Nonlinear Water Waves in Deep and Shallow Water

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Contents

It may be easy to run a computer model. One has to remember, however, that all that comes out of that is numbers. An enormous amount of numbers. They may be interpreted and plotted in diagrams to look like nearshore flow properties. But knowing/understanding the powers and limitations of models requires understanding the basis for the equations. Which features are represented in the equations, which not, why this or that effect is important, and when, etc., is a first condition for generating confidence in the results.

– Ib Svendsen (1937-2004)

At least in my own case, understanding mathematics doesn't come from reading or even listening. It comes from rethinking what I see or hear. I must redo the mathematics in the context of my particular background. And that background consists of many threads, some strong, some weak. My background is stronger in geometric analysis, but following a sequence of formulae gives me trouble. I tend to be slower than most mathematicians to understand an argument. The mathematical literature is useful in that it provides clues, and one can often use these clues to put together a cogent picture. When I have reorganized the mathematics in my own terms, then I feel an understanding, not before.

– Stephen Smale (b. 1930)

Preface

This is a set of notes for the course Nonlinear Water Waves in Deep and Shallow Water, last compiled on August 10, 2023. These notes are for personal use only and not for distribution. They represent a subjective snapshot of some interesting and also accessible portions of the vast literature on nonlinear water waves. No attempt is made to acknowledge the contributions of the many authors who developed the theory and ideas herein, and citations to the literature are inserted in a somewhat haphazard fashion. In particular, in writing these notes I have borrowed from other sources and undoubtedly introduced many mistakes and inaccuracies while doing so.

A partial list of books which contain relevant background material is:

- C.C. Mei, M. Stiassnie & D.K.-P. Yue, *Theory and Applications of Ocean Surface Waves*, World Scientific, 2005.
- J. Billingham & A. C. King, *Wave Motion*, Cambridge University Press, 2006.
- R. Dean & R. Dalrymple, *Water Wave Mechanics for Engineers and Scientists*, World Scientific, 1991.
- R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, 1997.
- R. S. Johnson, *Singular Perturbation Theory*, Springer, 2005.
- G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, 1974.

Chapter 1

Linear wave theory – fundamental concepts

1.1 Introduction

1.1.1 Basics of wave motion

The most general description of wave motion is that it involves *a disturbance propagating through space at a finite speed*. This immediately makes clear that wave motion must be connected intimately with PDEs, since both space and time must exist in the problem. It also excludes certain equations, like the heat equation, that support infinite propagation speed for disturbances.

The simplest PDE describing wave-like motion is known as the *transport equation*

$$u_t + c_0 u_x = 0.$$

This is one part of the well-known wave equation, and terms like $u_t + c_0 u_x$ occur in all manner of PDE, always denoting the transport of some quantity. Assuming that t is time and x is space, for dimensional equality we must have

$$[u][1/T] = [c_0][u][1/L],$$

where the square brackets denote the dimension of the quantity, and from which we immediately conclude that c_0 must denote a velocity with units L/T irrespective of the units of u.

In any PDE course we find that the solution to the transport equation is given by $u(x,t) = f(x - c_0 t)$ (*check this if unsure*). This solution is called a **traveling wave solution**, as it represents uniform translation of the initial condition $f(x)$ at $t = 0$ to the right with a velocity c_0 (see Figure 1.1). The most important feature of the traveling wave is that it does not change its shape during propagation.

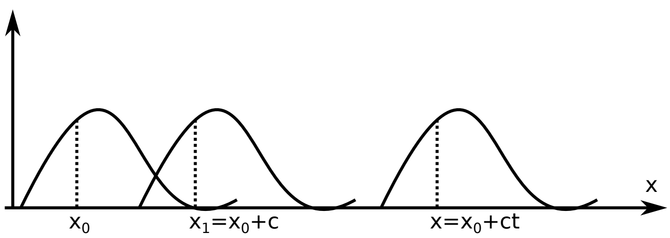


Figure 1.1: Traveling wave $f(x - c_0 t)$ with phase velocity c_0.

For essentially any PDE we can look for traveling wave solutions (though we may not find any) by making an ansatz of this form. Often periodic travelling waves are sought, in which case the ansatz (also called the **harmonic wave ansatz**)

$$\phi = A\exp(i(kx - \omega t))$$

is employed. For physically realistic waves, we need only the real part of any solution. However, calculations with the exponential function are much less laborious, so the real part can be taken after employing this simpler form of the solution.

If we use this harmonic wave ansatz in our transport equation, substituting $u = Ae^{i(kx-\omega t)}$ we find

$$\omega = c_0 k.$$

In general, linear equations modelling wave motion will have harmonic wave solutions with the relationship between ω and k determined by the number of temporal and spatial derivatives. The general scenario is that $\omega(k)$ is some real-valued function (what happens to the solution if it is not real valued?) of the wavenumber; then it is possible to determine a general phase velocity by the ratio

$$c(k) = \omega(k)/k.$$

Note that this can all be recast in terms of wavelength λ, if desired, using the relationship

$$k = \frac{2\pi}{\lambda}.$$

Likewise, it is sometimes useful to consider the wave period T, which is related to the radian frequency by

$$\omega = \frac{2\pi}{T}.$$

As a simple example, we consider the harmonic wave ansatz for a linearized form of the Korteweg de Vries equation

$$u_t + au_x = bu_{xxx}.$$

We recognize the transport equation on the left. Inserting $A\exp(i(kx - \omega t))$ into this equation, we find

$$-i\omega + aik = -ibk^3 \Rightarrow \omega = ak + bk^3.$$

This is known as the *dispersion relation* for this equation, and it shows that waves of different wavenumbers travel at different speeds. If an initial pattern contains different wavenumbers, this pattern will tend to disperse. The phase velocity in this example is

$$c = \omega(k)/k = a + bk^2,$$

which shows that shorter waves (larger k) travel faster than longer waves. This is in contrast to the transport equation, where all waves travel at the same, constant speed c_0.

We note (and will encounter again below) that it is possible to form another quantity with dimensions of velocity from ω and k, namely $d\omega/dk$. This is called the *group velocity,* and is the velocity at which the energy of a wave propagates.

We turn back to the governing equations for waves in water.

1.1.2 Governing equations for water waves

We recall the governing equations and boundary conditions relevant for the water wave problem, written in rectangular Cartesian coordinates, with constant densitey $\rho = 1$, and without surface tension:

$$u_t + uu_x + vu_y + wu_z = -P_x \tag{1.1}$$
$$v_t + uv_x + vv_y + wv_z = -P_y \tag{1.2}$$
$$w_t + uw_x + vw_y + ww_z = -P_z - g \tag{1.3}$$
$$u_x + v_y + w_z = 0 \tag{1.4}$$
$$P = P_0 \text{ on } z = \zeta(x, y, t) \tag{1.5}$$
$$w = \zeta_t + u\zeta_x + v\zeta_y \text{ on } z = \zeta(x, y, t) \tag{1.6}$$
$$w = 0 \text{ on } z = -h \tag{1.7}$$

The interface separating water and air is now denoted ζ, and we assume that it moves with the fluid (condition (1.6)). We shall also neglect the role of density, which will be taken constant for all subsequent examples. Non-constant density requires a separate equation of state, and leads to the propagation of sound waves, which we neglect here. If the water depth plays a role, the simplest scenario is that of a flat bed $z = -h$, and an accompanying condition that there is no normal velocity on this plane (1.7). If the water is sufficiently deep, it can be assumed that $h \to \infty$, and we assume that the fluid velocity field $\mathbf{u} \to \mathbf{0}$ there. Other scenarios such as a moving bed (tsunami) or a non-flat bed (shoaling and breaking of waves) may also be of interest.

The most important special case of these equations is when the flow can be assumed irrotational (which is always the case for flows started from rest by potential forces). In such a case, $\nabla \times \mathbf{u} = 0 \Rightarrow \mathbf{u} = \nabla\phi$, and we rewrite the equations using this velocity potential:

$$\Delta\phi = 0 \tag{1.8}$$
$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + P + gz = C \tag{1.9}$$
$$P = P_0 \text{ on } z = \zeta(x, y, t) \tag{1.10}$$
$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y \text{ on } z = \zeta(x, y, t) \tag{1.11}$$
$$\phi_z = 0 \text{ on } z = -h \tag{1.12}$$

The second equation can be recognized as Bernoulli's equation, where we can absorb the constant term into a redefined velocity potential if desired.

For a static fluid the solution of this problem is simply $\mathbf{u} = \mathbf{0}, \zeta = 0$ and $P = P_0 - gz$. The pressure distribution is fittingly called **hydrostatic,** and the pressure increases linearly with depth. In line with this, it is often convenient to measure the pressure P as a deviation from hydrostatic pressure by defining a new pressure variable $p = P - P_0 + gz$. Then our pressure boundary condition becomes

$$p = gz \text{ on } z = \zeta(x, y, t).$$

The Bernoulli condition then becomes

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + p = 0,$$

where we take the constant on the right to be zero without loss of generality. We can evaluate this condition on the free surface to obtain a system of equations from which the pressure has been eliminated:

$$\Delta\phi = 0 \tag{1.13}$$
$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + g\zeta = 0 \text{ on } z = \zeta(x, y, t) \tag{1.14}$$
$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y \text{ on } z = \zeta(x, y, t) \tag{1.15}$$
$$\phi_z = 0 \text{ on } z = -h \tag{1.16}$$

Notice that all nonlinearity is now contained in the boundary conditions, in contrast to the original Euler equations, and that we have been able to rewrite the problem in terms of two variables η and ϕ only.

1.2 Linear water waves

The simplest scenario we shall consider consists of linear water waves in finite depth. To this end, linearize the governing equations (1.13)–(1.16) by transferring the free boundary from $z = \zeta$ to $z = 0$ and discarding quadratic terms to leave

$$\Delta\phi = 0 \tag{1.17}$$
$$\phi_t + g\zeta = 0 \text{ on } z = 0 \tag{1.18}$$
$$\phi_z = \zeta_t \text{ on } z = 0 \tag{1.19}$$
$$\phi_z = 0 \text{ on } z = -h \tag{1.20}$$

For this simpler problem, posed on a strip between the two planes $z = 0$ and $z = -h$, we can eliminate ζ from the problem entirely (see Figure 1.2). Notice that $\phi_{tt} + g\zeta_t = 0$, and $\zeta_t = \phi_z$, so that we have $\phi_{tt} + g\phi_z = 0$ on $z = 0$.

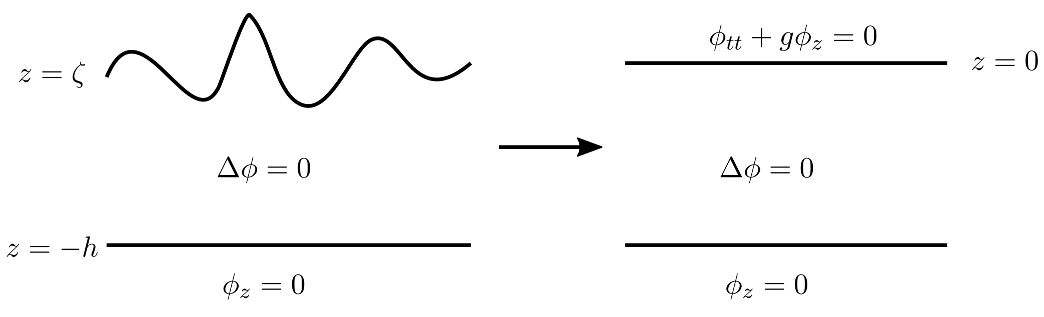


Figure 1.2: Transformation from the nonlinear problem on a fluid domain of unknown shape, with nonlinear boundary conditions (1.14) – (1.16) to the linear problem on a strip.

To cut down on the amount of algebra involved, let us assume that $\phi_y = 0$, so that the waves to be described are very nearly two dimensional (see Figure 1.3). Such waves are called *long-crested*, because the tops of the waves (the wave crests) form a long line in the water (as do the bottoms of the waves, called the wave troughs). The alternative are *short crested* waves, which have a more choppy appearance due to their greater two-dimensionality.

Solving this water wave problem is now a classical exercise in separation of variables for the Laplace equation. Insert a travelling wave ansatz $\phi(x, z, t) = F(x - ct)Z(z)$ into the equations

Figure 1.3: Long crested swell approaching the coast.

and separate[1]:

$$-\frac{F_{xx}}{F} = \frac{Z''}{Z} = k^2,$$

where we assume a positive constant k^2. Then

$$F_{xx} = -k^2 F,\; Z'' = k^2 Z,$$

so that we have the general solutions

$$\begin{cases} F(x-ct) = Ae^{ik(x-ct)} + Be^{-ik(x-ct)}, \\ Z(z) = C\cosh(kz) + D\sinh(kz). \end{cases}$$

The bottom boundary condition $\phi_z = 0$ on $z = -h$ then implies

$$C\sinh(-kh) + D\cosh(-kh) = 0 \Rightarrow \frac{D}{C} = -\tanh(-kh).$$

Substituting back into the expression for Z

$$\begin{aligned} Z &= C(\cosh(kz) - \tanh(-kh)\sinh(kz)) \\ &= C\left(\frac{\cosh(kz)\cosh(-kh) - \sinh(-kh)\sinh(kz)}{\cosh(-kh)}\right) \\ &= C\frac{\cosh(k(z+h))}{\cosh(kh)} \end{aligned}$$

having used some addition theorems for hyperbolic functions and the fact that the cosh is an even function.[2]

[1] You may prefer to make a more general ansatz. You may try $\phi(x,z,t) = X(x)Z(z)T(t)$, but will need to supplement the boundary value problem above with periodicity conditions in x and t, such that $\phi(x,z,t) = \phi(x+L,z,t)$ and $\phi(x,z,t) = \phi(x,z,t+T)$ for some period T and wavelength L.

[2] It is worth a pause to ask whether the trigonometric eigenfunctions could have been used in the vertical direction, and the hyperbolic in the horizontal. The answer is yes, but this is physically sensible only in the presence of a vertical boundary. The procedure can also be explored by asking what happens when k^2 is not a positive constant.

Inserting into the combined surface boundary condition $\phi_{tt} + g\phi_z = 0$ on $z = 0$ we have:

$$\begin{aligned}
&-k^2c^2FZ + gFZ' = 0 \text{ on } z = 0 \\
&\Leftrightarrow -k^2c^2C\frac{\cosh(kh)}{\cosh(kh)} + gkC\frac{\sinh(kh)}{\cosh(kh)} = 0 \\
&\Leftrightarrow c^2 = \frac{g}{k}\tanh(kh) \qquad (1.21)
\end{aligned}$$

We have derived the dispersion relation for linear water waves, giving the phase speed in terms of the wavenumber k, with the two possible roots indicating that waves can propagate either left or right, depending on whether c is positive or negative.

The two most useful limits of this dispersion relation occur for *deep water* ($kh \to \infty$) or *shallow water* ($kh \to 0$). Note:

- Waves are in deep water if kh is large, or equivalently h/λ is large.
  - Either the water depth h is very large, or the wavelength λ is very small.
  - The limit of very small wavelength requires that surface tension be accounted for, so requires a separate treatment.
- Waves are in shallow water if kh, or h/λ, is small.
  - Either we are dealing with shallow water, or very long waves.
  - In the open ocean, with $h = 4$ km, a tsunami wave is in shallow water, since $\lambda \approx 100$ km.

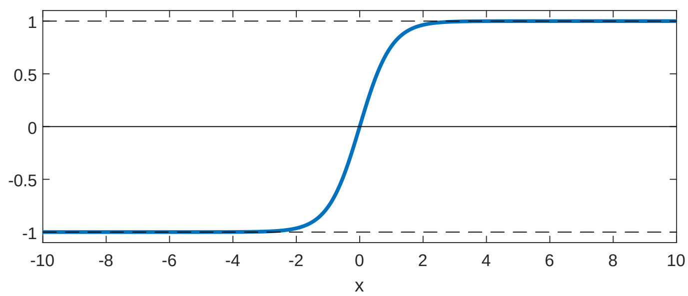


Figure 1.4: Plot of $\tanh(x)$.

For large values of the argument $\tanh(x) \to 1$, while for values near zero $\tanh(x) \sim x$ (as can be verified by a Taylor series expansion, or see Figure 1.4)

This leads to the dispersion relation for deep water

$$\boxed{c^2 = \frac{g}{k}}$$

and the dispersion relation for shallow water

$$\boxed{c^2 = gh}.$$

These yield a number of interesting qualitative (as well as quantitative) conclusions.

- Waves in shallow water are *not* dispersive, i.e. waves of all frequencies propagate with a speed dependent only on the water depth.

- As waves approach a beach (and h decreases), waves will slow down.
- Deep water waves are dispersive, and longer waves travel faster. This explains why the long swell (small k) generated in a storm arrives before the shorter waves (larger k).

Having obtained the potential for linear surface waves, we can recover the free surface elevation by substituting into the original boundary condition (1.18). The potential (absorbing the constant C into A and B) is

$$\phi(x,z,t) = F(x-ct)Z(z) = (Ae^{ik(x-ct)} + Be^{-ik(x-ct)})\frac{\cosh(k(z+h))}{\cosh(kh)},$$

so

$$g\zeta = ikc(Ae^{ik(x-ct)} - Be^{-ik(x-ct)}).$$

We are ultimately interested in the real parts of these expressions, and so write

$$\begin{aligned}\mathfrak{Re}(\phi) &= (A+B)\cos(k(x-ct))\frac{\cosh(k(z+h))}{\cosh(kh)}\\ \mathfrak{Re}(\zeta) &= -\frac{(A+B)kc}{g}\sin(k(x-ct))\end{aligned}$$

It is convenient to write

$$\zeta(x,t) = a\sin(k(x-ct)), \tag{1.22}$$

where a is an arbitrary amplitude, with

$$a = -\frac{(A+B)kc}{g},$$

and then

$$\phi = \frac{-ag}{kc\cosh(kh)}\cos(k(x-ct))\cosh(k(z+h)). \tag{1.23}$$

We conclude that, in linear theory, the wave profile is a sinusoid, which may moreover have arbitrary amplitude – the constant a may be chosen freely without altering the solution. This is in stark contrast to nonlinear waves, where the amplitude plays an important role in the dynamics of the waves and the dispersion relation.

