PHYS 5260 HW9

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Problem 1. Sakurai 5.24

Consider a particle bound in a simple harmonic-oscillator potential. Initially ($t < 0$), it is in the ground state. At $t = 0$ a perturbation of the form

$$H'(x,t) = Ax^2 e^{-t/\tau}$$

is switched on. Using time-dependent perturbation theory, calculate the probability that after a sufficiently long time ($t \gg \tau$), the system will have a transition to a given excited state. Consider all final states.

We can first expand the x^2 term in terms of raising and lowering operators, i.e.:

$$x^2 = \frac{\hbar}{2m\omega_0}\left(a^\dagger a^\dagger + a^\dagger a + aa^\dagger + aa\right)$$

Applying on the ground state:

$$\langle n| x^2 |0\rangle = \frac{\hbar}{2m\omega_0}\sqrt{2}\delta_{2,n}$$

Then, we can write the (first order) transition probability as:

$$c_n^{(1)} = \frac{-i}{\hbar}\int_0^t e^{i\omega_{n0}t'} \langle n| V |0\rangle \, dt'$$

$$c_2^{(1)} = \frac{-i}{\hbar} A \frac{\hbar}{2m\omega_0}\int_0^t e^{2i\omega_0 t'} e^{-t'/\tau} dt'$$

$$\left|c_2^{(1)}\right|^2 = \frac{A^2}{2m^2\omega_0^2}\frac{e^{2t/\tau} - 2e^{-t/\tau}\cos\omega_0 t + 1}{\omega_0^2 + 1/\tau^2}$$

$$\lim_{t\to\infty}\left|c_2^{(1)}\right|^2 = \frac{A^2}{2m^2\omega_0^2}\frac{\tau^2}{1+\omega_0^2\tau^2}$$

where $\omega_{20} = (E_2 - E_0)/\hbar = 2\omega_0$ and transitions only occur fron $|0\rangle$ to $|2\rangle$ in the first order. For higher order, transitions may occur from $|0\rangle \to |2n\rangle$.

*LaTeX source code: https://github.com/rstanuwijaya/hkust-advanced-qm/

Problem 2. Sakurai 5.27

Consider a particle in one dimension moving under the influence of some time-independent potential. The energy levels and the corrsponding eigenfunctions for this problem are assumed to be known. We now subject the particle to a travelling pulse represented by a time-dependent potential,

$$V(t) = A\delta(x - ct)$$

(a) Suppose that at $t = -\infty$ the particle is known to be in the ground state whose energy eigenfunction is $\langle x|i\rangle = u_i(x)$. Obtain the probability for finding the system in some excited state with energy eigenfunction $\langle x|f\rangle = u_f(x)$ at $t = \infty$.

The probability density for the particle to be in the excited state at $t- > \infty$ is given by:

$$\begin{aligned} c_f^{(1)} &= \frac{-i}{\hbar} \int_{-\infty}^{\infty} dt' e^{i\omega_{fi}t'} \langle f| V |i\rangle \\ &= \frac{-i}{\hbar} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx \, e^{i\omega_{fi}t'} A\delta(x - ct') u_f^*(x) u_i(x) \\ &= \frac{-iA}{c\hbar} \int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} cdt' \, e^{i\omega_{fi}t'} \delta(x - ct') \right) u_f^*(x) u_i(x) \\ &= \frac{-iA}{\hbar c} \int_{-\infty}^{\infty} dx \, e^{i\omega_{fi}x/c} u_f^*(x) u_i(x) \end{aligned}$$

where the probability to end up in state $|f\rangle$ is just $\left|c_f^{(1)}\right|^2$.

(b) Interpret your result in (a) physically by regarding the δ-function pulse as a superposition of harmonic perturbations; recall

$$\delta(x - ct) = \frac{1}{2\pi c} \int_{-\infty}^{\infty} d\omega \, e^{i\omega[(x/c) - t]}$$

Emphasize the role played by energy conservation which holds even quantum-mechanically as long as the perturbation has been on for a very long time.

