PHYS 5260 HW6

TANUWIJAYA, Randy Stefan *
(20582731)
rstanuwijaya@connect.ust.hk

Department of Physics - HKUST

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Problem 1. Sakurai 3.21

The goal of theis problem is to determine degenerate eigenstates of the three dimensional isotropic harmonic oscillator writeen as eigenstates of $\boldsymbol{L}^2$ and L_z, in terms of the Cartesian eigenstates $|n_x, n_y, n_z\rangle$.

(a) Show that the angular-momentum operators are given by:

$$L_i = i\hbar\epsilon_{ijk}a_j a_k^\dagger$$

$$\boldsymbol{L}^2 = \hbar^2\left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j\right]$$

where summation is implied over repeated indices, ϵ_{ijk} is the totally antisymmetric symbol, and $N \equiv a_j^\dagger a_j$ counts the total number of quanta.

Recall for L_z, we have $L_z = xp_y - yp_x$. In general we have the following relation:

$$L_i = \epsilon_{ijk}x_j p_k$$

Next, consider the following relations:

$$x_j = \sqrt{\frac{\hbar}{2m\omega}}(a_j^\dagger + a_j)$$

$$p_k = i\sqrt{\frac{\hbar m\omega}{2}}(a_k^\dagger - a_k)$$

$$x_j p_k = \frac{i\hbar}{2}(a_j^\dagger + a_j)(a_k^\dagger - a_k)$$

*LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Then, note that a_i, a_j commutes since they are independent. Therefore, we have:

$$\begin{aligned}
L_i &= \epsilon_{ijk} x_j p_k \\
&= \epsilon_{ijk} \frac{i\hbar}{2} (a_j^\dagger + a_j)(a_k^\dagger - a_k) \\
&= \frac{i\hbar}{2} [(a_j^\dagger + a_j)(a_k^\dagger - a_k) - (a_k^\dagger + a_k)(a_j^\dagger - a_j)] \\
&= i\hbar (a_j a_k^\dagger - a_k a_j^\dagger) \\
&= i\hbar \epsilon_{ijk} a_j a_k^\dagger
\end{aligned}$$

Next, consider the following relations:

$$\begin{aligned}
\boldsymbol{L^2} &= L_i L_i \\
&= -\hbar^2 \epsilon_{ijk} a_j a_k^\dagger \epsilon_{iuv} a_u a_v^\dagger \\
&= -\hbar^2 (a_j a_k^\dagger a_j a_k^\dagger - a_k a_j^\dagger a_j a_k^\dagger) \\
&= -\hbar^2 \left[(a_k^\dagger a_j + \delta_{jk})^2 - a_k a_j^\dagger (a_k^\dagger a_j + \delta_{jk}) \right] \\
&= -\hbar^2 \left[a_k^\dagger a_j a_k^\dagger a_j + 2 a_k^\dagger a_k + 3 - a_k a_j^\dagger a_k^\dagger a_j - a_k a_k^\dagger \right] \\
&= -\hbar^2 \left[a_k^\dagger (a_k^\dagger a_j + \delta_{jk}) a_j + 2 a_k^\dagger a_k + 3 - a_k a_j^\dagger a_k^\dagger a_j - a_k a_k^\dagger \right] \\
&= -\hbar^2 \left[a_k^\dagger a_k^\dagger a_j a_j + a_k^\dagger a_k + 2 a_k^\dagger a_k + 3 - a_k a_k^\dagger a_j^\dagger a_j - a_k a_k^\dagger \right] \\
&= -\hbar^2 \left[a_k^\dagger a_k^\dagger a_j a_j + 3 a_k^\dagger a_k + 3 - a_k a_k^\dagger (a_j^\dagger a_j - 1) \right] \\
&= -\hbar^2 \left[a_k^\dagger a_k^\dagger a_j a_j + 3 a_k^\dagger a_k + 3 - (a_k^\dagger a_k + 1)(a_j^\dagger a_j - 1) \right] \\
&= -\hbar^2 \left[a_k^\dagger a_k^\dagger a_j a_j + 3N + 3 - (N+3)(N-1) \right] \\
\boldsymbol{L^2} &= -\hbar^2 \left[a_k^\dagger a_k^\dagger a_j a_j + N(N+1) \right] \quad \text{(Q.E.D)}
\end{aligned}$$

(b) Use these relations to express the states $|qlm\rangle = |01m\rangle, m = 0, \pm 1$, in terms of the three eigenstates $|n_x n_y n_z\rangle$ that are degenerate in energy. Write down the representation of your answer in coordinate space, and check that the angular and radial dependences are correct.

