Assignment 10, Chapter 6 (**due on 26 Nov**): P1; P3; P5; P7

6.1 The Lippmann-Schwinger formalism can also be applied to a *one*-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

(a) Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \to E + i\varepsilon$ prescription still correct? Obtain an expression for the appropriate Green's function and write an integral equation for $\langle x|\psi^{(+)}\rangle$.

(b) Consider the special case of an attractive δ-function potential

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x) \quad (\gamma > 0).$$

Solve the integral equation to obtain the transmission and reflection amplitudes. Check your results with Gottfried 1966, p. 52.

(c) The one-dimensional δ-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of γ. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when k is regarded as a complex variable.

6.3 Estimate the radius of the ^{40}Ca nucleus from the data in Figure 6.6 and compare to that expected from the empirical value $\approx 1.4A^{1/3}$ fm, where A is the nuclear mass number. Check the validity of using the first-order Born approximation for these data.

6.5 A spinless particle is scattered by a weak Yukawa potential

$$V = \frac{V_0 e^{-\mu r}}{\mu r},$$

where $\mu > 0$ but V_0 can be positive or negative. It was shown in the text that the first-order Born amplitude is given by

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{[2k^2(1-\cos\theta)+\mu^2]}.$$

(a) Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for δ_l in terms of a Legendre function of the second kind,

$$Q_l(\zeta) = \frac{1}{2}\int_{-1}^{1} \frac{P_l(\zeta')}{\zeta - \zeta'} d\zeta'.$$

(b) Use the expansion formula

$$\begin{aligned} Q_l(\zeta) = {} & \frac{l!}{1\cdot 3\cdot 5\cdots(2l+1)} \\ & \times \left\{ \frac{1}{\zeta^{l+1}} + \frac{(l+1)(l+2)}{2(2l+3)} \frac{1}{\zeta^{l+3}} \right. \\ & \left. + \frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot 4\cdot (2l+3)(2l+5)} \frac{1}{\zeta^{l+5}} + \cdots \right\} \quad (|\zeta| > 1) \end{aligned}$$

to prove each assertion.

(i) δ_l is negative (positive) when the potential is repulsive (attractive).

(ii) When the de Broglie wavelength is much longer than the range of the potential, δ_l is proportional to k^{2l+1}. Find the proportionality constant.

6.7 Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} 0 & \text{for } r > a \\ \infty & \text{for } r < a. \end{cases}$$

(a) Derive an expression for the s-wave ($l = 0$) phase shift. (You need not know the detailed properties of the spherical Bessel functions to do this simple problem!)

(b) What is the total cross section $\sigma\,[\sigma = \int (d\sigma/d\Omega) d\Omega]$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section πa^2. You may assume without proof:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

$$f(\theta) = \left(\frac{1}{k}\right) \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta).$$