HW2

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Problem 1. Sakurai 1.21

Evaluate the $x - p$ uncertainty product $\left\langle(\Delta x)^2\right\rangle\left\langle(\Delta p)^2\right\rangle$ for a one-dimensional particle confined between two rigid walls,

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Do this for both the ground and excited states.

The wavefunction for the particle can be found by solving the Schrodinger equation:

$$\begin{aligned} H\psi(x) &= E_n\psi(x) \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= E_n\psi(x) \end{aligned}$$

Solving the differential equation and normalizing, we get the wavefunction for $0 < x < a$:

$$\psi(x) = A\sin(kx) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

and for $\psi(x) = 0$ for $x > a$:

The expectation value of any operator $\hat{A}$ is given by:

$$\begin{aligned} \left\langle\hat{A}\right\rangle &= \int_0^\infty \psi(x)^* A\psi(x)dx \\ &= \int_0^a \frac{2}{a}\sin\left(\frac{n\pi}{a}x\right)\hat{A}\sin\left(\frac{n\pi}{a}x\right)dx \end{aligned}$$

* LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

Thus the expectation value of the following operators are:

$$
\begin{aligned}
\langle x \rangle &= \frac{a}{2} \\
\left\langle x^2 \right\rangle &= \frac{2a^2}{6} \\
\langle p \rangle &= 0 \\
\left\langle p^2 \right\rangle &= \frac{-\hbar^2 n^2 \pi^2}{a^2}
\end{aligned}
$$

Substituting these values into the uncertainty product, we get:

$$
\begin{aligned}
\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle &= \left(\left\langle x^2 \right\rangle - \left\langle x^2 \right\rangle\right) \left(\left\langle p^2 \right\rangle - \langle p \rangle^2\right) \\
&= \frac{\hbar}{12}(-6 + n^2\pi^2)
\end{aligned}
$$

For ground state $n = 1$,

$$
\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle = \frac{\hbar}{12}(-6 + \pi^2) \approx 0.322\hbar^2
$$

which implies that the uncertainty principle holds, i.e. larger than $\hbar^2/4$. For excited states $n > 1$, the uncertainty principle also holds.

Problem 2. Sakurai 1.22

(a) Prove that $1/\sqrt{2}(1+\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by the angle of $\pi/2$. (The minus sign signifies that the rotation is clockwise)

The form of the rotation operator about the x-axis is:

$$
\begin{aligned}
D(\hat{\boldsymbol{x}}, \phi) &= \exp\left(-i\phi \frac{\hat{\boldsymbol{x}} \cdot \boldsymbol{S}}{\hbar}\right) \\
&= \exp\left(-i\phi \frac{S_x}{\hbar}\right) \\
&= \exp\left(-i\phi \frac{\sigma_x}{2}\right) \\
&= \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}
\end{aligned}
$$

Which is the same as the matrix representation of $1/\sqrt{2}(1 + \sigma_x)$ for $\phi = -\pi/2$, i.e.

$$
\frac{1}{\sqrt{2}}(1 + \sigma_x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = D(\hat{\boldsymbol{x}}, -\pi/2)
$$

(b) Construct the matrix representation of S_z when the eigekets of S_y are used as base vector

The eigenkets of S_y are:

$$
|\psi_{y+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}
$$
$$
|\psi_{y-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}
$$

Therefore to find the matrix representation of S_z, we need to find the projection of S_z onto the eigenkets:

$$
\begin{aligned}
\langle\psi_{y+}|\,\sigma_z\,|\psi_{y+}\rangle &= 0\\
\langle\psi_{y+}|\,\sigma_z\,|\psi_{y-}\rangle &= 1\\
\langle\psi_{y-}|\,\sigma_z\,|\psi_{y+}\rangle &= 1\\
\langle\psi_{y-}|\,\sigma_z\,|\psi_{y-}\rangle &= 0
\end{aligned}
$$

Therefore, the matrix representation of S_z in the basis of $|\psi_{y+}\rangle$ and $|\psi_{y-}\rangle$ is:

$$
\langle\psi_y|\,S_z\,|\psi_y\rangle = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}
$$

Which is the same as S_x. This is because we rotate the basis clockwise by $\pi/2$ about the x-axis. This implies under this basis rotation, the old z axis is now the new x axis.

Problem 3. Sakurai 1.27

(a) Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A\,|a'\rangle = a'\,|a'\rangle$. Evaluate $\langle b''|\,f(A)\,|b'\rangle$ when the transformation from the a' basis to the b' basis is known.

$$
\begin{aligned}
\langle b''|\,f(A)\,|b'\rangle &= \sum_{a''}\sum_{a'}\langle b''|a''\rangle\,\langle a''|\,f(A)\,|a'\rangle\,\langle a'|b'\rangle\\
&= \sum_{a''}\sum_{a'}\langle b''|a''\rangle\,f(a')\delta_{a,a''}\,\langle a'|b'\rangle\\
&= \sum_{a'} f(a')\,\langle b''|a'\rangle\,\langle a'|b'\rangle
\end{aligned}
$$

(b) Using the continuum analogue of the result obtained in (a), evaluate

$$
\langle \boldsymbol{p''}|\,F(r)\,|\boldsymbol{p'}\rangle
$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are *operators*.

We can start by applying continuum condition:

$$
\begin{aligned}
\langle \boldsymbol{p''} | \, F(r') \, | \boldsymbol{p'} \rangle &= \int F(r') \, \langle \boldsymbol{p''} | \boldsymbol{x'} \rangle \, \langle \boldsymbol{x'} | \boldsymbol{p'} \rangle \, d^3 r' \\
&= \frac{1}{(2\pi\hbar)^3} \int F(r') e^{i(\boldsymbol{p'} - \boldsymbol{p''}) \cdot \boldsymbol{x'}/\hbar} d^3 r'
\end{aligned}
$$

Then, we can use symmetry arguement to simplify the integral. Consider $\boldsymbol{q} \equiv \boldsymbol{p'} - \boldsymbol{p''}$, and $\boldsymbol{q'} \cdot \boldsymbol{x'} = q' r' \cos\theta$, where θ is the angle between $\boldsymbol{q}$ and $\boldsymbol{x'}$. Then the integral becomes:

$$
\begin{aligned}
\int F(r') e^{i\boldsymbol{q} \cdot \boldsymbol{x'}/\hbar} d^3 r' &= 2\pi \int_0^\infty dr' F(r') \int_0^\pi e^{iq'r'\cos\theta/\hbar} \sin\theta d\theta \\
&= 2\pi \int_0^\infty dr' F(r') \int_0^\pi e^{iq'r'\cos\theta/\hbar} d\theta \\
&= 2\pi \int_0^\infty dr' F(r') \frac{2\hbar}{q'r'} \sin\left(q'r'/\hbar\right)
\end{aligned}
$$

Thus the final result is:

$$
\langle \boldsymbol{p''} | \, F(r') \, | \boldsymbol{p'} \rangle = \frac{1}{2\pi^2\hbar^2} \int dr' F(r') \frac{\sin(q'r'/\hbar)}{q'r'}
$$