PHYS 5260 HW2

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Problem 1. Sakurai 1.21

Evaluate the $x - p$ uncertainty product $\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle$ for a one-dimensional particle confined between two rigid walls,

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Do this for both the ground and excited states.

The wavefunction for the particle can be found by solving the Schrodinger equation:

$$\begin{aligned} H\psi(x) &= E_n \psi(x) \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= E_n \psi(x) \end{aligned}$$

Solving the differential equation and normalizing, we get the wavefunction for $0 < x < a$:

$$\psi(x) = A \sin(kx) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

and for $\psi(x) = 0$ for $x > a$:

The expectation value of any operator $\hat{A}$ is given by:

$$\begin{aligned} \left\langle \hat{A} \right\rangle &= \int_0^\infty \psi(x)^* A \psi(x) dx \\ &= \int_0^a \frac{2}{a} \sin\left(\frac{n\pi}{a} x\right) \hat{A} \sin\left(\frac{n\pi}{a} x\right) dx \end{aligned}$$

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Thus the expectation value of the following operators are:

$$\langle x \rangle = \frac{a}{2}$$
$$\left\langle x^2 \right\rangle = \frac{2a^2}{6}$$
$$\langle p \rangle = 0$$
$$\left\langle p^2 \right\rangle = \frac{-\hbar^2 n^2 \pi^2}{a^2}$$

Substituting these values into the uncertainty product, we get:

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle &= \left(\left\langle x^2 \right\rangle - \left\langle x^2 \right\rangle \right) \left(\left\langle p^2 \right\rangle - \langle p \rangle^2 \right) \\ &= \frac{\hbar}{12}(-6 + n^2\pi^2)\end{aligned}$$

For ground state $n = 1$,

$$\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle = \frac{\hbar}{12}(-6 + \pi^2) \approx 0.322\hbar^2$$

which implies that the uncertainty principle holds, i.e. larger than $\hbar^2/4$. For excited states $n > 1$, the uncertainty principle also holds.

Problem 2. Sakurai 1.22

(a) Prove that $1/\sqrt{2}(1+\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by the angle of $\pi/2$. (The minus sign signifies that the rotation is clockwise)

The form of the rotation operator about the x-axis is:

$$\begin{aligned} D(\hat{\boldsymbol{x}}, \phi) &= \exp\left(-i\phi \frac{\hat{\boldsymbol{x}} \cdot \boldsymbol{S}}{\hbar}\right) \\ &= \exp\left(-i\phi \frac{S_x}{\hbar}\right) \\ &= \exp\left(-i\phi \frac{\sigma_x}{2}\right) \\ &= \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \end{aligned}$$

Which is the same as the matrix representation of $1/\sqrt{2}(1+\sigma_x)$ for $\phi = -\pi/2$, i.e.

$$\frac{1}{\sqrt{2}}(1+\sigma_x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = D(\hat{\boldsymbol{x}}, -\pi/2)$$

(b) Construct the matrix representation of S_z when the eigekets of S_y are used as base vector

The eigenkets of S_y are:

$$\begin{aligned} |\psi_{y+}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \\ |\psi_{y-}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned}$$

Therefore to find the matrix representation of S_z, we need to find the projection of S_z onto the eigenkets:

$$\begin{aligned} \langle\psi_{y+}|\, \sigma_z \,|\psi_{y+}\rangle &= 0 \\ \langle\psi_{y+}|\, \sigma_z \,|\psi_{y-}\rangle &= 1 \\ \langle\psi_{y-}|\, \sigma_z \,|\psi_{y+}\rangle &= 1 \\ \langle\psi_{y-}|\, \sigma_z \,|\psi_{y-}\rangle &= 0 \end{aligned}$$

Therefore, the matrix representation of S_z in the basis of $|\psi_{y+}\rangle$ and $|\psi_{y-}\rangle$ is:

$$\langle\psi_y|\, S_z \,|\psi_y\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which is the same as S_x. This is because we rotate the basis clockwise by $\pi/2$ about the x-axis. This implies under this basis rotation, the old z axis is now the new x axis.