We can also find some information about what individual water particles are doing in the linear theory, using the basic ideas presented in Section A.4. The particle paths satisfy

$$\mathbf{x}' = \mathbf{u}(\mathbf{x},t)$$

(see Definition A.4.1), and we can obtain the fluid velocity field via $\mathbf{u} = \nabla\phi$. To simplify the algebra, take without loss of generality the potential to be

$$\phi(x,z,t) = \tilde{a}\cos(k(x-ct))\cosh(k(z+h)), \tag{1.24}$$

where all constants have been absorbed into $\tilde{a}$ and we keep only one of the sinusoids of the real part of the solution.

Then

$$\begin{aligned}u = \phi_x &= -\tilde{a}k\sin(k(x-ct))\cosh(k(z+h)),\\ w = \phi_z &= \tilde{a}k\cos(k(x-ct))\sinh(k(z+h)).\end{aligned}$$

While this means that the particle trajectory ODEs $dx/dt = u$, $dz/dt = w$ form a system of coupled, nonlinear ODEs, there is an easy way forward, which dates back to the mid-19th century work by George Green.[3] Since we are in a linear scenario, assume that the particles do not move far from their initial position (x_0, z_0) over the course of a wave period. Then the system

$$\frac{dx}{dt} = -\tilde{a}k \sin(k(x_0 - ct)) \cosh(k(z_0 + h)),$$
$$\frac{dz}{dt} = \tilde{a}k \cos(k(x_0 - ct)) \sinh(k(z_0 + h)),$$

may be immediately integrated to yield

$$x = \frac{-\tilde{a}}{c} \cos(k(x_0 - ct)) \cosh(k(z_0 + h)),$$
$$z = \frac{-\tilde{a}}{c} \sin(k(x_0 - ct)) \sinh(k(z_0 + h)).$$

We then find that

$$\left(\frac{x}{\cosh(k(z_0 + h))}\right)^2 + \left(\frac{z}{\sinh(k(z_0 + h))}\right)^2 = \frac{\tilde{a}^2}{c^2},$$

the equation of an ellipse. As $z_0 \to -h$, $\sinh(k(z_0 + h)) \to 0$, flattening the semi-axis b. This

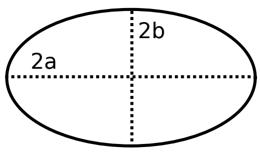


Figure 1.5: Plot of an ellipse $(x/a)^2 + (z/b)^2 = 1$, showing the semi-axes a and b.

means that the particle trajectories approach a line on the flat bed $z = -h$. Recalling that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

it is apparent that $\cosh(k(z_0 + h)) \approx \sinh(k(z_0 + h))$ for large values of the depth h, so that the particle trajectories are nearly circular. This fact can also be confirmed by considering the potential for infinite depth

$$\phi(x, z, t) = a \cos(k(x - ct))e^{kz}.$$

These conclusions are supported by experimental results (see Figure 1.6), but nonlinear effects modify the dynamics somewhat and must be taken into account for steeper waves.

The power of linear theory lies in the ability to superpose solutions, which allows us to construct all manner of surface waves. Thus, if

$$\phi_1(x, z, t) = a_1 \cos(k_1(x - c_1t) + \theta_1) \cosh(k_1(z + h))$$

and

$$\phi_2(x, z, t) = a_2 \cos(k_2(x - c_2t) + \theta_2) \cosh(k_2(z + h))$$

are two solutions to the linear system (1.17)–(1.20), where θ_i is an arbitrary phase shift, then $A\phi_1 + B\phi_2$ is also a solution for any scalars A, B. One special case occurs when both waves

[3] This approach persisted for more than 150 years. Only recently was the actual (nonlinear) system of particle trajectory ODEs analysed [CV08] and shown to have no closed paths.

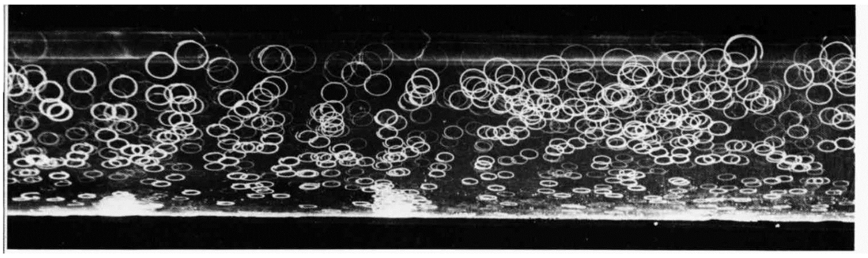


Figure 1.6: Particle trajectories in traveling water waves visualized via particles suspended in the water. From van Dyke, *An Album of Fluid Motion.*

have the same wavenumber and amplitude, but propagate in opposite directions (as occurs when waves are reflected fully by a wall). Then

$$\phi_1 = a\cos(k(x-ct)+\phi_1)\cosh(k(z+h))$$

is a right-moving wave,

$$\phi_2 = a\cos(k(x+ct)+\phi_1)\cosh(k(z+h))$$

a left-moving wave, and the superposition

$$\phi_1+\phi_2 = 2a\cos(kx)\cos(\omega t)\cosh(k(z+h))$$

is a *standing wave.* Recall that $c=\omega/k$.

1.2.1 Shallow-water waves

To set the scene for later developments, we will look at the most straightforward case of waves in shallow water. Since a great deal of human activity is concentrated near the coastline, we are often interested in describing waves in such settings. Moreover, we already expect (from the foregoing linear theory) that waves will be less dispersive (in a sense which should be specified) and trajectories of water particles flatter, when waves are in shallow depth.

Along these lines, and without being too systematic for the time being, two central simplifications allow some progress to be made with the equations:

1. As the waves are assumed very long compared to the water depth, we assume there is no vertical acceleration of the water;

2. Furthermore, we assume that the horizontal velocity is independent of the depth, so that the flow beneath the wave looks like a current.

We start from the full governing equations (1.1)–(1.7), which we reproduce here in 2D:

$$u_t + uu_x + wu_z = -P_x \tag{1.25}$$
$$w_t + uw_x + ww_z = -P_z - g \tag{1.26}$$
$$u_x + w_z = 0 \tag{1.27}$$
$$P = P_0 \text{ on } z=\zeta(x,t) \tag{1.28}$$
$$w = \zeta_t + u\zeta_x \text{ on } z=\zeta(x,t) \tag{1.29}$$
$$w = -uh_x \text{ on } z=-h(x) \tag{1.30}$$

On the sea-floor, which we now allow to vary with distance, writing $z = -h(x)$, we require there to be no normal velocity, so that the water particles must move along the surface $F(x, z) = z + h(x)$, implying $DF/Dt = 0$, which leads to (1.30)

The vertical acceleration is precisely Dw/Dt (see A.3), so the first condition may be substituted into (1.26) to yield

$$P = -g(z - \zeta) + P_0, \tag{1.31}$$

where we have used (1.28). Together with the second condition, that $u_z = 0$, this is substituted into (1.25):

$$u_t + uu_x = -g\zeta_x. \tag{1.32}$$

Now from integrating the equation of mass conservation (1.27), we find

$$w = -zu_x + C(x, t),$$

but since the boundary condition (1.30) must be fulfilled we fix the constant C and have

$$w = -(z + h)u_x - uh_x.$$

Substituting this into (1.29) yields

$$\zeta_t + u\zeta_x + (\zeta + h)u_x + uh_x = 0.$$

We thus have a system of two equations coupling u and ζ :

$$\begin{cases} \zeta_t + (u(\zeta + h))_x = 0, \\ u_t + uu_x + g\zeta_x = 0. \end{cases} \tag{1.33}$$

These are known as the *nonlinear shallow-water equations,* and are commonly used in a variety of applications in coastal engineering, including for models of tsunami propagation. If we assume that the depth h is constant, and linearise the equations (1.33), we find

$$\zeta_t + u_x h = 0,$$
$$u_t + g\zeta_x = 0,$$

and differentiating the first by t, the second by x, and combining these results we find

$$\zeta_{tt} = gh\zeta_{xx}. \tag{1.34}$$

So, after considerable labour, we have related the *wave equation* to the modelling of water waves, in shallow water over a flat bed. The factor gh corresponds to the square of the phase speed, which is consistent with what we found when deriving the dispersion relation for linear water waves in Section 1.2.

1.2.2 Energy and group velocity

We have mentioned the concepts of energy and group velocity in other contexts, but we shall set out here to give these a precise meaning and interpretation in the context of linear gravity waves. We mentioned in Section 1.1.1 that two definitions of velocity were possible in terms of the spatial and temporal parameters k and ω. The first, $c = \omega/k$ was established directly from the linear theory of waves in finite depth, and is given by the dispersion relation (1.21). Using this, we calculate the second:

$$\frac{d\omega}{dk} = \frac{d}{dk}\sqrt{gk\tanh kh} = \frac{1}{2}\sqrt{\frac{g}{k}\tanh kh}\left[1 + \frac{kh}{\sinh kh \cosh kh}\right] = \frac{c}{2}\left[1 + \frac{kh}{\sinh kh \cosh kh}\right].$$

In the limit of infinitely deep water, the second term in the brackets tends to zero, and we have

$$\frac{d\omega}{dk} = c/2,$$

so that the group velocity is exactly half the phase velocity.

From elementary physics, we know that the energy associated with a moving object under the force of gravity is partitioned into kinetic and potential energy, and

$$\begin{aligned}\text{Kinetic Energy} &= \frac{1}{2}mv^2\\ \text{Potential Energy} &= mgh\end{aligned}$$

where m is the mass of the object, v the velocity, g the gravitational acceleration, and h the height above some reference surface.

For water waves, it is not particularly enlightening to think about the energy of each individual water particle – rather, we want to have a measure for the energy "carried by the wave". The fact that energy is, in fact, propagated by waves is made clear when these waves dissipate that energy by crashing forcefully onto the coast. In analogy with the above, we write for the energy density per unit horizontal length[4]

$$\begin{aligned}\text{Kinetic Energy Density} &= \frac{1}{2}\rho\int_{-h}^{0}\mathbf{u}^2 dz,\\ \text{Potential Energy Density} &= \rho g\int_{0}^{\zeta} z dz.\end{aligned}$$

For the potential energy, we are interested only in the portion of the water that is elevated above the still-water level $z = 0$. For the kinetic energy, we must sum together all the square velocities in a vertical slice of the water – to leading order it suffices to consider the domain $z \in [-h, 0]$. Moreover, it is clear that the energy changes depending on the x-position of our slice of water: we may have more or less potential energy depending on whether we are closer to a wave crest or trough. We thus average over a wavelength by integrating in x and dividing by λ:

$$\bar{K} = \frac{\rho}{2\lambda}\int_0^{\lambda}\int_{-h}^{0}\mathbf{u}^2\, dzdx, \tag{1.35}$$

$$\bar{V} = \frac{\rho g}{\lambda}\int_0^{\lambda}\int_{0}^{\zeta} z\, dzdx. \tag{1.36}$$

Now recall the expressions for the potential and free surface elevation:

$$\zeta = a\sin(k(x-ct)) \tag{1.37}$$

$$\phi = B\cos(k(x-ct))\cosh(k(z+h)) \tag{1.38}$$

with $B = -ag/(kc\cosh(kh))$. Let us write $\xi = k(x-ct)$ and $\eta = k(z+h)$ for abbreviation. We recover

$$\begin{aligned}u &= \phi_x = -Bk\sin\xi\cosh\eta,\\ w &= \phi_z = Bk\cos\xi\sinh\eta.\end{aligned}$$

[4]We incorporate the density ρ here so that the physical units of the equations can be readily interpreted. This is lost when ρ is set to unity and quietly omitted.

To evaluate $\bar{K}$ it will be useful to recall the following indefinite integrals:

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4}$$
$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4}$$
$$\int \sinh^2 x\,dx = -\frac{x}{2} + \frac{\sinh x \cosh x}{2}$$
$$\int \cosh^2 x\,dx = \frac{x}{2} + \frac{\sinh x \cosh x}{2}$$

Then

$$\begin{aligned}
\bar{K} &= \frac{\rho}{2\lambda}\int_0^\lambda \int_{-h}^0 (u^2+w^2)dzdx \\
&= \frac{B^2k^2\rho}{2\lambda}\left(\int_0^\lambda \sin^2\xi dx \int_{-h}^0 \cosh^2 \eta dz + \int_0^\lambda \cos^2 \xi dx \int_{-h}^0 \sinh^2 \eta dz\right) \\
&= \frac{B^2\rho}{2\lambda}\left(\int_{-kct}^{k(\lambda-ct)} \sin^2\xi d\xi \int_0^{kh} \cosh^2\eta d\eta + \int_{-kct}^{k(\lambda-ct)} \cos^2\xi d\xi \int_0^{kh} \sinh^2 \eta d\eta\right) \\
&= \frac{B^2\rho}{2\lambda}\left(\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{-kct}^{2\pi-kct}\left[\frac{x}{2} + \frac{\sinh x \cosh x}{2}\right]_0^{kh}\right. \\
&\qquad \left. + \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-kct}^{2\pi-kct}\left[-\frac{x}{2} + \frac{\sinh x \cosh x}{2}\right]_0^{kh}\right) \\
&= \frac{B^2\rho\pi}{2\lambda}\sinh kh \cosh kh = \frac{a^2g^2}{4k^2c^2\cosh^2 kh}\rho k \sinh kh \cosh kh = \frac{a^2 g\rho}{4}.
\end{aligned}$$

where we have used the dispersion relation (1.21) in the last equality, and employed the fact that $k\lambda = 2\pi$.

We likewise compute

$$\begin{aligned}
\bar{V} &= \frac{\rho g}{\lambda}\int_0^\lambda \left[\frac{z^2}{2}\right]_0^\zeta dx = \frac{\rho g}{2\lambda}\int_0^\lambda a^2 \sin^2 \xi dx \\
&= \frac{\rho g a^2}{2\lambda k}\int_{-kct}^{2\pi-kct} \sin^2\xi d\xi = \frac{\rho g a^2}{4}.
\end{aligned}$$

which gives us a total mean energy density per wavelength

$$\boxed{\bar{V} + \bar{K} = \frac{\rho g a^2}{2}.}$$

This energy is equipartitioned between kinetic and potential energy.

In order to gain an appreciation for the energy carried by the waves, we will also calculate the amount of energy transported through a given control line over one wave period – this is what we refer to as the *energy flux,* and gives us a power (or energy/time) associated with a wave. We calculate it by considering the rate at which work is done by the fluid on one side of a vertical control line on the fluid on the other side. This rate is given by the dynamic pressure per unit width in the direction of propagation, or

$$E_F = \int_{-h}^0 pudz.$$

Averaging over a wave period, we find

$$\bar{E}_F = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-h}^{0} p\phi_x dzdt$$

and we use the linearised Bernoulli condition $\phi_t = -p/\rho$ to write

$$\begin{aligned}
\bar{E}_F &= \frac{-\rho\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-h}^{0}\phi_t\phi_x dzdt = \frac{\rho\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-h}^{0} B^2k^2c\sin^2\xi\cosh^2\eta dzdt \\
&= \frac{-\rho B^2k^2c\omega}{k^2c2\pi}\int_{kx}^{k(x-2\pi c/\omega)}\sin^2\tau d\tau\int_0^{kh}\cosh^2\eta d\eta \\
&= \frac{-\rho B^2\omega}{2\pi}\left[\frac{x}{2}+\frac{\sin 2x}{4}\right]_{kx}^{kx-2\pi}\left[\frac{x}{2}+\frac{\sinh x\cosh x}{2}\right]_0^{kh} \\
&= \frac{\rho B^2\omega}{2}\left[\frac{kh}{2}+\frac{\sinh kh\cosh kh}{2}\right] \\
&= \frac{\rho a^2g^2\omega}{4k^2c^2\cosh^2 kh}\left[kh+\sinh kh\cosh kh\right] = \frac{\rho a^2gc}{4}\left[1+\frac{kh}{\sinh kh\cosh kh}\right].
\end{aligned}$$

We recognize this last expression as being the total mean energy density per wavelength multiplied by the group velocity:

$$\bar{E}_F = \frac{d\omega}{dk}\left(\bar{V}+\bar{K}\right), \tag{1.39}$$

so that the rate of energy flow (with units [Energy]/[Time]) is equal to the energy density (with units [Energy]/[Length]) times the group velocity (with units [Length]/[Time]). The group velocity is thus the velocity at which the wave energy propagates, and for finite water depth it is *less* than the phase velocity of the individual waves.

This means that the overall energy of a group of waves is outrun by the waves themselves. Observe the waves created when a rock is thrown into a pond. These waves make up a *group* or *wave packet* forming a ring. An individual wave propagates outward from the inner edge of the group to the outer edge, where it disappears (see Figure 1.7), as it has outrun the energy of the group.

Figure 1.7: Concentric waves in water.

We can get some insight into the behaviour of such a wave group by considering the superposition of two wave-trains. Let

$$\zeta_1 = a\cos(k_1x-\omega_1t) \text{ and } \zeta_2 = a\cos(k_2x-\omega_2t),$$

and assume

$$k_1 = k - \frac{\Delta k}{2},\ k_2 = k + \frac{\Delta k}{2},\ \text{with } \Delta k \ll k.$$

Then, we compute the Taylor series expansion of ω:

$$\omega(k_1) = \omega(k) - \frac{\Delta k}{2}\omega'(k) + \dots$$

and

$$\omega(k_2) = \omega(k) + \frac{\Delta k}{2}\omega'(k) + \dots$$

Now assume that the free surface is composed of a superposition of the two waves $\zeta_1 + \zeta_2$, see Figure 1.8.

