Reference List on Discrete Mathematics for Systems Science

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1 Introduction: What is this odd document?

This document provides a summary reference list of some of the most fundamental concepts of discrete math relevant for the core methods in mathematical systems theory.

In its entirety, the scope of the material listed below could actually comprise a reasonable syllabus for an entire undergraduate course in discrete math. Moreover, each *section* below could *also* be its own course. Remarkably, discrete math, which is most appropriate for the systems, data and information, and computational sciences (as distinct from the "continuous" "calculus track" of standard undergraduate math which is most appropriate for the physics and engineering sciences), suffers (benefits?) from a large number of distinct topics which reference each other in complex ways. This results in a tremendous variation in how such syllabi are developed by different instructors and presented in different textbooks or other pedagogical corpora. As such, this document represents not only a distinct selection of only certain ideas, but also only one way of ordering them to maximize the "linearity" of the development to limit "fowrward" or "cross-" references.

Thus this is not intended to be comprehensive: many other concepts important for certain kinds of research are not included, and also there are many alternate ways of developing, structuring, and notating the concepts listed here.

Most significantly, this is only a *list* of topics, ideas, and terms. While examples are usually offered, there is effectively no exposition or discussion. The intended use is for a student to have a reference list of concepts they can then use to calibrate their self-exploration through other available pedagogical sources, for example wikipedia, online videos, courses, or texts.

The exception is Sec. 9, which has a slightly more discursive development of core ideas specifically in order and lattice theory. This is because it's derived from course notes developed for a different independent study, and just copied into this document.

Additionally, a course following this list or outline would best be done with a strong computational component. Not only do these topics have ample online pedagogical resources, but also effectively everything included here is supported by e.g. python language constructs (e.g. for collections) or libraries (e.g. NetworkX for graphs).

Even more strongly, my personal perspective and style is grounded in Sec. 2 to think of the two computer-science based concepts of **collection types**, specifically sets and lists (tuples), as being foundational to all other subseqent discrete math concepts. These are directly supported by primitive data types and operations in python.

The material of discrete math being so manifold results in dependence on notation which is both heavy and extensive. And because its components are so heavily interrelated, developing notation which can be used consistently from one topic to another becomes a challenge. The result is that notation is not always standardized, or, if there is standard notation used for one concept, the next topic to which it's closely related might have its own, inconsistent, but also standard, notation. Thus the sections below are not necessarily consistent in structure, example, or notation, although I've tried to be consistent within each section.

"$:=$" means "is defined as". And unless otherwise noted, all objects are finite and non-empty.

This document is not guaranteed to be free of errors.

Finally, in some cases I have developed or adopted notation, concepts, or structures distinctly from what I've typically seen in the literature. Where this is the case I've indicated so much by typesetting in red. Everything else in black should be all over the web pretty much as is here.

2 Collections: Sets and Tuples

Almost all discrete math concepts rely on expressions in terms of different kinds of collections, particularly sets and lists (vectors). See corresponding python structures.

	Duplicates not allowed	Duplicates allowed
Unordered	Set	Multiset, bag
Ordered	Totally ordered set (see Sec. 9)	List, vector, sequence, tuple