Problem 3. Sakurai 1.27

(a) Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A\left|a'\right\rangle = a'\left|a'\right\rangle$. Evaluate $\left\langle b''\right| f(A)\left|b'\right\rangle$ when the transformation from the a' basis to the b' basis is known.

$$\begin{aligned}
\left\langle b''\right| f(A)\left|b'\right\rangle &= \sum_{a''}\sum_{a'} \left\langle b''|a''\right\rangle \left\langle a''\right| f(A)\left|a'\right\rangle \left\langle a'|b'\right\rangle \\
&= \sum_{a''}\sum_{a'} \left\langle b''|a''\right\rangle f(a')\delta_{a,a''} \left\langle a'|b'\right\rangle \\
&= \sum_{a'} f(a') \left\langle b''|a'\right\rangle \left\langle a'|b'\right\rangle
\end{aligned}$$

(b) Using the continuum analogue of the result obtained in (a), evaluate

$$\left\langle \boldsymbol{p''}\right| F(r)\left|\boldsymbol{p'}\right\rangle$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are *operators.*

We can start by applying continuum condition:

$$\begin{aligned}
\left\langle \boldsymbol{p''}\right| F(r')\left|\boldsymbol{p'}\right\rangle &= \int F(r') \left\langle \boldsymbol{p''}|\boldsymbol{x'}\right\rangle \left\langle \boldsymbol{x'}|\boldsymbol{p'}\right\rangle d^3r' \\
&= \frac{1}{(2\pi\hbar)^3}\int F(r') e^{i(\boldsymbol{p'}-\boldsymbol{p''})\cdot\boldsymbol{x'}/\hbar} d^3r'
\end{aligned}$$

Then, we can use symmetry arguement to simplify the integral. Consider $\boldsymbol{q} \equiv \boldsymbol{p'} - \boldsymbol{p''}$, and $\boldsymbol{q'}\cdot\boldsymbol{x'} = q'r'\cos\theta$, where θ is the angle between $\boldsymbol{q}$ and $\boldsymbol{x'}$. Then the integral becomes:

$$\begin{aligned}
\int F(r') e^{i\boldsymbol{q}\cdot\boldsymbol{x'}/\hbar} d^3r' &= 2\pi\int_0^\infty dr' F(r') \int_0^\pi e^{iq'r'\cos\theta/\hbar}\sin\theta d\theta \\
&= 2\pi\int_0^\infty dr' F(r') \int_0^\pi e^{iq'r'\cos\theta/\hbar} d\theta \\
&= 2\pi\int_0^\infty dr' F(r') \frac{2\hbar}{q'r'}\sin\left(q'r'/\hbar\right)
\end{aligned}$$

Thus the final result is:

$$\left\langle \boldsymbol{p''}\right| F(r')\left|\boldsymbol{p'}\right\rangle = \frac{1}{2\pi^2\hbar^2}\int dr' F(r') \frac{\sin(q'r'/\hbar)}{q'r'}$$

Problem 4. Sakurai 1.30

The translation operator for a finite (spatial) displacement is given by:

$$\mathcal{T}(\boldsymbol{l}) = \exp\left(\frac{-i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right)$$

where $\boldsymbol{p}$ is the momentum operator.

(a) Evaluate

$$[x_i, \mathcal{T}(\boldsymbol{l})]$$

Note that $\mathcal{T}(\boldsymbol{l})$ is a power series of $\boldsymbol{p}$. First, note the following commutator relation:

$$\begin{aligned}
[x, p^n] &= p[x, p^{n-1}] + [x, p]p^{n-1} \\
&= p^2[x, p^{n-2}] + 2[x, p]p^{n-1} \\
&= \cdots \\
&= p^n[x, p^0] + n[x, p]p^{n-1} \\
&= ni\hbar p^{n-1} \\
&= i\hbar\frac{\partial p^n}{\partial p}
\end{aligned}$$