In this case, we can assume the travelling pulse as the superposition of harmonic perturbations $e^{i\omega x/c} e^{-i\omega t}$. Then after integrating with respect to t' (similar in (a)), we get:

$$\begin{aligned} \int_{-\infty}^{\infty} cdt' \, e^{i\omega_{fi}t'} \delta(x - ct') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega \, e^{i\omega[(x/c) - t']} e^{i\omega_{fi}t'} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt' \, e^{i\omega[(x/c) - t']} e^{i\omega_{fi}t'} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega x/c} \int_{-\infty}^{\infty} dt' \, e^{i(\omega_{fi} - \omega)t'} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega x/c} \left(2\pi \, \delta(\omega_{fi} - \omega)\right) \\ &= e^{i\omega_{fi}x/c} \end{aligned}$$

Which yield the same result as (a). In this case, we can see that the pulse transfers energy $\hbar omega_{fi}$ to the system, and the system will be in the excited state $|f\rangle$ with probability $\left|c_f^{(1)}\right|^2$.

Problem 3. Sakurai 5.30

Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the wto levels as follows:

$$V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}).$$

At t=0, it is known that only the lower level is populated - that is, $c_1(0) = 1$, $c_2(0) = 0$.

(a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by *exactly* solving the coupled differential equation

$$i\hbar \dot{c_k} = \sum_{n=1}^{2} V_{kn} e^{i\omega_{kn}t} c_n$$

$$i\hbar \dot{c_1} = V_{12}\gamma e^{-i\omega_0 t} c_2 = \gamma e^{i(\omega-\omega_0)t} c_2$$
$$i\hbar \dot{c_2} = V_{21}\gamma e^{i\omega_0 t} c_1 = \gamma e^{-i(\omega-\omega_0)t} c_1$$

where $\omega_0 \triangleq (E_2 - E_1)/\hbar$. To simplify the equations, we can substitue $c_1 = a_1 e^{i(\omega-\omega_0)t/2}$ and $c_2 = a_2 e^{i(\omega-\omega_0)t/2}$ to get:

$$i\hbar \dot{a_1} - \hbar a_1(\omega - \omega_0)/2 = \gamma a_2$$
$$i\hbar \dot{a_2} + \hbar a_2(\omega - \omega_0)/2 = \gamma a_1$$

Then, we can use the following substitution to solve the differential equations, $a_1 = b_1 e^{i\Omega t}$ and $a_2 = b_2 e^{i\Omega t}$, where b_1, b_2 are constants.

$$-\hbar(\Omega + (\omega - \omega_0)/2)b_1 = \gamma b_2$$
$$-\hbar(\Omega - (\omega - \omega_0)/2)b_2 = \gamma b_1$$

Solving for Ω, we get:

$$\Omega = \pm\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_0)^2}{4}}$$

Taking the positive solution for Ω, we get the following solution for a_1 and b_2:

$$a_1 = \alpha e^{i\Omega t} + \beta e^{-i\Omega t}$$
$$a_2 = r_\alpha \alpha e^{i\Omega t} + r_\beta \beta e^{-i\Omega t}$$

where r_α and r_β can be obtained by substituting each solution back to the coupled equations.

$$r_\alpha = \frac{b_2}{b_1} = -\frac{\Omega + (\omega - \omega_0)/2}{\gamma/\hbar} = -\frac{\gamma/\hbar}{\Omega - (\omega - \omega_0)/2}$$
$$r_\beta = \frac{b_2}{b_1} = \frac{\Omega - (\omega - \omega_0)/2}{\gamma/\hbar} = \frac{\gamma/\hbar}{\Omega + (\omega - \omega_0)/2}$$

Then, using the initial condition: $c_1(0) = a_1(0) = \alpha + \beta = 1$, and $c_2(0) = a_2(0) = r_\alpha \alpha + r_\beta \beta = 0$, we can solve for α and β:

$$\begin{aligned}
a_2(t) &= r_\alpha \alpha e^{i\Omega t} + r_\beta \beta e^{-i\Omega t} \\
&= 2ir_\alpha \alpha \sin(\Omega t) \\
&= 2i\frac{r_\alpha r\beta}{r_\beta - r_\alpha} \sin(\Omega t) \\
&= \frac{\gamma}{i\hbar\Omega} \sin(\Omega t)
\end{aligned}$$