- **Sets** are unordered collections without duplicates
  - Extensional (by roster): E.g. $A = \{x, y, 3\}; \mathbb{Z} = \{\dots, -2, -1; 0, 1, 2, \dots\}; \{2, 4, 6\} = \{2i\}_{i=1}^{3}$.
  - Elements: e.g. $x \in A, 4 \in \mathbb{Z}$.
  - Universal quantifier: $\forall x \in X :=$ for all $x \in X$.
  - Existential quantifier: $\exists x \in X :=$ there is an $x \in X$.
  - Intensional (by rule): e.g. evens $:= \{z \in \mathbb{Z} : \ z/2 \in \mathbb{Z}\} = \{\dots, -4, -2, 0, 2, 4, \dots\}; [n] = \{i \in \mathbb{Z} : \ 1 \leq i \leq n\} = \{1, 2, \dots, n\}$ for $n \in \mathbb{Z}, n \geq 1$.
  - Cardinality, size: $|A| = 3, |\mathbb{Z}| = \infty$.
  - Subsets: $\{y, 3\} \subseteq A; \{1, 2, 3\} \subseteq \mathbb{Z}$; evens $\subseteq \mathbb{Z}$.
  - Singleton subset: $|X| = 1$, e.g. $\{3\} \subseteq A$.
  - Union: $A \cup \{q, r\} = \{x, y, 3, q, r\}$.
  - Intersection: $A \cap \mathbb{Z} = \{3\}$.
  - Empty set: $\emptyset := \{\}, |\emptyset| = 0$; for all sets $X, \emptyset \subseteq X$.
  - Set difference: $X \setminus Y := \{x \in X : \ x \notin Y\}$.
  - Complementation: Given $X \subseteq Y$, then $\bar{X} := Y \setminus X$.
- **"Hyperset", Class:** A set of sets: $\{\{a, b, c\}, \{1, 2, 3\}\}; \{A, \mathbb{Z}\}$.
- **Multisets, bags:** Unordered collections with duplicates allowed. $A = \{a, b, b, c\} = \{a/1, b/2, c/1\}$.
- **Lists, a.k.a. vectors, tuples, sequences, ordered tuples (ordered pairs, ordered triples):** An ordered collection with duplicates allowed,
  - Extensional: $\langle a, b, c, a\rangle ; \langle \cos\theta\rangle_{\theta\in\{0,180,360\}} = \langle 1, -1, 1\rangle$.
  - Concatenation: $\langle a, 1, 2\rangle + \langle 3, 2, a, b\rangle = \langle a, 1, 2, 3, 2, a, b\rangle$.
- **Other observations:**
  - Set of sets: hyperset
  - Set of lists: $\{\langle 1, 2\rangle , \langle 3, 4\rangle\}$
  - List of sets: $\langle\{x, y, z\}, \mathbb{Z}, \mathbb{Z}\rangle$
  - List of lists: $\langle\langle x, y\rangle , \langle 1, 2\rangle\rangle$

3 Some Sets of Numbers

Booleans: $\mathbb{B} := \{0, 1\}$.

Natural Numbers: $\mathbb{N} := \{1, 2, \ldots\}$.

Whole Numbers: $\mathbb{W} := \{0, 1, 2, \ldots\}$.

Integers: $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Rational Numbers: $\mathbb{Q} := \{i/j : i \in \mathbb{W}, j \in \mathbb{N}\}$.

Real Numbers: $\mathbb{R}$.

$\mathbb{B} \subseteq \mathbb{W}$

$\mathbb{N} \subseteq \mathbb{W}$

$\mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

4 Operations of Linear Algebra

- **Scalar:** 0-dimensional, a thing, typically a number, $x = 3$.
- **(Row) Vector:** 1-dimensional, a list of numbers, $\mathbf{x} = \vec{x} = \langle x_1, x_2, \ldots, x_n \rangle = \langle x_i \rangle_{i=1}^{n}, n = |\vec{x}|$, e.g. $\vec{x} = \langle 1, 2, 3 \rangle, \vec{y} = \langle 4, 5, 6 \rangle, n = 3$.
  - Column vector: $\vec{x}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
  - Dot product: when $|\vec{x}| = |\vec{y}| = n$, then the dot product is the scalar $\vec{x} \cdot \vec{y}^T := \sum_{i=1}^{n} x_i y_i$.

$$\vec{x} \cdot \vec{y}^T = \langle 1, 2, 3 \rangle \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32.$$

- **Matrix:** 2-dimensional, a rectangular array of numbers with n rows and m columns. $A_{n \times m} = [a_{ij}], i \in [n], j \in [m]$.

$$A_{2 \times 3} = \begin{pmatrix} 1, 2, 3 \\ 4, 5, 6 \end{pmatrix}$$

  - Transpose: $A^T_{m \times n}$ where $a^T_{i,j} = a_{j,i}$.

$$A^T_{3 \times 2} = \begin{pmatrix} 1, 4 \\ 2, 5 \\ 3, 6 \end{pmatrix}$$

  - Row decomposition: A column vector of row vectors:

$$\vec{A}_{i,\bullet} := \langle a_{i,j} \rangle_{j=1}^{m}, \quad A = \begin{pmatrix} \vec{A}_{1,\bullet} \\ \vec{A}_{2,\bullet} \\ \cdots \\ \vec{A}_{n,\bullet} \end{pmatrix}$$

$$A = \begin{pmatrix} \langle 1,2,3 \rangle \\ \langle 4,5,6 \rangle \end{pmatrix}$$

