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ON CONSTANT VORTICITY FLOWS BENEATH TWO-DIMENSIONAL SURFACE SOLITARY WAVES

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We demonstrate that, for a two-dimensional, steady, solitary wave profile, a flow of constant vorticity beneath the wave must likewise be steady and two-dimensional, and the vorticity will point in the direction orthogonal to that of wave propagation. Constant vorticity is the hallmark of a harmonic velocity field, and the simplified vorticity equation is used along with maximum principles to derive the results.

Keywords: Euler equations; free boundary; solitary waves; vorticity.

Mathematics Subject Classification 2000: 76B25, 35Q31

1. Introduction

Vorticity is ubiquitous in fluid flows, and while often neglected can play an important structural role in the mathematical theory of water waves, e.g. in the construction of two-dimensional periodic waves by Constantin and Strauss [11], and in descriptions of wave-current interactions (see Constantin [6]). Vorticity describes the local rotation of a fluid element, which may be regarded as twice the angular velocity of a small element of the fluid — the simplest incarnation of which is seen in a linearly sheared current — exhibiting constant vorticity in two spatial dimensions. It is for this reason that vorticity is important in the study of tidal currents — modeled by such linear shear — and other currents as they interact with water waves (see Constantin and Varvaruca [13] and Wahlén [26] on water waves in flows with constant vorticity). We address the structure of the flow beneath a two-dimensional solitary wave under the assumption of constant nonzero vorticity — in particular, we show that this flow must necessarily be two-dimensional. A similar result for two-dimensional, periodic, gravity waves was obtained recently by Constantin [7].

We will assume a two-dimensional surface solitary wave which is symmetric, monotone, and has but a single crest. In the irrotational setting, results of Keady and Pritchard [22] indicate that symmetric, monotone solitary water waves must be waves of elevation and propagate with supercritical Froude number (see also the discussion by McLeod [24], who elucidates this condition on the Froude number). Amick and Toland [1, 2] address the existence of such irrotational solitary waves, establishing existence of waves of all amplitudes up to the wave of greatest height, all propagating with supercritical Froude number and decaying exponentially at infinity. Properties of such supercritical solitary waves are discussed by Craig and Sternberg [15], who find that they are necessarily positive, symmetric, and rising monotonically to the unique wave crest. The flow beneath an irrotational solitary wave was later investigated by Constantin along with Escher and Hsu [5, 8, 10]. It should be noted that the assumption of a solitary wave itself imposes certain restrictions, e.g. in irrotational flow all localized, steady, positive solitary waves are two-dimensional [14].

In the rotational case, early investigations focused on existence of solitary waves by means of asymptotic procedures, cf. Benjamin [3]. More recently, Hur [18] demonstrated existence of small amplitude solitary waves with supercritical velocity in water of arbitrary vorticity, which decay exponentially at infinity. The same author subsequently demonstrated *a priori* symmetry, monotonicity, and exponential decay of rotational solitary water waves [19], as well as real-analyticity of the streamlines [20] (further regularity and symmetry results were recently obtained by Matioc and Matioc [23]).

The importance of the flow beneath a wave is already underlined by considerations within the irrotational framework. It is interesting to note that the existence theory for irrotational solitary waves developed by Amick and Toland [2] shows that these wave profiles can be obtained by letting the wavelength of periodic waves go to infinity, while this limiting process seems to fail for the flow beneath the waves. Indeed the particle paths in a solitary water wave exhibit no motion opposite to that of wave propagation [5, 8], while results of Constantin and Strauss [4, 12] show that, in the absence of vorticity, the particles in a Stokes wave in fact move backwards. This serves to highlight some qualitative differences between solitary and periodic traveling waves.

