# PHYS 5260 HW5

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## Problem 1. Sakurai 3.3

Consider the $2 \times 2$ matrix defined by:

$$U = \frac{a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a}}{a_0 - i\boldsymbol{\sigma} \cdot \boldsymbol{a}}$$

where $a_0$ is a real number and $\boldsymbol{a}$ is a three-dimensional vector with real components.

(a) Prove that $U$ is unitary and unimodular.

First lets define a matrix $A$:

$$\begin{aligned} A &= a_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{a} \\ &= \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} \end{aligned}$$

And:

$$AA^\dagger = a_0^2 + (\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = a_0^2 + |\boldsymbol{a}|^2 \equiv \alpha^2$$

Note that since $\boldsymbol{\sigma}$ is symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, then $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^\dagger$ and $U^* = U^\dagger$. Then we can write $U$ as:

$$U = A(A^*)^{-1} = A(A^\dagger)^{-1}$$

First, to prove the unitary properties, we need to show that $UU^* = UU^\dagger = I$. We can expand $UU^\dagger$ as:

$$UU^\dagger = A(A^\dagger)^{-1}(A(A^\dagger)^{-1})^\dagger = A(A^\dagger)^{-1}A^{-1}A^\dagger = A(AA^\dagger)^{-1}A^\dagger = \alpha^2/\alpha^2 = I$$

Second, to prove the unimodular properties, we need to show that $|U| = 1$. We can expand $|U|$ as:

$$|U| = |A|/|A^\dagger|$$

Where:

$$\begin{aligned} |A| &= a_0^2 + a_3^2 + a_2^2 + a_1^2 = \alpha^2 \\ |A^\dagger| &= a_0^2 + a_3^2 + a_2^2 + a_1^2 = \alpha^2 \end{aligned}$$

*LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Thus:
$$|U| = |A|/|A^\dagger| = \alpha^2/\alpha^2 = 1$$

(b) In general, a $2 \times 2$ unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for $U$ in terms of $a_0, a_1, a_2$, and $a_3$.

Recall the general rotation operator along an axis $\hat{\boldsymbol{n}}$ is given by Eq (3.2.42) and Eq (3.2.45):

$$\exp\left(\frac{-i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{\hbar}\right) = \exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z \sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

We can write $U$ as:

$$\begin{aligned} U &= A(A^\dagger)^{-1}A^{-1}A = A(AA^\dagger)^{-1}A^\dagger = \frac{A^2}{\alpha^2} \\ &= \frac{1}{\alpha^2}\begin{pmatrix} 2a_0^2 + 2ia_0a_3 - \alpha^2 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & 2a_0^2 - 2ia_0a_3 - \alpha^2 \end{pmatrix} \end{aligned}$$

Matching with the previous equation, we can write:

$$\begin{aligned} \cos\left(\frac{\phi}{2}\right) &= 2a_0^2/\alpha^2 - 1 & \iff && \sin\left(\frac{\phi}{2}\right) &= 2a_0|\boldsymbol{a}|/\alpha^2 \\ n_x \sin\left(\frac{\phi}{2}\right) &= -2a_0a_1/\alpha^2 & \iff && n_x &= -a1/|\boldsymbol{a}| \\ n_y \sin\left(\frac{\phi}{2}\right) &= -2a_0a_2/\alpha^2 & \iff && n_y &= -a2/|\boldsymbol{a}| \\ n_z \sin\left(\frac{\phi}{2}\right) &= -2a_0a_3/\alpha^2 & \iff && n_z &= -a3/|\boldsymbol{a}| \end{aligned}$$

## Problem 2. Sakurai 3.6

Let the Hamiltonian of a rigid body be:

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right)$$

where $\boldsymbol{K}$ is the angular momentum in the body frame. From this expression obtain the Heisenberg equation for $K$, and then find Euler's equation of motion in the correspondence limit.

