PHYS 5260 HW3

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September 29, 2022

Problem 1. Sakurai 2.3

An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$ with eigenvalue $\hbar/2$, where $\hat{\boldsymbol{n}}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-axis.

(a) Obtain the probability for finding the electron in the $s_x = \hbar/2$ state as a function of time.

Similar to Problem 1.9, let the vector n and the spin operator S be given by:

$$\hat{\boldsymbol{n}} = \cos\alpha \sin\beta \hat{\boldsymbol{x}} + \sin\alpha \sin\beta \hat{\boldsymbol{y}} + \cos\beta \hat{\boldsymbol{z}}$$
$$\boldsymbol{S} = \frac{\hbar}{2} \left(\sigma_x \hat{\boldsymbol{x}} + \sigma_y \hat{\boldsymbol{y}} + \sigma_z \hat{\boldsymbol{z}}\right)$$

The inner product is thus given by:

$$\begin{aligned} \boldsymbol{S} \cdot \hat{\boldsymbol{n}} &= \frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & \cos(\alpha)\sin(\beta) - i\sin(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + i\sin(\alpha)\sin(\beta) & -\cos(\beta) \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & e^{-i\alpha}\sin(\beta) \\ e^{i\alpha}\sin(\beta) & -\cos(\beta) \end{pmatrix} \end{aligned}$$

Solving the eigenvalue problem for $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$, we obtain the condition:

$$|\boldsymbol{S} \cdot \hat{\boldsymbol{n}} - Iv| = 0 \iff v = \pm\frac{\hbar}{2}$$

To find the corresponding eigenvectors for the eigenvalue $+1$, we solve the following equation:

$$\frac{\hbar}{2} \begin{pmatrix} \cos(\beta) & e^{-i\alpha}\sin(\beta) \\ e^{i\alpha}\sin(\beta) & -\cos(\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} x \\ y \end{pmatrix}$$

We obtain the solution:

$$x(\cos\beta - 1) + ye^{-i\alpha}\sin\beta = 0$$
$$xe^{i\alpha}\sin\beta/2 + y\cos\beta/2 = 0$$

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Therefore:

$$\begin{aligned}
|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle &= \cos\beta/2\,|+\rangle + e^{-i\alpha}\sin\beta/2\,|-\rangle \\
|\alpha;t=0\rangle &= \cos\beta/2\,|+\rangle + \sin\beta/2\,|-\rangle && \text{since } \alpha=0
\end{aligned}$$

The Hamiltonian of our system is given by:

$$\begin{aligned}
\hat{H} = \hat{T} + \hat{V} = \hat{V} &= -\boldsymbol{\mu}\cdot\boldsymbol{B} \\
&= \frac{g_e e}{2m}\boldsymbol{S}\cdot\boldsymbol{B} \\
&\approx \frac{\hbar}{2}\frac{eB}{m}\sigma_z
\end{aligned}$$

Thus the time evolution operator is given by:

$$\begin{aligned}
U(t) = \exp\left(-i\hat{H}t/\hbar\right) &= \exp\left(-\frac{i}{2}\frac{eB}{m}\sigma_z t\right) \\
&= \begin{pmatrix} e^{-i\omega/2t} & 0 \\ 0 & e^{i\omega/2t} \end{pmatrix}
\end{aligned}$$

where $\omega = eB/m$

Therefore the state at time t is given by:

$$\begin{aligned}
|\alpha;t=t\rangle &= U(t)\,|\alpha;t=0\rangle \\
&= e^{-i\omega/2t}\cos\beta/2\,|+\rangle + e^{i\omega/2t}\sin\beta/2\,|-\rangle
\end{aligned}$$

And the state for the $s_x = \hbar/2$ state is given by:

$$|Sx;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

Therefore, the probability of finding the electron in the $s_x = \hbar/2$ state is given by at a given time t is given by:

$$\begin{aligned}
|\langle Sx;+|\alpha;t=t\rangle|^2 &= \frac{1}{2}\left(|\langle +|\alpha;t=t\rangle + \langle -|\alpha;t=t\rangle|^2\right) \\
&= \frac{1}{2}\left(\left|e^{-i\omega/2t}\cos\beta/2 + e^{i\omega/2t}\sin\beta/2\right|^2\right) \\
&= \frac{1}{2}\left(\cos^2\beta/2 + \sin^2\beta/2 + \cos\omega t\sin beta\right) \\
&= \frac{1+\cos\omega t\sin\beta}{2}
\end{aligned}$$