2. Equations and Boundary Conditions

We consider the water wave problem, which consists of finding a velocity field $\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$, pressure $P(\mathbf{x}, t)$, and free surface $H(x, y, t)$ satisfying the following nonlinear partial differential equations:

$$\frac{D\mathbf{u}}{Dt} := \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P - (0, 0, g), \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

as well as the boundary conditions

$$P = P_{\text{atm}} \quad \text{on the free surface}, \tag{2.3}$$

$$w = H_t + uH_x + vH_y \quad \text{on } z = 1 + H(x, y, t), \tag{2.4}$$

$$w = 0 \quad \text{on } z = 0. \tag{2.5}$$

The latter two, so-called kinematic boundary conditions, express the idea that the free surface $z = 1 + H(x, y, t)$ and the bed $z = 0$ are material surfaces, for which it is necessary and sufficient that the respective material derivatives vanish. This is to incorporate the

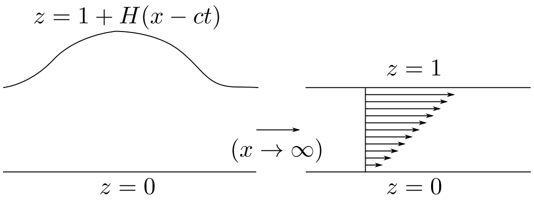

Fig. 1. Solitary wave decaying to asymptotically flat free-surface with constant vorticity (linear shear).

physical notion of separating the water from the air above and solid ground below, meaning that no water particles cross these interfaces. We will assume that the free surface is two-dimensional, so in fact $H(x, y, t)$ becomes $H(x, t)$, and additionally that the free surface represents a steady wave, which means that we can represent the elevation above still-water level as $H(x - ct)$, and the kinematic boundary condition then becomes

$$w = (u - c)H_x \quad \text{on } z = 1 + H(x - ct). \tag{2.6}$$

For a description of solitary waves, we need to supplement this with at least the following decay conditions (see Fig. 1), denoting $\xi = x - ct$:

$$H(\xi) \to 0 \quad \text{as } |\xi| \to \infty, \tag{2.7}$$

$$w \to 0 \quad \text{as } |x| \to \infty. \tag{2.8}$$

We also assume that the free surface, the pressure, and the velocity field are twice continuously differentiable, and that u, v and w are bounded.

2.1. *Vorticity*

In addition to the above governing equations, the vorticity

$$\Omega = \nabla \times \mathbf{u} = (w_y - v_z, u_z - w_x, v_x - u_y), \tag{2.9}$$

representing local rotation within the fluid, will play a central role. Its evolution in time is described by the vorticity equation

$$\frac{D\Omega}{Dt} = \Omega_t + (\mathbf{u} \cdot \nabla)\Omega = (\Omega \cdot \nabla)\mathbf{u} \tag{2.10}$$

(a derivation may be found in [6, p. 24]).

In the setting we will consider, namely that of constant vorticity, the left-hand side of (2.10) disappears and we are left with

$$(\Omega \cdot \nabla)\mathbf{u} = 0. \tag{2.11}$$

This implies that in the direction of Ω, $\mathbf{u}$ will be constant.

Furthermore, the assumption of constant vorticity entails that each velocity component is a harmonic function (as one can see using the equation of mass conservation) and therefore must be real-analytic (cf. Evans [16]). Moreover, in the absence of stagnation points, for traveling waves (whether periodic or solitary), the free surface is also real-analytic (cf. Constantin and Escher [9], Matioc and Matioc [23], and Hur [20]).

3. Main Result

We begin with the following lemma, which demonstrates that the flow is two-dimensional and yields the preferred direction of the vorticity.

Lemma 1. *A flow of constant, non-vanishing vorticity* Ω *above a flat bed and beneath a steady, two-dimensional solitary surface profile must also be two-dimensional. In particular, if* $z = 1 + H(x - ct)$ *is the nonflat free surface describing a monotonic solitary wave with a single crest, then the velocity* $\mathbf{u} = (u, 0, w)$ *is independent of the* y*-coordinate, and the vorticity* $\Omega = (0, \Omega_2, 0)$ *points in the direction orthogonal to that of the wave motion.*

Proof. We first demonstrate that the direction of Ω must be horizontal, by assuming towards a contradiction that $\Omega_3 \neq 0$. (See Fig. 2). We know that

$$w = 0 \quad \text{on } z = 0$$

and that w is constant in the direction of Ω by virtue of the vorticity equation (2.11). Since the vorticity has a non-vanishing vertical component Ω_3, in a region between a plane $z = \varepsilon < 1$ ($z = 1$ being the level of the undisturbed free surface) and the bed at $z = 0$, w must vanish, as Ω points into the fluid. Since it is zero on an open set, the real-analytic function w must vanish in the entire fluid domain. This straightforward argument will be repeated a number of times in what follows.