Therefore:

$$
\begin{aligned}
c_2(t) &= \frac{\gamma}{i\hbar\Omega} e^{i(\omega-\omega_0)t/2} \sin(\Omega t) \\
|c_2(t)|^2 &= \frac{\gamma^2}{\hbar^2\Omega^2} \sin^2(\Omega t) \\
|c_1(t)|^2 &= 1 - \frac{\gamma^2}{\hbar^2\Omega^2} \sin^2(\Omega t)
\end{aligned}
$$

(b) Do the same problem using time-dependent perturbation theory to lowest non-vanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21} and (ii) ω close to ω_{21}.

Using time-dependent perturbation theory, the transition probability is given by:

$$
\begin{aligned}
c_2(t) &= \frac{i}{\hbar} \int_0^t \langle n| V_i(t') |i\rangle \, dt' \\
&= \frac{i}{\hbar} \int_0^t e^{i\omega_0 t'} V_{ni}(t') dt' \\
&= \frac{i}{\hbar} \int_0^t e^{i\omega_0 t'} V_{ni}(t') dt' \\
&= \frac{i}{\hbar} \int_0^t e^{i\omega_0 t'} \gamma e^{i\omega t'} dt' \\
&= \frac{i}{\hbar} \frac{e^{i(\omega-\omega_0)t} - 1}{\omega - \omega_0} \\
|c_2(t)|^2 &= \frac{\gamma^2}{\hbar^2(\omega-\omega_0)/4} \sin^2\left[(\omega-\omega_0)t/2\right]
\end{aligned}
$$

For $|\omega - \omega_0|/2 \gg \gamma$, the exact solution agree to the perturbation solution. In contrast, for $|\omega - \omega_0|/2 \ll \gamma$ (near resonance), the perturbation solution is different from the exact solution.

Problem 4. Sakurai 5.33

Repeat Problem 5.32, but with the atomic hydrogen Hamiltonian

$$H = A\boldsymbol{S_1}\cdot\boldsymbol{S_2} + \left(\frac{eB}{m_e c}\right)\boldsymbol{S_1}\cdot\boldsymbol{B}$$

where in the hyperfine term, $A\boldsymbol{S_1}\cdot\boldsymbol{S_2}$, $\boldsymbol{S_1}$ is the electron spin and $\boldsymbol{S_2}$ is the proton spin. [Note that this problem has less symmetry than the positronium case].

The unperturbed energy levels are given by $H_0 = (A/2)\left(S^2 - S_1^2 - S_2^2\right)$:

$$E_{singlet} = \frac{-3\hbar^2 A}{4} \qquad |0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$E_{triplet} = \frac{-\hbar^2 A}{4} \qquad \begin{cases} |1,-1\rangle & = |\downarrow\downarrow\rangle \\ |1,0\rangle & = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \\ |1,1\rangle & = |\uparrow\uparrow\rangle \end{cases}$$

Assume the perturbation is in the form $V = \boldsymbol{B}\cdot\boldsymbol{S_1} = \omega S_{1z}$

$$\begin{aligned} V\,|0,0\rangle &= \omega\hbar/2\,|1,1\rangle \\ V\,|1,-1\rangle &= \omega\hbar/2\,|1,-1\rangle \\ V\,|1,0\rangle &= \omega\hbar/2\,|0,0\rangle \\ V\,|1,1\rangle &= \omega\hbar/2\,|1,1\rangle \end{aligned}$$

Therefore we can calculate the energy shift of first order perturbation for states $|1,\pm1\rangle$:

$$\begin{aligned} \Delta^{(1)}(|1,\pm1\rangle) &= \langle 1,\pm1|\,V\,|1,\pm1\rangle \\ &= \omega\hbar/2 \end{aligned}$$

