PHYS 5260 HW4

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Problem 1. Sakurai 2.21

Derive the normalization constant c_n by deriving the orthogonality relatioonship using generating functions. Start by working out the integral

$$I = \int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx$$

and then consider the integral again with the generating functions in terms of series with Hermite polynomials.

Using the definition of generating function in (2.5.17a), $g(x,t)$ is given by:

$$g(x,t) = e^{-t^2+2tx}$$

Substituting the generating function, the given integral yields:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx &= \int_{-\infty}^{\infty} e^{-t^2+2tx}e^{-s^2+2sx}e^{-x^2}dx \\
&= e^{2ts}\int_{-\infty}^{\infty} e^{-(x-(t+s))^2}dx \\
&= e^{2ts}\sqrt{\pi} \\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n t^n s^n}{n!} \qquad (1)
\end{aligned}$$

The second equation holds since $-t^2+2tx-s^2+2sx-x^2 = -(x^2-2(t+s)+(t+s)^2)+2ts$.

On the other hand, using the definition of generating function in (2.5.17b), $g(x,t)$ is given by:

$$g(x,t) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Substituting the generating function, the given integral yields:

$$\int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{n!}\frac{s^m}{m!}\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx \tag{2}$$

Comparing the two results for $m = n$, we have:

$$\frac{t^n s^n}{(n!)^2}\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}dx = 2^n\sqrt{\pi}\frac{t^n s^n}{n!}$$
$$\int_{-\infty}^{\infty} H_n(x)H_n(x)e^{-x^2}dx = 2^n\sqrt{\pi}n!$$

On the other hand, for $m \neq n$, the integral must be 0, i.e.:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0$$

Recalling the definition of the wave function:

$$u_n(x) = c_n H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right)e^{-m\omega x^2/2\hbar}$$

We can now write the normalization condition as:

$$\begin{aligned}\int_{-\infty}^{\infty} u_n^*(x)u_n(x)dx &= |c_n|^2\int_{-\infty}^{\infty} H_n^2\left(x\sqrt{\frac{m\omega}{\hbar}}\right)e^{-m\omega x^2/\hbar}dx \\ &= |c_n|^2 2^n\sqrt{\pi}n! = 1\end{aligned}$$

Which yields:

$$c_n = \left(\frac{m\omega}{2^{2n}\pi(n!)^2\hbar}\right)^{1/4}$$

Problem 2. Sakurai 2.24

Consider a particle in one dimension bound to a fixed centre by a δ-function potential of the form

$$V(x) = -v_0\delta(x), \; (v_0 \text{ real and positive})$$

Find the wave function and the binding energy of the ground state. Are there excited bound states?

The Schrodinger equation for the system is given by:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - v_0\delta(x)\right)\psi(x) = E\psi(x)$$

For bound states, we apply $E = -E_0 < 0$, i.e. the wave function decays exponentially. The wave function can be written as:

$$\begin{aligned}\psi(x) &= Ce^{-\kappa|x|} \\ &= \begin{cases} Ce^{-\kappa x}, & x < 0 \\ Ce^{\kappa x}, & x > 0 \end{cases}\end{aligned}$$

where $\kappa = ik = i\sqrt{2mE_0}/\hbar$.

The second boundary coindition usually can be obtained by matching the derivative of the wave function at $x = 0$. However, since there is a delta potential at the origin we cannot use this method. Instead, we integrate over a small distance ϵ from the origin and match the wave function:

$$\begin{aligned}\frac{-\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\psi''(x)dx - v_0\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx &= E\int_{-\epsilon}^{\epsilon}\psi(x)dx \\ \frac{-\hbar^2}{2m}(\psi'(\epsilon) - \psi'(-\epsilon)) - v_0\psi(0) &= E(\psi(\epsilon) - \psi(-\epsilon))\end{aligned}$$

Taking limit $\epsilon \to 0$ on both sides, we get:

$$\frac{-\hbar^2}{2m}(-2\kappa)C - v_0C = 0 \iff \kappa = mv_0/\hbar^2$$

If we solve the Schrodinger equation for any $x > 0$ or $x < 0$, we obtain the solution:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = -E_0\psi(x) \iff \psi(x) = Ce^{-\sqrt{2mE_0/\hbar^2}|x|}$$

Thus, matching the two solutions, we can obtain:

$$\frac{mv_0}{\hbar^2} = \frac{\sqrt{2mE_0}}{\hbar} \iff E_0 = \frac{mv_0^2}{2\hbar^2}$$

Since this energy is unique, there is no excited state of this system.

Problem 3. Sakurai 2.27

Derive an expression for the density of free-particle states in two dimensions, normalized with periodic boundary conditions inside a box of side length L. Your answer should be written as a function of k (or E) times $dEd\phi$, where ϕ is the polar angle that characterizes the momentum direction in two dimensions.

The wave function for a free particle in two dimensions is given by:

$$\psi(x, y) = \frac{1}{\sqrt{L^2}} e^{i(k_x x + k_y y)}$$

Where $k_x = \frac{2\pi}{L} n_x$ and $k_y = \frac{2\pi}{L} n_y$.

The energy of the particle is given by:

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \frac{2\pi^2\hbar^2}{mL^2}(n_x^2 + n_y^2)$$

Assume we do a transformation from a cartesian coordinate spanned by n_x, n_y to a polar coordinate spanned by n, ϕ, then the following relation holds:

$$n^2 = n_x^2 + n_y^2$$
$$\tan\phi = \frac{n_y}{n_x}$$
$$dN = dn_x dn_y = ndnd\phi$$

where dN is the density of states.

