# Ordinary Differential Equations - 104131

**H**omework No. 2

1. Plot schematically the direction fields and some typical solutions of the following differential equations:

   (a) $y' = y(y-1)(y+1)$.

   (b) $y' = y(y-1)^2(y+1)$.

2. Find families of curves which are orthogonal to the following given families:

   (a) The family of hyperbolas $xy = C$.

   (b) The family of circles $(x-C)^2 + y^2 = C^2$.

   Solution: (a) We differentiate and obtain $y + xy' = 0$ which is the differential equation corresponding to the family $xy = c$. Then the equation of the orthogonal family (obtained by replacing $y'$ by $-\frac{1}{y'}$) is $y + x\left(-\frac{1}{y'}\right) = 0$ or $y' = \frac{x}{y}$. This is a separable equation with no constant solutions and so

$$\int y dy = \int x dx$$
$$\frac{y^2}{2} = \frac{x^2}{2} + c$$
$$y^2 = x^2 + c$$
$$y = \pm\sqrt{x^2 + c}.$$

   (b) We rewrite the family as $x^2 - 2Cx + y^2 = 0$. We differentiate and obtain $2x - 2C + 2yy' = 0$ or $x - C + yy' = 0$ or $C = x + yy'$.

$$x^2 - 2Cx + y^2 = 0$$
$$C = x + yy'.$$

   Substituting the second equation into the first we obtain the differential equation of the family of circles $x^2 - 2(x + yy')x + y^2 = 0$ or $x^2 - 2x^2 - 2xyy' + y^2 = 0$ or $x^2 - 2xyy' + y^2 = 0$. Then the equation of the orthogonal family (obtained by replacing $y'$ by $-\frac{1}{y'}$) is $-x^2 - xy\left(-\frac{1}{y'}\right) + y^2 = 0$ or $y' = \frac{x^2-y^2}{xy} = \frac{x}{y} - \frac{y}{x}$. This is a (nonlinear) homogeneous equation and so we substitute $v = \frac{y}{x}$ or $y = xv$. Derivating this gives $y' = v + xv' = \frac{1}{v} + v$ or $v' = \frac{1}{xv}$. This is a separable equation with no constant solutions.

$$\begin{aligned}
&\int v dv = \int \frac{1}{x} dx \\
&\frac{y^2}{2} = \ln|x| + c \\
&y = \pm\sqrt{\ln x^2 + c}.
\end{aligned}$$

3. Without solving the initial value problem

$$\frac{dy}{dx} = \frac{(y^2-1)\sin y}{y^2+1}, \qquad y(5) = 3,$$

explain why its solution $y(x)$ is bounded from above and from below for all values of $x$. What are the bounds?

Solution: I will restate the existence and uniqueness theorem: Given an ODE $y' = f(x,y)$ and $(x_0, y_0) \in \mathbb{R}^2$, if $f(x,y), f'_y(x,y)$ are continuous in a neighbourhood of $(x_0, y_0)$, then there is exactly one solution whose graph passes through $(x_0, y_0)$ and is defined in $(x_0 - h, x_0 + h)$. Consequently, no two different solutions pass through $(x_0, y_0)$.
This means that if we look at an equation $y' = f(x,y)$ (no initial condition!) for whom $f, f'_y$ are continuous for all $(x,y) \in \mathbb{R}^2$, then no two solutions intersect at all. ever! Because if $y_1(x), y_2(x)$ intersect, then they intersect at some point $(x_0, y_0)$ meaning $y_0 = y_1(x_0) = y_2(x_0)$. However, $f, f'_y$ are continuous at a neighbourhood of $(x_0, y_0)$ and therefore no two solutions pass through $(x_0, y_0)$. meaning $y_1(x) = y_2(x)$.
Since $f(x,y) = \frac{(y^2-1)\sin y}{y^2+1}$ is continuous for all $(x,y) \in \mathbb{R}^2$ and similarly $f'_y(x,y) = \frac{(2y\sin y + (y^2-1)\cos y)(y^2+1) - (y^2-1)(\sin y)2y}{(y^2+1)^2}$ is continuous for all $(x,y) \in \mathbb{R}^2$, then no two different solutions intersect. ever! Now, it is clear that there are constant solutions to the equation and they are $y \equiv \pm 1$ and $y \equiv k\pi$ where $k \in \mathbb{Z}$. Denote $y_1(x) \equiv 1$ and $y_2(x) \equiv \pi$. Since the solution $y(x)$ to the equation and the initial condition satisfies $y_1(5) = 1 < y(5) = 3 < \pi =$