Note that the governing equations simplify drastically for vanishing w. In particular, we will use the surface kinematic boundary condition (2.6) in the form

$$0 = (u - c)H_x \quad \text{on } z = 1 + H(x - ct), \tag{3.1}$$

and the equation of mass conservation (2.2) in the form

$$u_x + v_y = 0. \tag{3.2}$$

If we assume that there are no stagnation points at the free surface, i.e. $u < c$ thereon, (3.1) implies $H_x = 0$, which indicates a flat free surface. This contradicts our assumption, hence $\Omega_3 = 0$.

If we are to allow for the possibility of stagnation points, Eq. (3.1) suggests at first sight that we lose a great deal of information regarding the free surface, opening up the

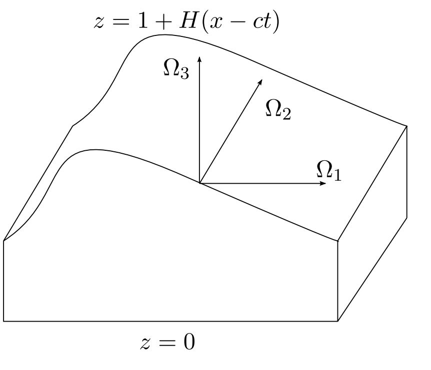

Fig. 2. Cross-section of a two-dimensional free surface solitary wave and the components of Ω.

possibility that the entire flow is stagnant and the wave motion is purely one of translation. We will see that this rather unphysical situation will lead to a contradiction. As in the paper of Constantin [7], we make use of the dynamic boundary condition on the pressure.

We see that (3.1) implies $u = c$ on the entire surface except at those points where H_x vanishes, but since this is solely the crest-line, by continuity $u = c$ all along the surface. Consider now the maximum and minimum of the bounded, harmonic (by virtue of mass conservation) function u. These must occur at the boundary of the fluid domain (see e.g. Protter and Weinberger [25] or Fraenkel [17]). If an extremum occurs on the bed, by the same argument used to show $w \equiv 0$ above, it is necessarily interior to the fluid domain, and $u \equiv c$ throughout the fluid. If both extrema occur on the free surface, the same conclusion holds.

The governing equations now simplify further, to

$$P_x = 0, \tag{3.3}$$

$$v_t + cv_x = -P_y, \tag{3.4}$$

$$P_z = -g, \tag{3.5}$$

$$v_y = 0, \tag{3.6}$$

$$P = P_{\mathrm{atm}} \quad \text{on the free surface.} \tag{3.7}$$

We can easily see that the pressure is then harmonic, has a minimum P_{atm} attained everywhere on the free surface. Unless the surface is flat, by (3.3) this minimum is interior to the fluid, and the pressure is constant, contradicting (3.5). We conclude that $\Omega_3 = 0$, and the vorticity is confined to a plane parallel to the flat bed.

The next step is to demonstrate that $\Omega_1 = 0$ throughout. We will in turn assume $\Omega_1 \neq 0$ and will demonstrate that this likewise implies $w \equiv 0$. Recall that the crest-line runs in the y-direction.

Again, consider the vertical velocity component w, which is harmonic, and bounded (and decays to zero), and whose maximum and minimum must thus be on the fluid boundary. Since we know $w = 0$ all along the bed, it cannot be the case that both maximum and minimum of w are at the bed, unless $w \equiv 0$.

It must thus be the case that at least one extremum of w is on the free surface. It is easy to see that both extrema cannot be on the crest line: this is distinguished by the fact that $H_x = 0$, hence the surface kinematic boundary condition implies $w = 0$ at the crest, and w would vanish identically. We have seen above that this leads to a contradiction.

