PHYS 5260 HW10

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November 30, 2022

Problem 1. Sakurai 6.1

The Lippmann-Schwinger formalism can also be applied to a one-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

(a) Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$ How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave only and the orifinal wave for $x < -a$? Is the $E \to E + i\epsilon$ prescription still correct? Obtain an expresssion for the appropriate Green's function and write an integral equation for $\left\langle x \middle| \psi^{(+)} \right\rangle$.

The wave function after scattering is given by:

$$
\begin{aligned}
\left|\psi^{(+)}\right\rangle &= |i\rangle + \frac{1}{E + i\epsilon - H_0} V \left|\psi^{(+)}\right\rangle \\
\left\langle x \middle| \psi^{(+)}\right\rangle &= \langle x|i\rangle + \int dx' \ \langle x| \frac{1}{E + i\epsilon - H_0} |x'\rangle \langle x'| V \left|\psi^{(+)}\right\rangle \\
\left\langle x \middle| \psi^{(+)}\right\rangle &= \langle x|i\rangle + \int dx' \ \langle x| \frac{1}{E + i\epsilon - H_0} |x'\rangle \langle x'| V \left|\psi^{(+)}\right\rangle \\
\psi^{(+)}(x) &= \phi(x) + \int dx' \ \langle x| \frac{1}{E + i\epsilon - H_0} |x'\rangle \langle x'| V \left|\psi^{(+)}\right\rangle \\
\psi^{(+)}(x) &= \phi(x) + \frac{2m}{\hbar^2} \int dx' \ G(x, x') V(x') \psi^{(+)}(x')
\end{aligned}
$$

where $G(x, x') = \frac{\hbar^2}{2m} \langle x| \frac{1}{E+i\epsilon-H_0} |x'\rangle$ is the Green's function.

*LaTeX source code: `https://github.com/rstanuwijaya/hkust-advanced-qm/`

The green function is given by:

$$\begin{aligned}
G(x,x') &= \frac{\hbar^2}{2m}\langle x|\,\frac{1}{E+i\epsilon-H_0}\,|x'\rangle \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dp'\ \langle x|p\rangle\langle p|\,\frac{1}{E-p^2/2m+i\epsilon}\,|p'\rangle\langle p'|x'\rangle \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dp'\ \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\frac{\langle p|p'\rangle}{E-p^2/2m+i\epsilon}\frac{e^{ip'x'/\hbar}}{\sqrt{2\pi\hbar}} \\
&= \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\ e^{ip(x-x')/\hbar}\frac{1}{E-p^2/2m+i\epsilon} \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}dq\ \frac{e^{iq(x-x')}}{(q-q_0)(q+q_0)}
\end{aligned}$$

where we have used the fact that $\langle p|p'\rangle=\delta(p-p')$, $E\equiv\hbar^2k^2/2m$, $p\equiv\hbar q$, $q_0\equiv k(1+i\epsilon)$. This integral can be solved by using contour integration method.

(i) For $x>x'$ and $q\to i\infty$, we have $e^{iq(x-x')}\to e^{-\infty}=0$. Therefore, we can do integration on the upper half plane of the complex q plane. Using the residue theorem, we have:

$$G(x,x') = -\frac{1}{2\pi}(2\pi i)\lim_{q\to+q_0}\frac{e^{iq(x-x')}}{(q+q_0)} = \frac{1}{2ik}e^{ik(x-x')}$$

(ii) Similarly, for $x<x'$:

$$G(x,x') = -\frac{1}{2\pi}(-2\pi i)\lim_{q\to-q_0}\frac{e^{iq(x-x')}}{(q-q_0)} = \frac{1}{2ik}e^{-ik(x-x')}$$

(b) Consider the special case of an attractive δ-function potential

$$V=-\frac{\gamma\hbar^2}{2m}\delta(x)\qquad(\gamma>0)$$

Solve the integral equation to obtain the transmission and reflection amplitudes.

For the given potential $V(x)=-\frac{\gamma\hbar^2}{2m}\delta(x)$, we have:

$$\begin{aligned}
\psi^{(+)}(x) &= \phi(x)+\frac{2m}{\hbar^2}\int dx'\ G(x,x;)V(x')\psi^{(+)}(x') \\
&= \phi(x)-\gamma\int dx'\ G(x,x')\delta(x')\psi^{(+)}(x') \\
&= \phi(x)-\gamma G(x,0)\psi^{(+)}(0) \\
&= \frac{e^{ikx}}{\sqrt{2\pi}}-\gamma G(x,0)\psi^{(+)}(0)
\end{aligned}$$

Where $\psi(0)=\phi(0)/(1+\gamma/2ik)$. Therefore:

$$\begin{aligned}
\psi^{(+)}(x) &= \frac{1}{\sqrt{2\pi}}\left[e^{ikx}-\frac{\gamma}{\gamma+2ik}e^{ikx}\right] \qquad \text{for } x>0 \\
\psi^{(+)}(x) &= \frac{1}{\sqrt{2\pi}}\left[e^{ikx}-\frac{\gamma}{\gamma+2ik}e^{-ikx}\right] \qquad \text{for } x<0
\end{aligned}$$

(c) The one-dimensional δ-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of γ. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when k is regarded as a complex variable.

