

# PHYS 5260 HW1

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## Problem 1. Sakurai 1.3

Show that the determinant of a  $2 \times 2$  matrix  $\boldsymbol{\sigma} \cdot \mathbf{a}$  is invariant under

$$\boldsymbol{\sigma} \cdot \mathbf{a} \rightarrow \boldsymbol{\sigma} \cdot \mathbf{a}' \equiv \exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} \boldsymbol{\sigma} \cdot \mathbf{a} \exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\}$$

Find  $a'_k$  in terms of  $a_k$  when  $\hat{\mathbf{n}}$  is in the positive  $z$ -direction and interpret your result.

Note the following properties of the matrix determinant:  $|ABC| = |A||B||C|$ . Therefore, to prove the invariance, we need to show that:  $|\boldsymbol{\sigma} \cdot \mathbf{a}| = |\boldsymbol{\sigma} \cdot \mathbf{a}'|$ . In this case,  $|\boldsymbol{\sigma} \cdot \mathbf{a}'|$  can be expanded as:

$$|\boldsymbol{\sigma} \cdot \mathbf{a}'| = \left| \exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} \right| |\boldsymbol{\sigma} \cdot \mathbf{a}| \left| \exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} \right|$$

Since the two exponential terms are the inverse of each others, it follows that:

$$\left| \exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} \right| \left| \exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} \right| = 1$$

Therefore, we have:

$$|\boldsymbol{\sigma} \cdot \mathbf{a}| = |\boldsymbol{\sigma} \cdot \mathbf{a}'| \quad (\text{Q.E.D})$$

For  $\hat{\mathbf{n}} = (0, 0, 1)$ , we have  $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Since  $\sigma_k$  is diagonal matrix, the exponential terms can be expressed as:

$$\begin{aligned} \exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} &= \begin{pmatrix} \exp\{i\phi/2\} & 0 \\ 0 & \exp\{-i\phi/2\} \end{pmatrix} \\ \exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\phi}{2}\right\} &= \begin{pmatrix} \exp\{-i\phi/2\} & 0 \\ 0 & \exp\{i\phi/2\} \end{pmatrix} \end{aligned}$$

Therefore:

$$\sigma_k a'_k = \begin{pmatrix} \exp\{i\phi/2\} & 0 \\ 0 & \exp\{-i\phi/2\} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \exp\{-i\phi/2\} & 0 \\ 0 & \exp\{i\phi/2\} \end{pmatrix} a_k$$

Solving for  $a'_k$ , we obtain  $a'_k = a_k$ . This means that  $\boldsymbol{\sigma} \cdot \boldsymbol{a}$  i.e., the angular momentum is invariant under rotation about the  $z$ -axis.

## Problem 2. Sakurai 1.6

Suppose  $|i\rangle$  and  $|j\rangle$  are eigenkets of some Hermitian operator  $A$ . Under what condition we conclude that  $|i\rangle + |j\rangle$  is also an eigenket of  $A$ ? Justify your answer.

Suppose the eigenvalues of each eigenkets are given by:  $E_i$  and  $E_j$ .

$$A|i\rangle = E_i|i\rangle$$

$$A|j\rangle = E_j|j\rangle$$

Summing the two equations, we have:

$$A|i\rangle + A|j\rangle = E_i|i\rangle + E_j|j\rangle = E(|i\rangle + |j\rangle)$$

The last equation only holds when  $E = E_i + E_j$ . This equation only holds when  $A$  is degenerate, and  $|i\rangle$  and  $|j\rangle$  are corresponding eigenkets.

### Problem 3. Sakurai 1.9

Construct  $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$  such that:

$$\mathbf{S} \cdot \hat{\mathbf{n}} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \left(\frac{\hbar}{2}\right) |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$$

where  $\hat{\mathbf{n}}$  is characterized by the angles shown in the accompanying figure. Express your answer as a linear combination of  $|+\rangle$  and  $|-\rangle$ .