Without loss of generality, let the maximum of w occur at some other point on the free surface. Denote this point by Q. By virtue of the vorticity equation, we know that w must be constant in the direction of Ω, whose x-component Ω_1 is nonzero. Because we are on a two-dimensional free surface (and by monotonicity $H_x = 0$ only at the crest), w is constant on the line through Q in the direction of Ω, which necessarily passes through the fluid domain. The same line of reasoning as above now yields a contradiction, and we establish that $\Omega_1 = 0$.

Our assumption of constant, non-vanishing vorticity, (2.11) now necessitates that $\mathbf{u}$ be independent of y. The fact that $\Omega_1 = \Omega_3 = 0$ and the Euler equation (2.1) then immediately imply that $v'(t) = -P_y$, which can only be the case if v is in fact constant, which we may choose to be zero. □

Proposition 1. *Assume that the free surface of a constant vorticity flow describes a two-dimensional, steady, symmetric solitary wave of elevation $z = 1 + H(x - ct)$, rising monotonically to a single crest and tending asymptotically to the undisturbed water level $z = 1$. Further assume that the vertical velocity and its gradient decay: $w, \nabla w \to 0$ for $|x| \to \infty$. Then this flow is necessarily two-dimensional, independent of the y-coordinate, and the associated constant vorticity is one-dimensional, and points in the direction orthogonal to the fluid flow. Furthermore, the flow is steady, u is symmetric and decays to a linear shear as $|x| \to \infty$, and w is antisymmetric.*

Proof. The first part of the assertion was demonstrated in the lemma above. We proceed to show steadiness, decay, and symmetry of the flow. Using the two-dimensionality of the velocity $\mathbf{u}$, we may define a stream function via the equation of mass conservation (2.2) satisfying

$$u - c = \psi_z, \quad w = -\psi_x, \tag{3.8}$$

where ψ is defined uniquely up to an additive function of time. The stipulation of constant vorticity $\Omega_2 = u_z - w_x = \text{const.}$, along with the kinematic boundary conditions (2.4) and (2.5) yield

$$\begin{cases} \psi(x - ct, 1 + H(x - ct), t) = 0 \quad \text{on } z = 1 + H(x - ct), \\ \Delta\psi(x, z, t) = \Omega_2 \quad \text{in the fluid domain } 0 < z < 1 + H(x - ct), \quad \text{and} \\ \psi(x, 0, t) = -m(t) \quad \text{on the bed } z = 0, \end{cases} \tag{3.9}$$

where

$$m(t) = \int_0^{1+H(x-ct)} (u - c)dz \tag{3.10}$$

is independent of x.[a] A stream function ψ as defined by (3.8) must be constant on the free surface and on the bottom. We may choose $\psi = 0$ on the surface $z = 1 + H(x - ct)$, which implies immediately that ψ on the bottom is equal to the negative mass flux.

Note that, by uniqueness for the boundary value problem, given a solution ψ_0 to (3.9) at $t = 0$,

$$\psi(x - ct, z, t) = \psi_0(x - ct, z, 0)$$

defines the unique, steady solution at any time $t \geq 0$ iff m is independent of t, and the problem becomes one of demonstrating this time-independence.

[a]Note

$$\begin{aligned} \frac{dm}{dx} &= \int_0^{1+H(x-ct)} u_x dz + (u(x, 1 + H(x - ct), t) - c) \cdot H_x(x - ct) \\ &= -w(x, 1 + H(x - ct), t) + w(x, 0, t) + (u(x, 1 + H(x - ct), t) - c) \cdot H_x(x - ct) = 0 \end{aligned}$$

having used mass conservation (2.2) and the kinematic boundary conditions (2.5) and (2.4).