The energy relation can be rewritten in polar coordinate as:

$$dE = \frac{4\pi^2\hbar^2}{mL^2} ndn$$

Finally, the density of states is given by:

$$dN = ndnd\phi = \frac{mL^2}{4\pi^2\hbar^2} dEd\phi$$

Problem 4. Sakurai 2.30

Using sphreical coordinates, obtain an expression for $\boldsymbol{j}$ for the ground and excites states of the hydrogen atom. Show, in particular for $m_i \neq 0$ states, there is a circulating flux in the sense that $\boldsymbol{j}$ is in the direction of increasing or decreasing ϕ depending on whether m_i is positive or negative.

Recall the probability current is given by:

$$\boldsymbol{j} = \frac{1}{2m}(\psi^* p\psi - \psi p\psi^*) = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\nabla\psi)$$

Where the ∇ operator in spherical coordinate is defined as:

$$\nabla = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}$$

The wave function of a hydrogen atom is given by:

$$\begin{aligned}\psi(r,\theta,\phi) &= C_{nlm}R_{nl}(r)Y_l^m(\theta,\phi)\\ &= C_{nlm}R_{nl}(r)P_l^m(\cos\theta)e^{im\phi}\end{aligned}$$

Where R_{nl} is the radial part of the wave function and Y_l^m is the spherical harmonic function. Note that C_{nlm}, R_{nl} and Y_l^m are all real numbers.

Note that since we are only interested on the imaginary part of $\psi * \nabla\psi$, we can only work on the $\hat{\psi}$ part of the wave function. Thus, we can write:

$$\begin{aligned}\boldsymbol{j} &= \frac{\hbar}{m_e}\,\mathrm{Im}(\psi^*\nabla\psi)\\ &= \frac{\hbar}{m_e}\,\mathrm{Im}(\psi^*\left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\phi}\frac{\partial}{\partial\phi}\right)\psi)\\ &= \frac{\hbar}{m_e}\,\mathrm{Im}(\psi^*\frac{im}{r\sin\phi}\psi)\hat{\boldsymbol{\phi}}\\ &= \frac{\hbar}{m_e}\frac{m}{\sin\phi}|\psi^2|\hat{\boldsymbol{\phi}}\end{aligned}$$

Therefore, we show that the flux is in direction of ϕ, depending of the magnetic quantum number m.

Problem 5. Sakurai 2.33

The propagator in momentum space analogous to (2.6.26) is given by $\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle$. Derive an explicit expression for $\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle$ for the free-particle case.

For a free particle, the hamiltonial is given by:

$$H = \frac{p^2}{2m}$$

Thus, the propagator is given by:

$$\begin{aligned}
\langle \boldsymbol{p''}, t | \boldsymbol{p'}, t_0 \rangle &= \langle \boldsymbol{p''} | \, e^{-iHt} e^{iHt_0} \, | \boldsymbol{p'}, t_0 \rangle \\
&= \exp\left(\frac{-i\boldsymbol{p'}^2}{2m\hbar}(t - t_0) \right) \langle \boldsymbol{p''} | \boldsymbol{p'}, t_0 \rangle \\
&= \exp\left(\frac{-i\boldsymbol{p'}^2}{2m\hbar}(t - t_0) \right) \delta(\boldsymbol{p'} - \boldsymbol{p'})
\end{aligned}$$

Problem 6. Sakurai 2.36

Show that wave-mechanical approach to the gravity-induced problem discussed in Section 2.7 also leads to phase-difference expression (2.7.17).

In the experiment, the phase of the neutron wave function depends on the length of the path and the time it takes to travel the path i.e., $\phi \sim kx - \omega t$. Thus, the phase difference of the two path is given by:

$$\begin{aligned}
\phi_{BD} - \phi_{AC} &= \left(\frac{p_{BD} - p_{AC}}{\hbar}\right) l_1 - \omega \left(\frac{l_1}{v_{BD}} - \frac{l_1}{v_{AC}}\right) \\
&= \left(\frac{p_{BD} - p_{AC}}{\hbar}\right) l_1 \left(1 + \frac{\hbar \omega m_n}{p_{AC} p_{BD}}\right) \\
&= \left(\frac{p_{BD} - p_{AC}}{\hbar}\right) l_1 \left(1 + \frac{E m_n}{p^2}\right) \\
&= \left(\frac{p_{BD} - p_{AC}}{\hbar}\right) \frac{3}{2} l_1
\end{aligned}$$

Since $E = \hbar\omega = \frac{p^2}{2m}$.

Then we can make the following approximation and apply the conservation law of energy:

$$\begin{aligned}
p_{BD} - p_{AC} &\approx \left(\frac{p_{BD}^2}{2m} - \frac{p_{AC}^2}{2m}\right) \frac{2m}{p} \\
&= -mgz \frac{2m}{p} = -\frac{2m^2 g l_2 \sin\theta}{p}
\end{aligned}$$

Therefore the phase difference is given by:

$$\begin{aligned}
\phi_{BD} - \phi_{AC} &= \left(\frac{p_{BD} - p_{AC}}{\hbar}\right) \frac{3}{2} l_1 \\
&= -\frac{2m^2 g l_2 \sin\theta}{p\hbar} \frac{3}{2} l_1 \\
&= -\frac{3m^2 g \lambda}{2\pi\hbar^2} l_1 l_2 \sin\theta
\end{aligned}$$