PHYS 5260 HW7

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Problem 1. Sakurai 4.3

A quantum-mechanical state Ψ is known to be a simultaneous eigenstate of two Hermitian operators A and B that *anticommute*

$$AB + BA = 0$$

What can you say about the eigenvalues of A and B? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi$) and the momentum operator.

We can first denote the simultaneous eigenstate as $|\Psi\rangle = |a, b\rangle$. Then, using the anticommutation relation, we have:

$$(AB + BA)|a, b\rangle = (ab + ba)|a, b\rangle = 0$$

which implies either $a = 0$ or $b = 0$.

For $A = \pi$ and $B = \boldsymbol{p}$, then we have:

$$\begin{aligned} \langle\Psi|\pi^\dagger \boldsymbol{p}\pi|\Psi\rangle &= -\langle\Psi|\boldsymbol{p}|\Psi\rangle \\ \langle\Psi|\boldsymbol{p}\pi|\Psi\rangle &= -\langle\Psi|\pi\boldsymbol{p}|\Psi\rangle \\ 0 &= -\langle\Psi|\boldsymbol{p}\pi - \pi\boldsymbol{p}|\Psi\rangle \end{aligned}$$

Then in this case, p must be 0.

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Problem 2. Sakurai 4.6

Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} \infty & \text{for } |x| > a+b \\ 0 & \text{for } a < |x| < a+b \\ V_0 > 0 & \text{for } |x| < a \end{cases}$$

Assuming that V_0 is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between the two lowest-lying states.

Without loss of generality, we can only consider the case where $x > 0$. Denote $\psi^{(L)}(x)$ as the wavefunction inside finite potential barrier, i.e. $|x| < a$, and $\psi^{(R)}(x)$ as the wavefunction at $a < |x| < a+b$. Then, we have the following boundary conditions:

$$\psi_s^{(L)}(a) = \psi_s^{(R)}(a)$$
$$\psi_s^{(R)}(a+b) = \psi_a^{(R)}(a+b) = 0$$

We can guess the form of the wavefunction as:

$$\psi_s^{(R)} = A_s \sin(k_s(x-a-b))$$
$$\psi_a^{(R)} = A_a \sin(k_a(x-a-b))$$
$$\psi_s^{(L)} = B_s \cosh(\kappa_s x)$$
$$\psi_a^{(L)} = B_a \sinh(\kappa_a x)$$

Which gives us the matching condition at $x = a$ and its derivative:

$$0 = A_s \sin(k_s b) + B_s \cosh(\kappa_s a)$$
$$0 = k_s A_s \cos(k_s b) - \kappa_s B_s \cosh(\kappa_s a)$$
$$0 = A_a \sin(k_a b) + B_a \sinh(\kappa_a a)$$
$$0 = k_a A_a \cos(k_a b) + \kappa_a B_a \sinh(\kappa_a a)$$

Using the approximation $E \ll V_0$, we have $\kappa \equiv \sqrt{2mV_0}/\hbar = \kappa_s = \kappa_a$. Which gives:

$$k_s \cot(k_s b) = -\kappa \tanh(\kappa a)$$
$$k_s \cot(k_s b) = -\kappa \coth(\kappa a)$$

Since $E \ll V_0$, we can expect $\lambda \approx 2b = (1+\epsilon)2b$ or $k = \frac{2\pi}{\lambda} = \pi(1-\epsilon)$. Then, we have:

$$\begin{aligned} \tan(kb) &= \frac{\sin(kb)}{\cos(kb)} = \frac{\sin(\pi)\cos(\pi\epsilon) - \cos(\pi)\sin(\pi\epsilon)}{\cos(\pi)\cos(\pi\epsilon) + \sin(\pi)\sin(\pi\epsilon)} \\ &= \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} = \tan(\pi\epsilon) \\ &\approx \pi\epsilon = kb - \pi \end{aligned}$$

Therefore the previous condition yields:

$$
\begin{aligned}
\frac{k_s}{k_s b - \pi} &= -\kappa \tanh(\kappa a) \\
\frac{k_a}{k_a b - \pi} &= -\kappa \coth(\kappa a)
\end{aligned}
$$

And the energy splitting is given by $E = \hbar^2 k^2/2m$:

$$
\begin{aligned}
\Delta E &= \frac{\hbar^2}{2m}(k_a^2 - k_s^2) \\
&= \frac{\hbar^2\pi^2\kappa^2}{2m}\left(\frac{(\coth(\kappa a))^2}{(1+\kappa b\coth(\kappa a))^2} - \frac{(\tanh(\kappa a))^2}{(1+\kappa b\tanh(\kappa a))^2}\right) \\
&= \frac{\hbar^2\pi^2\kappa^2}{2m}\left(\frac{1}{(\tanh(\kappa a)+\kappa b)^2} - \frac{1}{(\coth(\kappa a)+\kappa b)^2}\right)
\end{aligned}
$$

Problem 3. Sakurai 4.9

Let $\phi(p')$ be the momentum-space wave function for state $|\alpha\rangle$ - that is, $\phi(p') = \langle p'|\alpha\rangle$ Is the momentum-space wave function for the time-reversed state $\theta|\alpha\rangle$ given by $\phi(p')$, by $\phi(-p')$, by $\phi^*(p')$, or by $\phi^*(-p')$? Justify your answer.

