Assignment 5, Chapter 3 (**due on 13 Oct**): P3; P6; P9; P12;P15

3.3 Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\mathbf{a}}{a_0 - i\boldsymbol{\sigma}\cdot\mathbf{a}},$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

(a) Prove that U is unitary and unimodular.

(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

3.6 Let the Hamiltonian of a rigid body be

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right),$$

where $\mathbf{K}$ is the angular momentum in the body frame. From this expression obtain the Heisenberg equation of motion for $\mathbf{K}$, and then find Euler's equation of motion in the correspondence limit.

3.9 Consider a sequence of Euler rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

3.12 Consider an ensemble of spin 1 systems. The density matrix is now a 3×3 matrix. How many independent (real) parameters are needed to characterize the density matrix? What must we know in addition to $[S_x]$, $[S_y]$, and $[S_z]$ to characterize the ensemble completely?

3.15 **(a)** Let **J** be angular momentum. (It may stand for orbital **L**, spin **S**, or $\mathbf{J}_{\text{total}}$.) Using the fact that $J_x, J_y, J_z (J_\pm \equiv J_x \pm i J_y)$ satisfy the usual angular-momentum commutation relations, prove

$$\mathbf{J}^2 = J_z^2 + J_+ J_- - \hbar J_z.$$

(b) Using (a) (or otherwise), derive the "famous" expression for the coefficient c_- that appears in

$$J_- \psi_{jm} = c_- \psi_{j,m-1}.$$