– Column decomposition: A row vector of column vectors:

$$\vec{A}_{\bullet,j} := \langle a_{i,j} \rangle_{i=1}^{n}, \quad A = \left\langle \vec{A}_{\bullet,1}^T, \vec{A}_{\bullet,2}^T, \ldots, \vec{A}_{\bullet,m}^T \right\rangle$$

$$A = \left\langle \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\rangle$$

- **Matrix multiplication:** Assuming $A_{n\times m}$ and $B_{m\times p}$. Then

$$A \times B = AB := C_{n,p} = [c_{ik}], i \in [n], k \in [p],$$

where

$$c_{ik} = A_{i,\bullet} \cdot B_{\bullet,k}^T = \sum_{j=1}^{m} a_{i,j} b_{j,k}. \tag{1}$$

Let $B_{3\times 2} = \begin{pmatrix} 1,0 \\ 2,1 \\ 1,2 \end{pmatrix}$. Then using A from above, we have

$$C_{2\times 2} = A \times B = AB = \begin{pmatrix} 1,2,3 \\ 4,5,6 \end{pmatrix} \times \begin{pmatrix} 1,0 \\ 2,1 \\ 1,2 \end{pmatrix} = \begin{pmatrix} 7,20 \\ 8,17 \end{pmatrix}.$$

Note that $AB \neq BA$.

- **Vector-array multiplication:** Assume $\vec{x}$ with $|\vec{x}| = n$, and $A_{n\times m}$. Then $\vec{x}A$ and $A\vec{x}^T$ are derived as expected, given that $\vec{x}_{n\times 1}$ is also an $n \times 1$ matrix. And again, $\vec{x}A \neq A\vec{x}^T$.

- **Tensor:** an N-dimensional array of numbers, for $N \geq 0$. See multi-dimensional arrays in python.

  – A scalar is a 0-tensor. It cannot be decomposed into anything.
  – A vector is a 1-tensor. It can be decomposed into one set of scalars (0-dimensional tensors).
  – A matrix is a 2-tensor. It can be decomposed into two sets of vectors (1-dimensional tensors).
  – A 3-tensor is a cubic array of numbers. It can be decomposed into three sets of matrices (2-dimensional tensors).
  – A (general) tensor is an N-dimensional rectangular array of scalars, for $N \geq 0$. It can be decomposed into N sets of $N-1$-dimensional tensors. Thus it's safe to say that a -1-dimensional tensor is nothing.

5 Relations

Assume a hyperset (set of sets) $\mathcal{X} = \{X_l\}_{l=1}^N$, e.g. $\mathcal{X} = \{X, Y, Z\}$ so $N = 3$ with $X = \{x_i\}_{i=1}^n, Y = \{y_j\}_{j=1}^m, Z = \{z_k\}_{k=1}^p$. In the actual examples, let

$$X = \{a, b, c\}, \quad Y = \{1, 2\}, \quad Z = \{\alpha, \beta\}$$

so that $n = 3$ and $m = p = 2$.

- **Cartesian product, cross-product:**
  - Binary case: $X \times Y = \{\langle x, y\rangle \, : \, x \in X, y \in Y\}$.
    $Y \times Z = \{\langle 1, \alpha\rangle, \langle 1, \beta\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle\}$.
  - Ternary case: $X \times Y \times Z = \{\langle x, y, z\rangle \, : \, x \in X, y \in Y, z \in Z\}$
  - General case:

    $$\mathbf{X} := \bigtimes_{l=1}^{N} X_l = \bigtimes_{X_l \in \mathcal{X}} X_l = X_1 \times X_2 \times \ldots \times N_N = \{\vec{x}\}, \quad \text{where } \vec{x} = \langle x_l\rangle_{l=1}^N \in \mathbf{X}, x_l \in X_l.$$

- **Self-product:** Ordered pairs $X \times X = X^2 = \{\langle x, x'\rangle \, : \, x, x' \in X\}$.
  $Z \times Z = Z^2 = \{\langle \alpha, \alpha\rangle, \langle \alpha, \beta\rangle, \langle \beta, \alpha\rangle, \langle \beta, \beta\rangle\}$.
- **Binary relation:** $R = \{\langle x, y\rangle\} \subseteq X \times Y$ e.g.

  $$R = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle a, 2\rangle\} \subseteq X \times Y.$$

