Exercises for Nonlinear Waves in Deep and Shallow Water

The following is a short series of exercises which complements the course *Nonlinear Water Waves in Deep and Shallow Water* held at Technion – Israel Institute of Technology in the Summer Semester 2023. You are asked to show the steps necessary to arrive at your result, including stating the assumptions you make to get there. The completed set of exercises should be submitted as a single, scanned PDF file (handwritten is fine, please ensure your scan is legible and of a suitable quality to allow for printing and subsequent marking) by email to `raphaels@technion.ac.il` (cc: `raphael.stuhlmeier@plymouth.ac.uk`) no later than 23:59 UTC on September 14th, 2023.

Exercises

Chapter 1

1. For waves with a length of several millimetres surface tension is an important restoring force. The linear equations for such capillary-gravity waves are

$$
\begin{aligned}
&\Delta\phi = 0,\\
&\phi_{tt} + g\phi_z + \frac{\gamma}{\rho}\phi_{zzz} = 0 \text{ on } z = 0,\\
&\phi_z = 0 \text{ on } z = -h,
\end{aligned}
$$

where γ is the coefficient of surface tension and ρ is the density of water. Make a harmonic wave ansatz and derive the dispersion relation for capillary-gravity waves. Are these waves dispersive? What is the group velocity?

(7 Points

2. Surface waves generated by a storm arrive at the Israeli coast with a measured period of 15 seconds. A day later the period of the arriving waves has dropped to 12.5 seconds. Roughly how far away did the storm occur? *Hint: the storm generates waves of many different frequencies/periods, which arrive at different times due to dispersion. What quantity are you tracking?*

(5 points)

Chapter 2

3. Consider the linear part of the KdV

$$u_t + u_{xxx} = 0$$

and insert the following similarity ansatz:

$$u_s(x,t) = \frac{1}{(3t)^p} f\left(\frac{x}{(3t)^q}\right)$$

This assumes that there is a solution that can be written in terms of the similarity variable $\eta = x/(3t)^q$. There are two parameters left to tune: choose q so that your resulting ordinary differential equation (in η) has no explicit t dependence. The choice $p = q$ further simplifies your equation. Can you integrate once in η to obtain a (well-known) ODE?

(5 points)

4. Derive the dimensional dispersion relation associated with the Boussinesq equation

$$\eta_t + \delta\eta_x u_0 + (\delta\eta + 1)u_{0x} - \frac{\mu^2}{6}u_{0xxx} = 0, \tag{1}$$

$$\eta_x + u_{0t} - \frac{\mu^2}{2}u_{0xxt} + \frac{1}{2}\delta(u_0^2)_x = 0. \tag{2}$$

This means reversing the nondimensionalisation undertaken at the beginning of Chapter 2

$$x = \lambda x',\ z = hz',\ \eta = A\eta'\ t = \frac{\lambda}{c_0}t',\ \phi = \frac{Ac_0}{\mu}\phi',$$

and linearising the equation. For the linear system you should try the following procedure: write the equations in matrix form using differential operators. For example, the system of linear shallow water equations

$$\zeta_t + u_x h = 0, \quad u_t + g\zeta_x = 0$$

can be written compactly as

$$\begin{pmatrix} h\partial_x & \partial_t \\ \partial_t & g\partial_x \end{pmatrix} \begin{pmatrix} u \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the compact notation $\partial_x = \partial/\partial x$ has been used for derivatives. Now, in accordance with the harmonic wave ansatz (or Fourier transforms), replace $\partial_x \longrightarrow ik$ and $\partial_t \longrightarrow -i\omega$. Finally set the determinant of the resulting matrix equal to zero. *Note: it doesn't matter at what point you reintroduce dimensional variables, the result should be the same.*

(5 marks)

Chapter 3

5. What is the maximum amplification factor of the Kuznetsov-Ma breather

$$u(x,t) = e^{2it}\frac{\cos(\Omega t - 2i\phi) - \cosh(\phi)\cosh(px)}{\cos(\Omega t) - \cosh(\phi)\cosh(px)}?$$

Plot the solution using your favourite computer programme or sketch it by hand.

(3 marks)

6. In an ad hoc way we can see that a dispersion relation corresponds to a differential equation. For example, given the dispersion relation

$$\omega = -k^3$$

and using the correspondence $\omega = i\partial_t$ and $k = -i\partial_x$ we see (applying the corresponding operators to a function) that

$$\left(i\frac{\partial}{\partial t} + i\frac{\partial^3}{\partial x^3}\right)u = i(u_t + u_{xxx}) = 0,$$

so the corresponding equation is the linear KdV.

Now assume a hypothetical dispersion relation where ω depends on the squared amplitude of the function, i.e. $\omega = \omega(k_0 + k, |u|^2)$. Taylor expand ω about the point $(k_0, 0)$, retaining terms quadratic in the variable k and linear in $|u|^2$. Your expanded dispersion relation should be of the form

$$\omega(k_0 + k, |u|^2) = \omega(k_0, 0) + \ldots.$$

Can you find a corresponding PDE? *Note: you will have to leave derivative terms like ω_{kk} as-is. There is no corresponding differential operator. Recall that ω_k has a specific physical meaning.*

(5 marks)

Chapter 4

7. What equation can you get from considering the variation of the following Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{\infty}\left(q_x q_x^* - \frac{1}{2}\sigma_1 q^2 q^{*2}\right)dx,$$

via

$$i\frac{\partial q}{\partial t} = \frac{\delta\mathcal{H}}{\delta q^*}?$$

Show your work.

(4 marks)