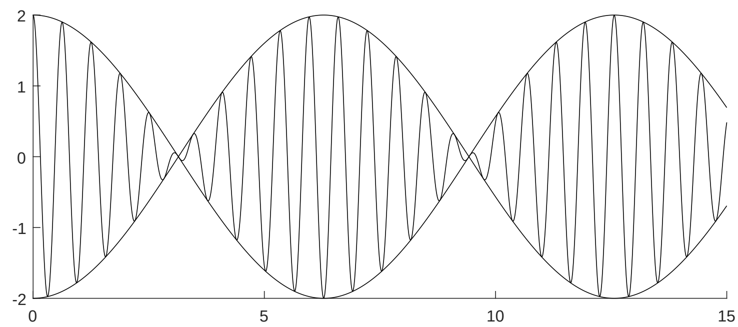


Figure 1.8: Superposition of two waves with nearby frequencies, plotted together with the envelope.

We compute, using some addition theorems for trigonometric functions

$$\begin{aligned}\zeta &= \zeta_1 + \zeta_2 = a(\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t))\\ &= 2a\cos\left(\frac{(k_1+k_2)x - (\omega_1+\omega_2 t)}{2}\right)\cos\left(\frac{(k_1-k_2)x - (\omega_1-\omega_2 t)}{2}\right)\end{aligned}$$

and find, to leading order in Δk

$$\zeta = 2a\cos(kx - \omega t))\underbrace{\cos(\frac{\Delta k}{2}(x - \omega' t))}_{\text{envelope}}.$$

So the resulting free surface consists of a wave $\cos(kx - \omega t)$ modulated by an envelope that moves with speed $\omega'(k)$, exactly the group velocity.

A slightly more sophisticated approach to the same problem involves considering a superposition of a continuum of waves with some narrow band of wavelengths centred around $k = k_0$. This means moving to a Fourier integral description, and writing

$$\zeta(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k)\exp(i(kx - \omega(k)t)dk, \tag{1.40}$$

where we will assume $\Delta k/k_0 \ll 1$ in accordance with our "narrowness" assumption. Then we can expand the dispersion relation about the carrier wavenumber k_0 as follows:

$$\omega(k) = \omega(k_0 + (k - k_0)) = \omega(k_0) + (k - k_0)\omega'(k_0) + \dots, \tag{1.41}$$

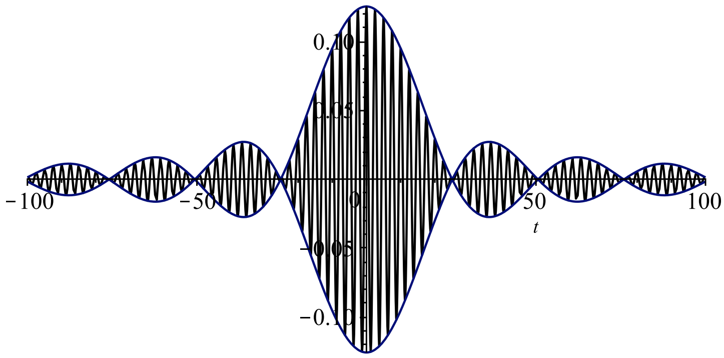


Figure 1.9: Plot of the free surface versus time t at $x = 0$ from (1.42), with $A(k_0) = 1$, $\lambda_0 = 10$, $\Delta k = 0.1k_0$.

where we will call $\omega(k_0) = \omega_0$ and recognize that $\omega'(k_0) =: c_{g,0}$ is the group velocity associated with mode k_0. We will now use a dirty trick, in assuming that the amplitude spectrum $A(k)$ in the integrand doesn't vary very much over the domain of integration, and can be extracted from the integral. This is a crude approximation, and should not be taken too seriously! We write, then (using $k = k_0 + (k - k_0)$)

$$\begin{aligned}
\zeta(x,t) &\approx A(k_0)\int_{k_0-\Delta k}^{k_0+\Delta k} \exp(i((k_0 + (k-k_0))x - (\omega_0 + (k-k_0)c_{g,0})t)dk \\
&= A(k_0)e^{i(k_0x-\omega_0 t)}\int_{k_0-\Delta k}^{k_0+\Delta k} \exp(i(k-k_0)(x - c_{g,0}t)dk \\
&= A(k_0)e^{i(k_0x-\omega_0 t)}\int_{-\Delta k/k_0}^{\Delta k/k_0} k_0\exp(ik_0\xi(x - c_{g,0}t))d\xi
\end{aligned}$$

where we use the substitution $(k - k_0)/k_0 = \xi$ in the last equality. Evaluating the elementary integral yields

$$\zeta(x,t) = \underbrace{\frac{2A(k_0)}{x - c_{g,0}t}\sin(\Delta k(x - c_{g,0}t))}_{:=\tilde{A}}\, e^{i(k_0x-\omega_0 t)}. \tag{1.42}$$

This clearly needs some interpretation! The first term $\tilde{A}$ can be viewed as a slowly varying envelope which multiplies the sinusoidal wave $\exp(i(k_0x - \omega_0 t)$. The speed of propagation of the envelope is clearly the group velocity $c_{g,0}$, and the distance between adjacent nodes of the envelope is approximately $\pi/\Delta k$, much greater than the length of the constituent (carrier) waves $2\pi/k_0$. See Figure 1.9 for a plot for certain parameter values.

Chapter 2

The Korteweg de Vries equation and shallow water waves

2.1 Governing equations in shallow water

We have seen in dealing with the linear problem that the assumption of shallow water means we must make some mathematical restrictions on the flow. In particular, we assumed in the derivation of the shallow water equations that the horizontal velocity component was independent of the depth z. This means that we can identify the horizontal velocity with the values it takes on the (flat) bed, or any other convenient level.

In contrast to the development of the linear theory in Chapter 1, we will work with the potential-flow problem throughout for ease of use. In this case our irrotationality and mass conservation conditions are encapsulated in the Laplace equation

$$\phi_{xx} + \phi_{zz} = 0,$$

which we can integrate over the water depth

$$-\int_{-h}^{z} \phi_{xx} dz + C(x,t) = \phi_z,$$

and use the kinematic boundary condition on the bed $\phi_z = 0$ to eliminate $C(x,t)$. A second integration yields

$$-\int_{-h}^{z}\int_{-h}^{z'} \phi_{xx} dz dz' + C(x,t) = \phi. \tag{2.1}$$

So far no approximations have been made, and we can observe that $\phi^b(x,t) := \phi(x, z = -h, t) = C(x,t)$ for the potential at the bed.

If we now impose the assumption that the horizontal velocity does not vary with depth, we can take the term ϕ_{zz} out of the integral, to obtain a first approximation:

$$\phi(x,z,t) \approx \phi^b - \frac{\phi^b_{xx}}{2}(z+h)^2.$$

There is another, more synthetic approach, which has the advantage of making the reasoning above watertight: the potential ϕ is harmonic (meaning it satisfies the Laplace equation), therefore also *real analytic* (meaning it is locally represented by a convergent power series). This means that we can simply elect to expand ϕ in a power series as

$$\phi(x,z,t) = \sum_{n=0}^{\infty}(z+h)\phi_n$$

with $\phi_n = \phi_n(x,t)$. This already shows how the next terms in the Boussinesq approximations can be expected to look!

No matter how we approach it, this is the main idea behind the so-called Boussinesq equations: obtain a simple structure for the depth-dependence of the flow, and thereby find simplified model equations which can be employed in shallow waters. This also lays bare why such models are called *depth-integrated:* the specifics of the vertical velocity are integrated (or averaged).

2.1.1 Derivation of a Boussinesq equation

We reproduce the governing equation for two dimensional potential flow:

$$\phi_{xx} + \phi_{zz} = 0, \tag{2.2}$$
$$\phi_z = 0 \text{ on } z = -h, \tag{2.3}$$
$$\zeta_t + \phi_x \zeta_x - \phi_z = 0 \text{ on } z = \zeta, \tag{2.4}$$
$$g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \phi_t = 0 \text{ on } z = \zeta. \tag{2.5}$$

In order to progress with this system we will try to write everything in terms of two dimensionless parameters $\mu = h/\lambda$ and $\delta = A/h$. The former is a kind of "long wavelength" or "shallowness" parameter, while the latter is a kind of "amplitude" parameter.

We will insert the following nondimensionalisation:

$$x = \lambda x',\ z = hz',\ \zeta = A\zeta'\ t = \frac{\lambda}{c_0}t',\ \phi = \frac{Ac_0}{\mu}\phi',$$

where we (looking forward to treating shallow water waves) will use a characteristic speed $c_0 = \sqrt{gh}$. Inserting this into the governing equations (2.2)–(2.5) yields a dimensionless system of the form

$$\phi_{xx} + \frac{1}{\mu^2}\phi_{zz} = 0, \tag{2.6}$$
$$\phi_z = 0 \text{ on } z = -1, \tag{2.7}$$
$$\zeta_t + \delta\phi_x\zeta_x - \frac{1}{\mu^2}\phi_z = 0 \text{ on } z = \delta\zeta, \tag{2.8}$$
$$\zeta + \delta\frac{1}{2}\left(\phi_x^2 + \frac{1}{\mu^2}\phi_z^2\right) + \phi_t = 0 \text{ on } z = \delta\zeta. \tag{2.9}$$

We have suppressed the primes throughout for readability. Note that the dimensionless, flat bed is now denoted by $z = -1$.

To progress, we will expand the potential about the bed as

$$\phi(x,z,t) = \sum (z+1)^n \phi_n(x,t) = \phi_0(x,t) + (z+1)\phi_1(x,t) + \dots \tag{2.10}$$

We insert this expansion into the Laplace equation (2.6) to find, order by order in the polynomial

$(z+1)$

$$
\begin{aligned}
&\phi_{0xx} + \frac{2}{\mu^2}\phi_2 = 0\\
&\phi_{1xx} + \frac{6}{\mu^2}\phi_3 = 0\\
&\phi_{2xx} + \frac{12}{\mu^2}\phi_4 = 0\\
&\vdots\\
&\phi_{(n-2)xx} + \frac{n(n-1)}{\mu^2}\phi_n = 0
\end{aligned}
$$

Inserting into the bottom boundary condition (2.7) we find

$$\phi_1 + 2(z+1)\phi_2 + 3(z+1)^2\phi_3 + \ldots = 0 \text{ on } z = -1.$$

This means that ϕ_1, and therefore all odd terms ϕ_{2n+1} are zero in the recursion relation, we therefore obtain

$$
\begin{aligned}
\phi_2 &= -\frac{\mu^2}{2}\phi_{0xx}\\
\phi_4 &= -\frac{\mu^2}{12}\phi_{2xx} = \frac{\mu^4}{24}\phi_{0xxxx}\\
&\vdots
\end{aligned}
$$

and thus

$$\phi = \phi_0 - \frac{\mu^2}{2}(z+1)^2\phi_{0xx} + \frac{\mu^4}{24}(z+1)^4\phi_{0xxxx} + \ldots \tag{2.11}$$

specifies the vertical structure of the potential explicitly. Note that this is one of many possible assumptions, and the expansion about the bed $z = -1$ is only one possibility.

We need to insert this expansion into the two boundary conditions at the free surface (2.8)–(2.9). To this end, it is useful to gather the identities

$$
\begin{aligned}
\phi_t &= \phi_{0t} - \frac{\mu^2}{2}(z+1)^2\phi_{0xxt} + \frac{\mu^4}{24}(z+1)^4\phi_{0xxxxt} + \ldots\\
\phi_x &= \phi_{0x} - \frac{\mu^2}{2}(z+1)^2\phi_{0xxx} + \frac{\mu^4}{24}(z+1)^4\phi_{0xxxxx} + \ldots\\
\phi_z &= -\mu^2(z+1)\phi_{0xx} + \frac{\mu^4}{6}(z+1)^3\phi_{0xxxx} + \ldots
\end{aligned}
$$

This gives the boundary conditions

$$\begin{aligned}
&\zeta_t + \delta\zeta_x \left(\phi_{0x} - \frac{\mu^2}{2}(\delta\zeta+1)^2\phi_{0xxx} + \frac{\mu^4}{24}(\delta\zeta+1)^4\phi_{0xxxxx} \right) \\
&\quad - \frac{1}{\mu^2}\left(-\mu^2(\delta\zeta+1)\phi_{0xx} + \frac{\mu^4}{6}(\delta\zeta+1)^3\phi_{0xxxx} \right) = 0, \qquad (2.12) \\
&\zeta + \phi_{0t} - \frac{\mu^2}{2}(\delta\zeta+1)^2\phi_{0xxt} + \frac{\mu^4}{24}(\delta\zeta+1)^4\phi_{0xxxxt} \\
&\quad + \frac{1}{2}\delta\left\{ \left(\phi_{0x} - \frac{\mu^2}{2}(\delta\zeta+1)^2\phi_{0xxx} + \frac{\mu^4}{24}(\delta\zeta+1)^4\phi_{0xxxxx} \right)^2 \right. \\
&\quad \left. \frac{1}{\mu^2}\left(-\mu^2(\delta\zeta+1)\phi_{0xx} + \frac{\mu^4}{6}(\delta\zeta+1)^3\phi_{0xxxx} \right)^2 \right\} = 0. \qquad (2.13)
\end{aligned}$$

The situation looks complicated because of the two ordering parameters μ and δ which appear in the problem, but we are tacitly thinking of a relationship $\delta = O(\mu^2)$, and will thus neglect all terms $O(\mu^4, \delta\mu, \delta^2)$. This assumption gets rid of the vast majority of terms, and leaves us with a relatively simple system of equations in two variables ζ and ϕ_0 only:

$$\zeta_t + \delta\zeta_x\phi_{0x} + (\delta\zeta+1)\phi_{0xx} - \frac{\mu^2}{6}\phi_{0xxxx} = 0, \qquad (2.14)$$

$$\zeta + \phi_{0t} - \frac{\mu^2}{2}\phi_{0xxt} + \frac{1}{2}\delta\phi_{0x}^2 = 0. \qquad (2.15)$$

This is one form of the famous *Boussinesq equations*. Differentiating the second equation in x allows us also to write this equation in terms of the bottom velocity u_0:

$$\zeta_t + \delta\zeta_x u_0 + (\delta\zeta+1)u_{0x} - \frac{\mu^2}{6}u_{0xxx} = 0, \qquad (2.16)$$

$$\zeta_x + u_{0t} - \frac{\mu^2}{2}u_{0xxt} + \frac{1}{2}\delta(u_0^2)_x = 0. \qquad (2.17)$$

Either formulation shows that, to lowest order in μ and δ the system is simply the linear shallow water equations. Indeed, this has the consequence that

$$\zeta_t = -\phi_{0xx} + O(\delta, \mu^2),$$

and

$$\zeta = -\phi_{0t} + O(\delta, \mu^2),$$

which allows us to trade derivatives in x and t on ϕ and ζ, and so give rise to numerous different (but asymptotically equivalent) Boussinesq equations! Not to mention the fact that our expansion about the bed was an arbitrary choice, and also expanding about the mean water level $z = 0$ leads to equations called Boussinesq equations! To make a long story short: there is a zoo of Boussinesq equations, and a small industry devoted to deriving them.

Before moving on, we note that the Boussinesq equations (at least in the current form) can also be written as a single equation in either ϕ or ζ. Since both u_0 and ζ satisfy a wave equation to lowest order, if we restrict ourselves to waves travelling in a single direction, e.g.

$$u_0(x,t) = f(x-t),$$

we know that $u_{0x} = -u_{0t}$ to lowest order, and likewise for ζ. In particular, this implies $\zeta = u_0 + O(\delta, \mu^2)$. Differentiating (2.16) with respect to t, and differentiating (2.17) with respect to x yields the system

$$\begin{aligned}
&\zeta_{tt} + \delta\zeta_{xt}u_0 + \delta\zeta_x u_{0t} + (\delta\zeta + 1)u_{0xt} + \delta\zeta_t u_{0x} - \frac{\mu^2}{6}u_{0xxxt} = 0,\\
&\zeta_{xx} + u_{0xt} - \frac{\mu^2}{2}u_{0xxxt} + \frac{1}{2}\delta(u_0^2)_{xx} = 0.
\end{aligned}$$

Subtracting the equations leads to

$$\zeta_{tt} - \zeta_{xx} + \delta(u_0\zeta)_{xt} - \frac{1}{2}\delta(u_0^2)_{xx} + \frac{1}{3}\mu^2 u_{0xxxt} = 0,$$

where we can swap out the remaining u_0 terms (to this order of approximation) with ζ to obtain

$$\zeta_{tt} - \zeta_{xx} + \delta(\zeta^2)_{xt} - \frac{1}{2}\delta(\zeta^2)_{xx} + \frac{1}{3}\mu^2\zeta_{0xxxx} = 0, \tag{2.18}$$

a Boussinesq equation in the single variable ζ.

2.1.2 Derivation of the KdV

We have seen that there is a large class of related Boussinesq equations, and a huge literature devoted to their study. We want to pick out a special case of an equation arising from the Boussinesq equations, which has a particularly nice structure. Our starting point is the equation (2.18). Recall that we are in the setting of unidirectional propagation, and that we will therefore move to a coordinate $\xi = x - t$ moving with the wave. Moreover, we introduce a slow time scale $\tau = \delta t$, so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \delta\frac{\partial}{\partial \tau}.$$

Inserting this into (2.18) and integrating once with respect to ξ leads to the equation

$$\zeta_\tau + \frac{3}{2}\zeta\zeta_\xi - \frac{\mu^2}{6\delta}\zeta_{\xi\xi\xi} = 0, \tag{2.19}$$

named after Korteweg and de Vries, who first derived it in [KD95]. Note that we can return to a fixed reference frame using the transformation

$$x = \xi + \tau/\delta,\ t = \tau/\delta,$$

which gives us the KdV in the form

$$\zeta_t + \zeta_x + \frac{3}{2}\delta\zeta\zeta_x - \frac{\mu^2}{6}\zeta_{xxx} = 0.$$

In all cases, we should remember that this equation was derived under assumptions of shallow water (depth integration), retaining terms such that nonlinearity and dispersion balance one another out, and on a long time-scale. It is possible to derive the KdV directly from the governing equation with these assumptions, without the scenic detour into the Boussinesq equations.