(b) Find the expectation value of S_x as a function of time

Therefore, the expectation value $\langle S_x(t) \rangle$ is given by:

$$\begin{aligned}
\langle \alpha; t = t | \, S_x \, | \alpha; t = t \rangle &= \frac{\hbar}{2} \langle \alpha; t = t | \, \sigma_x \, | \alpha; t = t \rangle \\
&= \frac{\hbar}{2} (e^{i\omega t} + e^{-i\omega t}) \cos \beta/2 \sin \beta/2 \\
&= \frac{\hbar}{2} \cos \omega t \sin \beta
\end{aligned}$$

(c) For your own peace of mind, show that your answers make good sense in the extreme cases (i) $\beta \to 0$ (ii) $\beta \to \pi/2$

For $\beta \to 0$, $\langle S_x(t) \rangle = 0$. This is what we are expected since there is no precession, and the electron is always in the $s_z = \hbar/2$ state.

For $\beta \to \pi/2$, $\langle S_x(t) \rangle = \frac{\hbar}{2} \cos \omega t$, which is similar to classical precession.

Problem 2. Sakurai 2.6

Consider a particle in one dimension whose Hamiltonial is given by:

$$H = \frac{p^2}{2m} + V(x)$$

By calculating $[[H, x], x]$, prove:

$$\sum_{a'} |\langle a''| x |a'\rangle|^2 (E_{a'} - E_{a''}) = \frac{\hbar^2}{2m}$$

where $|a'\rangle$ is an energy eigenket with eigenvalue $E_{a'}$

First, we calculate the following commutators:

$$\begin{aligned}
[H, x] &= \frac{1}{2m}[p^2, x] = \frac{1}{2m}(p[p, x] + [p, x]p) \\
&= \frac{1}{2m}(p(-i\hbar) + (-i\hbar)p) = \frac{-i\hbar p}{m}
\end{aligned}$$

$$\begin{aligned}
[[H, x], x] &= \frac{-i\hbar}{m}[p, x] = \frac{-i\hbar}{m}(-i\hbar) \\
&= -\frac{\hbar^2}{m}
\end{aligned}$$

On the other hand, we can expand the commutator relation to be:

$$\begin{aligned}
[[H, x], x] &= [H, x]x - x[H, x] \\
&= Hx^2 - xHx - xHx + x^2H \\
&= Hx^2 + x^2H - 2xHx
\end{aligned}$$

The expectation value of the above expression is given by:

$$\begin{aligned}
\langle a'| (Hx^2 + x^2H - 2xHx) |a'\rangle &= 2E' \langle a'| x^2 |a'\rangle + -2 \langle a'| xHx |a'\rangle \\
&= \sum_{a''} 2E' \langle a'| x |a''\rangle \langle a''| x |a'\rangle - 2 \langle a'| x |a''\rangle \langle a''| Hx |a'\rangle \\
-\frac{\hbar^2}{m} &= 2\sum_{a''}(E' - E'')|\langle a''| x |a'\rangle|^2
\end{aligned}$$

Without loss of generality, we can change the sum basis from a'' to a', and we can obtain:

$$\begin{aligned}
\sum_{a''}(E' - E'')|\langle a''| x |a'\rangle|^2 &= -\frac{\hbar^2}{2m} \\
\sum_{a'}(E'' - E')|\langle a''| x |a'\rangle|^2 &= -\frac{\hbar^2}{2m} \\
\sum_{a'}(E' - E'')|\langle a''| x |a'\rangle|^2 &= \frac{\hbar^2}{2m} \quad \text{(Q.E.D)}
\end{aligned}$$

Problem 3. Sakurai 2.9

Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of a Hermition operator A with eigenvalues a' and a'' respectively ($a' \neq a''$). The Hamiltonian operator is given by:

$$H = |a'\rangle \, \delta \, \langle a''| + |a''\rangle \, \delta \, \langle a'|$$

where δ is just a real number.

(a) Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?

First, we can construct the Hamiltonian for $|a'\rangle$, $|a''\rangle$ basis, which is:

$$H = \begin{pmatrix} \langle a'| H |a'\rangle & \langle a''| H |a'\rangle \\ \langle a'| H |a''\rangle & \langle a''| H |a''\rangle \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}$$

The eigenstates and the corresponding eigenvalues for this Hamiltonian are:

$$|\alpha_+\rangle = \frac{1}{\sqrt{2}}(|a'\rangle + |a''\rangle) \qquad E_+ = \delta$$
$$|\alpha_-\rangle = \frac{1}{\sqrt{2}}(|a'\rangle - |a''\rangle) \qquad E_- = -\delta$$

(b) Suppose the system is known to e in state $|a'\rangle$ at $t = 0$. Write down the state vector in the Schroedinger picture for $t > 0$.

