PHYS 5260 HW1

TANUWIJAYA, Randy Stefan *
(20582731)
rstanuwijaya@connect.ust.hk

Department of Physics - HKUST

September 15, 2022

Problem 1. Sakurai 1.3

Show that the determinant of a 2×2 matrix $\boldsymbol{\sigma} \cdot \boldsymbol{a}$ is invariant under

$$\boldsymbol{\sigma} \cdot \boldsymbol{a} \to \boldsymbol{\sigma} \cdot \boldsymbol{a'} \equiv \exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\} \boldsymbol{\sigma} \cdot \boldsymbol{a} \exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\}$$

Find a'_k in terms of of a_k when $\hat{\boldsymbol{n}}$ is in the positive z-direction and interpret your result.

Note the following properties of the matrix determinant: $|ABC| = |A||B||C|$. Therefore, to prove the invariance, we need to show that: $|\boldsymbol{\sigma} \cdot \boldsymbol{a}| = |\boldsymbol{\sigma} \cdot \boldsymbol{a'}|$. In this case, $|\boldsymbol{\sigma} \cdot \boldsymbol{a'}|$ can be expanded as:

$$|\boldsymbol{\sigma} \cdot \boldsymbol{a'}| = \left|\exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\}\right| |\boldsymbol{\sigma} \cdot \boldsymbol{a}| \left|\exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\}\right|$$

Since the two exponential terms are the inverse of each others, it follows that:

$$\left|\exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\}\right| \left|\exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\}\right| = 1$$

Therefore, we have:

$$|\boldsymbol{\sigma} \cdot \boldsymbol{a}| = |\boldsymbol{\sigma} \cdot \boldsymbol{a'}| \quad \text{(Q.E.D)}$$

For $\hat{\boldsymbol{n}} = (0, 0, 1)$, we have $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since σ_k is diagonal matrix, the expononential terms can be expressed as:

$$\exp\left\{\frac{i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\} = \begin{pmatrix} \exp\{i\phi/2\} & 0 \\ 0 & \exp\{-i\phi/2\} \end{pmatrix}$$
$$\exp\left\{\frac{-i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}\phi}{2}\right\} = \begin{pmatrix} \exp\{-i\phi/2\} & 0 \\ 0 & \exp\{i\phi/2\} \end{pmatrix}$$

* $\LaTeX$ source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Therefore:

$$\sigma_k a_k' = \begin{pmatrix} \exp\{i\phi/2\} & 0 \\ 0 & \exp\{-i\phi/2\} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \exp\{-i\phi/2\} & 0 \\ 0 & \exp\{i\phi/2\} \end{pmatrix} a_k$$

Solving for a_k', we obtain $a_k' = a_k$. This means that $\boldsymbol{\sigma} \cdot \boldsymbol{a}$ i.e., the angular momentum is invariant under rotation about the z-axis.

Problem 2. Sakurai 1.6

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

Suppose the eigenvalues of each eigenkets are given by: E_i and E_j.

$$A|i\rangle = E_i|i\rangle$$
$$A|j\rangle = E_j|j\rangle$$

Summing the two equations, we have:

$$A|i\rangle + A|j\rangle = E_i|i\rangle + E_j|j\rangle = E(|i\rangle + |j\rangle)$$

The last equation only holds when $E = E_i + E_j$. This equation only holds when A is degenerate, and $|i\rangle$ and $|j\rangle$ are corresponding eigenkets.

Problem 3. Sakurai 1.9

Construct $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle$ such that:

$$\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\,|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \left(\frac{\hbar}{2}\right)|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle$$

where $\hat{\boldsymbol{n}}$ is characterized by the angles shown in the accompanying figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$.