2.2 Properties of the KdV

2.2.1 Effects of quadratic nonlinearity

The nonlinear terms in the KdV are of the form

$$u_t + uu_x = 0, \tag{2.20}$$

which is exactly the so-called *inviscid Burgers equation* (also called Euler's equation). We can see how this behaves explicitly using the method of characteristics, and determine from it one of the key properties of nonlinearity.

Recall that a general first-order quasi-linear[1] PDE (here in 2D for geometric accessibility) can be written as

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

This is equivalent to the scalar product

$$(a, b, c) \cdot (u_x, u_y, -1) = 0,$$

and we can recall that the solution surface $z = u(x, y)$ has normal vector $(u_x, u_y, -1)$ the first vector (a, b, c) must be tangent to the solution surface everywhere. This means we have the system of characteristic equations

$$x'(t) = a, \tag{2.21}$$

$$y'(t) = b, \tag{2.22}$$

$$z'(t) \equiv u'(t) = c. \tag{2.23}$$

These represent families of curves along the solution surface, and we should specify parametrically where the curves start by giving initial conditions

$$x(t = 0, s) = x_0(s),$$
$$y(t = 0, s) = y_0(s),$$
$$u(t = 0, s) = u_0(s).$$

One simple case is the linear transport equation, which we write as

$$U_T + cU_X = 0. \tag{2.24}$$

We have used upper case variables to avoid confusion. Typically we want to specify an initial profile, something like $U(T = 0, X) = f(X)$. The system of characteristics for the equation is

$$T'(t) = 1, \tag{2.25}$$

$$X'(t) = c, \tag{2.26}$$

$$U'(t) = 0. \tag{2.27}$$

The initial condition (written parametrically) is

$$T(t = 0, s) = 0,$$
$$X(t = 0, s) = s,$$
$$U(t = 0, s) = f(s).$$

[1] This is quasi-linear since the derivatives of the unknown u appear only linearly.

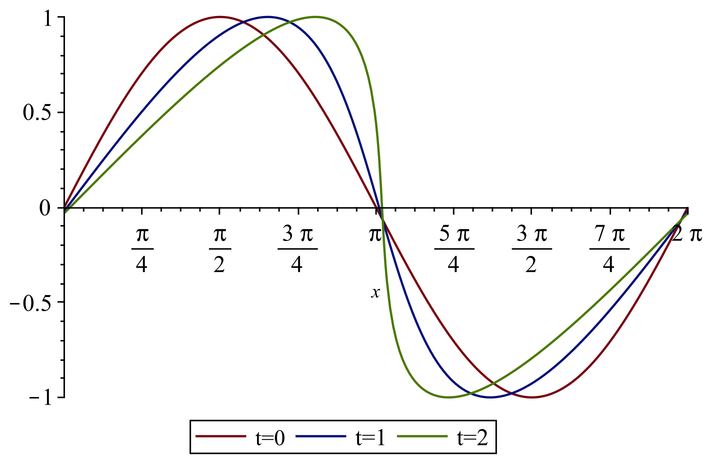


Figure 2.1: Solutions of the nonlinear shallow water equations with (dimensionless) time t, starting from the initial condition $\sin(x)$ at $t = 0$. Typical steepening of the wave profile is observed as time progresses.

Solving the system of characteristics gives $T(t) = t + f(s)$, $X(t) = ct + g(s)$ and $U(t) = h(s)$. Now we insert the initial condition in parametric form to find $f(s) = 0$, $g(s) = s$ and $f(s) = h(s)$. Therefore $T = t$, $X = ct + s$, and $s = X - cT$. Finally, we can invert to obtain the expected solution

$$U(T, X) = f(s) = f(X - cT).$$

The inviscid Burgers equation (2.20) is formally similar, with characteristics

$$T'(t) = 1, \tag{2.28}$$
$$X'(t) = U, \tag{2.29}$$
$$U'(t) = 0, \tag{2.30}$$

and identical initial data. Here we find $X(t) = Ut + s$, so that $s = X - UT$ and we have the implicit solution

$$U(T, X) = f(X - UT).$$

If we think of $U(T, X)$ as the profile of a wave, we can see that this solution will generically steepen and eventually lead to a "breaking wave" where the slope becomes infinite: the profile slope is given by $U_X = f'(1 - U_x T)$, where $f' = f'(X - UT)$ is the total derivative. Rearranging gives

$$U_X = \frac{f'}{1 + Tf'}.$$

We see that the denominator tends to zero (the solution blows up) at the critical time $T_c = -1/f'$. (See Figure 2.1.)

2.2.2 The effects of linear, dispersive terms

If we remove the nonlinear terms from the KdV (written in a moving coordinate frame) we are left with the simple dispersive linear equation

$$u_t + u_{xxx} = 0. \tag{2.31}$$

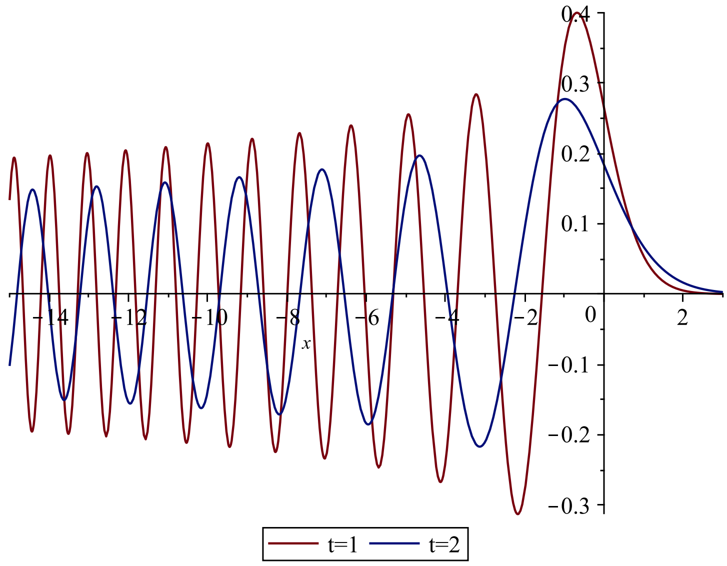


Figure 2.2: Dispersion of initial data from the linear KdV with dimensionless time. Initial data is a Dirac delta distribution.

Inserting the harmonic wave ansatz $\exp(i(kx + \omega t))$ immediately shows that the equation is dispersive with dispersion relation

$$\omega = k^3.$$

We can understand the consequences of dispersion by solving the equation for sufficiently localised initial data using Fourier transforms. One example is shown in Figure 2.2, where the equation is solved for a Dirac delta initial condition.

2.3 Solutions of the KdV

2.3.1 The travelling wave ansatz

We can seek travelling wave solutions of the KdV

$$2\zeta_t + 3\zeta\zeta_x + \frac{1}{3}\zeta_{xxx} = 0$$

simply by plugging in the appropriate Ansatz $\zeta = f(x - c_0 t)$ and checking to see whether the equation can be satisfied. If it succeeds – great! If not – i.e. there are no solutions – then we know that the KdV will not support waves of permanent form travelling at a fixed speed.

Conveniently, the travelling wave ansatz turns our KdV into an ordinary differential equation in the unknown function $f(\xi)$, with $\xi = x - c_0 t$:

$$-2c_0 f' + 3ff' + \frac{1}{3}f''' = 0. \tag{2.32}$$

Notice that

$$ff' = \frac{d}{d\xi}\left(\frac{f^2}{2}\right),$$

so that each term in the equation is a derivative, and the equation can be immediately integrated to yield

$$-2c_0 f + \frac{3}{2}f^2 + \frac{1}{3}f'' = C. \tag{2.33}$$

After multiplying this equation by f' another integration in ξ is possible, yielding

$$-c_0 f^2 + \frac{1}{2} f^3 + \frac{1}{6} (f')^2 = Cf + D, \tag{2.34}$$

where C and D are the undetermined constants of integration. Reordering means that we find

$$\frac{1}{6} (f')^2 = c_0 f^2 - \frac{1}{2} f^3 + Cf + D, \tag{2.35}$$

which we write as $(f')^2 = p_3(f)$ where $p_3(f)$ denotes a cubic polynomial in the unknown function f.

This result is a differential equation involving the root of a polynomial. It looks terrible, but is actually good news! The equation is separable, and can be written as

$$\int_{f_0}^{f} \frac{df}{\sqrt{p_3(f)}} = \xi - \xi_0, \tag{2.36}$$

and experience teaches us that solutions can be found using elliptic functions. Before we do so, it is worth thinking about what the equation is telling us. Let's write the cubic $p_3(f)$ as

$$p_3(f) = (f-a)(f-b)(f-c)$$

for three (possibly repeated, possibly complex) roots. Consider such a cubic polynomial (figure 2.3) Since f represents the free surface elevation (a physical quantity) only the regions where p_3

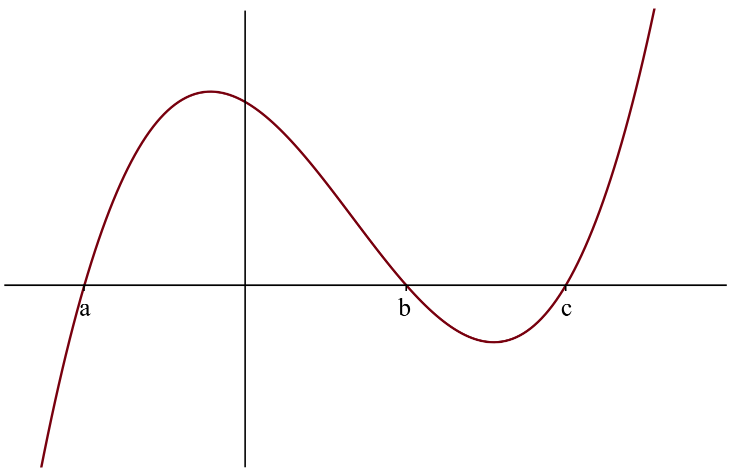


Figure 2.3: An example of a cubic polynomial with three distinct real roots.

is non-negative are relevant (between a and b and beyond c in figure 2.3).

If we start with an initial condition $a < f_0 < b$ at ξ_0,

$$f' = \pm\sqrt{p_3(f_0)}.$$

Let us choose the positive root, so that f increases. If the integral from f_0 to b is finite, we will reach the right fixed point b at a finite value of ξ.[2] What happens when we get there?

Notice that we have $(f')^2 = p_3(f)$, so that

$$f'' = \frac{1}{2} p_3'(f),$$

[2] This is different from the standard situation with smooth right-hand side, where the approach to a fixed point is asymptotic and would require $\xi \to \infty$. Here we reach the fixed point b for a finite value of ξ.

which indicates that $f''(b) \neq 0$ iff $p_3'(b) \neq 0$. If p_3 has three simple roots, such that a, b and c are distinct, then $p_3(b) = 0 \Longrightarrow p_3'(b) \neq 0$, so that our solution cannot stay at the fixed point b for all subsequent times ξ. On the other hand, if b is a repeated root of the polynomial, the second derivative vanishes and we have no such restriction. This gives us important qualitative information: *the constants of integration in the KdV determine a cubic polynomial. If the polynomial has three distinct real roots, we expect periodic solutions. If the polynomial has a repeated real root we can expect a solution with infinite period.*

We now proceed to the classification of these solutions based on assumptions on the waves.

2.3.2 Special solutions for periodic and solitary waves

Cnoidal wave solutions of the KdV We will look first at the more generic-seeming case where the polynomial p_3 has three distinct roots. We call these roots a, b and c and assume they are ordered as in Figure 2.3, $a < b < c$, with $p_3(f) > 0$ for $f \in (a, b)$. In this case, we can find the wavelength (or period) by considering the (bounded) integral

$$\int_a^b \frac{df}{\sqrt{(f-a)(f-b)(f-c)}} = \xi_L. \tag{2.37}$$

This is the distance to go from root a to root b, and twice the distance, i.e. 2ξ is the wavelength of the solution. The amplitude of the motion can be denoted

$$\alpha = b - a, \tag{2.38}$$

recalling that f is the free surface variable.

We already know that Jacobian elliptic functions will be involved in the representation of the solution, and the smart move is to consult a textbook (e.g. Byrd & Friedman [BF13]). It may be useful to look (at least once in a lifetime) at the transformations necessary to go from the general equation involving the root of a cubic polynomial to the standard form where the denominator is $\sqrt{1 - k^2 \sin^2 x}$.

To this end, we introduce a new variable

$$g = f/a,$$

where the choice of normalisation is not random (i.e. we should not select b or c). Moreover, we want to introduce a phase-shift (change the function g by a constant, which will have no effect on the differential equation) such that $g(\xi = 0) = 1$, i.e. at $\xi = 0$ we are at the point $f = b$. Then we obtain the normalised differential equation for g

$$(g')^2 = a(g-1)(g-g_1)(g-g_2), \qquad g_1 = \frac{b}{a},\ g_2 = \frac{c}{a}. \tag{2.39}$$

Introduce a new variable h, related to g via

$$g = 1 + (g_1 - 1)\sin^2 h. \tag{2.40}$$

Note that $g(0) = 1$ implies $h(0) = 0$. Plugging this in to the preceding equation and collecting terms gives

$$(h')^2 = \frac{-a(1-g_2)}{4}\left(1 - \frac{1-g_1}{1-g_2}\sin^2 h\right), \tag{2.41}$$

from which the outlines of the standard form are visible. Defining

$$l := \frac{-a(1-g_2)}{4} = \frac{c-a}{4} > 0, \quad k^2 := \frac{1-g_1}{1-g_2} = \frac{b-a}{c-a} \leq 1$$

we finally have

$$(h')^2 = l(1 - k^2 \sin^2 h) \Longrightarrow \sqrt{l}\xi = \int_0^h \frac{ds}{\sqrt{1 - k^2 \sin^2(s)}}. \tag{2.42}$$

We have already normalised such that $h(\xi = 0) = 0$.

Recalling that the elliptic function $y = \mathrm{sn}(x; k)$ is defined via

$$x = \int_0^{\arcsin(y)} \frac{ds}{\sqrt{1 - k^2 \sin^2 s}}$$

we immediately see that

$$\sin(h) = \mathrm{sn}(\sqrt{l}\xi; k).$$

Transforming back from h to g and from g to f allows this to be written in the form

$$f = a + (b - a)\mathrm{sn}^2(\sqrt{l}\xi; k),$$

or, using the identity $\mathrm{sn}^2(x; k) + \mathrm{cn}^2(x; k) = 1$,

$$f = b + (a - b)\mathrm{cn}^2(\sqrt{l}\xi; k), \tag{2.43}$$

where we recall $l = (c - a)/4$.

What does this mean? f is the free surface elevation written in terms of the travelling wave coordinate ξ, and the cnoidal functions cn can be computed explicitly, but the solution is still hard to interpret. It helps if we relate this to physically meaningful quantities. We recall that the wave amplitude was defined as the distance between the two fixed points a and b, so this provides one scale. Another scale is provided by the wavelength ξ_L, which is the integral of the polynomial from $f = a$ to $f = b$. In terms of transformed variables we find that this is the integral from $g = 1$ to $g = g_1$ or the integral from $h = 0$ to $h = \arcsin(1) = \pi/2$. In standard form this means that

$$\sqrt{l}\xi_L = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - k^2 \sin^2(s)}},$$

where we recognise the complete elliptic integral $K(k)$. The wavelength, being

$$\lambda = 2\xi = \frac{2K(k)}{\sqrt{l}} = \frac{4K(k)}{\sqrt{c - a}},$$

where $K(k)$ is known as the quarter period of the Jacobian elliptic function with modulus k.

Finally, it is useful to look back at the derivation of the polynomial expression $p_3(f)$, where we find that the wave speed in the KdV c_0 is related to the three roots a, b and c as :

$$6c_0 = -(a + b + c) \tag{2.44}$$

The outcome of our work is therefore: we have **three roots** a, b and c and **three physical quantities**: wave amplitude, wavelength (or period), and wave speed. Together these define everything we need to know about our periodic, travelling wave solutions.

The small amplitude limit of this nonlinear theory is obtained when $\alpha = b - a \ll 1$, or when the two roots a and b are very close. This implies that $k = \sqrt{(b - a)/(c - a)}$ is very small, and that the complete elliptic integral $K(k) \longrightarrow K(0) = \pi/2$, while the Jacobi elliptic function $\mathrm{sn}(x; k) \longrightarrow \mathrm{sn}(x; 0) = \sin(x)$. This behaviour can also be seen in Figure 2.4, where the parameter k in the solution is varied synthetically to demonstrate the linear/nonlinear regimes.