- **Boolean (characteristic) matrix form:** $R_{n\times m} = [r_{ij}], n = |X|, m = |Y|$

  $$r_{i,j} = \begin{cases} 1, & \text{if } \langle x_i, y_j\rangle \in R \\ 0, & \text{if } \langle x_i, y_j\rangle \notin R \end{cases} \in \mathbb{B}.$$

  $$R_{3\times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{array}{c|cc} & 1 & 2 \\ \hline a & 1 & 1 \\ b & 0 & 1 \\ c & 0 & 0 \end{array}$$

- **Binary relation composition:** Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then $R \circ S \subseteq X \times Z$ where $\langle x, z\rangle \in R \circ S$ when there is a $y \in Y$ such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$. E.g. let

  $$S = \{\langle 1, \alpha\rangle, \langle 2, \alpha\rangle, \langle 2, \beta\rangle\} \subseteq Y \times Z$$

  so that

  $$S_{2\times 2} = \begin{array}{c|cc} & \alpha & \beta \\ \hline 1 & 1 & 0 \\ 2 & 1 & 1 \end{array}.$$

  Then using R from above, we have

  $$(R \circ S)_{3\times 2} = \begin{array}{c|cc} & \alpha & \beta \\ \hline x & 1 & 1 \\ y & 1 & 1 \\ z & 0 & 0 \end{array}$$

Note that in terms of their characteristic matrices, $(R \circ S) = RS$, except that instead of (1), we use

$$(R \circ S)_{ik} = \operatorname{sgn}\left(\sum_{j=1}^{m} r_{ij} s_{jk}\right), \tag{2}$$

that is, $(R \circ S)_{ik} = 1$ if $RS_{ik} > 0$.

- **General relation, multi-relation, a.k.a. system:** $R \subseteq \mathbf{X}$

  Representable as a Boolean characteristic N-tensor.

- **Self relation, homogeneous relation:** Binary relation when $Y = X$, so that $S = \{\langle x, x' \rangle\} \subseteq X \times X = X^2$, e.g.

  $$S = \{\langle a, a\rangle, \langle b, c\rangle, \langle a, b\rangle\} \subseteq X^2.$$

  For a self-relation $S \subseteq X^2$, we can also write xSy for $\langle x, y\rangle \in S$. E.g.

  $$S_{3\times 3} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 1 & 0 \\ b & 0 & 0 & 1 \\ c & 0 & 0 & 0 \end{array}$$

  Characteristic matrices of self-relations are always square.

  - Reflexive: For all $x \in X$, $\langle x, x\rangle \in S$.
  - Symmetric: For all $x, y \in X$, if $\langle x, y\rangle \in S$, then $\langle y, x\rangle \in S$.
  - Transitive: For all $x, y, z \in X$, if $\langle x, y\rangle, \langle y, z\rangle \in S$, then $\langle x, z\rangle \in S$.
  - Equivalence relation: Reflexive, symmetric, and transitive.
  - Partial order: Reflexive, anti-symmetric, transitive.
  - See Table 1 for more types of self-relations, including graphs.

6 Functions

A binary relation $R \subseteq X \times Y$ is a function from X to Y when for each $x \in X$ there is always exactly one $y \in Y$ such that $\langle x, y\rangle \in R$. We then call X the **domain**, Y the **co-domain**, write $R\colon X \to Y$ or $X \xrightarrow{R} Y$, and write $R(x) = y$ when $\langle x, y\rangle \in R$.

Relations are commonly notated in symbols or Roman capitals, while functions are typically notated in Roman lowercase. E.g. consider the binary relation $f \subseteq X \times Y$ with

$$f = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 2\rangle\} \subseteq X \times Y$$

and matrix form

$$f_{3\times 2} = \begin{array}{c|cc} & 1 & 2 \\ \hline a & 1 & 0 \\ b & 0 & 1 \\ c & 0 & 1 \end{array}$$

then $f\colon X \to Y$ and $f(a) = 1, f(b) = f(c) = 2$.

- **Injective:** one-to-one: Not only is there exactly one $y \in Y$ for each $x \in X$, but also at most one x for each y. f above is not injective, since $f(b) = f(c) = 1$.

  For $f: X \to Y$ to be injective, $m \geq n$.

- **Surjective:** onto: For all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. f above is surjective.

- **Bijective:** Both injective and surjective.