First, consider the following relation:

$$\langle n_x n_y n_z | L_z |qlm\rangle = m\hbar \langle n_x n_y n_z | qlm\rangle = i\hbar \langle n_x n_y n_z | (a_x a_y^\dagger - a_y a_x^\dagger) |qlm\rangle$$

Which yield:

$$\begin{aligned}
m \langle n_x n_y n_z | qlm\rangle = i\sqrt{(n_x+1)n_y} \langle n_x + 1, n_y - 1, n_z | qlm\rangle \\
- i\sqrt{n_x(n_y+1)} \langle n_x - 1, n_y + 1, n_z | qlm\rangle
\end{aligned} \tag{1}$$

$$|qlm\rangle = \sum_{n_x n_y n_z} |n_x n_y n_z\rangle \langle n_x n_y n_z | qlm\rangle$$

For $N = 1$, we have:

$$\begin{aligned} m \langle 100|01m\rangle &= -i \langle 010|01m\rangle \\ m \langle 010|01m\rangle &= i \langle 100|01m\rangle \\ m \langle 001|01m\rangle &= 0 \end{aligned}$$

Therefore:

$$\begin{aligned} |0,1,\pm 1\rangle_q &= \langle 100|0,1,\pm 1\rangle |100\rangle_n + \langle 010|0,1,\pm 1\rangle |010\rangle_n + \langle 001|0,1,\pm 1\rangle |001\rangle_n \\ &= \langle 100|0,1,\pm 1\rangle (|100\rangle_n \pm i |010\rangle_n) \end{aligned}$$

$$\boxed{|0,1,\pm 1\rangle_q = \frac{1}{\sqrt{2}} (|100\rangle_n \pm i |010\rangle_n)}$$

$$\boxed{|0,1,0\rangle_q = |001\rangle}$$

(c) Repeat for $|qlm\rangle = |200\rangle$.

Using the Equation 1 we have the following relations:

$$\begin{aligned} m \langle 110|200\rangle &\to \langle 200|200\rangle - \langle 020|200\rangle = 0 \\ m \langle 101|200\rangle &\to \langle 011|200\rangle = 0 \\ m \langle 011|200\rangle &\to \langle 101|200\rangle = 0 \\ m \langle 200|200\rangle &\to \langle 110|200\rangle = 0 \\ m \langle 020|200\rangle &\to \langle 110|200\rangle = 0 \end{aligned}$$

Using the given form of $\boldsymbol{L^2}$ we have:

$$\begin{aligned} \langle 002| L^2 |200\rangle = 0 &= 6 \langle 002|200\rangle - 2 |200\rangle |200\rangle - 2 |020\rangle |200\rangle - 2 |002\rangle |200\rangle \\ &= 4 \langle 002|200\rangle - 2 |200\rangle |200\rangle - 2 |020\rangle |200\rangle \end{aligned}$$

Combining with the previous result, we have:

$$\langle 002|200\rangle = \langle 020|200\rangle = \langle 200|200\rangle$$

which implies these three states have the same probability amplitude, whereas the others are 0. Therefore, we have:

$$\boxed{|200\rangle_q = \frac{1}{\sqrt{3}} (|200\rangle_n + |020\rangle_n + |002\rangle_n)}$$

(d) Repeat for $|qlm\rangle = |02m\rangle$, with $m = 0, 1, 2$.

Using the Equation 1 we have the following relations:

$$\begin{aligned} m\langle 110|02m\rangle &= i\sqrt{2}(\langle 200|02m\rangle - \langle 020|02m\rangle) \\ m\langle 101|02m\rangle &= -i\langle 011|02m\rangle \\ m\langle 011|02m\rangle &= i\langle 101|02m\rangle \\ m\langle 200|02m\rangle &= -i\sqrt{2}\langle 110|02m\rangle \\ m\langle 020|02m\rangle &= i\sqrt{2}\langle 110|02m\rangle \\ m\langle 002|02m\rangle &= 0 \end{aligned}$$

