CMSC 666 Spring 2007 | **Homework 5** | Roman Stanchak May 8, 2007

Problem 1: If $e^{(0)} = \hat{e}^{(0)}$ and $\rho(S_1) < \rho(S_2)$, is it the case that $\left\|e^{(k)}\right\| < \left\|\hat{e}^{(k)}\right\|$? If so, prove it, otherwise provide a counter example.

Solution 1: Consider the following values for A, b, Q_1, Q_2:

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\
b &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Trivially, the true solution is $x = [\frac{1}{2}, 1]$. Consider the following splittings:

$$
\begin{aligned}
Q_1 &= \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix} \\
Q_2 &= \begin{bmatrix} 200 & 0 \\ 0 & 1 + \frac{1}{99} \end{bmatrix}
\end{aligned}
$$

The associated iteration matrices are:

$$
\begin{aligned}
S_1 &= \begin{bmatrix} .9 & 0 \\ 0 & .9 \end{bmatrix} \\
S_2 &= \begin{bmatrix} .99 & 0 \\ 0 & .01 \end{bmatrix}
\end{aligned}
$$

Trivially,

$$
\begin{aligned}
\rho(S_1) &= .9 \\
\rho(S_2) &= .99 \\
\rho(S_2) > \rho(S_1)
\end{aligned}
$$

Starting with initial estimate $x^{(0)} = [00]$, have $e^{(0)} = [\frac{1}{2}, 1]$. After applying one iteration:

$$
\begin{aligned}
e^{(1)} &= \begin{bmatrix} .45 \\ .9 \end{bmatrix} \\
\hat{e}^{(1)} &= \begin{bmatrix} .495 \\ .01 \end{bmatrix} \\
\left\|e^{(1)}\right\| &\approx 1.062 \\
\left\|\hat{e}^{(1)}\right\| &\approx 0.49510
\end{aligned}
$$

Thus, it is not the case that $\left\|e^{(k)}\right\| < \left\|\hat{e}^{(k)}\right\|$.

Problem 2: Implement Jacobi method, the Gauss-Seidel method, the SOR method and the conjugate gradient method.

Solution 2: See attached for source code. As can be seen in Figures1 and 2, as n increases, the number of iterations and execution time for Jacobi, GS and SOR seem to increase on the order of n^2.

On the other hand, Conjugate Gradient, and the built-in 'solve' function seem to increase linearly with n.

Problem 3a: Show that the Chebyshev polynomials $\tau_k(t)$ satisfy the three term recurrence:

$$\tau_{k+1}(t) = 2t\tau_k(t) - \tau_{k-1}(t)$$

Solution 3a: Proof by Induction on k.

Proof of Base Cases, $k = 0, 1, 2; t \leq |1|$.

$$\begin{aligned}
T_0(t) &= 1 \\
T_1(t) &= \cos\cos^{-1} t \\
&= t \\
T_2(t) &= \cos 2\cos^{-1} t \\
\text{let:} & \\
\theta &= \cos^{-1} t \\
T_2(\theta) &= \cos 2\theta \\
&= 2\cos^2\theta - 1 \\
&= 2t^2 - 1 \\
T_2(t) &= 2t \cdot T_1(t) - T_0(t)
\end{aligned}$$

$\square$

Inductive Case. Assume true for all $j \leq k$, prove for $k+1$.

$$\begin{aligned}
T_{k+1} &= \cos\left((k+1)\cos^{-1}(t)\right) \\
&= \cos(k+1)\theta \\
&= \cos k\theta\cos\theta - \sin k\theta \sin\theta \\
&= t \cdot \cos k\theta - \sin k\theta \sin\theta \\
&= t \cdot T_k(t) - \sin k\theta\sin\theta \\
&= t \cdot T_k(t) - \frac{1}{2}\left[\cos(k\theta - \theta) - \cos(k\theta + \theta)\right] \\
&= t \cdot T_k(t) - \frac{1}{2}T_{k-1}(t) + \frac{1}{2}T_{k+1}(t) \\
\frac{1}{2}T_{k+1}(t) &= t \cdot T_k(t) - \frac{1}{2}T_{k-1}(t) \\
T_{k+1}(t) &= 2t \cdot T_k(t) - T_{k-1}(t)
\end{aligned}$$

$\square$

When $t > |1|$: *see attached*

Problem 3b: Prove that:

$$\tau_k(t) = \frac{1}{2}\left[\left(t + \sqrt{t^2 - 1}\right)^k + \left(t - \sqrt{t^2-1}\right)^k\right]$$