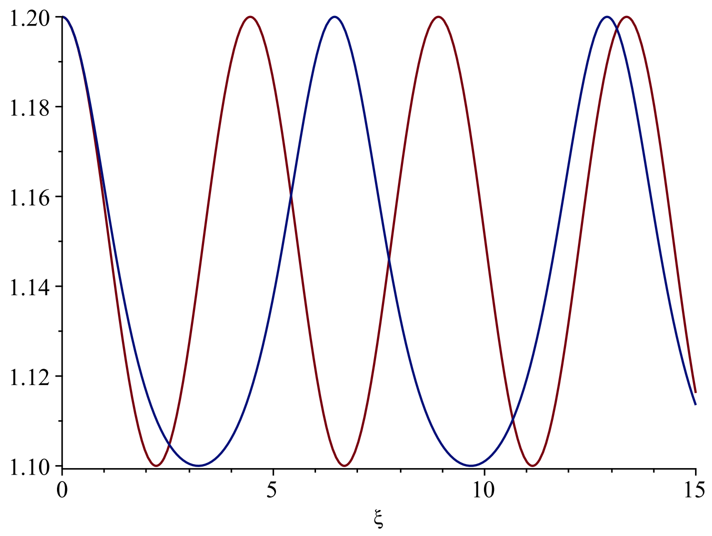


Figure 2.4: Plot of a cnoidal wave solution with wavelength k varied synthetically. Red curve: $k \ll 1$, the shape approaches a sinusoid. Blue curve: $k = O(1)$, the wave shape has distinctly flatter troughs and sharper crests, as expected in nonlinear theory.

2.3.3 Soliton solutions of the KdV

We have already mentioned above that we can expect (purely from the dynamics of the resulting ODE) to find non-periodic travelling-wave solutions of the KdV when two roots coincide. In the notation of the above section, this means that $b = c$ is a double root, and we have

$$\int_{f_0}^{f} \frac{df}{(f-b)\sqrt{(f-a)}} = \xi - \xi_0, \tag{2.45}$$

which can be integrated explicitly without trouble (for example by Maple or Mathematica, or using suitable substitutions). Unfortunately such integration leaves a large number of parameters, and it is difficult to get insight into which parameter combinations give sensible behaviour.

The more appropriate strategy is to start with some physical considerations: let us specify that we are looking for non-periodic waves which are "solitary", i.e. present as an isolated hump on the surface of the water. Such waves were observed by J. Scott Russell in 1834 [Rus44] and caused considerable uproar in the fluid dynamics community of the time. To this end, let us specify that we would like the water to be at rest "at infinity", i.e. far away from the wave form. This is sensible, since we are travelling with the wave in a moving frame ξ, so "away from the wave" is well defined. If the water is at rest we would like the surface disturbance f, or equivalently ζ, to vanish, as should all derivatives. In the derivation leading to (2.35) this means that the constants of integration D and C must disappear, leaving

$$(f')^2 = 6c_0 f^2 - 3f^3 = f^2(6c_0 - 3f). \tag{2.46}$$

From elementary calculus (or the plot Figure 2.6) we see that a real solution – with f positive between the two roots at 0 and $2c_0$ – requires $c_0 > 0$. Inserting the change of variables

$$f = \frac{2c_0}{g^2}$$

into (2.46) we find, after some simple manipulations

$$(g')^2 = \frac{3c_0}{2}\left(g^2 - 1\right),$$

which leads to the elementary integral

$$\int_{g_0}^{g} \frac{dg}{\sqrt{g^2-1}} = \sqrt{\frac{3c_0}{2}} \int_{\xi_0}^{\xi} d\xi,$$

whose left-hand side immediately yields $\operatorname{arccosh}(g) - \operatorname{arccosh}(g_0)$. Taking $\xi_0 = 0$, $g_0 = g(0) = 1$ and so $f(0) = 2c_0$ we find

$$f = 2c_0 \operatorname{sech}^2\left(\sqrt{\frac{3c_0}{2}}\xi\right). \tag{2.47}$$

This is the famous solitary wave solution of the KdV (see Figure 2.5), with a characteristic dependence of the celerity on the wave amplitude.

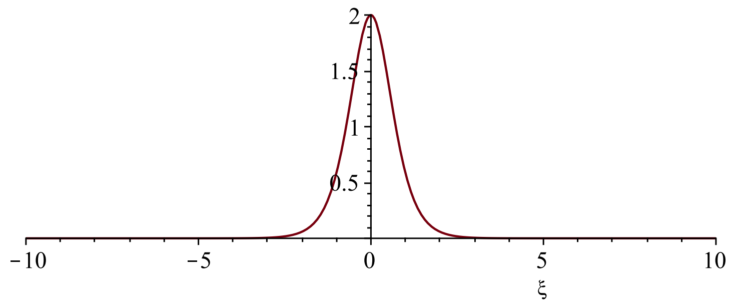


Figure 2.5: Plot of the sech^2 soliton solution of the KdV.

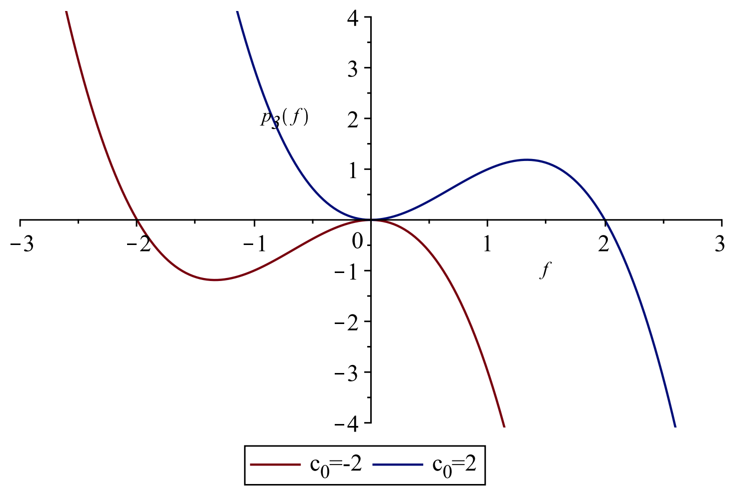


Figure 2.6: Polynomial $f^2(c_0 - f)$ for c_0 positive/negative.

2.3.4 Uses of soliton theory

There is more to the soliton theory of the KdV than this: not only does the equation have simple, closed–form travelling–wave solutions which arise in a transparent manner (this is already an exceptional situation for a nonlinear PDE), but it can be solved *analytically* for arbitrary initial data, provided it is localised and decently behaved (i.e. it doesn't extend to infinity and is reasonably smooth). This major breakthrough, due Gardner, Greene, Kruskal and Miura

[GGKM67], and later to Ablowitz, Kaup, Newell and Segur (see [AS81]) – the so-called inverse scattering theory – provided a huge impetus to the study of integrable systems and the KdV in particular. It also forms the basis of a kind of "nonlinear Fourier transform" based on the idea of decomposing signals into KdV solitons instead of sine waves (see [OPB83], [TPL$^+$17]).

This idea has also been explored experimentally. Since the work of Hammack & Segur [HS74] the solitary wave solutions of the KdV have been used extensively in modelling tsunami. They set up an experimental apparatus depicted in Figure 2.7, where a bottom-mounted piston was used to generate impulsive displacements in a wave flume, akin to the situation encountered during earthquakes.

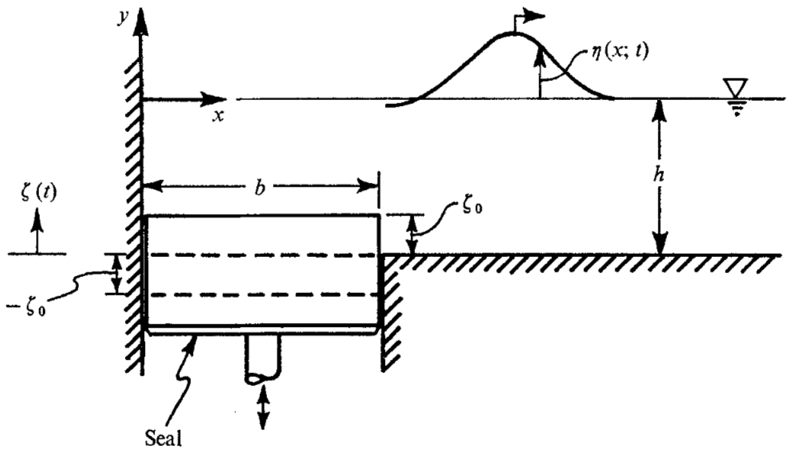


FIGURE 2. Schematic drawing of the wave generator.

Figure 2.7: Schematic from experiments of Hammack & Segur (1974).

Since then soliton theory has commonly used in discussions of tsunami, in numerical as well as experimental work. The sech^2 soliton solution is easy to visualise and to generate, and has been used many studies of tsunami run-up, propagation, etc. The foregoing theory of the Korteweg de Vries equation can also be extended to weakly two-dimensional propagation (the Kadomtsev-Petviashvili equation, which also supports certain soliton solutions), as well as other settings (shear flows, concentric propagation).

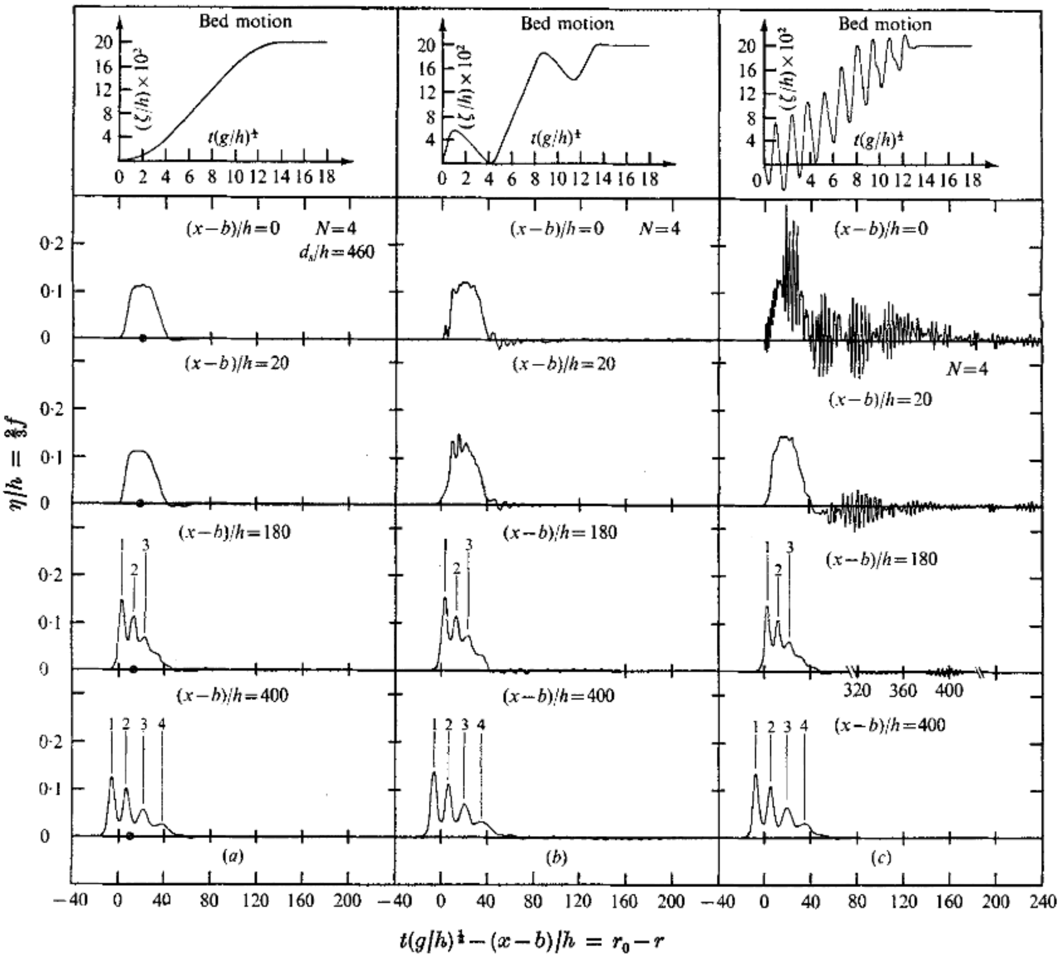


FIGURE 6. Time-displacement histories of the wave generator and the resulting wave systems: $h = 5$ cm, $b = 61$ cm, $V = 61$ cm^3/cm. (*a*) Mean motion; (*b*), (*c*) mean motion with superposed oscillation. ◑, location of centroid.

Figure 2.8: Results about the emergence of solitons from experiments of Hammack & Segur (1974).

Chapter 3

The nonlinear Schrödinger equation and deep water waves

3.1 Nonlinear waves in deep water

We have previously looked at the nonlinear problem

$$\phi_{xx} + \frac{1}{\mu^2}\phi_{zz} = 0, \tag{3.1}$$

$$\phi_z = 0 \text{ on } z = -1, \tag{3.2}$$

$$\zeta_t + \delta\phi_x\zeta_x - \frac{1}{\mu^2}\phi_z = 0 \text{ on } z = \delta\zeta, \tag{3.3}$$

$$\zeta + \delta\frac{1}{2}\left(\phi_x^2 + \frac{1}{\mu^2}\phi_z^2\right) + \phi_t = 0 \text{ on } z = \delta\zeta. \tag{3.4}$$

from the perspective of shallow water ((2.6)–(2.9), reproduced above). In deep water, the water depth plays no role (the wave motion does not reach the water bed – see Figure 1.6 and recall the particle paths) so the depth of the fluid can be effectively infinite. To this end, we will set $\mu = O(1)$ and therefore $h = O(\lambda)$. We notice that the amplitude parameter δ multiplies all the nonlinear terms in the equations, and the one important way to make progress is to use a perturbation expansion in terms of this parameter. We now write $\delta = A/\lambda$ in conjuction with $\mu = 1$ and find

$$\phi_{xx} + \phi_{zz} = 0, \tag{3.5}$$

$$\phi_z = 0 \text{ on } z = -1, \tag{3.6}$$

$$\zeta_t + \delta\phi_x\zeta_x - \phi_z = 0 \text{ on } z = \delta\zeta, \tag{3.7}$$

$$\zeta + \delta\frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \phi_t = 0 \text{ on } z = \delta\zeta. \tag{3.8}$$

If we were truly interested in deep water, we might replace the bottom boundary condition by $\nabla\phi \to 0$ as $z \to -\infty$, and this would simplify our calculations somewhat.

In what follows we will describe the procedure to carry out, and give the main results obtained when we look for travelling waves of steady form. The actual algebra is somewhat tedious, but provides a good exercise. Noting the δ dependence in our equations, we make the ansatz of a perturbation expansion

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \ldots, \quad \zeta = \zeta_0 + \delta\zeta_1 + \delta^2\zeta_2 + \ldots$$

which we insert into the governing equations. The idea is to separate each order of δ and solve the resulting systems at $O(1)$, $O(\delta), \ldots$. The only fly in the ointment is that the surface boundary conditions (3.7)–(3.8) are evaluated at $z = \delta\zeta$, so that the functions, say ϕ_z appearing therein are actually $\phi_z(x, \delta\zeta, t)$. This could confound our ordering!

The way around it is to perform a Taylor expansion of all functions in the boundary conditions about a fixed level $z = 0$ (say – also other fixed levels are possible and sometimes more convenient). For example

$$\phi_z(x, z = \delta\zeta, t) = \phi_z(x, 0, t) + \delta\zeta\phi_{zz}(x, 0, t) + \frac{\delta^2\zeta^2}{2!}\phi_{zzz}(x, 0, t) + \ldots.$$

At the very lowest order we get the usual linear wave system, which is equivalent to simply setting (formally) $\delta = 0$ in (3.5)–(3.8). This system can be solved to yield ϕ_0 and ζ_0 as shown in Section 1.2. At each order, the Laplace equation and bottom boundary condition retain their form, so that at $O(\delta)$ we expect a system that looks like

$$\begin{aligned}
&\phi_{1xx} + \phi_{1zz} = 0,\\
&\phi_{1z} = 0 \text{ on } z = -1,\\
&\zeta_{1t} - \phi_{1z} = F_1(\zeta_0, \phi_0) \text{ on } z = 1,\\
&\zeta_1 + \phi_{1t} = G_1(\zeta_0, \phi_0) \text{ on } z = 1.
\end{aligned}$$

The important insight is that this is linear system of equations forced by (or including inhomogeneous terms from) the lower orders. The functions F_1 and G_1 are known combinations of known quantities ϕ_0 and ζ_0. In this way, like climbing a staircase, we solve one order after another, with increasingly complex inhomogeneous terms.

Sometimes, however, the climb stops and we need to descend and re-evaluate. This happens for the first time when we are at the third order $O(\delta^2)$. We notice that the inhomogeneous terms F_2 and G_2 which appear in the boundary conditions contain contributions which are equal to solutions of the homogeneous equations. That this leads to resonance and secular growth is known from the theory of the simple harmonic oscillator, and clearly requires some intervention. The key is to retrace our steps and add an additional free variable to our expansion. Note that we should not accept the secular growth, since we have no evidence to support the idea that this is a physically realistic phenomenon. Several possibilities exist, including an expansion of the time-scales $t = t_0 + \delta t_1 + \delta^2 t_2 + \ldots$ or of the frequency $\omega = \omega_0 + \delta\omega_1 + \ldots$. Choosing the latter of these, the problem can be solved without further ado:

$$\phi = \frac{Ag}{\omega}e^{kz}\sin(kx - \omega t) \tag{3.9}$$

$$\zeta = A\left[\left(1 + \frac{k^2A^2}{8}\right)\cos(kx - \omega t) + \frac{kA}{2}\cos(2(kx - \omega t)) + \frac{3}{8}k^2A^2\cos(3(kx - \omega t))\right] \tag{3.10}$$

$$\omega = \sqrt{gk}\left(1 + \frac{1}{2}A^2k^2\right) \tag{3.11}$$

up to third order and in deep water. Note that the potential (in deep water only) does not change from the linear potential until $O(\delta^3)$. Three important lessons can be drawn from the form of the solution:

1. The speed of the waves depends on their amplitude A, with higher waves travelling faster.

2. The shape of the waves adjusts based on nonlinear interactions: bound waves of the form $\cos(2\xi)$ and $\cos(3\xi)$, for $\xi = kx - \omega t$ that travel with the wave lead to sharper crests and flatter troughs (see Figure 3.1).