- **Inverse:** if $f: X \to Y$ is bijective, then define $f^{-1}: Y \to X$ where $f^{-1}(y) = x$ such that $f(x) = y$. The characterstic matrix of the inverse of a function is the transpose of its charactestic matrix:
$$f^{-1}_{n\times n} = (f_{n\times n})^T.$$

- **Composition:** If $f: X \to Y$ and $g: Y \to Z$ then $g \circ f: X \to Z$ where $(g \circ f)(x) = g(f(x)) \in Z$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

  The characteristic matrix of the composition of two functions is the composition of the two characteristic matrices:
$$(g \circ f)_{n\times p} = f_{n\times m} \circ g_{m\times p}.$$
  Note that the composition is $g \circ f$, not, as one might think, $f \circ g$, because.

- **Boolean functions:**

  - Not: $\neg : \mathbb{B} \to \mathbb{B}, \neg(b) = 1 - b$
  - Conjunction: $\text{and}: \mathbb{B}^2 \to \mathbb{B}, \text{and}(b_1, b_2) = b_1 \text{ and } b_2 = \min(b_1, b_2)$
  - Disjunction: $\text{or}: \mathbb{B}^2 \to \mathbb{B}, \text{or}(b_1, b_2) = b_1 \text{ or } b_2 = \max(b_1, b_2)$
  - Many more by composition, see https://en.wikipedia.org/wiki/Boolean_function, e.g. nand = "not and", $\text{nand}: \mathbb{B}^2 \to \mathbb{B}, \text{nand}(b_1, b_2) = \neg(b_1 \text{ and } b_2) = 1 - \min(b_1, b_2)$

7 Graphs

Assume a finite, non-empty set of vertices $V = \{v\}$, and a self-relation $A \subseteq V^2$.

- **Directed graph:** Draw an arrow from tail v to head u when $\langle v, u\rangle \in A$. Call each such ordered pair $\langle v, u\rangle$ an **arc**, and say $G = \langle V, A\rangle$.

- **Undirected graph, or just "graph":** A directed graph where A is symmetric, so that $\langle v, u\rangle \in A$ if and only if $\langle u, v\rangle \in A$. Now call the set $e = \{v, u\} \subseteq V$ an **edge**, and say $G = \langle V, E\rangle$ where $E = \{e\}$. We draw a line segment from v to u with no arrowheads.

- **Adjacency:** Vertices $v, u \in V$ are adjacent when $\exists e = \{v, u\} \in E$.

- **Incidence:** Edges $e, f \in E$ are incident or overlapping when $e \cap f = \{v\} \neq \emptyset$ for some $v \in V$.

- **Walk:** A sequence of vertices $v_1, v_2, \ldots, v_n$ is a walk when $\forall 1 \leq i \leq n-1, \{v_i, v_{i+1}\} \in E$.
- **Cycle:** A walk where $v_1 = v_n$.
- **Tree:** An undirected graph with no cycles.
- **Clique, complete graph:** Contains all possible edges, $\forall u, v \in V, \{v, u\} \in E$.

8 Set Systems

Assume a set X of points, e.g. $X = \{a, b, c\}, n = |X| = 3$.

- **Power Set:** A hyperset, set of all subsets of another set X, $2^X := \{Y : Y \subseteq X\}$
  - $\emptyset \in 2^X; X \in 2^X; \forall x \in X, \{x\} \in 2^X$; etc.
  - 2^X is isomorphic to a Boolean lattice: $X \vee Y = X \cup Y, X \wedge Y = X \cap Y, \bar{X}$, see Sec. 9.
- **Class, Undirected Hypergraph:** Set of subsets $\mathcal{C} \subseteq 2^X$, e.g. $\mathcal{C} = \{C_1, C_2\} = \{\{a\}, \{a, b\}\}$, where $C_j \subseteq X, m = |\mathcal{C}| = 2$.
- **Incidence relation, matrix:** A class $\mathcal{C} \subseteq 2^X$ determines a Boolean incidence relation $R \subseteq X \times \mathcal{C}$ on points and subsets, with matrix $R_{n \times m} = [r_{i,j}], i \in [n], j \in [m] = [|\mathcal{C}|]$, where

$$r_{i,j} = \begin{cases} 1, & \text{if } x_i \in C_j \\ 0, & \text{if } x_i \notin C_j \end{cases} .$$

- **Cover:** Class which covers X, $\mathcal{C} \subseteq 2^X$ where $\bigcup_{C_j \in \mathcal{C}} C_j = X$. E.g. $\mathcal{C} = \{\{a\}, \{a, b, c\}\}$.
- **Partition:** Cover with all disjoint sets. $\mathcal{C} \subseteq 2^X$ where $\bigcup_{C_j \in \mathcal{C}} C_j = X$ and $\forall C, C' \in \mathcal{C}, C \cap C' = \emptyset$. E.g. $\mathcal{C} = \{\{a\}, \{b, c\}\}$.

