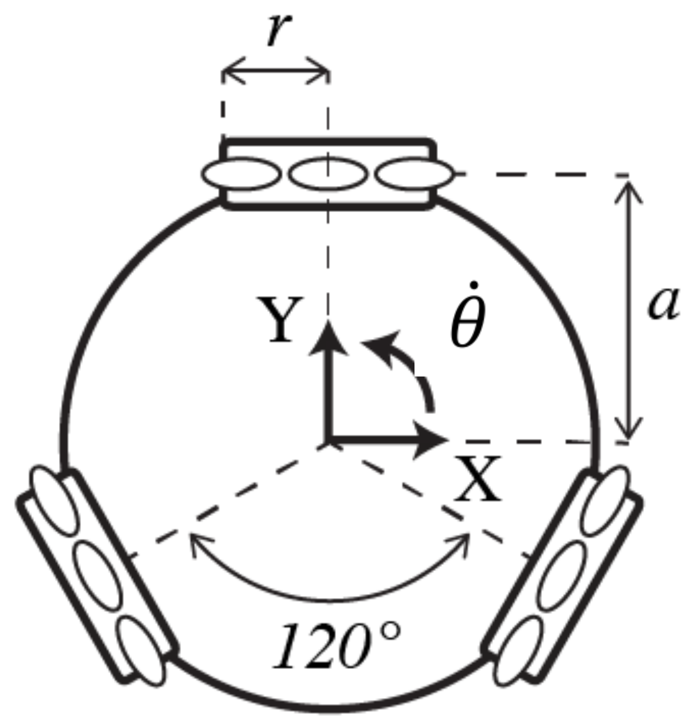




$$\mathbf{J} = \frac{1}{2a}\begin{bmatrix} -ras_3 & -ras_3 & 0 \\ rac_3 & rac_3 & 0 \\ -r & r & 2a \end{bmatrix}$$

Mobility Ellipsoid





Mobility Ellipsoid