The transmission and reflection amplitudes are given by:

$$T(k) = \frac{2ik}{\gamma + 2ik}$$
$$R(k) = \frac{-\gamma}{\gamma + 2ik}$$

Problem 2. Sakurai 6.3

Estimate the radius of the ^{40}Ca nucleus from the data in Figure 6.6. and compare to that expected from the empirical value $\approx 1.4A^{1/3}$fm where A is the nuclear mass number. Check the validity of using the first-order Born approximation using this data.

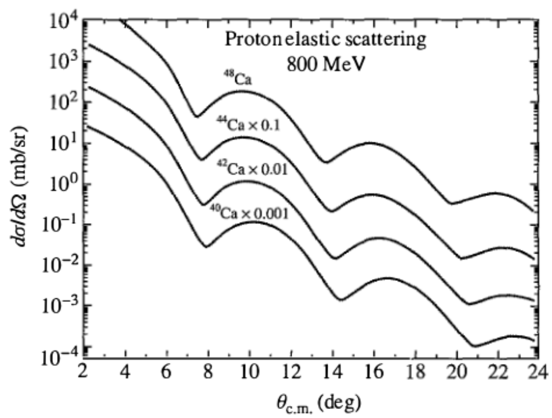

FIGURE 6.6 Data on elastic scattering of protons from the nuclei of four different isotopes of calcium. The angles at which the cross sections show minima decrease consistently with increasing neutron number. Therefore, the radius of the calcium nucleus increases as more neutrons are added, as one expects. From L. Ray et al., *Phys. Rev.* **C23** (1980) 828.

For a square well, the scattering amplitude obtained by using the first-order Born approximation is given by Eq (6.3.7):

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{V_0 a^3}{(qa)^2}\left[\frac{\sin qa}{qa} - \cos qa\right]$$

The first three zeros of this function can be solved numerically, which are: 4.49341, 7.72525, 10.9041