$y_2(5)$ then it is clear that $y(x)$ is not the solution $y \equiv 1$ nor the solution $y \equiv \pi$. $y(x)$ also doesnt intersect them and since at $x = 5$ it is bounded by $y_1$ from below and by $y_2$ from above, then $1 = y_1(x) < y(x) < y_2(x) = \pi$.

4. Find explicitly two different solutions of the initial value problem

$$\frac{dy}{dx} = \sqrt{1-y^2}, \qquad y(0) = 1.$$

Why this does not contradict the existence and uniqueness theorem?

Solution: This is a separable equation and the constant solutions are $y \equiv \pm 1$. Clearly $y \equiv 1$ satisfies the initial condition.

$$\begin{aligned} &\int \frac{1}{\sqrt{1-y^2}} dy = \int 1 dx \\ &\arcsin y = x + c \\ &y = \sin(x + c). \end{aligned}$$

From the initial condition we get $1 = y(0) = \sin(0 + c)$ and so $c = \frac{\pi}{2} + k2\pi$ and so $y = \sin(x + \frac{\pi}{2} + k\pi)$ are solutions. However, since the $\sin x$ function has period $2\pi$ then all the solutions are really one solution $y(x) = \sin(x+\frac{\pi}{2})$. This is in addition to the solution $y \equiv 1$.

This doesnt contradict existence and uniqueness because $f'_y(x, y) = -\frac{2y}{\sqrt{1-y^2}}$ is not continuous at $(0, 1)$.

For advanced students: In fact, $\sin(x+\frac{\pi}{2})$ is not a solution because when we substitute it into the equation we obtain $\cos(x + \frac{\pi}{2}) = \sqrt{1 - \sin^2(x + \frac{\pi}{2})} = \sqrt{\cos^2(x + \frac{\pi}{2})} = |\cos(x + \frac{\pi}{2})|$. The two sides of the equation are not equal when $x > 0$. What we do is sew two functions: Define

$$y(x) = \begin{cases} 1 & x \geq 0 \\ \sin(x + \frac{\pi}{2}) & x < 0. \end{cases}$$

It is easy (though long) to check that this is in fact a solution and in addition to $y \equiv 1$ give us two different solution for the initial value problem.

5. In a previous exercise we solved the equation $y' = \dfrac{4y - 3x}{2x - y}$.

(a) Show (again) that it has two singular solutions $y = x$ and $y = -3x$.

(b) The two singular solutions intersect at $(0, 0)$. Does this fact contradict the existence and uniqueness theorem?

(c) Show that every solution which starts at an initial point between $y = x$ and $y = -3x$, stays always between these two straight lines.

Solutions: (b) It doesn not contradict because $f(x, y) = \frac{4y-3x}{2x-y}$ is not continuous at $(0, 0)$.
Alternatively, we can say the initial condition $y(0) = 0$ is not legal/valid because the function $f(x, y)$ is not defined there and so no solution passes through $(0, 0)$. So while as functions they intersect, as solutions they are not defined for $x = 0$.
(c) On the lines $y = x$ and $y = -3x$, excluding the origin, the conditions of the theorem of existence and uniqueness hold. Meaning for every point $(x_0, y_0)$ on either of the lines (excluding the origin) there is only one solution passing through the point $(x_0, y_0)$. Therefore no solution intersects with either of the solutions $y = x$ and $y = -3x$. And also no solution passes through $(0, 0)$ (see (b)) which proves any solution which starts between them, stays between them.