  A partition $\mathcal{C} \subseteq 2^X$ establishes an equivalence relation $\sim \subseteq X^2$, where $x \sim y$ when $\exists C_j \in \mathcal{C}, x, y \in C_j$.

9 Partial orders

This section was derived from a prior source using a different style.

Posets: Let P be a finite, non-empty set with $|P| \geq 2$, and let $\leq \subseteq P^2$ be a partial order on P: a reflexive, anti-symmetric, transitive binary relation. Then $\mathcal{P} = \langle P, \leq \rangle$ is a poset. For any pair of elements $a, b \in P$, we say $a \leq b \in P$ to mean that $a, b \in P$ and $a \leq b$. $a < b$ means that $a \leq b$ and $a \neq b$.

Dual Order: Saying $a \geq b$ means that $b \leq a$. Given a poset $\mathcal{P} = \langle P, \leq \rangle$, then the poset $\mathcal{P}^* = \langle P, \geq \rangle$ is its **dual**.

Antichains: For $a, b \in P$, if neither $a \leq b$ nor $b \leq a$ then we say that a and b are **incomparable**, denoted $a \| b$. A set of elements $A \subseteq P$ is an **anti-chain** if $\forall a, b \in A, a \| b$.

Chains: If $a \leq b$ or $b \leq a$ then we say that a and b are **comparable**, denoted $a \sim b$. A set of elements $C \subseteq P$ is a **chain** if $\forall a, b \in C, a \sim b$. If P is a chain, then $\mathcal{P}$ is called a **total** or **linear order**. A chain $C \subseteq P$ is **maximal** if there is no other chain $C' \subseteq P$ with $C \subseteq C'$. The **height** $\mathcal{H}(\mathcal{P})$ of a poset is the size of its largest maximal chain.

Cover Relations: For $a, b \in P$, let $a \prec b$ be the **covering relation** (**immediate predecessor**) when $a < b$ and there is no $c \in P$ with $a < c < b$. A chain is called **saturated** when $C = \{a_i\}_{i=1}^{|C|} \subseteq P$ can be sorted by $<$ and written as $C = a_1 \prec a_2 \prec \ldots \prec a_{|C|}$.

Bounds: For any subset of elements $Q \subseteq P$, let its maximal and minimal elements, defined as

$$\mathrm{Max}(Q) := \{a \in Q : \nexists b \in Q, a < b\} \subseteq Q$$

$$\mathrm{Min}(Q) := \{a \in Q : \nexists b \in Q, a > b\} \subseteq Q,$$

be called the **roots** and **leaves** respectively. When $\mathcal{P}$ has a single root (resp. leaf) so that $|\mathrm{Max}(P)| = 1$ (resp. $|\mathrm{Min}(P)| = 1$), then let $\mathrm{Max}(P) = \{\top\}$ (resp. $\mathrm{Min}(P) = \{\bot\}$) so that $\mathcal{P}$ is called **top-bounded** with **top** $\top \in P$ being that single root (resp. **bottom-bounded** with **bottom** $\bot \in P$ being that single leaf). If $\top, \bot \in P$ then well call $\mathcal{P}$ **bounded**.

Meet Sets and Join Sets: Let $J \colon 2^P \to 2^P$ be the **join set**, and $M \colon 2^P \to 2^P$ be the **meet set**, function with, for $Q \subseteq P$,

$$J(Q) := \mathrm{Min}\left(\bigcap_{a \in Q} \uparrow a\right) \subseteq P, \quad M(Q) := \mathrm{Max}\left(\bigcap_{a \in Q} \downarrow a\right) \subseteq P.$$

For $Q = \{a, b\} \subseteq P$, then we say

$$J(a, b) := \mathrm{Min}(\uparrow a \cap \uparrow b) \subseteq P, \quad M(a, b) := \mathrm{Max}(\downarrow a \cap \downarrow b) \subseteq P.$$

Lattices: If for any $a, b \in P, J(a, b) = \{c\}$ (resp. $M(a, b) = \{c\}$) for some unique $c \in P$, then we denote $J(a, b) = a \vee b = c$ (resp. $M(a, b) = a \wedge b = c$), we say that a and b "have a join" (resp. "have a meet"), and we call c just the **join** (resp. **meet**) of a and b. These relationships for $a \| b \in P$ in a lattice are illustrated in an abstraction in Fig. 1.