First we will demonstrate symmetry properties of the flow, which are independent of the steadiness and decay results below. In the setting of the Poisson problem, it is simple to show symmetry of ψ, as long as we suppose the free surface to be symmetric about the crest at zero i.e. $H(\alpha) = H(-\alpha)$ (cf. Hur [19], where the *a priori* symmetry of supercritical solitary waves with vorticity is discussed). At a fixed time t consider the map

$$(x, z) \to \psi(x - ct, z, t) - \psi(-x + ct, z, t),$$

which we will show to vanish identically. Now

$$(x, 0) \to \psi(x - ct, 0, t) - \psi(-x + ct, 0, t) = -m(t) + m(t) = 0 \quad \text{on the bed } z = 0.$$

In the interior of the domain the map is harmonic, since $\Delta\psi(x, z, t) = \Omega_2 = \Delta\psi(-x, z, t)$. Finally, on the surface we have

$$\begin{aligned}
&\psi(x - ct, 1 + H(x - ct), t) - \psi(-x + ct, 1 + H(x - ct), t) \\
&\quad = \psi(x - ct, 1 + H(x - ct), t) - \psi(-x + ct, 1 + H(-x + ct), t) = 0,
\end{aligned}$$

making use of (3.9) and the symmetry of H. Then, since the map is bounded by assumption, and by the Hopf Maximum Principle it can have neither interior minima nor maxima, it must vanish identically in the fluid domain. Consequently we see that u is symmetric and w antisymmetric, cf. the results of Constantin and Escher [8, p. 426] for irrotational solitary water waves.

Under the assumption that $w, \nabla w \to 0$, we will show steadiness and decay of the velocity field to a constant, time independent linear shear. Steadiness is immediate if the mass flux can be shown to be time independent, as noted above, and this in turn is a consequence of the decay to a linear shear.

We have already seen that $\Delta u = \Delta w = 0$ throughout the fluid domain. Note that $u - \Omega_2 z$ is harmonic conjugate of w. It is then easy to see that $\nabla w = (w_x, w_z) \to (0, 0)$ entails $((u - \Omega_2 z)_x, (u - \Omega_2 z)_z) \to (0, 0)$, meaning that $u_x \to 0, u_z \to \Omega_2$, or $\mathbf{u} \to (\Omega_2 z + C, 0, 0)$.

This yields the decay of $\mathbf{u}$ to a linear shear flow. In particular, C does not depend on t, since letting $|x| \to \infty$ and using the Euler equation we can see that $C'(t) = -P_x$, which, along with the surface dynamic boundary condition $P = P_{\text{atm}}$ on the surface, implies $C'(t) = 0$ thereon. Since C is independent of x and z, we see that $C'(t) = 0$ everywhere, hence the shear flow is time-independent. It is intuitively and physically clear that this is necessary for the mass flux to be time-independent.

Considering now the mass flux, which is independent of x, we see

$$m(t) = \int_0^{1+H(x-ct)} (u(x, z, t) - c)dz = \int_0^1 (\Omega_2 z + C - c)dz \tag{3.11}$$

since $H(x - ct) \to 0$ and $u \to \Omega_2 z + C$ for $|x| \to \infty$, cf. [21, p. 178]; thus the mass flux is time independent. Based on this, the unique solution to the Poisson boundary value problem (3.9) is given by $\psi_0(x - ct, z, 0)$. Therefore $u = \psi_z + c$ and $w = -\psi_x$ are likewise steady. Again denoting $\xi = x - ct$, since

$$-P_\xi = (u - c)u_\xi + wu_z$$

and $u_\xi \to 0, w \to 0$ as $|\xi| \to \infty$, then also $P_\xi \to 0$ as $|\xi| \to \infty$. Since we are dealing with steady flow, we have Bernoulli's equation

$$\frac{\mathbf{u} \cdot \mathbf{u}}{2} + P + gz = \text{constant on streamlines.}$$

This implies that on the streamlines asymptotically $P \to \text{Const.} - gz - \frac{(\Omega_2 z + C)^2}{2}$. □

4. Comments

We have demonstrated two-dimensionality, decay, and symmetry of the flow for a steady, surface solitary wave that is monotonic and symmetric about a single crest, under the assumption of constant vorticity. Note that Lemma 1 demonstrating two-dimensionality does not require symmetry, and that these results were obtained along different lines in the periodic setting by Constantin [7]. By restriction to a single wavelength, the above proof of the lemma may also be used for periodic waves.