Recall the unitary time evolution operator:

$$\begin{aligned} U(t) &= e^{-iHt/\hbar} \\ &= \begin{pmatrix} \cos \delta t/\hbar & -i \sin \delta t/\hbar \\ -i \sin \delta t/\hbar & \cos \delta t/\hbar \end{pmatrix} \end{aligned}$$

Therefore, for the given initial state $|a', t = 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the state vector in the Schroedinger picture for $t > 0$ is:

$$\begin{aligned} |a', t\rangle &= U(t) \, |a', t = 0\rangle \\ &= \begin{pmatrix} \cos \delta t/\hbar & -i \sin \delta t/\hbar \\ -i \sin \delta t/\hbar & \cos \delta t/\hbar \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos \delta t/\hbar \\ -i \sin \delta t/\hbar \end{pmatrix} \end{aligned}$$

In the energy eigenstates basis, we have:

$$\begin{aligned} \sum_{\pm} |\alpha_\pm\rangle \, \langle \alpha_\pm | a', t\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \delta t/\hbar - i \sin \delta t/\hbar \\ \cos \delta t/\hbar + i \sin \delta t/\hbar \end{pmatrix} \\ |\alpha', t\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\delta t/\hbar} \\ e^{\delta t/\hbar} \end{pmatrix} \end{aligned}$$

(c) What is the probability for finding the system in $|a''\rangle$ for $t > 0$? if the system is known to be in state $|a'\rangle$ at $t = 0$?

The probability for finding the system in $|a''\rangle$ for $t > 0$ is:

$$|\langle a''|a', t\rangle|^2 = |-i \sin \delta t/\hbar|^2 = \sin^2 \delta t/\hbar$$

(d) Can you think of a physical situation corresponding to this problem?

Spin 1/2 particle, i.e, electron.

Problem 4. Sakurai 2.12

Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose that at $t = 0$ the state vector is given by:

$$\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

where p is the momentum operator and a is some number with dimension of length. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$

In the Heisenberg picture, the time evolution of the operator is given by:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{i\hbar}[x, H] = \frac{1}{i\hbar}[x, p^2/2m + m\omega^2 x^2/2] \\
&= \frac{1}{2mi\hbar}(p[x,p] + [x,p]p) = \frac{1}{2mi\hbar}2i\hbar p \\
&= \frac{p}{m} \\
\frac{dp}{dt} &= \frac{1}{i\hbar}[p, H] = \frac{1}{i\hbar}[p, p^2/2m + m\omega^2 x^2/2] \\
&= \frac{m\omega^2}{2i\hbar}(x[p,x] + [p,x]x) = \frac{m\omega^2}{2i\hbar}(-2i\hbar x) \\
&= -m\omega^2 x
\end{aligned}$$

Solving the above differential equations with the initial condition $x(0)$, we have:

$$\begin{aligned}
x(t) &= x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) \\
p(t) &= p(0)\cos(\omega t) + \frac{m\omega^2 x(0)}{\omega}\sin(\omega t)
\end{aligned}$$

Then, the expectation value for the posision is:

$$\begin{aligned}
\langle x\rangle &= \langle 0|\exp\left(\frac{ipa}{\hbar}\right) x \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle \\
&= \cos(\omega t)\langle 0|\exp\left(\frac{ipa}{\hbar}\right) x(0)\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle + \frac{\sin(\omega t)}{m\omega}\langle 0|\exp\left(\frac{ipa}{\hbar}\right) p(0)\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle \\
&= \langle 0|(x(0)+a)|0\rangle\cos(\omega t) + \langle 0|p(0)|0\rangle\sin(\omega t)
\end{aligned}$$

If we define the ground state as: $\langle 0|x|0\rangle = 0$ and $\langle 0|p|0\rangle = 0$, then:

$$\begin{aligned}
\langle x\rangle &= \langle 0|(x(0)+a)|0\rangle\cos(\omega t) + \langle 0|p(0)|0\rangle\sin(\omega t) \\
&= a\cos(\omega t)
\end{aligned}$$

Which is similar to the classical simple harmonic oscillator with the initial displacement $x(0) = a$.