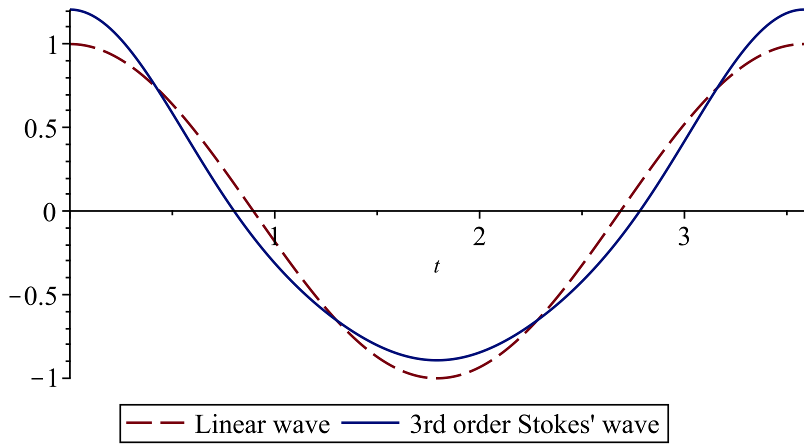


Figure 3.1: Figure depicting changes in surface elevation due to bound mode contributions in third-order Stokes theory.

3. All these changes are ordered in terms of a small parameter Ak – the wave slope. The dominant effects are captured by the linear theory, and each higher contribution is one order smaller.

The foregoing theory was first developed by G.G. Stokes and is known as the *Stokes wave theory.*

3.2 Envelope evolution equations and cubic nonlinearity

The nonlinear Schrödinger equation (NLS) describes the evolution of envelopes of waves in water of deep or intermediate depth. We can get some insight into the structure of this equation by looking

We write the free surface, according to linear theory, as

$$\zeta(x,t) = \int_{-\infty}^{\infty} A(k)\exp(i(kx - \omega(k)t)dk, \tag{3.12}$$

where $A(k)$ is a given amplitude spectrum. We have encountered this situation before in Section 1.2, equation (1.40). To make our life easier, we will assume that the amplitude spectrm $A(k)$ is Gaussian, corresponding to an initial condition

$$\zeta(x,0) = A_0 \exp(-ik_0 x)\exp\left(\frac{-x^2}{4\sigma^2}\right),$$

and therefore (recalling that the Fourier transform of a Gaussian is also a Gaussian)

$$A(k) = \frac{A_0\sigma}{\sqrt{\pi}}\exp(-(k-k_0)^2\sigma^2).$$

This choice of Gaussian initial condition is essentially arbitrary, but we make it so that we can assume (with good reason) that the integrand in

$$\zeta(x,t) = \frac{A_0\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-(k-k_0)^2\sigma^2)\exp(i(kx-\omega(k)t)dk$$

will be localised around (or decay quickly away from) $k = k_0$.

As we did previously, we will therefore Taylor expand the frequency ω about the carrier wavenumber k_0 as

$$\omega(k) = \omega(k_0) + \underbrace{(k - k_0)}_{=:\xi}\omega'(k_0) + \frac{(k-k_0)^2}{2}\omega''(k_0) + \dots$$

Substituting this into $\zeta(x,t)$ gives a chunky approximation

$$\begin{aligned}
\zeta(x,t) &\approx \frac{A_0\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2\sigma^2} e^{i[(k_0+\xi)x-(\omega_0+\xi\omega_0'+\frac{\xi^2}{2}\omega_0'')t]}d\xi \\
&= \frac{A_0\sigma}{\sqrt{\pi}} e^{i(k_0x-\omega_0 t)}\int_{-\infty}^{\infty} e^{-\xi^2(\sigma^2+i\frac{\omega_0'' t}{2})} e^{i\xi(x-\omega_0' t)}d\xi. \qquad (3.13)
\end{aligned}$$

The integrand here can be written as

$$\exp(-(a\xi^2 + b\xi)), \text{ with } a = \sigma^2 + \frac{i\omega_0''}{2},\ b = i(\omega_0' t - x).$$

Completing the square

$$a\xi^2 + b\xi = (\sqrt{a}\xi + \frac{b}{2\sqrt{a}})^2 - \frac{b^2}{4a}$$

and substituting $v = \sqrt{a}\xi + \frac{b}{2\sqrt{a}}$, $dv = \sqrt{a}d\xi$ into the integral leaves us with the exponential integral

$$\int_{-\infty}^{\infty}\exp(-v^2)dv = \sqrt{\pi}$$

and some auxiliary terms such that

$$\zeta(x,t) = \frac{A_0\sigma}{\sqrt{\sigma^2 + \frac{i\omega_0'' t}{2}}}\exp\left(\frac{\omega_0' t - x}{4(\sigma^2 + \frac{i\omega_0'' t}{2}}\right)\exp(i(k_0 x - \omega_0 t)). \qquad (3.14)$$

Once again, the first term is an envelope which multiplies the sinusoidal carrier wave. The propagation speed of the envelope is the group velocity ω_0'. Having retained higher order terms in the expansion of the dispersion relation, we see that the envelope amplitude also decays with time. Moreover, the length of the envelope, increases with time. This means that the phase (or frequency) of the waves is varying, in a way that depends on the second derivative $\omega''(k_0)$. One can show that that the slowly varying amplitude is governed by a transport equation whose speed of propagation is the group velocity, but we will use a more general formulation to arrive at the same equation in the next section.

3.3 Slow evolution of the envelope

The starting point of this section, which sets out to derive an equation for the evolution of a slowly varying wave amplitude, is the linear water wave problem in finite depth:

$$\begin{aligned}
\Delta\phi &= 0, &(3.15)\\
\phi_{tt} + g\phi_z &= 0 \text{ on } z = 0, &(3.16)\\
\phi_z &= \text{ on } z = -h. &(3.17)
\end{aligned}$$

In order to capture slow variations of the wave envelope, we introduce new variables

$$x_0 = x,\ x_1 = \mu x,\ \dots t_0 = t,\ t_1 = \mu t, \dots.$$

This means that our derivatives must be rewritten as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \mu\frac{\partial}{\partial t_1} + \ldots$$
$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t_0^2} + 2\mu\frac{\partial^2}{\partial t_0 \partial t_1} + \ldots.$$

Inserting this into the linear equations and retaining terms of order $O(\mu)$ leaves us with

$$\begin{aligned} &\phi_{x_0x_0} + 2\mu\phi_{x_0x_1} + \phi_{zz} = 0, \\ &\phi_{t_0t_0} + 2\mu\phi_{t_0t_1} + g\phi_z = 0 \text{ on } z = 0, \\ &\phi_z = 0 \text{ on } z = -h. \end{aligned}$$

We further insert the expansion

$$\phi = e^{i\xi}\left(\phi_0 + \mu\phi_1 + \ldots\right)$$

with $\xi = kx_0 - \omega t_0$, where $\phi_i = \phi_i(x_1, \ldots, t_1, \ldots, z)$ is independent of the (fast) variables x_0, t_0. This leads to the equations (keeping terms of $O(\mu)$ only)

$$\begin{aligned} &-k^2\left(\phi_0 + \mu\phi_1\right) + 2\mu ik\phi_{0x_1} + \phi_{0zz} + \mu\phi_{1zz} = 0 \\ &-\omega^2\left(\phi_0 + \mu\phi_1\right) - 2\mu i\omega\phi_{0t_1} + g\left(\phi_{0z} + \mu\phi_{1z}\right) = 0 \text{ on } z = 0, \\ &\phi_{0z} + \mu\phi_{1z} = 0. \end{aligned}$$

Separating these equations order by order in the small parameter μ gives:

$$\mathcal{O}(1):$$
$$\phi_{0zz} - k^2\phi_0 = 0 \tag{3.18}$$
$$g\phi_{0z} - \omega^2\phi_0 = 0 \text{ on } z = 0 \tag{3.19}$$
$$\phi_{0z} = 0 \text{ on } z = -h \tag{3.20}$$
$$\mathcal{O}(\mu):$$
$$\phi_{1zz} - k^2\phi_1 = -2ik\phi_{0x_1} \tag{3.21}$$
$$g\phi_{1z} - \omega^2\phi_1 = 2i\omega\phi_{0t_1} \text{ on } z = 0 \tag{3.22}$$
$$\phi_{1z} = 0 \text{ on } z = -h. \tag{3.23}$$

We notice that the lowest order equations for ϕ_0 are simply the well-known linear system we started with. This raises questions about the whole enterprise: why not simply stick with what we already have? In fact, the solution

$$\phi_0 = \frac{-igA}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)} \tag{3.24}$$

has already been derived in Chapter 1 (remember that we have taken out $\exp(i\xi)$ when considering our expansion). The only free parameter here is the amplitude A, which can be arbitrary. Now, however, we have extra parameters in the $\mathcal{O}(1)$ problem: all of the additional slow time and space scales, so that the amplitude $A = A(x_1, x_2, \ldots, t_1, t_2, \ldots)$ can depend parametrically on all of these. This is exactly what we want: a slowly varying amplitude A. The hope is that the equations at $\mathcal{O}(\mu)$ will be able to tell us something about its behaviour!

We inspect (3.21)–(3.23), and notice that the right-hand side is completely specified by the lower order solution (3.24) ϕ_0. However, the homogeneous equation has a solution of the same form as ϕ_0, which tells us that we will need to be careful with the solution of the inhomogeneous equation. Various possibilities exist, but we will consider the following solvability condition: take the $O(\mu)$ equation

$$\phi_{1zz} - k^2\phi_1 + 2ik\phi_{0x_1} = 0$$

and multiply by ϕ_0:

$$\phi_0\left(\phi_{1zz} - k^2\phi_1 + 2ik\phi_{0x_1}\right) = 0.$$

Integrate across the domain from $z = -h$ to $z = 0$

$$\int_{-h}^{0} \phi_0\left(\phi_{1zz} - k^2\phi_1 + 2ik\phi_{0x_1}\right)\mathrm{d}z = 0. \tag{3.25}$$

Using the first term in the integrand, integrate by parts twice:

$$\int_{-h}^{0}\phi_0\phi_{1zz}\mathrm{d}z = [\phi_0\phi_{1z}]_{-h}^{0} - \int_{-h}^{0}\phi_{0z}\phi_{1z}\mathrm{d}z = [\phi_0\phi_{1z}]_{-h}^{0} - [\phi_{0z}\phi_1]_{-h}^{0} + \int_{-h}^{0}\phi_{0zz}\phi_1\mathrm{d}z.$$

With this we can rewrite (3.25) as follows:

$$[\phi_0\phi_{1z} - \phi_{0z}\phi_1]_{-h}^{0} + \int_{-h}^{0}\phi_1\underbrace{\left(\phi_{0zz} - k^2\phi_0\right)}_{=0} + 2ik\phi_{0x_1}\phi_0\mathrm{d}z = 0.$$

where we have used the $O(1)$ equation which appears in the integrand. Notice that both bottom boundary conditions are identical $\phi_{0z} = \phi_{1z} = 0$, which eliminates the term in square brackets evaluated at $z = -h$. We are left with

$$[\phi_0\phi_{1z} - \phi_{0z}\phi_1]_{z=0} + 2ik\int_{-h}^{0}\phi_{0x_1}\phi_0\mathrm{d}z = 0.$$

Using (3.22) and (3.19) together with the expression (3.24) for ϕ_0 we obtain

$$A_{t_1} + c_g A_{x_1} = 0, \tag{3.26}$$

where $c_g = d\omega/dk$ is the finite depth group velocity. This allows us to establish the following important fact: the slowly varying amplitude propagates at the group velocity. Indeed, the equation (3.26) is the *linear Schrödinger equation*, and it will occur as an important constituent of the NLS.

3.4 The nonlinear Schrödinger equation

The simplest setting of the NLS is in deep water, where the equation is written in dimensional form in a fixed reference frame as

$$i\left(a_x + \frac{2k_0}{\omega_0}a_t\right) - \frac{1}{4k_0}a_{xx} - k_0^3|a|^2a = 0. \tag{3.27}$$

The dependent variable $a(x,t)$ is the wave envelope of a slowly varying wave-train. This envelope is related to the free surface elevation exactly as we would expect:

$$\zeta(x,t) = \mathfrak{Re}\left\{a(x,t)\exp(i(k_0x - \omega_0 t)\right\} \tag{3.28}$$

Moreover, we recognise the term multiplying a_t as the reciprocal of the deep-water group velocity c_g, so the envelope can be said to move at approximately the group velocity.

We can see this more clearly by changing coordinates to a moving reference frame to move with the group velocity,

$$\xi(x,t) = 2k_0\left(x - \frac{\omega_0}{2k_0}t\right), \tag{3.29}$$

$$\tau(x,t) = \omega_0 t, \tag{3.30}$$

which gives

$$ia_\tau - \frac{1}{2}a_{\xi\xi} - \frac{k_0^2}{2}|a|^2 a = 0, \tag{3.31}$$

and we rescale the amplitude by a factor of $k_0/\sqrt{2}$ which gives a standard form of the NLS

$$ia_\tau - \frac{1}{2}a_{\xi\xi} - a|a|^2 = 0. \tag{3.32}$$

Some immediate insight into the what the NLS is telling us can be had by looking for the simplest possible solutions. In particular, we can insert the ansatz for a monochromatic wave train

$$a(x,t) = a_0 \exp(-i\alpha t) \tag{3.33}$$

into the dimensional equation (3.27). Many terms cancel, and we are left with

$$\alpha = \frac{\omega_0 k_0^2 a_0^2}{2}.$$

The significance of this is clear once we look at the free surface, by inserting into (3.28):

$$\zeta(x,t) = \mathfrak{Re}\left[a_0 \exp\left(i\left(k_0 x - \omega_0\left(1 + \frac{k_0^2 a_0^2}{2}\right)\right)\right)\right].$$

This is precisely the third-order Stokes wave found in (3.11), albeit without the higher harmonic contributions to the free surface.

The NLS often appears with different coefficients, depending on the context of the derivation (deep water, shallow water, nonlinear optics, etc.). It is useful to note that, for α, β, $\gamma \in \mathbb{R}^+$ we can always rescale the equation

$$i\alpha u_t + \beta u_{xx} \pm u|u|^2 = 0$$

to the form with coefficients equal to unity

$$iu_t + u_{xx} \pm u|u|^2 = 0$$

via a transformation

$$t \longrightarrow At,\ x \longrightarrow Bx,\ u \longrightarrow Cu$$

where $A = \alpha$, $B^2 = \beta$ and $C^2 = 1/\gamma$. Of course, we can also change to or from a moving frame by introducing a characteristic coordinate as above. The most important feature of the NLS is the sign preceding the nonlinear term:

"+" if the nonlinear term in the NLS has the same sign as the dispersive term, this is called the focusing case. In water waves derivations of the NLS inevitably lead to the focusing version.

"-" if the nonlinear term has a sign opposite that of the dispersive term, this is called the defocusing case. This arises naturally in nonlinear optics.

3.5 Modulational instability

We have seen that the NLS models the Stokes' wave to third order, and so contains some important germs of the water wave theory. More importantly, still, it can be used to investigate the stability of the Stokes' wave to small disturbances, which we shall demonstrate below.

We write the NLS for unidirectional propagation in the general form

$$iA_\tau + \lambda A_{\xi\xi} - \nu|A|^2 A = 0. \tag{3.34}$$

There are several ways to progress, but we will first separate the equation into real and imaginary parts by making a general complex exponential ansatz of the form

$$A = a(\xi,\tau)\exp(i\chi(\xi,\tau)),$$

where both the amplitude a and phase χ (both real functions) are allowed to vary with ξ and τ. It is useful to gather the expressions for the derivatives:

$$\begin{aligned} A_\tau &= e^{i\chi}\left(a_\tau + ia\chi_\tau\right) \\ A_\xi &= e^{i\chi}\left(a_\xi + ia\chi_\xi\right) \\ A_{\xi\xi} &= e^{i\chi}\left(a_{\xi\xi} + 2ia_\xi\chi_\xi + ia\chi_{\xi\xi} - a(\chi_\xi)^2\right) \end{aligned}$$

Inserting this into (3.34) yields, after separating the real and imaginary parts

$$a_\tau + \lambda\left(2a_\xi\chi_\xi + a\chi_{\xi\xi}\right) = 0, \tag{3.35}$$

$$-a\chi_\tau + \lambda\left(a_{\xi\xi} - a(\chi_\xi)^2\right) - \nu a^3 = 0. \tag{3.36}$$

Now comes the perturbation ansatz: we will assume

$$\begin{aligned} a(\xi,\tau) &= a_0 + \epsilon b(\xi,\tau) \\ \chi(\xi,\tau) &= \chi_0(\tau) + \epsilon\theta(\xi,\tau) \end{aligned}$$

where ϵ is a small number $\epsilon \ll 1$, which indicates that we are looking for a small departure or perturbation from a known solution. In particular, if ϵ vanishes identically, we know there is a solution with $\chi_0(\tau) = -\nu a_0^2\tau$ – this is just the Stokes' wave we have already encountered. This also explains why the lowest order a_0 term is independent of ξ and τ, while the lowest order phase term depends on τ. The philosophy is now as follows: we will insert the perturbation ansatz into the NLS system (3.35)–(3.36) and neglect terms of order ϵ^2 and above. This is the classical *linear stability analysis* of the system.