Thus for a power series $\mathcal{T}$, the commutation relation is:

$$\begin{aligned}
[x_i, \mathcal{T}(\boldsymbol{l})] &= i\hbar\frac{\partial}{\partial p_i}\exp\left(\frac{-i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right) \\
&= l_i\exp\left(\frac{-i\boldsymbol{p}\cdot\boldsymbol{l}}{\hbar}\right) \\
&= l_i\mathcal{T}(\boldsymbol{l})
\end{aligned}$$

(b) Using (a) (or otherwise), demonstrate how the expectation value of $\langle\boldsymbol{x}\rangle$ changes under translation:

Using Heisenberg picture, the operator x_i changes to $x_i' = \mathcal{T}(\boldsymbol{l})^\dagger x_i'\mathcal{T}(\boldsymbol{l})$ under translation. The expectation value is given by:

$$\begin{aligned}
\langle x_i'\rangle &= \langle\alpha| x_i' |\alpha\rangle \\
&= \langle\alpha| \mathcal{T}(\boldsymbol{l})^\dagger x_i'\mathcal{T}(\boldsymbol{l}) |\alpha\rangle \\
&= \langle\alpha| \mathcal{T}(\boldsymbol{l})^\dagger[x, \mathcal{T}(\boldsymbol{l})] + x |\alpha\rangle \\
&= l_i + \langle x_i\rangle
\end{aligned}$$

Problem 5. Sakurai 1.33

(a) Prove the following:

(i) $\langle p'| x |\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p'|\alpha\rangle$

(ii) $\langle\beta| x |\alpha\rangle = \int dp' \varphi_\beta^*(p') i\hbar\frac{\partial}{\partial p'}\varphi_\alpha(p')$

where $\varphi_\alpha(p') = \langle p'|\alpha\rangle$ and $\varphi_\beta(p') = \langle p'|\beta\rangle$ are momentum space wavefunction.

First, lets find the projection of $x|p\rangle$ onto the different momentum space:

$$\begin{aligned}
\langle p'| x |p''\rangle &= \int \langle p'| x |x'\rangle \langle x'|p''\rangle \, dx' \\
&= \int x' \langle p'|x'\rangle \langle x'|p''\rangle \, dx' \\
&= \int x' \frac{1}{2\pi\hbar} \exp\left(-i\frac{(p'-p'')\cdot x}{\hbar}\right) dx' \\
&= \frac{1}{2\pi\hbar} i\hbar \frac{\partial}{\partial p'} \int \exp\left(-i\frac{(p'-p'')\cdot x'}{\hbar}\right) dx' \\
&= i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')
\end{aligned}$$

Then we can expand the operator x

$$\begin{aligned}
x &= \iint dp' dp'' |p'\rangle \langle p'| x |p''\rangle \langle p''| \\
&= i\hbar \int dp'' |p''\rangle \frac{\partial}{\partial p''} \langle p''|
\end{aligned}$$

Then:

$$\begin{aligned}
\langle p'| x |\alpha\rangle &= \int dp'' \langle p'|p''\rangle \frac{\partial}{\partial p''} \langle p''|\alpha\rangle \\
&= \frac{\partial}{\partial p'} \langle p'| \alpha
\end{aligned}$$

Similarly:

$$\begin{aligned}
\langle\beta| x |\alpha\rangle &= \int dp' \langle\beta|p'\rangle \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \\
&= \int dp' \varphi_\beta^*(p') i\hbar\frac{\partial}{\partial p'}\varphi_\alpha(p')
\end{aligned}$$

(b) What is the signifiance of

$$\exp\left(\frac{ix\ \Theta}{\hbar}\right)$$

where x is the position operator and Θ is some number with the dimension of momentum? Justify your answer.

This is the momentum translation operator. Similar to the position translation operator, we can define a momentum translation operator $T(dp)$:

$$T(dp) = \left(1 - \frac{x\ dp}{\hbar}\right)$$

Over a finite distance in the momentum space, the momentum translation operator is given by:

$$\begin{aligned} T(dp) &= \lim_{N\to\infty}\left(1 - \frac{x\ \Theta}{N\hbar}\right)^N \\ &= \exp\left(\frac{ix\ \Theta}{\hbar}\right) \end{aligned}$$