Problem 5. Sakurai 2.15

(a) Using

$$\langle x'|p'\rangle = (2\pi\hbar)^{-1/2} \exp\left(\frac{ip'x'}{\hbar}\right)$$

prove:

$$\langle p'| x |\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

First note that we can represent the position operator as:

$$x = \int dx'' |x''\rangle x'' \langle x''| = i\hbar \int dp'' |p''\rangle \frac{\partial}{\partial p''} \langle p''|$$

Then,

$$\langle p'| x |\alpha\rangle = i\hbar \int dp'' \langle p'|p''\rangle \frac{\partial}{\partial p''} \langle p''|\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \quad \text{(Q.E.D)}$$

(b) Consider a one-dimensional simple harmonic oscillator. Starting with the Schrodinger equation for the state vector, derive the Schrodinger equation for the *momentum-space* wave function. (Make sure to distinguish the operator p from the eigenvalue p'.) Can you guess the energy eigenfunction in momentum space?

Recall the hamiltonial for the simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

We can write the Schrodinger equation for the momentum space vector as:

$$\begin{aligned} \langle p'| H |\alpha\rangle &= E \langle p'|\alpha\rangle \\ \frac{1}{2m} \langle p'| p^2 |\alpha\rangle + \frac{1}{2} m\omega^2 \langle p'| x^2 |\alpha\rangle &= E\, \phi(p') \\ \left(\frac{1}{2m} p'^2 - \frac{m\omega^2\hbar^2}{2} \frac{\partial^2}{\partial p'^2}\right) \phi(p') &= E\, \phi(p') \end{aligned}$$

The energy eigenfunction must be of the similar form with the one derived using the ladder operator method. i.e.:

$$E_n = \hbar\omega(n + 1/2)$$

And the corresponding eigenfunction must be of the form of the Hermite polynomial with some normalization constant.

Problem 6. Sakurai 2.18

Show that for the one-dimensional simple harmonic osciallator,

$$\langle 0| e^{ikx} |0\rangle = \exp\left[-k^2 \langle 0| x^2 |0\rangle /2\right]$$

where x is the position operator

We can directly calculate the right hand side:

$$\begin{aligned}
x^2 &= \frac{\hbar}{2m\omega}(a^\dagger a^\dagger + a^\dagger a + a a^\dagger + aa) \\
x^2 |0\rangle &= \frac{\hbar}{2m\omega}(\sqrt{2}\,|2\rangle + |0\rangle) \\
\langle 0| x^2 |0\rangle &= \frac{\hbar}{2m\omega} \\
\exp\left(-\frac{k^2}{2}\langle 0| x^2 |0\rangle\right) &= \exp\left(-\frac{k^2\hbar}{4m\omega}\right) = \exp\frac{-\beta^2}{2} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{n!}\frac{1}{2^n}
\end{aligned}$$

where $\beta = k\sqrt{\frac{\hbar}{2m\omega}}$.

Then we can also write the left hand side as:

$$\begin{aligned}
\exp(ikx) &= \exp\bigl(i\beta(a^\dagger + a)\bigr) \\
&= \sum_{m=0}^{\infty}\frac{(i\beta)^m}{m!}(a^\dagger + a)^m
\end{aligned}$$

Note that when we take the expectation with $|0\rangle$, the nonzero term are $m = 2n$:

$$\langle 0| \exp(ikx) |0\rangle = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{n!}\frac{n!}{(2n)!}\langle 0| (a^\dagger + a)^{2n} |0\rangle$$

We can try the first few terms:

$$\begin{aligned}
n = 1 \rightarrow \frac{1!}{2!}\langle 0| (a^\dagger + a)^2 |0\rangle & & = \frac{1}{2^1} \\
n = 2 \rightarrow \frac{2!}{4!}\langle 0| (a^\dagger + a)^4 |0\rangle & & \\
= \frac{1}{12}\langle 0| (aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + \dots) & & \\
= \frac{1}{12}(2+1) & & = \frac{1}{2^2} \\
n = 3 \rightarrow \frac{3!}{6!}\langle 0| (a^\dagger + a)^6 |0\rangle & & \\
= \frac{1}{12}\langle 0| (aaaa^\dagger a^\dagger a^\dagger + aaa^\dagger aa^\dagger a^\dagger + aa^\dagger aaa^\dagger a^\dagger + aaa^\dagger a^\dagger aa^\dagger + aa^\dagger aa^\dagger aa^\dagger + \dots) |0\rangle & & \\
= \frac{1}{120}(6+4+2+2+1) & & = \frac{1}{2^3}
\end{aligned}$$

For higher n, we can also prove that the two sides are equal.