Meanwhile, using the given form of $\boldsymbol{L^2}$, we have:

$$\begin{aligned} \langle 200|L^2|02m\rangle \to 6\langle 200|02m\rangle &= 6\langle 200|02m\rangle + 2(\langle 200|02m\rangle + \langle 020|02m\rangle + \langle 002|02m\rangle) \\ 0 &= \langle 200|02m\rangle + \langle 020|02m\rangle + \langle 002|02m\rangle \end{aligned}$$

$\langle 020|L^2|02m\rangle$ and $\langle 002|L^2|02m\rangle$ will yield similar result.
For $m = 0$, we have:

$$\begin{aligned} 0 &= \langle 200|020\rangle + \langle 020|020\rangle + \langle 002|020\rangle \\ 0 &= \langle 200|002\rangle - \langle 020|002\rangle \end{aligned}$$

While the rest are 0. Therefore, for $|020\rangle$, we have:

$$\boxed{|200\rangle_q = \frac{1}{\sqrt{6}}(|200\rangle_n + |020\rangle_n - 2|002\rangle_n)}$$

For $m = 1$, we have:

$$\begin{aligned} \langle 002|021\rangle &= 0 \\ \langle 200|021\rangle &= -i\sqrt{2}\langle 110|021\rangle \\ \langle 020|021\rangle &= i\sqrt{2}\langle 110|021\rangle \\ \langle 110|021\rangle &= i\sqrt{2}(\langle 200|021\rangle - \langle 020|021\rangle) \\ \langle 011|021\rangle &= i\langle 101|021\rangle \end{aligned}$$

Which implies:

$$0 = \langle 110|021\rangle = \langle 200|021\rangle = \langle 020|021\rangle = \langle 002|021\rangle$$

Thus, for $|021\rangle$, we have:

$$\begin{aligned} |021\rangle_q &= (\langle 101|021\rangle|101\rangle_n + \langle 011|021\rangle|011\rangle_n) \\ &= \langle 101|021\rangle(|101\rangle_n + i|011\rangle_n) \end{aligned}$$

$$\boxed{|021\rangle_q = \frac{1}{\sqrt{2}}(|101\rangle_n + i|011\rangle_n)}$$

For $m = 2$, we also have:

$$\begin{aligned} 0 &= \langle 101|021\rangle = \langle 011|021\rangle = \langle 002|021\rangle \\ 0 &= \langle 200|021\rangle + \langle 020|021\rangle \end{aligned}$$

$$\begin{aligned} \langle 200|022\rangle &= -i\sqrt{2}\langle 110|022\rangle \\ \langle 020|022\rangle &= i\sqrt{2}\langle 110|022\rangle \end{aligned}$$

Therefore, for $|022\rangle$, we have:

$$\begin{aligned}
|022\rangle_q &= (\langle 110|022\rangle\,|110\rangle_n + \langle 200|022\rangle\,|200\rangle_n + \langle 020|022\rangle\,|020\rangle_n) \\
&= \langle 110|022\rangle\,(|110\rangle_n + -i\sqrt{2}\,|200\rangle_n + i\sqrt{2}\,|020\rangle_n)
\end{aligned}$$

$$\boxed{|022\rangle_q = \frac{1}{\sqrt{5}}(|110\rangle_n + -i\sqrt{2}\,|200\rangle_n + i\sqrt{2}\,|020\rangle_n)}$$

Problem 2. Sakurai 3.24

We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2, 1$, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j, m\}$ eigenkets in terms of $|j_1 j_2; m_1, m_2\rangle$. Write your answer as:

$$|j = 1, m = 1\rangle = \frac{1}{\sqrt{2}} |+, 0\rangle - \frac{1}{\sqrt{2}} |0, +\rangle$$

where + and 0 stand for $m_{1,2} = 1, 0$ respectively.

First, recall the ladder operator method:

$$J_{\pm} |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$$

For $j = j_1 + j_2 = 2$, we have:

$$\boxed{|2, 2\rangle = |{+}{+}\rangle}$$

$$J_- |2, 2\rangle = (J_{1-} \otimes 1 + 1 \otimes J_{2-}) |{+}{+}\rangle$$

$$\sqrt{6-2}\, |2, 1\rangle = \sqrt{2}\, |0{+}\rangle + \sqrt{2}\, |{+}0\rangle$$

$$\boxed{|2, 1\rangle = \frac{1}{\sqrt{2}} |0{+}\rangle + \frac{1}{\sqrt{2}} |{+}0\rangle}$$

$$J_- |2, 1\rangle = (J_{1-} \otimes 1 + 1 \otimes J_{2-}) \frac{1}{\sqrt{2}} (|0{+}\rangle + |{+}0\rangle)$$

$$\sqrt{6-0}\, |2, 0\rangle = \frac{1}{\sqrt{2}} \left(\sqrt{2}\, |{-}0\rangle + \sqrt{2}\, |00\rangle + \sqrt{2}\, |00\rangle + \sqrt{2}\, |0{-}\rangle\right)$$

$$\boxed{|2, 0\rangle = \frac{1}{\sqrt{6}} (|{-}0\rangle + 2\, |00\rangle + |0{-}\rangle)}$$

