HW2

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Problem 1. Sakurai 1.21

Evaluate the $x - p$ uncertainty product $\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle$ for a one-dimensional particle confined between two rigid walls,

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Do this for both the ground and excited states.

The wavefunction for the particle can be found by solving the Schrodinger equation:

$$\begin{aligned} H\psi(x) &= E_n \psi(x) \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= E_n \psi(x) \end{aligned}$$

Solving the differential equation and normalizing, we get the wavefunction for $0 < x < a$:

$$\psi(x) = A \sin(kx) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

and for $\psi(x) = 0$ for $x > a$:

The expectation value of any operator $\hat{A}$ is given by:

$$\begin{aligned} \left\langle \hat{A} \right\rangle &= \int_0^\infty \psi(x)^* A \psi(x) dx \\ &= \int_0^a \frac{2}{a} \sin\left(\frac{n\pi}{a}x\right) \hat{A} \sin\left(\frac{n\pi}{a}x\right) dx \end{aligned}$$

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Thus the expectation value of the following operators are:

$$
\begin{aligned}
\langle x \rangle &= \frac{a}{2} \\
\left\langle x^2 \right\rangle &= \frac{2a^2}{6} \\
\langle p \rangle &= 0 \\
\left\langle p^2 \right\rangle &= \frac{-\hbar^2 n^2 \pi^2}{a^2}
\end{aligned}
$$

Substituting these values into the uncertainty product, we get:

$$
\begin{aligned}
\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle &= \left(\left\langle x^2 \right\rangle - \left\langle x^2 \right\rangle \right) \left(\left\langle p^2 \right\rangle - \langle p \rangle^2 \right) \\
&= \frac{\hbar}{12}(-6 + n^2\pi^2)
\end{aligned}
$$

For ground state $n = 1$,

$$
\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle = \frac{\hbar}{12}(-6 + \pi^2) \approx 0.322\hbar^2
$$

which implies that the uncertainty principle holds, i.e. larger than $\hbar^2/4$. For excited states $n > 1$, the uncertainty principle also holds.