CMSC 666
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Homework I

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Problem 1: Show that there exist unique numbers $\gamma_0, \gamma_1 \ldots \gamma_n$ s.t.:

$$\sum_{j=0}^{n} \gamma_j p(x_j) = \int_a^b p(x)dx$$

for all polynomials $p(x)$ for degree $\leq n$.

Solution 1: $p(x)$ is a polynomial of degree $\leq n$, so:

$$\begin{aligned} p(x) &= \sum_{i=0}^{n} \alpha_i x^i \\ \int_a^b p(x)dx &= \sum_{i=0}^{n} \frac{\alpha_i(b^{i+1} - a^{i+1})}{(i+1)} \\ \sum_{j=0}^{n} \gamma_j p(x_j) &= \sum_{j=0}^{n} \gamma_j \left[\sum_{i=0}^{n} \alpha_i x_j^i \right] \end{aligned}$$

This must generalize to any polynomial, and therefore any choice of α_i's. Thus, for each α_i, the component on the left is equivalent to that on the right, and:

$$\sum_{j=0}^{n} \alpha_i x_j^i = \frac{(b^{i+1} - a^{i+1})}{(i+1)}$$

Each of the n α's provides a constraint on the γ_i's leading to the linear system $A\Gamma = B$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_0 & x_1 & x_2 & \ldots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \ldots & x_n^2 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ x_0^n & x_1^n & x_2^n & \ldots & x_n^n \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ . \\ . \\ . \\ \gamma_n \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{(b-a)}{2} \\ \frac{(b^2-a^2)}{3} \\ . \\ . \\ . \\ \frac{(b^{n+1}-a^{n+1})}{(n+1)} \end{bmatrix}$$

Since the x_i's are unique, then this matrix A is the transpose of the Vandermonde matrix. It has been shown that this is invertible, and in general, if a matrix is invertible, so is its transpose[3]. Thus, there is unique solution to the set of γ_i's.

Problem 2: Define a continuous function $g(x)$ s.t.

$$\|p_n\|_\infty = \|\gamma_n\|_\infty \|g\|_\infty$$

Solution 2: Expanding the above:

$$\begin{aligned}
\left\|\sum_{j=0}^{n} g(x_j) l_j(x)\right\|_\infty &= \left\|\sum_{j=0}^{n} l_j(x)\right\|_\infty \|g\|_\infty \\
\max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} g(x_j) l_j(x^*)\right| &= \max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} l_j(x^*)\right| \cdot \max_{x^* \in [a,b]} |g(x^*)|
\end{aligned}$$

Suppose g is constructed s.t. for some fixed y, $\forall i = 0, 1 \ldots n, g(x_i) = y$ and $\forall x \in [a,b], g(x) \leq |y|$. Thus, the above becomes:

$$\begin{aligned}
\max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} y \cdot l_j(x^*)\right| &= \max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} l_j(x^*)\right| \cdot |y| \\
\max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} l_j(x^*)\right| \cdot |y| &= \max_{x^* \in [a,b]} \left|\sum_{j=0}^{n} l_j(x^*)\right| \cdot |y|
\end{aligned}$$

And g as described fits the requirements. Constructing a non-trivial g (i.e. not the uniform function) requires satisfying three conditions: that g is uniform at the x_i's, that is, $g(x_i) = y$; the x_i's are peaks, that is: $g'(x_i) = 0$ and the x_i's are also maxima: $g(x_i) \geq |g(x)|$.
I was not able to come up with a way of constructing g... though it seems related to the Chebyshev polynomial and optimal uniform polynomial.

Problem 3: Prove the error bound for the piecewise linear spline that interpolates a function f.

Solution 3: Take the Taylor expansion of $f(x_j)$ and $f(x_{j+1})$ at the point x:

$$\begin{aligned}
f(x_j) &= f(x) - f'(x)(x_j - x) + R_1 \qquad (1) \\
f(x_{j+1}) &= f(x) - f'(x)(x_{j+1} - x) + R_2 \qquad (2)
\end{aligned}$$

Where the remainders R_1, R_2 are given by [2]:

$$\begin{aligned}
R_1 &= \frac{(x_j - x)^2 f''(x_j^+)}{2} \text{ for some } x_j^+ \in [x_j, x] \qquad (3) \\
R_2 &= \frac{(x_{j+1} - x)^2 f''(x_{j+1}^-)}{2} \text{ for some } x_{j+1}^- \in [x, x_{j+1}] \qquad (4)
\end{aligned}$$