Let the vector n and the spin operator S be given by:

$$\begin{aligned}\hat{\boldsymbol{n}} &= \cos\alpha\sin\beta\hat{\boldsymbol{x}} + \sin\alpha\sin\beta\hat{\boldsymbol{y}} + \cos\beta\hat{\boldsymbol{z}}\\ \boldsymbol{S} &= \frac{\hbar}{2}\left(\sigma_x\hat{\boldsymbol{x}} + \sigma_y\hat{\boldsymbol{y}} + \sigma_z\hat{\boldsymbol{z}}\right)\end{aligned}$$

The inner product is thus given by:

$$\begin{aligned}\boldsymbol{S}\cdot\hat{\boldsymbol{n}} &= \frac{\hbar}{2}\begin{pmatrix}\cos(\beta) & \cos(\alpha)\sin(\beta) - i\sin(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta) + i\sin(\alpha)\sin(\beta) & -\cos(\beta)\end{pmatrix}\\ &= \frac{\hbar}{2}\begin{pmatrix}\cos(\beta) & e^{-i\alpha}\sin(\beta)\\ e^{i\alpha}\sin(\beta) & -\cos(\beta)\end{pmatrix}\end{aligned}$$

Solving the eigenvalue problem for $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$, we obtain the condition:

$$|\boldsymbol{S}\cdot\hat{\boldsymbol{n}} - Iv| = 0 \iff v = \pm\frac{\hbar}{2}$$

which is consistent with the problem statement.

To find the corresponding eigenvectors for the eigenvalue $+1$, we solve the following equation:

$$\frac{\hbar}{2}\begin{pmatrix}\cos(\beta) & e^{-i\alpha}\sin(\beta)\\ e^{i\alpha}\sin(\beta) & -\cos(\beta)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}x\\ y\end{pmatrix}$$

We obtain the solution:

$$\begin{aligned}x(\cos\beta - 1) + ye^{-i\alpha}\sin\beta = 0\\ xe^{i\alpha}\sin\beta/2 + y\cos\beta/2 = 0\end{aligned}$$

Therefore:

$$|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \cos\beta/2\,|+\rangle + e^{-i\alpha}\sin\beta/2\,|-\rangle$$

Problem 4. Sakurai 1.12

A spin $\frac{1}{2}$ system to be in an eigenstate of $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$ with eigenvalue $\hbar/2$, where $\hat{\boldsymbol{n}}$ is a unit vector lying in xz-plane that makes an angle γ with the positive z-axis.

(a) Suppose S_x is measured. WHat is the probability of getting $+\hbar/2$?

The state of the given system and the eigenstate of spin x with the eigenvalue of $\hbar/2$ can be written as:

$$\begin{aligned} |\psi\rangle = |\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle &= \cos\gamma/2\,|+\rangle + \sin\gamma/2\,|-\rangle \\ |\boldsymbol{S}\cdot\hat{\boldsymbol{x}};+\rangle &= \frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle \end{aligned}$$

Therefore, the probability of getting $|\boldsymbol{S}\cdot\hat{\boldsymbol{x}};+\rangle$ is given by:

$$\begin{aligned} |\langle\boldsymbol{S}\cdot\hat{\boldsymbol{x}};+|\psi\rangle|^2 &= \left(\frac{1}{\sqrt{2}}(\cos\gamma/2+\sin\gamma/2)\right)^2 \\ &= \frac{1+\sin\gamma}{2} \end{aligned}$$

(b) Evaluate the dispersion in S_x - that is,

$$\left\langle (S_x - \langle S_x\rangle)^2 \right\rangle$$

The expectation values of S_x and S_x^2 are given by:

$$\begin{aligned} \langle S_x\rangle &= \langle\psi|\,S_x\,|\psi\rangle = \frac{\hbar}{2}\sin\gamma \\ \left\langle S_x^2\right\rangle &= \langle\psi|\,S_x^2\,|\psi\rangle = \frac{\hbar^2}{4} \end{aligned}$$

Therefore, the dispersion is given by:

$$\left\langle (S_x - \langle S_x\rangle)^2 \right\rangle = \left\langle S_x^2\right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}\cos^2\gamma$$

Problem 5. Sakurai 1.15

Let A and B be observables. Suppose the simultaneous eigekets of A and B $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0$$

Yes. Suppose $|a', b'\rangle$ is the simultaneous eigenstates of A and B, then:

$$A\,|a', b'\rangle = a'\,|a', b'\rangle$$
$$B\,|a', b'\rangle = b'\,|a', b'\rangle$$

Consider the following equations:

$$BA\,|a', b'\rangle = a'B\,|a', b'\rangle \qquad = a'b'\,|a', b'\rangle$$
$$AB\,|a', b'\rangle = b'A\,|a', b'\rangle \qquad = b'a'\,|a', b'\rangle$$

Therefore, subtracting the two equations above, we obtain the commutator:

$$[A, B]\,|a', b'\rangle = BA\,|a', b'\rangle - AB\,|a', b'\rangle = 0$$

Problem 6. Sakurai 1.18

(a) The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceeding inequality reduces to the Schwarz inequality.

