Homework III

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Problem 1: Derive an error bound for Simpson's rule.

Solution 1: The error for a general quadrature rule is as follows:

$$\begin{aligned} E(f) &= I(f) - Q(f) \\ &= \int_a^b f(x)dx - \int_a^b p_n(x)dx \\ &= \int_a^b \left[f(x) - p_n(x)\right] dx \\ &= \int_a^b \left[(x-x_0)\cdot(x-x_1)\ldots(x-x_n)\frac{f^{(n+1)}(x)}{(n+1)!}\right] dx \end{aligned}$$

For Simpson's rule, $n=2$, so $x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$, and:

$$E(f) = \int_a^b \left[(x-a)\left(x - \frac{a+b}{2}\right)(x-b)\frac{f^{(3)}(x)}{3!}\right] dx$$

Let $w(x) = (x-a)\left(x-\frac{a+b}{2}\right)(x-b)$. Then

$$E(f) = \int_a^b w(x)\frac{f^{(3)}(x)}{6}dx$$

If $w(x)$, can be shown to be one sign, then the mean value theorem can be applied to pull $f[a,b,x]$ out of the integral. However:

$$w(x) = \begin{cases} \leq 0 & a \leq x \leq \frac{a+b}{2} \\ \geq 0 & \frac{a+b}{2} \leq x \leq b \end{cases}$$

So $w(x)$ is not of one sign, but by splitting the integral into two parts:

$$E(f) = \int_a^{\frac{a+b}{2}} w(x)f^{(3)}dx + \int_{\frac{a+b}{2}}^b w(x)f^{(3)}dx$$

$w(x)$ is one sign on the new bounds of integration. Thus, the mean value theorem applies to each part and:

$$E(f) = f^{(3)}(\xi)\int_a^{\frac{a+b}{2}} w(x)dx + f^{(3)}(\xi')\int_{\frac{a+b}{2}}^b w(x)dx$$

For some $\xi, \xi' \in [a,b]$. Thus, it also true that $\exists\zeta \in [a,b]$:

$$E(f) \leq f^{(3)}(\zeta)\left[\int_a^{\frac{a+b}{2}} w(x)dx + \int_{\frac{a+b}{2}}^b w(x)dx\right]$$

Doing some algebra:

$$\begin{aligned} \int w(x)dx &= \int (x-a)(x-(a+b)/2)(x-b)dx \\ &\ldots \end{aligned}$$