By symmetry arguement, we have:

$$\boxed{|2, -1\rangle = \frac{1}{\sqrt{2}} |0{-}\rangle + \frac{1}{\sqrt{2}} |{-}0\rangle}$$

$$\boxed{|2, -2\rangle = |{-}{-}\rangle}$$

For $j = j_1 + j_2 = 1$, first note that: $\langle 2, 1|1, 1\rangle = 0$, and $|1, 1\rangle$ must be normalizeable. Thus, we have:

$$\boxed{|1, 1\rangle = \frac{1}{\sqrt{2}} (|{+}0\rangle - |0{+}\rangle)}$$

$$J_- |1, 1\rangle = (J_{1-} \otimes 1 + 1 \otimes J_{2-}) \left(\frac{1}{\sqrt{2}} (|{+}0\rangle - |0{+}\rangle)\right)$$

$$\sqrt{2}\, |1, 0\rangle = \frac{1}{\sqrt{2}} \left(\sqrt{2}\, |00\rangle - \sqrt{2}\, |{-}{+}\rangle + \sqrt{2}\, |{+}{-}\rangle - \sqrt{2}\, |0\rangle\right)$$

$$\boxed{|1, 0\rangle = \frac{1}{\sqrt{2}} (|{+}{-}\rangle - |{-}{+}\rangle)}$$

$$\boxed{|1, -1\rangle = \frac{1}{\sqrt{2}} (|{-}0\rangle - |0{-}\rangle)}$$

For $j = j_1 + j_2 = 0$, we have:

$$|0,0\rangle = a\,|-+\rangle + b\,|00\rangle + c\,|+-\rangle$$

Taking inner product $\langle 00|10\rangle$ and $\langle 00|20\rangle$ gives:

$$\langle 00|10\rangle = a - c = 0$$
$$\langle 00|20\rangle = a + 2b + c = 0$$

Which yields: $a = c$ and $b = -a = -c$. Therefore:

$$\boxed{|0,0\rangle = \frac{1}{\sqrt{3}}\left(|-+\rangle - |00\rangle + |+-\rangle\right)}$$

Problem 3. Sakurai 3.27

Express the matrix element $\langle\alpha_2\beta_2\gamma_2|\,\boldsymbol{J}_3^2\,|\alpha_1\beta_1\gamma_1\rangle$ in terms of a series in

$$\mathcal{D}^j_{mn}(\alpha\beta\gamma) = \langle\alpha\beta\gamma|jmn\rangle$$

We can simply insert an identity operator as follows:

$$\begin{aligned}
\langle\alpha_2\beta_2\gamma_2|\,\boldsymbol{J}_3^2\,|\alpha_1\beta_1\gamma_1\rangle &= \sum_{jmn}\sum_{j'm'n'} \langle\alpha_2\beta_2\gamma_2|jmn\rangle\,\langle jmn|\,\boldsymbol{J}_3^2\,|j'm'n'\rangle\,\langle j'm'n'|\alpha_1\beta_1\gamma_1\rangle \\
&= \sum_{jmn}\sum_{j'm'n'} \langle\alpha_2\beta_2\gamma_2|jmn\rangle\, n^2\delta_{jj'}\delta_{mm'}\delta_{nn'}\,\langle j'm'n'|\alpha_1\beta_1\gamma_1\rangle \\
&= \sum_{jmn} n^2\,\langle\alpha_2\beta_2\gamma_2|jmn\rangle\,\langle j'm'n'|\alpha_1\beta_1\gamma_1\rangle \\
&= \sum_{jmn} n^2\mathcal{D}^j_{mn}(\alpha_2\beta_2\gamma_2)\left(\mathcal{D}^j_{mn}(\alpha_1\beta_1\gamma_1)\right)^*
\end{aligned}$$