To lowest order in ϵ, therefore, we find

$$\begin{aligned} & b_\tau + \frac{\lambda}{2}a_0\theta_{\xi\xi} = 0, \\ & a_0\theta_\tau - \lambda b_{\xi\xi} + 2\nu a_0^2 b = 0, \end{aligned}$$

where we need to use $\chi_0(\tau) = -\nu a_0^2\tau$ to eliminate $O(1)$ terms in the second equation. Between these two equations we can eliminate one variable, say θ, and obtain only a single equation in b

$$b_{\tau\tau} = \frac{-\lambda^2}{2}b_{\xi\xi\xi\xi} + \lambda\nu a_0^2 b_{\xi\xi}.$$

Solving this equation will tell us *how the amplitude perturbation will evolve.* How should we do this? The natural choice, given that we are dealing with a wave problem, is to assume that both b and θ themselves have a wave character. In this case, we are dealing with *linear* waves, which makes our life much easier: we assume $b = b_0 \exp i(\kappa\xi - \Omega\tau)$ and $\theta = \theta_0 \exp i(\kappa\xi - \Omega\tau)$. Substituting this yields the following relationship between Ω, κ and the parameters appearing in the problem:

$$\Omega^2 = \lambda\kappa^2 \left(\frac{\lambda\kappa^2}{2} + \nu a_0^2 \right).$$

To interpret this we need to think of the possible cases: if Ω^2 is positive, then the perturbation of the form $\exp i(\kappa\xi - \Omega\tau)$ will remain purely oscillatory. On the other hand, if $\Omega^2 < 0$, then Ω will be imaginary, and multiplication by i will yield a real growth rate (both plus and minus signs occur). This will mean that the perturbation is growing with time, and so is termed *linearly unstable.*

The condition for linear instability can therefore be written as

$$\lambda^2 \frac{\kappa^2}{2} + \lambda\nu a_0^2 < 0, \tag{3.37}$$

and in deep water we have

$$\lambda = -\frac{1}{8}\frac{\omega_0}{k^2}, \quad \nu = \frac{1}{2}\omega_0 k^2,$$

which allows us to rewrite ultimately as

$$\frac{\kappa}{k} < 2\sqrt{2}ak. \tag{3.38}$$

Interpreting k as the wavenumber of the fundamental (Stokes) wave, and κ as the wavenumber of the perturbation, we see that instability (for a given steepness ak of the Stokes' wave) requires that the perturbation be sufficiently close in wavenumber space. Alternatively, for a given separation in wavenumber space between perturbation and Stokes wave the steepness of the Stokes wave must be large to induce instability.

3.6 Soliton solutions of the NLS

Starting with the cubic NLS

$$-i\frac{\partial A}{\partial \tau} + \frac{1}{8}\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}|A|^2 A = 0 \tag{3.39}$$

we can look for permanent envelope solutions, which are the travelling wave solutions of the NLS. To this end, we make an ansatz

$$A = a\exp(ir(\xi - V\tau - \delta),$$

where $a = a(\xi - U\tau)$ is the (real) wave envelope, δ is a constant phase shift which we are free to adjust, and r is a real constant to be determined. Substituting this ansatz into the NLS turns the equation from a PDE into a (nonlinear) ODE,

$$a'' + ia'(8U + 2r) + 4a^3 - ar(8V + r) = 0. \tag{3.40}$$

Since this is an ODE in a real variable we can ask for the real and imaginary parts to be satisfied separately. The imaginary part is simple, and immediately implies that

$$4U = -r.$$

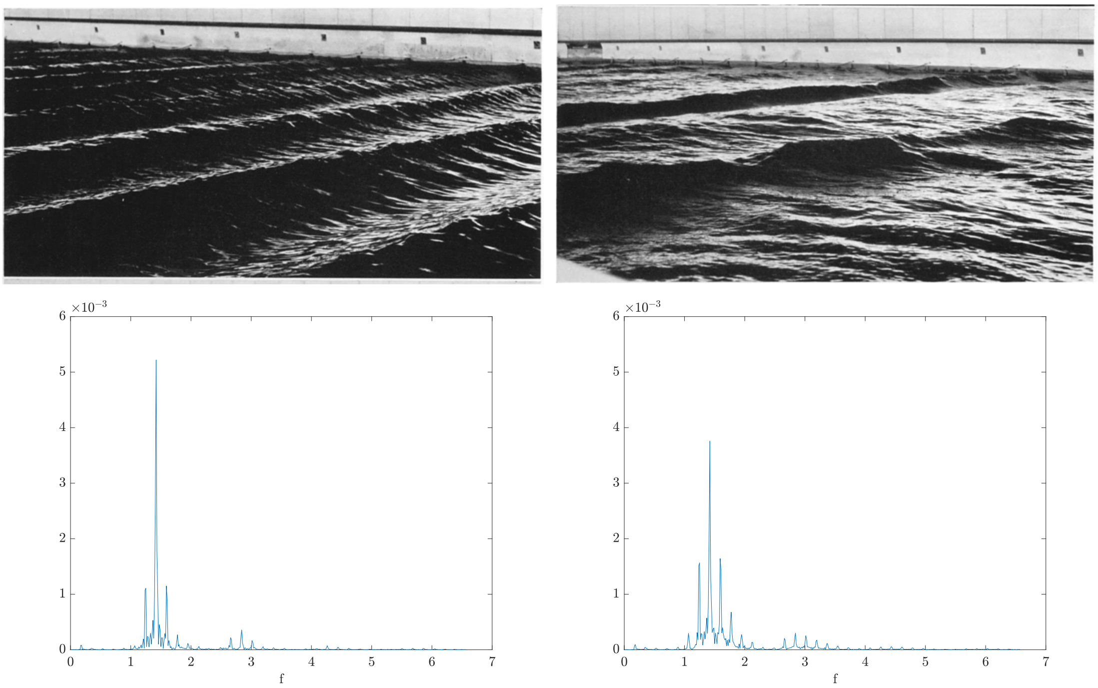


Figure 3.2: Benjamin-Feir instability of a Stokes' wave in the tank (above) and in Fourier space (below). Notice how a very nearly monochromatic wave with small side-bands disintegrates into a disordered pattern as the side-bands grow. (Fourier amplitude spectra are taken from unidirectional flume measurements, and therefore given in terms of frequency f rather than wavenumber k. Nevertheless these are representative of the main mechanism of Benjamin-Feir instability).

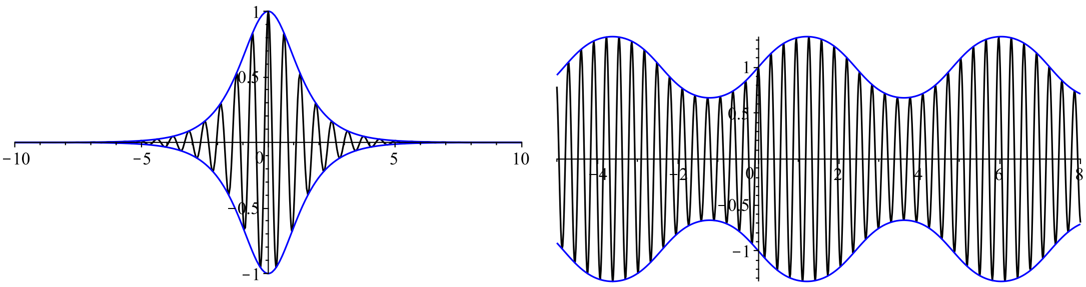


Figure 3.3: Solitary sech envelope solution (left) and periodic, cnoidal envelope solution (right) to the NLS.

Consequently the real part of the equation becomes

$$a'' + 4a^3 + 16aU(2V - U) = 0.$$

We have encountered a situation like this before, with the KdV. We can multiply by a' and integrate once to obtain

$$(a')^2 + 2a^4 + 16a^2U(2V - U) = C.$$

If we now multiply by a^2 and define $E := a^2$, we can rewrite this as

$$(E')^2 + 8E^3 + 64E^2U(2V - U) = 4CE,$$

where we recognize yet another cubic polynomial in the new variable E! We conveniently write this as

$$(E')^2 = 8(E_{max} - E)(E - E_{min})E = P_3(E), \tag{3.41}$$

where $8(E_{max} + E_{min}) = -64U(2V - U)$. Note that the cubic polynomial in E corresponds to a sixth degree polynomial in a. As we have already seen, a solution exists only in regions where P_3 is positive, and the non-periodic case corresponds to a double root. If we are thinking of soliton solutions, we naturally set the constant of integration $C \equiv 0$, so that E_{min} vanishes and we are left with the simplified equation

$$(E')^2 = 8E^2(E_{max} - E) \tag{3.42}$$

We have been in this situation before with (2.46), and we find again

$$E = E_{max}\mathrm{sech}^2\left(\sqrt{2E_{max}}(\xi - U\tau)\right), \tag{3.43}$$

and in the amplitude variable

$$a = a_m\mathrm{sech}\left(\sqrt{2}a_m(\xi - U\tau)\right). \tag{3.44}$$

Visualising the envelope soliton of the NLS in Figure 3.3, we see that it represents a modulated pulse of waves. Of course, we already know that (3.41) will support periodic solutions in the form of elliptic functions. We see (without going into the algebraic details) that these represent modulated wave trains, as shown on the right-hand side of Figure 3.3.

As was the case for the KdV, the soliton theory of the NLS goes far beyond the simple solutions sketched here. There is a scattering theory, developed by Shabat and Zakharov [SZ72] which parallels the developments by Gardner et al, and allows for the NLS to be solved for suitable initial data. More background and some details can be found in the book by Drazin & Johnson [DJ89].

3.7 Breather solutions of the NLS

For ease of presentation we will discuss some remarkable solutions of the NLS in the simplified form

$$iu_t + u_{xx} + 2u|u|^2 = 0.$$

In this case the Stokes' wave solution with unit amplitude has the simple form

$$u(x,t) = \exp(2it),$$

while the envelope soliton can be written (setting variable coefficients to unity) as

$$u(x,t) = \frac{\exp(it)}{\cosh(x)}.$$

Both of these are particular cases of solutions with a steady (in time) envelope $|u|$. In fact, other such solutions exist, and they are described by the two families

$$u_1(x,t) = \exp(i(2-m^2)t)\mathrm{dn}(x,m), \quad u_2(x,t) = m\exp(i(2m^2-1)t)\mathrm{cn}(x,m),$$

with $m \in [0,1]$ and cn, dn denoting Jacobi elliptic functions. Note that

$$\lim_{m\to 1}\mathrm{cn}(x,m) = \lim_{m\to 1}\mathrm{dn}(x,m) = \mathrm{sech}(x),\ \lim_{m\to 0}\mathrm{cn}(x,m) = \cos(x),\ \lim_{m\to 0}\mathrm{dn}(x,m) = 1.$$

Consequently the envelope soliton is the limit of either family of solutions as $m \to 1$.

In terms of our equation, numerous other solutions have been found, with a range of interesting behaviours in space and time.

The spatially-periodic breathers, first found by Akhmediev et al [AEK87] are a one parameter family of solutions which approach the plane wave as $t \longrightarrow \pm\infty$. They can be written as

$$u(x,t) = e^{2it}\frac{\cosh(\Omega t - 2i\phi) - \cos(\phi)\cos(px)}{\cosh(\Omega t) - \cos(\phi)\cos(px)} \tag{3.45}$$

with

$$p = 2\sin(\phi), \quad \Omega = 2\sin(2\phi)$$

and $\phi \in \mathbb{R}$. This solution is spatially periodic with period $T = 2\pi/p$ and obtains its maximum at $t=0$, $x=0$. The maximum of the envelope is given by

$$|u(0,0)| = 2\cos(\phi) + 1.$$

These are now known as *Akhmediev breathers.* If we take the limit of the period tending towards infinity in (3.45), we obtain a solution which "breathes" once in space and in time

$$u(x,t) = \frac{\left(4x^2 - 3 - 16it + 16t^2\right)e^{2it}}{16t^2 + 4x^2 + 1}, \tag{3.46}$$

first discovered by Peregrine [Per83] and called the *Peregrine breather.* We can also construct a breather solution which is periodic in time rather than space by transforming $\phi \longrightarrow i\phi$. This gives the so-called *Kuznetsov-Ma breather* solution

$$u(x,t) = e^{2it}\frac{\cos(\Omega t - 2i\phi) - \cosh(\phi)\cosh(px)}{\cos(\Omega t) - \cosh(\phi)\cosh(px)}. \tag{3.47}$$

Why are these breathers special? It seems like they are a (perhaps slightly modified) type of solitary wave solution for the NLS. The reason for the intense interest in breathers in recent years

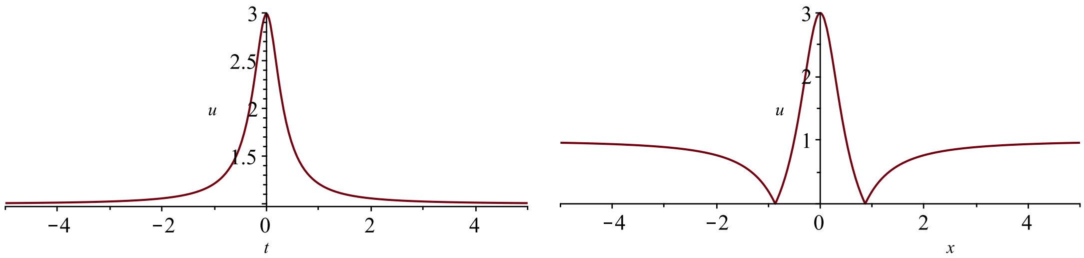


Figure 3.4: Plot of the envelope Peregrine breather solution at $x = 0$ with time (left), and at $t = 0$ with space (right). This is a limiting case of the Akhmediev breather where the amplification is maximal.

Figure 3.5: Stereotypical rogue wave as imagined by Katsushika Hokusai.

is that they present a possible mechanism for sudden wave amplification. The usual envelope solitons of the NLS (shown in Figure 3.3) represent wave packets propagating into still water – this is clear because the envelope tends to zero away from the disturbance. (Remember that the envelope is the curve drawn around the crests and troughs of the wave).

In contrast, the breather solutions tend towards a *constant, nonzero* background. This means (physically) that we are looking at a periodic train of waves of constant amplitude, which suddenly changes in time and/or space, giving a dramatic amplification of crests and also troughs of the waves. This makes breathers an attractive mechanism for the formation of rogue waves (no discussion of which is complete without the woodblock print of the Great Wave of Kanagawa, shown in Figure 3.5). Breathers present an attractive connection between an elegant mathematical theory and a dramatic physical phenomenon – whether or not they are the definitive solution to the problem of understanding rogue waves remains to be established.

Chapter 4

Variational methods in water waves

4.1 Introduction to variational principles

Water wave theory is simply Newtonian mechanics, and it is surprising that it continues to supply interesting problems nearly 300 years after Isaac Newton's death and 250 years after the governing equations were formulated by Leonhard Euler. It has been known since work by Lagrange and Hamilton that Newtonian mechanics can also be derived from the principle of least action, so it should be no surprise that associated variational principles can also be used to describe fluid motion.

In typical Newtonian mechanics we follow the motion of particles under the action of forces. We define the kinetic energy of a particle as

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

where $s(t) = (x(t), y(t), z(t))$ is a parametrisation of the particle path, m is the mass of the particle, and we use the dot to represent the time derivative. The total kinetic energy

$$T = \frac{1}{2}\sum_j m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2)$$

Of course, the path of a particle in an interesting physical system is not usually totally arbitrary, with the particle zooming off wildly in any x, y or z direction. Instead, the particle path is constrained in some way, and these constraints give rise to new, generalised coordinates. For example, considering the motion of a point mass on a rigid, massless rod of length R (the classical situation of a pendulum) the particle motion may be constrained to a circle.

Recall that the path of a circle is parametrised by

$$s(\phi) = (R\cos\phi, R\sin\phi),$$

where $s : \mathbb{R} \to \mathbb{R}^2$, $s : \phi \mapsto (x(\phi), y(\phi))$. Here the parametrising variable ϕ represents the angle measured from the positive x-axis counterclockwise. In fact, how this angle evolves with time is the principle problem we are trying to solve. We relate the kinetic energy of the system to the velocity at which the path is traversed – considering $\phi = \phi(t)$

$$\dot{s}(t) = (-R\dot{\phi}\sin\phi, R\dot{\phi}\cos\phi)$$

gives the velocity along the path, and

$$\dot{s}^2 = R^2\dot{\phi}^2.$$

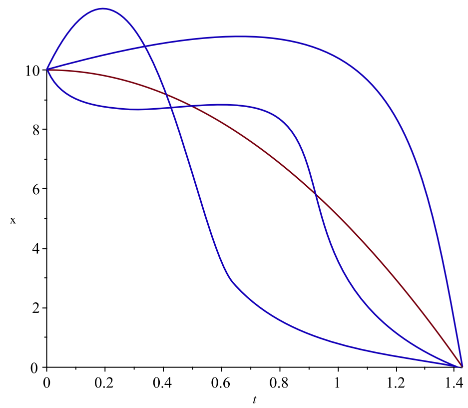


Figure 4.1: The trajectory of a falling mass $x(t) = -gt^2/2 + x_0$ in (x, t)-space (red) and other possible trajectories which are not observed in nature (blue). The vertical is x and the horizontal is time t.