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In[93]:= f[x_] := -1/x^2 (Sin[x]/x - Cos[x])
         NSolveValues[f[x] == 0 && 0 < x < 12, x, Reals]
Out[94]= {4.49341, 7.72525, 10.9041}
```

For ^{4}0Ca, The three minimas are at $\theta = 7.95°, 14.2°, 20.4°$. For 800 MeV energy: $\hbar^2k^2/2m = 800\,\text{MeV} \iff k = 4.32\,(/\text{fm})$. The corresponding scattering radius are:

$$\begin{aligned} q &= 2k\sin\frac{\theta}{2} && \iff q = (1.20, 2.12, 3.08)1/\text{fm} \\ a &= (4.49/1.20, 7.73/2.12, 10.90/3.08)\text{fm} && \iff a = 3.74, 3.64, 3.54\text{fm} \end{aligned}$$

Comparing with the empiricial value $a_{empirical} = 1.4A^{1/3} = 4.79$fm, we see that the first-order Born approximation is not a good approximation. Nevertheless, it is still useful to explain the scattering phenomenon.

Problem 3. Sakurai 6.5

A spinless particle is scattered by a weak Yukawa potential

$$V = \frac{V_0 e^{-\mu r}}{\mu r}$$

where $\mu > 0$ but V_0 can be positive or negative. It was shown in the text that the first-order Born amplitude is given by:

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{V_0}{\mu}\left[\frac{1}{2k^2(1-\cos\theta)+\mu^2}\right]$$

(a) Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for δ_l in terms of a Legendre function of the second kind,

$$Q_l(\zeta) = \frac{1}{2}\int_{-1}^{1}\frac{P_l(\zeta')}{\zeta-\zeta'}d\zeta'$$

First we can begin with Eq.(6.4.40) define $x \equiv \cos\theta$ and integrate both sides to get the orthogonality condition:

$$\begin{aligned}
f(x) &= \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta l}\sin\delta_l P_l(x) \\
\int_{-1}^{1} f(x)P_l(x)dx &= \frac{1}{k}\sum_{l'=0}(2l'+1)e^{i\delta l'}\int_{-1}^{1}\sin\delta_l P_l(x)P_{l'}(x)dx \\
&= \frac{1}{k}\sum_{l'=0}(2l'+1)e^{i\delta l'}\frac{2}{2l'+1}\delta_{ll'} \\
&= \frac{2}{k}e^{i\delta l}\sin\delta_l \\
\int_{-1}^{1} f(x)P_l(x)dx &\approx \frac{2}{k}\delta_l
\end{aligned}$$

On the other hand, we can substitute from the given expression of $f^{(1)}(\theta)$:

$$\begin{aligned}
\frac{2}{k}\delta_l &= -\frac{2m}{\hbar^2}\frac{V_0}{\mu}\int_{-1}^{1}\frac{P_l(x)}{2k^2(1-\cos\theta)+\mu^2}dx \\
&= -\frac{2m}{\hbar^2 k^2}\frac{V_0}{\mu}\frac{1}{2}\int_{-1}^{1}\frac{P_l(x)}{1-x+\mu^2/2k^2}dx \\
\frac{2}{k}\delta_l &= -\frac{V_0}{E\mu}Q_l(\alpha) \\
\delta_l &= -\frac{k}{2\mu}\frac{V_0}{E}Q_l(\alpha)
\end{aligned}$$

where $E \equiv \hbar^2k^2/2m$ and $\alpha \equiv 1 + \mu^2/2k^2$

(b) Use the expansion formula to prove each assertion.

(i) δ_l is negative (positive) when the potential is repulsive (attractive)

(ii) When the de Broglie wavelength is much longer than the range of the potential, δ_l is proportional to k^{2l+1}. Find the proportionality constant.

For the first assertion: we can see it clearly that $\delta_l \propto -V_0$. When $V_0 > 0$, the potential is repulsive, and $\delta_l < 0$. When $V_0 < 0$, the potential is attractive, and $\delta_l > 0$.

For the second assertion: If $k \ll \mu \rightarrow \mu/k \gg 1$. Then $\alpha \approx \mu^2/2k^2 \gg 1$. We can use the expansion formula of $Q_l(\alpha)$:

$$\begin{aligned}
\delta_l &= -\frac{k}{2\mu}\frac{V_0}{E}Q_l(\alpha) \\
&= -\frac{1}{2\mu}\frac{V_0}{E}\frac{l!}{(2l+1)!}\frac{k}{\alpha^{l+1}} \\
&= -\frac{1}{2\mu}\frac{V_0}{E}\frac{l!}{(2l+1)!}\frac{2^{l+1}}{\mu^{2l+2}}k^{2l+3} \\
&= -\frac{2^l l!}{(2l+1)!}\frac{2mV_0}{\hbar^2\mu^{2l+2}}\frac{k^{2l+3}}{k^2} \\
&= -\frac{2^{l+1}l!}{(2l+1)!}\frac{mV_0}{\hbar^2\mu^{2l+3}}k^{2l+1}
\end{aligned}$$

Problem 4. Problem 6.7

Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} 0 & r \geq a \\ \infty & r < a \end{cases}$$

(a) Derive an expression for the s-wave (l=0) phase shift.

(b) What is the total cross section $\sigma = \int (d\sigma/d\Omega) d\Omega$ in the extreme low-energy limit $k \to 0$? Compare your anser with the geometric cross section πa^2.

The condition implies that all wave function must vanish at $r > a$. The wavefunction at $r > a$ is given by Eq.(6.4.52):

$$A_l(r) = e^{i\delta_l} \left[\cos\delta_l j_l(ka) - \sin\delta_l n_l(ka)\right]$$

The condition $A_l(r) = 0$ gives:

$$\tan\delta_l = \frac{j_l(ka)}{n_l(ka)}$$

For s-wave ($l = 0$), we have:

$$\tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = \frac{\sin(ka)/ka}{-\cos(ka)/ka} = -\tan(ka)$$

Therefore $\delta_0 \propto -ka$:

$$\begin{aligned} A_{l=0}(r) &= e^{i\delta_0} \left(\frac{\sin(kr)}{kr} \cos(kr) + \frac{\cos(kr)}{kr} \sin(kr) \right) \\ &\propto \frac{e^{i\delta_0}}{kr} \left(\sin(kr + \delta_0)\right) \\ &\propto \frac{e^{i\delta_0}}{kr} \left(\sin(kr - ka)\right) \end{aligned}$$

In the extreme low-energy limit. For $k \ll 1$:

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} \approx a^2$$

The total cross section is:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi a^2$$

which is exactly four times the geometric cross section.