For the states $|0,0\rangle$ and $|1,0\rangle$, we need to go to the second order perturbation theory to get the energy shift:

$$\begin{aligned} \Delta^{(2)}(|0,0\rangle) &= \frac{|\,\langle 1,0|\,V\,|0,0\rangle\,|^2}{E_{singlet} - E_{triplet}} \\ &= \frac{\omega^2\hbar^2/4}{-\hbar^2 A} = -\frac{\omega^2}{4A} \\ \Delta^{(2)}(|1,0\rangle) &= \frac{|\,\langle 0,0|\,V\,|1,0\rangle\,|^2}{E_{triplet} - E_{singlet}} \\ &= \frac{\omega^2\hbar^2/4}{\hbar^2 A} = \frac{\omega^2}{4A} \end{aligned}$$

We can see that the $|1,\pm1\rangle$ states dont get mixed. The $|0,0\rangle$ and $|1,0\rangle$ states get mixed. The eigenvectors to the first order are:

$$\begin{aligned} |0,0\rangle &\to |0,0\rangle + \frac{\langle 1,0|\,V\,|0,0\rangle}{E_{singlet} - E_{triplet}}\,|1,0\rangle \\ |0,0\rangle &\to |0,0\rangle - \frac{\omega}{2\hbar A}\,|1,0\rangle \\ |1,0\rangle &\to |1,0\rangle + \frac{\omega}{2\hbar A}\,|0,0\rangle \end{aligned}$$

For time-dependent magnetic field, we can write the Hamiltonian as:

$$V = e^{i\Omega t}\omega S_{1z}$$

Here we assumed that the magnetic field must be on the z-axis direction in order to have the mixing between $|0,0\rangle$ and $|1,0\rangle$.

Problem 5. Sakurai 5.36

Show that $\boldsymbol{A}_n(\boldsymbol{R})$ defined in (5.6.23) is a purely real quantity.

Recall $\boldsymbol{A}_n(\boldsymbol{R})$

$$\boldsymbol{A}_n(\boldsymbol{R}) = i \langle n; t| \left(\nabla_{\boldsymbol{R}} |i; t\rangle\right)$$

Therefore, we just need to show that $\alpha \triangleq \langle n; t| \left(\nabla_{\boldsymbol{R}} |i; t\rangle\right)$ is a purely imaginary quantity. We can use the following identity:

$$\begin{aligned}
\nabla_{\boldsymbol{R}} \langle n; t|n; t\rangle &= \left(\nabla_{\boldsymbol{R}} \langle n; t|\right) |n; t\rangle + \langle n; t| \left(\nabla_{\boldsymbol{R}} |n; t\rangle\right) \\
0 &= \left(\nabla_{\boldsymbol{R}} \langle n; t|\right) |n; t\rangle + \langle n; t| \left(\nabla_{\boldsymbol{R}} |n; t\rangle\right) \\
0 &= \left[\langle n; t| \left(\nabla_{\boldsymbol{R}} |n; t\rangle\right)\right]^* + \langle n; t| \left(\nabla_{\boldsymbol{R}} |n; t\rangle\right)
\end{aligned}$$

Therefore, $\alpha = -\alpha^*$. Thus, α is purely imaginary.

Problem 6. Sakurai 5.39

A particle of mass m constrained to move in one dimension is confined within $0 < x < L$ by an infinite-wall potential

$$\begin{aligned} V &= \infty \qquad \text{for } x < 0,\ x > L \\ V &= 0 \qquad \text{for } 0 \leq x \leq L \end{aligned}$$

Obtain an expression for the density of states (that is, the number of states per unit energy interval) for *high* energies as a function of E. (Check your dimension!)

For particle in a box, the energy level is given by:

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The density of states is given by:

$$\begin{aligned} \frac{dn}{dE} &= \frac{mL^2}{\pi^2\hbar^2}\frac{1}{n} \\ &= \frac{mL^2}{\pi^2\hbar^2}\frac{\pi\hbar}{L}\sqrt{\frac{1}{2mE}} \\ &= \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}} \end{aligned}$$