Let $\lambda = x + iy$. Then:

$$\begin{aligned}
&0 \leq (\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \\
&0 \leq \langle\alpha|\alpha\rangle + \lambda \langle\alpha|\beta\rangle + \lambda^* \langle\beta|\alpha\rangle + |\lambda|^2 \langle\beta|\beta\rangle \\
&0 \leq \langle\alpha|\alpha\rangle + \lambda \langle\beta|\alpha\rangle^* + \lambda^* \langle\beta|\alpha\rangle + |\lambda|^2 \langle\beta|\beta\rangle \\
&0 \leq \frac{\langle\alpha|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{2x \operatorname{Re}\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{2y \operatorname{Im}\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + (x^2 + y^2) \\
&0 \leq \left(x + \frac{\operatorname{Re}\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\right)^2 + \left(y + \frac{\operatorname{Im}\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}\right)^2 + \frac{\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle - |\langle\beta|\alpha\rangle|^2}{\langle\beta|\beta\rangle^2}
\end{aligned}$$

Note that the first two terms must be larger than 0. Then we can see the Schwarz inequality on the last term, which is:

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle - |\langle\beta|\alpha\rangle|^2 \geq 0$$

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A |\alpha\rangle = \lambda \Delta B |\beta\rangle$$

with λ purely imaginary.

Recall the generalized uncertainty principle:

$$\left\langle(\Delta A)^2\right\rangle \left\langle(\Delta B)^2\right\rangle \geq \frac{1}{4} |\langle[A, B]\rangle|^2$$

The expectation value of commutator $[A, B] = [\Delta A, \Delta B]$ is given by:

$$\begin{aligned}
\langle[\Delta A, \Delta B]\rangle &= \langle\alpha| \Delta A \Delta B - \Delta B \Delta A |\alpha\rangle \\
&= (\lambda^* - \lambda) \langle\alpha| (\Delta B)^2 |\alpha\rangle \\
&= -2\lambda \left\langle(\Delta B)^2\right\rangle
\end{aligned}$$

since $\lambda^* = -\lambda$.

Then, note that $\left\langle(\Delta A)^2\right\rangle = |\lambda|^2 \left\langle(\Delta B)^2\right\rangle$. Therefore, both sides of the inequality are equal to $|\lambda|^2 \left\langle(\Delta B)^2\right\rangle^2$.

(c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet is given by:

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} \exp\left[\frac{i \langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle^2)}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$$

Prove that the requirement:

$$\langle x'|\,\Delta x\,|\alpha\rangle = (\text{imaginary number})\,\langle x'|\,\Delta p\,|\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

Begin with expanding $\langle x'|\,\Delta x\,|\alpha\rangle$ and $\langle x'|\,\Delta p\,|\alpha\rangle$

$$\langle x'|\,\Delta x\,|\alpha\rangle = (x' - \langle x\rangle)\,\langle x'|\alpha\rangle$$
$$\langle x'|\,\Delta p\,|\alpha\rangle = \left(\frac{\hbar}{i}\frac{d}{dx} - \langle p\rangle\right)\langle x'|\alpha\rangle$$

where

$$\frac{\hbar}{i}\frac{d}{dx}\,\langle x'|\alpha\rangle = \left(\langle p\rangle - \frac{\hbar}{i}\frac{1}{2d^2}(x' - \langle x\rangle)^2\right)\langle x'|\alpha\rangle$$

Therefore:

$$\langle x'|\,\Delta x\,|\alpha\rangle = -\frac{\hbar}{i}\frac{1}{2d^2}\,\langle x'|\,\Delta p\,|\alpha\rangle$$

which is consistent with the result in (b) as the factor is purely imaginary.