Without loss of generality we can only consider the time evolution of $K_1$. Using the Heisenberg picture, we can write the the time evolution of $K_1$ as:

$$\frac{dK_1}{dt} = \frac{i}{\hbar}[H, K_1] = \frac{i}{2\hbar}\left(\frac{[K_2^2, K_1]}{I_2} + \frac{[K_3^2, K_1]}{I_3}\right)$$

Using the commutation relation for $K_i$:

$$[K_i, K_j] = -i\hbar\epsilon_{ijk}K_k$$

We can write:

$$\begin{aligned}
[K_2^2, K_1] &= K_2[K_2, K_1] + [K_2, K_1]K_2 = i\hbar(K_2K_3 + K_3K_2) \\
[K_3^2, K_1] &= K_3[K_3, K_1] + [K_3, K_1]K_3 = -i\hbar(K_2K_3 + K_3K_2)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dK_1}{dt} &= \frac{-1}{2}(K_2K_3 + K_3K_2)\left(\frac{1}{I_2} - \frac{1}{I_3}\right) \\
\frac{dK_1}{dt} &= \frac{I_2 - I_3}{2I_2I_3}(K_2K_3 + K_3K_2)
\end{aligned}$$

Similarly, we can write the time evolution of $K_2$ and $K_3$:

$$\begin{aligned}
\frac{dK_2}{dt} &= \frac{I_3 - I_1}{2I_1I_3}(K_3K_1 + K_1K_3) \\
\frac{dK_3}{dt} &= \frac{I_1 - I_2}{2I_1I_2}(K_1K_2 + K_2K_1)
\end{aligned}$$

In the correspondence/classical limit, the operators $K_i$ are replaced by the corresponding classical variables $K_i = I_i\omega_i$. Thus, we can write:

$$\begin{aligned}
\frac{dK_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 \\
\frac{dK_2}{dt} &= (I_3 - I_1)\omega_3\omega_1 \\
\frac{dK_3}{dt} &= (I_1 - I_2)\omega_1\omega_2
\end{aligned}$$

## Problem 3. Sakurai 3.9

Consider a sequence of Euler rotations represented by:

$$\begin{aligned}\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) &= \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right)\end{pmatrix}\end{aligned}$$

Because of the group properties of rotations, we exprect that this sequence of operations is equivalent to a single rotation about some axis by an angle $\theta$. Find $\theta$.

Recall the general rotation operator along an axis $\hat{\boldsymbol{n}}$ is given by Eq (3.2.42) and Eq (3.2.45):

$$\exp\left(\frac{-i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{\hbar}\right) = \exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\theta}{2}\right) = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) & (-in_x - n_y)\sin\left(\frac{\theta}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + in_z\sin\left(\frac{\theta}{2}\right)\end{pmatrix}$$

We can match the equations to obtain $\hat{\boldsymbol{n}}$ and $\theta$, i.e.:

$$\begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right)\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) & (-in_x - n_y)\sin\left(\frac{\theta}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + in_z\sin\left(\frac{\theta}{2}\right)\end{pmatrix}$$

In particular, we can solve for $\theta$ by observing the real part along the diagonal:

$$\begin{aligned}\cos\left(\frac{\theta}{2}\right) &= \cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right) \\ &= 2\cos^{-1}\left(\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)\end{aligned}$$

## Problem 4. Sakurai 3.12

Consider an ensemble of spin 1 systems. The density matrix is now a $3 \times 3$ matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to $[S_x]$, $[S_y]$, and $[S_z]$ to characterize the ensemble completely?