While we have made no assumptions on the wave speed, and only implicitly assumed that we have a wave of elevation, Hur [19] and Motioc *et al.* [23] have demonstrated, with certain assumptions concerning the wave speed and the streamlines, respectively, that a solitary wave of elevation is *a priori* symmetric about a single crest from which the wave profile falls monotonically. We make use of the symmetry of the free surface solely to demonstrate symmetry properties of the flow.

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References

[1] C. J. Amick and J. F. Toland, On solitary water-waves of finite amplitude, *Arch. Ration. Mech. Anal.* **76**(1) (1981) 9–95.
[2] C. J. Amick and J. F. Toland, On periodic water-waves and their convergence to solitary waves in the long-wave limit, *Philos. Trans. R. Soc. Lond., Math. Phys. Eng. Sci.* **303** (1981) 633–669.
[3] T. B. Benjamin, The solitary wave on a stream with an arbitrary distribution of vorticity, *J. Fluid Mech.* **12**(1) (1962) 97–116.
[4] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.* **166**(3) (2006) 523–535.
[5] A. Constantin, On the particle paths in solitary water waves, *Quart. Appl. Math.* **68** (2010) 81–90.
[6] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 81 (SIAM, 2011).
[7] A. Constantin, Two-dimensionality of gravity water flows of constant nonzero vorticity beneath a surface wave train, *Eur. J. Mech. B Fluids* **30** (2011) 12–16.
[8] A. Constantin and J. Escher, Particle trajectories in solitary water waves, *Bull. Amer. Math. Soc.* **44**(3) (2007) 423–431.
[9] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math.* **173**(1) (2011) 559–568.

[10] A. Constantin, J. Escher and H.-C. Hsu, Pressure beneath a solitary water wave: Mathematical theory and experiments, *Arch. Ration. Mech. Anal.* **201** (2011) 251–269.

[11] A. Constantin and W. Strauss, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.* **57**(4) (2004) 481–527.

[12] A. Constantin and W. Strauss, Pressure beneath a Stokes wave, *Comm. Pure Appl. Math.* **63** (2010) 533–557.

[13] A. Constantin and E. Varvaruca, Steady periodic water waves with constant vorticity: Regularity and local bifurcation, *Arch. Ration. Mech. Anal.* **199** (2011) 33–67.

[14] W. Craig, Non-existence of solitary water waves in three dimensions, *Philos. Trans. R. Soc. Lond. Math., Phys. Eng. Sci.* **360**(1799) (2002) 2127–2135.

[15] W. Craig and P. Sternberg, Symmetry of solitary waves, *Comm. Partial Differential Equations* **13**(5) (1988) 603–633.

[16] L. C. Evans, *Partial Differential Equations* (American Mathematical Society, Providence, RI, 1998).

[17] L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems* (Cambridge University Press, Cambridge, 2000).

[18] V. M. Hur, Exact solitary water waves with vorticity, *Arch. Ration. Mech. Anal.* **188** (2008) 213–244.

[19] V. M. Hur, Symmetry of solitary waves with vorticity, *Math. Res. Lett.* **15**(3) (2008) 491–509.

[20] V. M. Hur, Analyticity of rotational flows beneath solitary water waves, *Inst. Math. Res. Not.* **2012**(11) (2012) 2550–2570.

[21] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves* (Cambridge University Press, Cambridge, 1997).

[22] G. Keady and W. G. Pritchard, Bounds for surface solitary waves, *Math. Proc. Cambridge Philos. Soc.* **76** (1973) 345–358.

[23] A.-V. Matioc and B.-V. Matioc, Regularity and symmetry properties of rotational solitary waves, *J. Evol. Equ.* **12** (2012) 481–494.

[24] J. McLeod, The Froude number for solitary waves, *Proc. Roy. Soc. Edinburgh Sect. A* **97** (1984) 193–197.

[25] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Partial Differential Equations Series (Prentice-Hall, 1967).

[26] E. Wahlén, Steady water waves with a critical layer, *J. Differential Equations* **246**(6) (2009) 2468–2483.