First note the time-reversal operator satisfies anitunitary property the time-reversal property of the momentum operator:

$$\theta(c|\alpha\rangle) = c^*\theta|\alpha\rangle$$
$$\theta p \theta^{-1} = -p \iff \theta|p'\rangle = |-p'\rangle$$

Then, we can expand the time-reversed state in terms of the momentum eigenkets:

$$\begin{aligned}\Theta|\alpha\rangle &= \int d^3p''\Theta\left(|p''\rangle\langle p''|\alpha\rangle\right) \\ &= \int d^3p''\,|-p''\rangle\langle p''|\alpha\rangle^* \\ &= \int d^3p''\,|p''\rangle\langle -p''|\alpha\rangle^*\end{aligned}$$

Thus, the time-reversed momentum-space wave function is given by:

$$\begin{aligned}\langle p'|\,\Theta\,|\alpha\rangle &= \langle p'|\int d^3p''\,|p''\rangle\langle -p''|\alpha\rangle^* \\ &= \langle -p'|\alpha\rangle^* = \phi^*(-p')\end{aligned}$$

Problem 4. Sakurai 4.12

The Hamiltonian for a spin 1 system is given by:

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

Solve this problem exactly to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind appears in crystal physics.) Is this Hamiltonian invariant under the time-reversal? How do the normalized eigenstates you obtained transform under time reversal?

Recall the spin operators for spin-1 system:

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Therefore the Hamiltonian is given by:

$$H = AS_z^2 + B(S_x^2 - S_y^2) = \hbar \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

The eigenvalues and corresponding eigenvectors are given by:

$$\lambda_1 = A + B \qquad |\Psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (|1,1\rangle + |1,-1\rangle)$$

$$\lambda_2 = A - B \qquad |\Psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = |1,1\rangle - |1,-1\rangle$$

$$\lambda_3 = 0 \qquad |\Psi_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1,0\rangle$$

We can check that the Hamiltonian is invariant under time-reversal by checking the eigenstate under time-reversal transformation, i.e. $\Theta\,|\Psi\rangle$:

$$\Theta\,|\Psi_1\rangle = -\,|1,1\rangle - |1,-1\rangle = -\,|\Psi_1\rangle$$
$$\Theta\,|\Psi_2\rangle = -\,|1,1\rangle + |1,-1\rangle = -\,|\Psi_2\rangle$$
$$\Theta\,|\Psi_3\rangle = |1,0\rangle = |\Psi_3\rangle$$

Suppose $|\Psi\rangle = \alpha\,|\Psi_1\rangle + \beta\,|\Psi_2\rangle + \gamma\,|\Psi_3\rangle$, e.g., $H\,|\Psi\rangle = \lambda_1\,|\Psi_1\rangle + \lambda_2\,|\Psi_2\rangle + \lambda_3\,|\Psi_3\rangle$. Thus,

$$\begin{aligned}
\Theta H \Theta^{-1}\,|\Psi\rangle &= \Theta H \Theta^{-1}\left(\alpha\,|\Psi_1\rangle + \beta\,|\Psi_2\rangle + \gamma\,|\Psi_3\rangle\right) \\
&= \Theta H \left(\alpha^*\,|-\Psi_1\rangle + \beta^*\,|-\Psi_2\rangle + \gamma^*\,|\Psi_3\rangle\right) \\
&= \Theta H \left(\alpha^*\,|-\Psi_1\rangle + \beta^*\,|-\Psi_2\rangle + \gamma^*\,|\Psi_3\rangle\right) \\
&= \Theta\left(\lambda_1\alpha^*\,|-\Psi_1\rangle + \lambda_2\beta^*\,|-\Psi_2\rangle + \lambda_3\gamma^*\,|\Psi_3\rangle\right) \\
&= \left(\lambda_1^*\alpha\,|\Psi_1\rangle + \lambda_2^*\beta\,|\Psi_2\rangle + \lambda_3\gamma\,|\Psi_3\rangle\right)
\end{aligned}$$

But λ_1, λ_2 are real numbers, i.e., $\lambda^* = \lambda$. Thus, $\Theta H \Theta^{-1} = H$ or H is invariant under time reversal transformation.