Multiply equation (1) by $(x_{j+1}-x)$ and equation (2) by $(x_j - x)$, and subtract them:

$$
\begin{aligned}
f(x_j)(x_{j+1}-x) - f(x_{j+1})(x_j+1) = \ldots & \\
& f(x)(x_{j+1}-x-x_j+x) + (f'(x)-f'(x))(x_j-x)(x_{j+1}-x) + R_1(x_{j+1}-x) - R_2(x_j-x) \\
&= f(x)(x_{j+1}-x_j) + R_1(x_{j+1}-x) - R_2(x_j-x) \\
&= f(x)(x_{j+1}-x_j) + R_1(x_{j+1}-x) - R_2(x_j-x) \\
f(x) &= \frac{f(x_j)(x_{j+1}-x) - f(x_{j+1})(x_j-x)}{(x_{j+1}-x_j)} + \frac{R_1(x_{j+1}-x) - R_2(x_j-x)}{(x_{j+1}-x_j)} \\
&= \frac{f(x_j)(x_{j+1}-x+x_j-x_j) - f(x_{j+1})(x_j-x)}{(x_{j+1}-x_j)} + \ldots \\
&= \frac{f(x_j)(x_{j+1}-x_j) + f(x_j)(x_j-x) - f(x_{j+1})(x_j-x)}{(x_{j+1}-x_j)} + \ldots \\
&= f(x_j) + \frac{\left(f(x_j)-f(x_{j+1})\right)(x_j-x)}{(x_{j+1}-x_j)} + \ldots \\
&= p_n(x) + \ldots \\
f(x) - p_n(x) &= \frac{R_1(x_{j+1}-x) + R_2(x_j-x)}{(x_{j+1}-x_j)} \\
&= \left[\frac{(x_j-x)^2 f''(x_j^+)\cdot(x_{j+1}-x)}{2} + \frac{(x_{j+1}-x)^2 f''(x_{j+1}^-)\cdot(x_j-x)}{2}\right]\frac{1}{(x_{j+1}-x_j)} \\
&\le \max_{x_j^*\in[x_j,x_{j+1}]}\left[\frac{(x_j-x)^2 f''(x_j^*)\cdot(x_{j+1}-x)}{2} + \frac{(x_{j+1}-x)^2 f''(x_j^*)\cdot(x_j-x)}{2}\right]\frac{1}{(x_{j+1}-x_j)} \\
&= \max_{x_j^*\in[x_j,x_{j+1}]}\frac{f''(x_j^*)}{2}\cdot\left[\frac{(x_{j+1}-x)(x_j-x)(x_{j+1}-x-x_j+x)}{(x_{j+1}-x_j)}\right] \\
&= \max_{x_j^*\in[x_j,x_{j+1}]}\frac{f''(x_j^*)}{2}\cdot(x_{j+1}-x)(x_j-x) \\
& (x_{j+1}-x)(x_j-x) \text{ is bounded by } -1/4(x_{j+1}-x_j)^2\text{, so:} \\
|f(x)-p_n(x)| &\le \max_{x_j^*\in[x_j,x_{j+1}]}\frac{f''(x_j^*)}{8}\cdot(x_{j+1}-x_j)^2
\end{aligned}
$$

Expanding this definition to the entire range of $[a,b]$:

$$
|f(x)-p_n(x)| \quad \le \quad \frac{h^2}{8}\max_{\xi\in[a,b]} f''(\xi)
$$

Where $h = max_j(x_{j+1}-x_j)$.

Note: I had some help from [1], but the proof there is very terse, so filling in the steps was non-trivial. It was most helpful in leading me to realize I could take a Taylor expansion of $f(x_j)$ and $f(x_{j+1})$ about x instead of $f(x)$ about x_j, x_{j+1}.

References

[1] C.L. Seebeck Jr. Note on linear interpolation error. *The American Mathematical Monthly*, 62(1):35–36, jan 1955.

[2] Eric W. Weisstein. Taylor series. From MathWorld–A Wolfram Web Resource http://mathworld.wolfram.com/TaylorSeries.html .

[3] Eric W. Weisstein. Transpose. From MathWorld–A Wolfram Web Resource http://mathworld.wolfram.com/Transpose.html .