6. Find two solutions for $y' = 2 + \sqrt{9 - (y - 2x - 1)^2}$ $y(0) = 4$.

Solution: We substitute $v = y - 2x - 1$ and then

$$v' = y' - 2 = \sqrt{9 - (y - 2x - 1)^2} - 2 = \sqrt{9 - v^2}.$$

This is a separable equation and so $v = \pm 3$ are constant solutions. Then

$$\begin{aligned}
&\int \frac{1}{\sqrt{9 - v^2}} dv = \int 1 dx \\
&\arcsin \frac{v}{3} = x + c \\
&v = 3 \sin(x + c) \\
&y - 2x - 1 = 3 \sin x + c \\
&y = 2x + 1 + 3 \sin x + c.
\end{aligned}$$

It is clear that $v = \pm 3$ or $y = 2x + 4$ and $y = 2x - 2$ are singular solutions. The initial condition is satisfied by $y = 2x+4$ and $y = 2x+1+3\sin(x+\frac{\pi}{2})$.

For advanced students: Compare with previous exercise $y = \sqrt{1-y^2}$ with $y(0) = 1$.

7. Solve $(xy + x^2y^3)y' = 1$.

   Solution: We consider instead $(xy + x^2y^3)\frac{1}{x'} = 1$ or $x' - xy = x^2y^3$ which is a bernouli equation with $n = 2$. Then $x \equiv 0$ is a solution. We substitute $z = y^{1-2} = y^{-1}$ and then $z' = -y^{-2}y'$. we rewrite the equation $x^{-2}x' - yx^{-1} = y^3$ and then $-z' - yz = y^3$ or $z' + yz = -y^3$. The solution is $z = ce^{-\frac{y^2}{2}} - y^2 + 2$. Therefore the general solution to the original equation is $y = frac1ce^{-\frac{y^2}{2}} - y^2 + 2$. We add $y \equiv 0$ as a singular solution.

8. Solve $(x - y + 1)dy + (2x - 2y - 1)dx = 0$

   Solution: $y' = \frac{2x-2y-1}{x-y+1}$. We first solve

$$2x - 2y - 1 = 0$$
$$x - y + 1 = 0.$$

there is no solution which means the lines are parallel and so this is a homogeneous equation.

$$y' = \frac{2x - 2y - 1}{x - y + 1} = \frac{2(x - y + 1 - \frac{3}{2})}{x - y + 1} = 2 - \frac{3}{x - y + 1}.$$

This is an equation which is a function of a line. $v = x - y + 1$ and then $v' = 1 - y' = 2 - \frac{3}{v} = \frac{2v-3}{v}$. This is a separable equation and the constant solutions are $v \equiv -\frac{3}{2}$. The general solution in implicit form is

$$\int \frac{v}{2v-3}dv = \int 1dx$$
$$\frac{1}{2}\int \frac{2v-3+3}{2v-3}dv = x + c$$
$$\frac{v}{2} + \frac{3}{4}\ln|2v-3| = x + c$$
$$v + \frac{3}{2}\ln|2v-3| = x + c$$
$$x - y + 1 + \frac{3}{2}\ln|2(x-y+1) - 3| = x + c.$$

Recall $v \equiv -\frac{3}{2}$ as a solution which corresponds to $x-y+1 = -\frac{3}{2}$ or $y = x+\frac{5}{2}$. Is this a singular solution? substituting it into the general solution gives

$$x - (x + \frac{5}{2}) + 1 + \frac{3}{2} \ln |2(x - (x + \frac{5}{2}) + 1) - 3| = x + c$$
$$x - (x + \frac{5}{2}) + 1 + \frac{3}{2} \ln |0| = x + c.$$

and so it is singular.