Let the vector  $\mathbf{n}$  and the spin operator  $\mathbf{S}$  be given by:

$$\begin{aligned}\hat{\mathbf{n}} &= \cos \alpha \sin \beta \hat{\mathbf{x}} + \sin \alpha \sin \beta \hat{\mathbf{y}} + \cos \beta \hat{\mathbf{z}} \\ \mathbf{S} &= \frac{\hbar}{2} (\sigma_x \hat{\mathbf{x}} + \sigma_y \hat{\mathbf{y}} + \sigma_z \hat{\mathbf{z}})\end{aligned}$$

The inner product is thus given by:

$$\begin{aligned}\mathbf{S} \cdot \hat{\mathbf{n}} &= \frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & \cos(\alpha) \sin(\beta) - i \sin(\alpha) \sin(\beta) \\ \cos(\alpha) \sin(\beta) + i \sin(\alpha) \sin(\beta) & -\cos(\beta) \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & e^{-i\alpha} \sin(\beta) \\ e^{i\alpha} \sin(\beta) & -\cos(\beta) \end{pmatrix}\end{aligned}$$

Solving the eigenvalue problem for  $\mathbf{S} \cdot \hat{\mathbf{n}}$ , we obtain the condition:

$$|\mathbf{S} \cdot \hat{\mathbf{n}} - Iv| = 0 \iff v = \pm \frac{\hbar}{2}$$

which is consistent with the problem statement.

To find the corresponding eigenvectors for the eigenvalue  $+1$ , we solve the following equation:

$$\frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & e^{-i\alpha} \sin(\beta) \\ e^{i\alpha} \sin(\beta) & -\cos(\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} x \\ y \end{pmatrix}$$

We obtain the solution:

$$\begin{aligned}x(\cos \beta - 1) + ye^{-i\alpha} \sin \beta &= 0 \\ xe^{i\alpha} \sin \beta/2 + y \cos \beta/2 &= 0\end{aligned}$$

Therefore:

$$|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \cos \beta/2 |+\rangle + e^{-i\alpha} \sin \beta/2 |-\rangle$$

## Problem 4. Sakurai 1.12

A spin  $\frac{1}{2}$  system to be in an eigenstate of  $\mathbf{S} \cdot \hat{\mathbf{n}}$  with eigenvalue  $\hbar/2$ , where  $\hat{\mathbf{n}}$  is a unit vector lying in  $xz$ -plane that makes an angle  $\gamma$  with the positive  $z$ -axis.

- (a) Suppose  $S_x$  is measured. What is the probability of getting  $+\hbar/2$ ?

The state of the given system and the eigenstate of spin  $x$  with the eigenvalue of  $\hbar/2$  can be written as:

$$\begin{aligned} |\psi\rangle &= |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \cos \gamma/2 |+\rangle + \sin \gamma/2 |-\rangle \\ |\mathbf{S} \cdot \hat{\mathbf{x}}; +\rangle &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \end{aligned}$$

Therefore, the probability of getting  $|\mathbf{S} \cdot \hat{\mathbf{x}}; +\rangle$  is given by:

$$\begin{aligned} |\langle \mathbf{S} \cdot \hat{\mathbf{x}}; + | \psi \rangle|^2 &= \left( \frac{1}{\sqrt{2}} (\cos \gamma/2 + \sin \gamma/2) \right)^2 \\ &= \frac{1 + \sin \gamma}{2} \end{aligned}$$

- (b) Evaluate the dispersion in  $S_x$  - that is,

$$\langle (S_x - \langle S_x \rangle)^2 \rangle$$

The expectation values of  $S_x$  and  $S_x^2$  are given by:

$$\begin{aligned} \langle S_x \rangle &= \langle \psi | S_x | \psi \rangle = \frac{\hbar}{2} \sin \gamma \\ \langle S_x^2 \rangle &= \langle \psi | S_x^2 | \psi \rangle = \frac{\hbar^2}{4} \end{aligned}$$

Therefore, the dispersion is given by:

$$\langle (S_x - \langle S_x \rangle)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} \cos^2 \gamma$$

## Problem 5. Sakurai 1.15

Let  $A$  and  $B$  be observables. Suppose the simultaneous eigekets of  $A$  and  $B$   $\{|a', b'\rangle\}$  form a complete orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0$$

Yes. Suppose  $|a', b'\rangle$  is the simultaneous eigenstates of  $A$  and  $B$ , then:

$$A |a', b'\rangle = a' |a', b'\rangle$$

$$B |a', b'\rangle = b' |a', b'\rangle$$

Consider the following equations:

$$BA |a', b'\rangle = a' B |a', b'\rangle = a' b' |a', b'\rangle$$

$$AB |a', b'\rangle = b' A |a', b'\rangle = b' a' |a', b'\rangle$$

Therefore, subtracting the two equations above, we obtain the commutator:

$$[A, B] |a', b'\rangle = BA |a', b'\rangle - AB |a', b'\rangle = 0$$

## Problem 6. Sakurai 1.18

- (a) The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle \alpha | + \lambda^* \langle \beta |) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0$$

for any complex number  $\lambda$ ; then choose  $\lambda$  in such a way that the preceding inequality reduces to the Schwarz inequality.