Solution 3b:

$$
\begin{aligned}
\tau_k(t) &= \cos k \cos t^{-1} \\
&= \frac{1}{2}e^{ik\cos t^{-1}} + \frac{1}{2}e^{-ik\cos t^{-1}} \\
\cos t^{-1} &= -i\ln(t + i\sqrt{1-t^2}) \\
&= \frac{1}{2}e^{ik(-i\ln(t+i\sqrt{1-t^2}))} + \frac{1}{2}e^{-ik(-i\ln(t+i\sqrt{1-t^2}))} \\
&= \frac{1}{2}e^{k\ln(t+i\sqrt{1-t^2})} + \frac{1}{2}e^{-k\ln(t+i\sqrt{1-t^2})} \\
&= \frac{1}{2}e^{\ln(t+i\sqrt{1-t^2})^k} + \frac{1}{2}e^{\ln(t+i\sqrt{1-t^2})^{-k}} \\
&= \frac{\left(t+i\sqrt{1-t^2}\right)^k}{2} + \frac{1}{2\left(t+i\sqrt{1-t^2}\right)^k} \\
&= \frac{\left(t+i\sqrt{1-t^2}\right)^{2k}+1}{2\left(t+i\sqrt{1-t^2}\right)^k}
\end{aligned}
$$

Proof by induction on k

Base case, $k = 1$.

$$
\begin{aligned}
\tau_1(t) &= \frac{\left(t+i\sqrt{1-t^2}\right)^2+1}{2\left(t+i\sqrt{1-t^2}\right)} \\
&= \frac{t^2+2it\sqrt{1-t^2}-(1-t^2)+1}{2\left(t+i\sqrt{1-t^2}\right)} \\
&= \frac{2t^2+2it\sqrt{1-t^2}+0}{2\left(t+i\sqrt{1-t^2}\right)} \\
&= t \\
&= \frac{1}{2}(t+\sqrt{t^2-1}) + \frac{1}{2}(t-\sqrt{t^2-1})
\end{aligned}
$$

□

Inductive case, assume for $j \leq k$*, prove for* $k+1$.

$$
\begin{aligned}
\tau_{k+1}(t) &= 2t \cdot \tau_k(t) - \tau_{k-1}(t) \\
&= \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^k + \left(t-\sqrt{t^2-1}\right)^k\right] \cdot 2t - \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^{k-1} + \left(t-\sqrt{t^2-1}\right)^{k-1}\right] \\
&= \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^{k-1}\left[\left(t+\sqrt{t^2-1}\right)\cdot 2t - 1\right] + \left(t-\sqrt{t^2-1}\right)^{k-1}\left[\left(t-\sqrt{t^2-1}\right)\cdot 2t - 1\right]\right] \\
&= \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^{k-1}\left[2t^2+2t\sqrt{t^2-1}-1\right] + \left(t-\sqrt{t^2-1}\right)^{k-1}\left[2t^2-2t\sqrt{t^2-1}-1\right]\right]
\end{aligned}
$$

Note that:

$$
\begin{aligned}
(t+\sqrt{t^2-1})^2 &= t^2+2t\sqrt{t^2-1}+t^2-1 \\
&= 2t^2+2t\sqrt{t^2-1}-1
\end{aligned}
$$

And:

$$\begin{aligned}(t-\sqrt{t^2-1})^2 &= t^2-2t\sqrt{t^2-1}+t^2-1\\ &= 2t^2-2t\sqrt{t^2-1}-1\end{aligned}$$

So:

$$\begin{aligned}\tau_{k+1}(t) &= \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^{k-1}(t+\sqrt{t^2-1})^2+\left(t-\sqrt{t^2-1}\right)^{k-1}(t-\sqrt{t^2-1})^2\right]\\ &= \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^{k+1}+\left(t-\sqrt{t^2-1}\right)^{k+1}\right]\end{aligned}$$

□

Problem 4a: Given a symmetric positive-definite matrix A of order n, show that $\langle x,y\rangle_A$ defines an inner product on $\mathcal{R}_n$.

Solution 4a: Show $\langle x,y\rangle_A = \langle y,x\rangle_A$.

Proof.

$$\begin{aligned}\langle x,y\rangle_A &= (x,Ay)\\ &= x^TAy\\ &= x^TA^Ty\\ &= y^TAx\\ &= \langle y,x\rangle\end{aligned}$$

□

Show $\langle \alpha x,y\rangle_A = \alpha\langle x,y\rangle_A$.