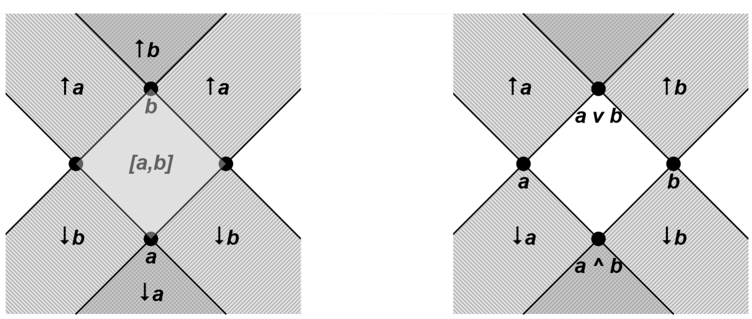

Figure 1: Abstraction of relationships between filters, ideals, joins, and meets of two noncomparable elements of a lattice.

If for all $a, b \in P$, a and b have a join (resp. meet), then we have $\vee \colon P^2 \to P$ (resp. $\wedge \colon P^2 \to P$) as a binary operator on P, and $\mathcal{P}$ is called a **join** (resp. **meet**) **semi-lattice**. If $\mathcal{P}$ is both a join and a meet semi-lattice, then we call $\mathcal{P} = \langle P, \leq \rangle$ a **lattice**, denoted $\mathcal{L} = \langle \mathcal{P}, \vee, \wedge \rangle$.

	Properties				Symbol	Example
	Reflexivity	Symmetry	Transitivity	Connectedness		
Directed graph					$\to$	
Undirected graph		✓				
Preorder	✓		✓		$\leq$	Preference
Total preorder	✓		✓	✓	$\leq$	
Partial order	✓	×	✓		$\leq$	Subset
Strict partial order	×	×	✓		$<$	Strict subset
Total order	✓	×	✓	✓	$\leq$	Alphabetical order
Strict total order	×	×	✓	✓	$<$	Strict alphabetical order
Partial equivalence relation		✓	✓			
Equivalence relation	✓	✓	✓		$\sim, \equiv$	Equality

Table 1: Types of self-relations (homogeneous relations). Adapted by claude.ai from https://en.wikipedia.org/wiki/Homogeneous_relation#Operations. $\surd$ = has that property, $\times$ = has that anti-property.

Boolean Lattice: Assume a finite lattice $\mathcal{L} = \langle \mathcal{P}, \vee, \wedge \rangle$. Every finite lattice is bounded, so $\bot, \top \in P$.

- Complement: An element $x \in P$ has a complement $\bar{x} \in P$ if $x \vee \bar{x} = \top$ and $x \wedge \bar{x} = \bot$.
- Complemented lattice: A lattice $\mathcal{L}$ is complemented if every element has a complement.
- Distributive lattice: A lattice $\mathcal{L}$ is distributed if $\forall x, y, z \in P$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It follows that also $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- Boolean lattice: A lattice $\mathcal{L}$ is Boolean if it is both complemented and distributive. In a Boolean lattice, all complements are unique.

Concepts from Boolean functions, set systems, and logic unify in a Boolean lattice.

	Boolean functions	Power set 2^X	Logic	Boolean lattice
True	1	X	true	$\top$
False	0	$\emptyset$	false	$\bot$
Negation	$1 - x$	$\bar{A}$	not	$\bar{x}$
Disjunction	$\max(x, y)$	$A \cup B$	or	$x \vee y$
Conjunction	$\min(x, y)$	$A \cap B$	and	$x \wedge y$
If-then	$\max(1 - x, y)$	$A \subseteq B$	implies	$x \leq y$

And for binary relation composition, (1) and (2) generalize to

$$(R \circ S)_{ik} = \bigvee_{j=1}^{m} r_{ij} \wedge s_{jk}$$

where now $\vee$ is taken as max and $\wedge$ as min.