Let  $\lambda = x + iy$ . Then:

$$\begin{aligned} 0 &\leq (\langle \alpha | + \lambda^* \langle \beta |) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \\ 0 &\leq \langle \alpha | \alpha \rangle + \lambda \langle \alpha | \beta \rangle + \lambda^* \langle \beta | \alpha \rangle + |\lambda|^2 \langle \beta | \beta \rangle \\ 0 &\leq \langle \alpha | \alpha \rangle + \lambda \langle \beta | \alpha \rangle^* + \lambda^* \langle \beta | \alpha \rangle + |\lambda|^2 \langle \beta | \beta \rangle \\ 0 &\leq \frac{\langle \alpha | \alpha \rangle}{\langle \beta | \beta \rangle} + \frac{2x \operatorname{Re} \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle} + \frac{2y \operatorname{Im} \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle} + (x^2 + y^2) \\ 0 &\leq \left( x + \frac{\operatorname{Re} \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle} \right)^2 + \left( y + \frac{\operatorname{Im} \langle \beta | \alpha \rangle}{\langle \beta | \beta \rangle} \right)^2 + \frac{\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle - |\langle \beta | \alpha \rangle|^2}{\langle \beta | \beta \rangle^2} \end{aligned}$$

Note that the first two terms must be larger than 0. Then we can see the Schwarz inequality on the last term, which is:

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle - |\langle \beta | \alpha \rangle|^2 \geq 0$$

- (b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A |\alpha\rangle = \lambda \Delta B |\beta\rangle$$

with  $\lambda$  purely imaginary.

Recall the generalized uncertainty principle:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

The expectation value of commutator  $[A, B] = [\Delta A, \Delta B]$  is given by:

$$\begin{aligned} \langle [\Delta A, \Delta B] \rangle &= \langle \alpha | \Delta A \Delta B - \Delta B \Delta A | \alpha \rangle \\ &= (\lambda^* - \lambda) \langle \alpha | (\Delta B)^2 | \alpha \rangle \\ &= -2\lambda \langle (\Delta B)^2 \rangle \end{aligned}$$

since  $\lambda^* = -\lambda$ .

Then, note that  $\langle (\Delta A)^2 \rangle = |\lambda|^2 \langle (\Delta B)^2 \rangle$ . Therefore, both sides of the inequality are equal to  $|\lambda|^2 \langle (\Delta B)^2 \rangle^2$ .

- (c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet is given by:

$$\langle x' | \alpha \rangle = (2\pi d^2)^{-1/4} \exp \left[ \frac{i \langle p \rangle x'}{\hbar} - \frac{(x' - \langle x \rangle)^2}{4d^2} \right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$$

Prove that the requirement:

$$\langle x' | \Delta x | \alpha \rangle = (\text{imaginary number}) \langle x' | \Delta p | \alpha \rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

Begin with expanding  $\langle x' | \Delta x | \alpha \rangle$  and  $\langle x' | \Delta p | \alpha \rangle$

$$\langle x' | \Delta x | \alpha \rangle = (x' - \langle x \rangle) \langle x' | \alpha \rangle$$

$$\langle x' | \Delta p | \alpha \rangle = \left( \frac{\hbar}{i} \frac{d}{dx} - \langle p \rangle \right) \langle x' | \alpha \rangle$$

where

$$\frac{\hbar}{i} \frac{d}{dx} \langle x' | \alpha \rangle = \left( \langle p \rangle - \frac{\hbar}{i} \frac{1}{2d^2} (x' - \langle x \rangle)^2 \right) \langle x' | \alpha \rangle$$

Therefore:

$$\langle x' | \Delta x | \alpha \rangle = -\frac{\hbar}{i} \frac{1}{2d^2} \langle x' | \Delta p | \alpha \rangle$$

which is consistent with the result in (b) as the factor is purely imaginary.