Proof.

$$\begin{aligned}\langle \alpha x,y\rangle_A &= (x,\alpha Ay)\\ &= \alpha(x,Ay)\\ &= \alpha\langle x,y\rangle_A\end{aligned}$$

□

Show $\langle x+z,y\rangle_A = \langle x,y\rangle_A + \langle z,y\rangle_A$

Proof.

$$\begin{aligned}\langle x+z,y\rangle_A &= \langle y,x+z\rangle_A\\ &= (y,A(x+z))\\ &= (y,Ax+Az)\\ &= (y,Ax)+(y,Az)\\ &= \langle y,x\rangle_A+\langle y,z\rangle_A\\ &= \langle x,y\rangle_A+\langle z,y\rangle_A\end{aligned}$$

□

Show $\langle x, x\rangle_A \geq 0,\ 0 \iff x = 0$

Proof.

$$\begin{array}{rcl} \langle x, x\rangle_A & = & (x, Ax) = x^T Ax \\ x^T Ax > 0 & \iff & x > 0 \text{ By definition of positive definite} \\ \implies & & \\ \langle x, x\rangle_A \geq 0, & & = 0 \iff x = 0 \end{array}$$

□

Problem 4b: A matrix B is symmetric with respect to an inner product $\langle , \rangle$ if $\langle Bx, y\rangle = \langle x, By\rangle$. Show that if A and Q are symmetric positive-definite, then $Q^{-1}A$ is symmetric with respect to the inner product $\langle , \rangle_Q$.

Solution 4b:

Proof.

$$\begin{array}{rcl} \left\langle Q^{-1}Ax, y\right\rangle_Q & = & (QQ^{-1}Ax, y) \\ & = & (x, Ay) \\ & = & x^T A^T y \\ & = & x^T Q^T Q^{-T} A^T y \\ & = & x^T Q^T Q^{-1} Ay \\ & = & (Qx, Q^{-1}Ay) \\ & = & \left\langle x, Q^{-1}Ay\right\rangle \end{array}$$

□

Problem 5a: Show that

$$AR_k = R_k S_k - \frac{1}{\alpha_k}[0, \ldots, 0, r_k].$$

where

$$S_k = tridiag\left[-\frac{1}{\alpha_{j-1}}, -\frac{1}{\alpha_j} + \frac{\beta_{j-1}}{\alpha_{j-1}}, -\frac{\beta_j}{\alpha_j}\right]$$

Solution 5a: The step to update the residual r goes as follows:

$$r^{(j+1)} = r^{(j)} - \alpha_j Ap^{(j)}$$

The step to update the 'update' vector p:

$$p^{(j)} = r^{(j)} + \beta_j p^{(j-1)}$$

Substituting $p^{(j)}$:

$$r^{(j+1)} = r^{(j)} - \alpha_j Ar^{(j)} - \alpha_j \beta_j Ap^{(j-1)}$$

Rewriting the residual step for j:

$$Ap^{(j-1)} = \frac{r^{(j-1)} - r^{(j)}}{\alpha_{j-1}}$$

And plugging back in above for $Ap^{(j-1)}$:

$$r^{(j+1)} = r^{(j)} - \alpha_j A r^{(j)} - \frac{\alpha_j \beta_j}{\alpha_{j-1}} \left(r^{(j-1)} - r^{(j)} \right)$$

Rearranging terms:

$$A r^{(j)} = \frac{1}{\alpha_j} (r^{(j)} - r^{(j+1)}) - \frac{\beta_j}{\alpha_{j-1}} \left(r^{(j-1)} - r^{(j)} \right)$$

For $j = 0, 1 \ldots k-1$:

$$\begin{aligned}
A r^{(0)} &= \frac{1}{\alpha_0} (r^{(0)} - r^{(1)}) \\
A r^{(1)} &= \frac{1}{\alpha_1} (r^{(1)} - r^{(2)}) - \frac{\beta_1}{\alpha_0} \left(r^{(0)} - r^{(1)} \right) \\
A r^{(2)} &= \frac{1}{\alpha_2} (r^{(2)} - r^{(3)}) - \frac{\beta_2}{\alpha_1} \left(r^{(1)} - r^{(2)} \right) \\
\ldots & \\
A r^{(k-1)} &= \frac{1}{\alpha_{k-1}} (r^{(k-1)} - r^{(k)}) - \frac{\beta_{k-1}}{\alpha_{k-2}} \left(r^{(k-2)} - r^{(k-1)} \right)
\end{aligned}$$

Collecting the common coefficients in $r^{(0)} \ldots r^{(k-1)}$ this forms the linear system:

$$A[r^{(0)}, r^{(1)} \ldots r^{(k-1)}] = [r^{(0)}, r^{(1)} \ldots r^{(k-1)}] \begin{bmatrix} \frac{1}{\alpha_0} & -\frac{\beta_1}{\alpha_0} & 0 & \ldots & 0 \\ -\frac{1}{\alpha_0} & \frac{1}{\alpha_1} + \frac{\beta_1}{\alpha_0} & -\frac{\beta_2}{\alpha_1} & 0 & \ldots \\ \ldots & & & & \\ \ldots & & & & \\ 0 & \ldots & 0 & -\frac{1}{\alpha_{k-2}} & \frac{1}{\alpha_{k-1}} + \frac{\beta_{k-1}}{\alpha_{k-2}} \end{bmatrix} + \frac{1}{\alpha_{k-1}} r^{(k)} \mathbf{e}^{(k)}$$

Where $\mathbf{e}^{(k)}$ is the $k+1$th column of the identity matrix.
This is exactly the form desired:

$$A R_k = R_k S_k - \frac{1}{\alpha_k} [0, \ldots, 0, r_k].$$

Problem 5b: Show that S_k is similar to a symmetric matrix T_k . Where have you seen the matrix T_k before?