For spin 1 particles, there are three basis states, so the density matrix is a $3 \times 3$ matrix. Note that the density matrix is Hermitian ($\rho = \rho^\dagger$) and the trace is equal to 1 ($tr(\rho) = 1$), so we can write:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

Following the two conditions above, we can write:

$$\rho_{11} = \rho_{11}^* \qquad \rho_{22} = \rho_{22}^* \qquad \rho_{33} = \rho_{33}^*$$
$$\rho_{12} = \rho_{32}^* \qquad \rho_{23} = \rho_{32}^* \qquad \rho_{31} = \rho_{13}^*$$
$$\rho_{11} + \rho_{22} + \rho_{33} = 1$$

Threfore, the diagonals must be real number and have sum 1, and the off diagonal elements are complex conjugates of each other, for example:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{21}^* & \rho_{31}^* \\ \rho_{21} & \rho_{22} & \rho_{32}^* \\ \rho_{31} & \rho_{32} & 1 - \rho_{11} - \rho_{22} \end{pmatrix} = \begin{pmatrix} u & a - i\alpha & c - i\gamma \\ a + i\alpha & v & b - i\beta \\ c + i\gamma & b + i\beta & 1 - u - v \end{pmatrix}$$

Where $a, b, c, \alpha, \beta, \gamma, u, v$ are real numbers, i.e. there are 8 degrees of freedom. In addition to $[S_x]$, $[S_y]$, and $[S_z]$, we need to know the linear combinations of the spin operators, i.e., $[S_x^2]$, $[S_y^2]$, $[S_x S_y]$, $[S_y, S_z]$, $[S_z S_x]$ (as $[S_z^2] = 2\hbar^2 - [S_x^2] - [S_y^2]$).

## Problem 5. Sakurai 3.15

(a) Let $J$ be angular momentum. (It may stand for orbital $L$, spin $S$, or $J_{total}$.) Using the fact that $J_x, J_y, J_z$ ($J_\pm \equiv J_x \pm iJ_y$) satisfy the usual angular-momentum relations, prove:

$$J^2 = J_z^2 + J_+J_- - \hbar J_z$$

Using the definition of $J_\pm$, we can compute $J_+J_-$:

$$\begin{aligned} J_+J_- &= J_x^2 + J_y^2 - i[J_x, J_y] \\ &= J_x^2 + J_y^2 + \hbar J_z \end{aligned}$$

Therefore, we have:

$$\begin{aligned} J^2 &= J_z^2 + J_+J_- - \hbar J_z \\ &= J_z^2 + J_x^2 + J_y^2 + \hbar J_z - \hbar J_z \\ &= J_z^2 + J_x^2 + J_y^2 \end{aligned}$$

(b) Using (a) (or otherwise), derive the famous expression for the coefficient $c_-$ that appears in:

$$J_-\psi_{jm} = c_-\psi_{j,m-1}$$

First, we can rewrite the equation as:

$$\begin{aligned} J_-\left|j,m\right\rangle &= c_-\left|j,m-1\right\rangle \\ \left\langle j,m-1\right| J_-\left|j,m\right\rangle &= c_- \\ \left\langle j,m\right| J_+J_-\left|j,m\right\rangle &= |c_-|^2 \end{aligned}$$

Using the previous result, we have $J_+J_- = J^2 - J_z^2 + \hbar J_z$. Note the following identities:

$$\begin{aligned} J^2\left|j,m\right\rangle &= \hbar^2 j(j+1)\left|j,m\right\rangle \\ J_z^2\left|j,m\right\rangle &= \hbar^2 m^2\left|j,m\right\rangle \\ \hbar J_z\left|j,m\right\rangle &= \hbar^2 m\left|j,m\right\rangle \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \left\langle j,m\right| J_+J_-\left|j,m\right\rangle &= \left\langle j,m\right| \left(J^2 - J_z^2 + \hbar J_z\right)\left|j,m\right\rangle \\ |c_-|^2 &= \left\langle j,m\right| \left(\hbar^2 j(j+1) - \hbar^2 m^2 + \hbar^2 m\right)\left|j,m\right\rangle \\ |c_-|^2 &= \hbar^2\left(j(j+1) - m(m+1)\right) \\ c_- &= \hbar\sqrt{j(j+1) - m(m+1)} \end{aligned}$$