Solution 5b: Need to show that there exists an invertible $k \times k$ matrix U s.t.

$$U S_k U^{-1} = T_k$$

or perhaps that S_k and T_k have the same eigenvalues. T_k is most likely the matrix of coefficients from the Lanczos procedure, but α and β are probably different. I'm fairly certain that if $v^{(1)} = x^{(0)}$, both the set of $\{r^{(j)}\}$ and the Lanczos basis vectors $\{v^{(j)}\}$ span the Krylov subspace $K(A, x^{(0)}$, but I'm not sure how this helps.

Problem 6: Show the PCG algorithm can be interpreted as an implementation of solving the normal CG on the following system:

$$\left[L^{-1} A L^{T} \right] \hat{x} = \left[L^{-1} b \right]$$

where $Q = LL^T$ and $\hat{x} = L^{-T} x$.

Solution 6: The PCG algorithm proceeds as follows:

$$\begin{aligned}
x^{(0)} &= \textit{arbitrary} \\
r^{(0)} &= b - Ax^{(0)} \\
\tilde{r}^{(0)} &= Q\left(b - Ax^{(0)}\right) \\
p^{(0)} &= \tilde{r}^{(0)} \\
x^{(j+1)} &= x^{(j)} + \alpha_j p^{(j)} \\
r^{(j+1)} &= r^{(j)} + \alpha_j A p^{(j)} \\
\tilde{r}^{(j+1)} &= Q r^{(j+1)} \\
\beta_j &= ?? \\
p^{(j+1)} &= \tilde{r}^{(j)} + \beta_j p^{(j)}
\end{aligned}$$

TODO: Finish this.

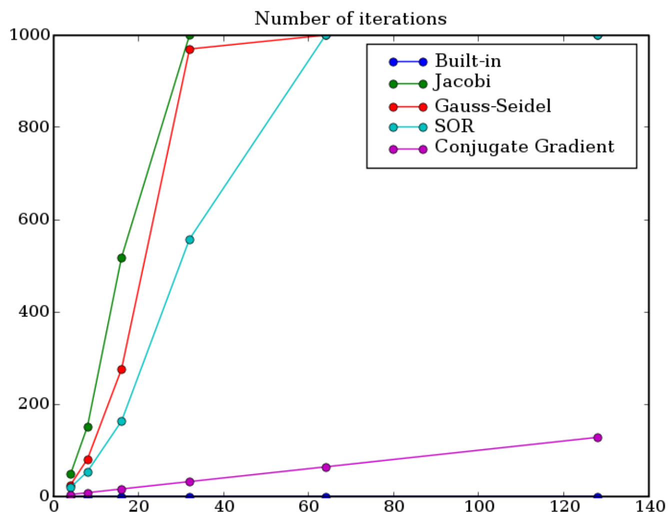

Figure 1: Number of iterations to convergence (Max is 10000)

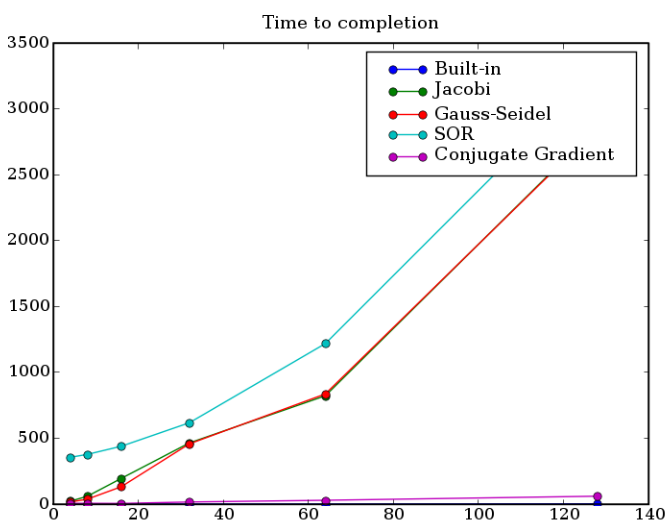

Figure 2: Amount of time until convergence (ms) (Max 10000 iterations)