The kinetic energy is therefore

$$T = \frac{1}{2} m R^2 \dot{\phi}^2.$$

We prescribe the potential energy (which depends only on position of particles and not on their velocities) as

$$V = mgy = mgR \sin\phi.$$

The Lagrangian of the system is defined as

$$L = T - V = \frac{1}{2} mR \left(R\dot{\phi}^2 - g \sin\phi \right). \tag{4.1}$$

This is a function $L = L(t, \phi, \dot{\phi})$, which depends on another function ϕ that we are looking for. In fact, Hamilton's principle tells us that the ϕ which describes the motion is such as to render the integral

$$I = \int_{t_1}^{t_2} L(t, \phi, \dot{\phi}) dt$$

minimal, for arbitrary values of time t_1 and t_2. The functions ϕ which are eligible are those with prescribed values at t_1 and t_2. We thus *vary* these functions, by introducing a perturbation $\phi(t) + \epsilon\psi(t)$, and then take the derivative with respect to the coefficient ϵ (effectively linearising in the perturbation) and setting this coefficient equal to zero. This is known as the *variational derivative:*

$$\delta I = \frac{d}{d\epsilon} \int_{t_1}^{t_2} L(t, \phi + \epsilon\psi, \dot{\phi} + \epsilon\dot{\psi}) \mathrm{d}t|_{\epsilon=0} = \int_{t_1}^{t_2} \left(\frac{dL}{d\phi}\psi + \frac{dL}{d\dot{\phi}}\dot{\psi} \right) \mathrm{d}t.$$

The second term in the integrand on the right can be treated by integration by parts:

$$\int_{t_1}^{t_2} \frac{dL}{d\dot{\phi}} \frac{d\psi}{dt} \mathrm{d}t = \left[\frac{dL}{d\dot{\phi}} \psi \right]_{t=t_1}^{t=t_2} - \int_{t=t_1}^{t=t_2} \frac{d}{dt} \left(\frac{dL}{d\dot{\phi}} \right) \psi \mathrm{d}t,$$

and assuming the arbitrary function ψ vanishes at the endpoints of the integration t_1 and t_2 we have

$$\delta I = \int_{t_1}^{t_2} \psi \left(\frac{dL}{d\phi} - \frac{d}{dt} \left(\frac{dL}{d\dot{\phi}} \right) \right) \mathrm{d}t \tag{4.2}$$

which disappears exactly when the Euler-Lagrange equation

$$\left(\frac{dL}{d\phi} - \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right)\right) = 0 \tag{4.3}$$

is satisfied. If we now wish to perform this procedure for our Lagrangian (4.1), we find the corresponding Euler-Lagrange equation is

$$-\frac{1}{2}mR\left(g\cos(\phi) + R\ddot{\phi}\right) = 0.$$

This gives almost the standard form of the pendulum equation. If we recall that our equilibrium position $\phi = 0$ is along the positive x-axis we can define a new angle $\theta = \phi + \pi/2$ which transforms our equation into standard form

$$\ddot{\theta} = -\frac{g}{R}\sin(\theta).$$

A few comments: the function ψ introduced is some (suitable) neighbouring function. It has no specific meaning, except as a foil when we must set certain integrals equal to zero. The notation used throughout is "physics-y", and care must be taken with derivatives: in the Lagrangian $L(t, \phi, \dot{\phi})$ we have ϕ and $\dot{\phi}$ as separate functions with respect to which we can take partial derivatives. Of course both $\phi = \phi(t)$ and $\dot{\phi} = \dot{\phi}(t)$ are functions of time.

The approach pursued here is for a Lagrangian which depends on a single variable, but in principle arbitrarily many generalised coordinates $L(t, \phi_1, \phi_2, \ldots, \phi_n, \dot{\phi}_1, \ldots, \dot{\phi}_n)$ may be involved in a Lagrangian and will produce a system of Euler-Lagrange equations analogous to (4.3). Extensions to multiple independent variables are also possible, and will be used below. In this case, the Lagrangian $L(x, y, f, f_x, f_y)$ must be varied in two dimensions, and we obtain PDEs.

The principle problem in applying the classical variational calculus to fluid dynamics is that the most common description of fluid motion forgets about the fluid particles entirely. We do not track any particles, but only velocities at specific points in space. This difference (between Eulerian and Lagrangian fluid dynamics (mentioned also in Section A.2) makes the variational approach to fluid mechanics somewhat tricky.

4.2 Luke's variational principle

Luke [Luk67] formulated a variational principle for the full water wave problem, including the boundary conditions in compact form. Here we are dealing with a problem in three dimensions: two spatial x and y and one temporal t. The Lagrangian in this case is the pressure written in terms of the Bernoulli equation

$$L = \int_0^{\zeta(x,t)} \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \phi_t + gy\,dy, \tag{4.4}$$

where y is used as the vertical coordinate, and the free surface is located at $y = \zeta(x, t)$. Note that a bed can be incorporated by replacing the lower boundary of the integration with $-d$. If the bed is flat (d constant) there is no additional variation, and this contributes nothing to the subsequent discussion. If the bed varies (possibly in space and time, such that $d = d(x, t)$) we will need to account for additional variations in d, leading to a bottom boundary condition. In what follows we illustrate the procedure for a flat bed at $y = 0$ for simplicity.

The integral whose variation must be considered is

$$I = \int_{t_1}^{t_2}\int_{x_1}^{x_2} L\,dx\,dt, \tag{4.5}$$

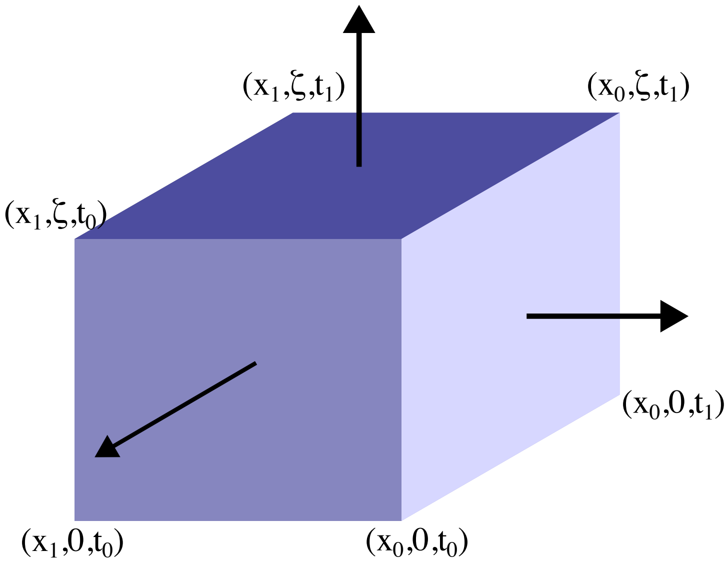


Figure 4.2: Depiction of the (rectangular) integration domain $\mathcal{D} = [x_0, x_1] \times [0, \zeta] \times [t_0, t_1]$. Arrows depict some of the outward-pointing normal vectors.

and we note that both ϕ and ζ must be varied. We could do this separately (considering $\delta L/\delta\phi$ and then $\delta L/\delta\zeta$), but elect to proceed with the total variation. We insert the usual ansatz:

$$\begin{aligned}
\delta I &= \frac{d}{d\epsilon} \iiint_0^{\zeta+\epsilon\eta} \frac{1}{2}\left(\phi_x + \epsilon\psi_x\right)^2 + \frac{1}{2}\left(\phi_y + \epsilon\psi_y\right)^2 + \left(\phi_t + \epsilon\psi_t\right) + gy \, \mathrm{d}y\mathrm{d}x\mathrm{d}t\bigg|_{\epsilon=0} \\
&= \iint \left[\frac{1}{2}\left(\phi_x + \epsilon\psi_x\right)^2 + \frac{1}{2}\left(\phi_y + \epsilon\psi_y\right)^2 + \left(\phi_t + \epsilon\psi_t\right) + gy\right]_{y=\zeta+\epsilon\eta} \eta \, \mathrm{d}x\mathrm{d}t\Bigg|_{\epsilon=0} \\
&+ \iiint_0^{\zeta+\epsilon\eta} \left(\phi_x + \epsilon\psi_x\right)\psi_x + \left(\phi_y + \epsilon\psi_y\right)\psi_y + \psi_t \, \mathrm{d}y\mathrm{d}x\mathrm{d}t\big|_{\epsilon=0} \\
&= \iint \left[\frac{1}{2}\left(\phi_x\right)^2 + \frac{1}{2}\left(\phi_y\right)^2 + \left(\phi_t\right) + gy\right]_{y=\zeta} \eta \, \mathrm{d}x\mathrm{d}t \\
&+ \iiint_0^{\zeta} \left(\phi_x\right)\psi_x + \left(\phi_y\right)\psi_y + \psi_t \, \mathrm{d}y\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.6}$$

where we have employed some tricks of differentiating integrals with variable boundaries (see Appendix A.1). The first integral over x and t is in the form we want: for the variation δI to vanish for arbitrary choices of η, the expression in the square brackets must vanish – this is exactly the Bernoulli condition at the free surface $y = \zeta(x, t)$. The second integral must be treated carefully to extract the remaining equations.

To this end, it is convenient to rewrite it in slightly greater generality as an integral over a 3D region $\mathcal{D}$ as shown in Figure 4.2:

$$\iiint_{\mathcal{D}} \left(\phi_x\right)\psi_x + \left(\phi_y\right)\psi_y + \psi_t \, \mathrm{d}y\mathrm{d}x\mathrm{d}t$$

Our desire to transfer the derivatives onto the arbitrary auxiliary function ψ means we need to integrate by parts; in higher dimensions this is accomplished using an application of the divergence theorem (see Theorem A.1.2):

$$\begin{aligned}
&\iiint_{\mathcal{D}} (\phi_x)\,\psi_x + (\phi_y)\psi_y + 1\psi_t \,\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&= -\iiint \psi\left(\phi_{xx} + \phi_{yy}\right) dydxdt + \underbrace{\iint_{\partial\mathcal{D}} \psi(\phi_x, \phi_y, 1)\cdot \mathbf{n}dS}_{(*)}
\end{aligned} \tag{4.7}$$

The integral (*) is taken over all six faces of the cube depicted in Figure 4.2:

- At x_0 and x_1 (the left and right faces in Figure 4.2) the outward normal vectors are $(x, y, t) = (1, 0, 0)$ and $(-1, 0, 0)$.
- At t_0 and t_1 (the front and back faces in Figure 4.2) the outward normal vectors are $(x, y, t) = (0, 0, 1)$ and $(0, 0, -1)$.
- At 0 and $\zeta(x, t)$ (the top and bottom faces in Figure 4.2) the outward normal vectors are $(x, y, t) = (0, -1, 0)$ and $(-\zeta_x, 1, -\zeta_t)$.[1]

With these calculations in hand, we find:

$$\iint_{\partial\mathcal{D}} \psi(\phi_x, \phi_y, 1)\cdot \mathbf{n}dS = \iint -\psi\phi_y|_{y=0}\mathrm{d}x\mathrm{d}t + \iint \psi\left(-\zeta_x\phi_x + \phi_y - \zeta_t\right)|_{y=\zeta}\mathrm{d}x\mathrm{d}t, \tag{4.8}$$

with all other integrals disappearing. If we now gather what we have found in (4.6), (4.7) and (4.8), with the understanding that the functions η and ψ are arbitrary auxiliary functions, the equations and boundary conditions of the water wave problem emerge:

$$\begin{aligned}
&\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + gy = 0 \text{ on } y = \zeta \\
&\phi_{xx} + \phi_{yy} = 0 \\
&\phi_y = 0 \text{ on } y = 0 \\
&\phi_y = \zeta_t + \phi_x\zeta_x \text{ on } y = \zeta(x, t)
\end{aligned}$$

4.2.1 Reduced variational principles

Luke's Lagrangian gives important structural information about the water wave problem, and the variational approach is a very useful tool for accurate numerical modelling of water waves (see Bokhove [Bok22] for an overview of recent work connected to finite-element methods). However, variational principles can also be used to derive consistent reduced equations. In this sense it is a technique that stands alongside perturbation theory. We illustrate one example, where we assume that the potential $\phi(x, z, t)$ is replaced by the first term in the shallow water/Boussinesq expansion (2.11) $\phi(x, z, t) = \phi_0(x, t)$. Inserting this into Luke's Lagrangian (4.4) with lower boundary $y = -d$ allows the integral over the depth y to be evaluated immediately, yielding

$$L = (\zeta + d)\left(\phi_{0t} + \frac{1}{2}\phi_{0x}^2\right) + \frac{g}{2}\left(\zeta^2 - d^2\right). \tag{4.9}$$

[1] Recall that the surface $y = \zeta(x, t)$ can be rewritten as the set $F(x, y, t) := y - \zeta(x, t) = 0$. The normal to a surface defined as a zero level set is $\nabla F = (-\zeta_x, 1, -\zeta_t)$.

We consider the variations

$$\begin{aligned}\frac{\delta L}{\delta \phi_0} &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \iint (\zeta + d) \left[\phi_{0t} + \epsilon \psi_{0t} + \frac{1}{2} \left(\phi_{0x} + \epsilon \psi_{0x} \right)^2 \right] \mathrm{d}x\mathrm{d}t \\ &= - \iint \psi_0 \left(\zeta_t + ((\zeta + d)\phi_{0x})_x \right) \mathrm{d}x\mathrm{d}t, \\ \frac{\delta L}{\delta \zeta} &= \iint \eta \left(\phi_{0t} + \frac{1}{2}\phi_{0x}^2 + g\zeta \right) \mathrm{d}x\mathrm{d}t,\end{aligned}$$

which lead (since η and ψ_0 are arbitrary functions) immediately to the system of equations

$$\begin{aligned}\zeta_t + (\phi_{0x}(\zeta + d))_x &= 0, \\ \phi_{0t} + \frac{1}{2}\phi_{0x}^2 + g\zeta &= 0.\end{aligned}$$

Taking the x derivative of the second equation and writing $\phi_{0x} = u$ yields the familiar system

$$\eta_t + (u(\zeta + d))_x = 0 \tag{4.10}$$

$$u_t + uu_x + g\zeta_x = 0 \tag{4.11}$$

of shallow water equations (1.33) derived in Chapter 1. More sophisticated modifications of the variational principle can be found in Clamond & Dutykh [CD12].

4.3 The averaged Lagrangian idea

In this section we will explore another powerful application of variational principles due to Whitham [Whi74]. This technique, and the ideas that go with it have found applications in the propagation of waves in inhomogeneous media, in particular situations with currents, bathymetry, and related areas. These continue to be explored to the present day.

4.3.1 Fundamentals of slowly varying waves

Before we jump into an exploration of Whitham's ideas, we will set the scene with some basic ideas that go back to linear wave dynamics. In the classical linear setting we may seek solutions which are superpositions of sinusoids, of the form

$$\zeta(\mathbf{x}, t) = \mathfrak{Re} \left(a \exp(i\theta(\mathbf{x}, t)) \right), \tag{4.12}$$

where a and θ are real. We call $\theta(\mathbf{x}, t)$ the *phase function*, and it satisfies

$$\frac{\partial \theta}{\partial t} = -\omega, \quad \nabla \theta = \mathbf{k}. \tag{4.13}$$

Now we are used to a frequency ω and a wavenumber k which are fixed and related to one another by a function called the *dispersion relation*, but there is no reason why both cannot change with time and/or space. For example, if we are thinking of the usual linear, finite-depth dispersion relation $\omega^2 = gk \tanh(kh)$, we may find that the depth changes with distance so that $h = h(x)$. This naturally introduces a spatial variation. Likewise, we may find that time-dependent (unsteady) currents introduce a temporal variation. Or we may find that the amplitude $a = a(\mathbf{x}, t)$ is itself slowly varying (see (3.14)).

Provided the variations are slow, it still makes sense to use (4.13) to describe the *local* frequency and wavenumber. In this context it is worth pointing out a few other facts: the (local) wavenumber $\mathbf{k}$ is formally defined as a gradient $\mathbf{k} = \nabla\theta$. This means that $\mathbf{k}$ is rotation free, i.e.

$$\nabla \times \mathbf{k} = \frac{\partial}{\partial y}k_x - \frac{\partial}{\partial x}k_y = 0,$$

where we denote the components $\mathbf{k} = (k_x, k_y)$. Furthermore, by cross differentiating in (4.13) we find

$$\frac{\partial}{\partial t}\mathbf{k} + \nabla\omega = 0.$$

4.3.2 The averaged Lagrangian

We have seen above that the water wave problem can be seen as a variational principle, where we try to minimise the variations of a Lagrangian L in an integral like

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} L\mathrm{d}x\mathrm{d}t.$$

Here we will restrict to a single spatial dimension for ease of use. In Luke's Lagrangian, the functions appearing are the potential ϕ and the free surface elevation ζ. Whitham proposed the following ad hoc procedure: insert a travelling wave ansatz of a certain type for ϕ and ζ, or more generally assume that all the generalised coordinates in the Lagrangian $q_i(x,t) = f_i(\theta)$, for θ a suitable phase-function – for example $\theta = kx - \omega t$. Subsequently average over the phase of the wave while keeping other parameters fixed, so forming a new Lagrangian which depends on the amplitudes in the ansatz, the frequency, and the wavenumber.

More concretely, we want to replace

$$L(t, x, q, q_x, q_t) = L(\theta, f, kf', -\omega f'),$$

where $f'(\theta)$ is the total derivative. This gives a new Lagrangian, which is then averaged over θ to yield

$$L(\omega, k, a) = \frac{1}{2\pi}\int_0^{2\pi} L(\theta, f, kf', -\omega f')\mathrm{d}\theta.$$

It is clear that the θ dependence is being integrated out, so only dependence on ω, k and a remains, where a is the amplitude in the periodic wave ansatz. Whitham's *averaged variational principle* then consists of looking at the variational derivative

$$\delta \iint L(\omega, k, a)\mathrm{d}x\mathrm{d}t = 0,$$

bearing in mind that generally ω and k are connected by $k_t + \omega_x = 0$ (through $\theta_{tx} = \theta_{xt}$).

Looking separately at the variation

$$\frac{\delta I}{\delta a} = 0$$

leads to

$$\frac{\partial L}{\partial a} = 0,$$

while the variation

$$\frac{\delta I}{\delta\theta} = \frac{d}{d\epsilon}\iint L(-\theta_t - \epsilon\phi_t, \theta_x + \epsilon\phi_x, a)\mathrm{d}x\mathrm{d}t\bigg|_{\epsilon=0}$$

leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial k}\right) = 0.$$

This is a conservation law for the quantity $\frac{\partial L}{\partial \omega}$, which can be identified with the wave action. The foregoing is rather abstract, and it is useful to see what it means via a series of practical calculations of increasing complexity.

In our first example, we shall take Luke's Lagrangian

$$L = \int_{-h_0}^{\zeta(x,t)} \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \phi_t + gy dy$$

and insert the linear expressions

$$\phi(x,y,t) = \frac{\omega a}{k}\sin(\theta)\exp(ky), \quad \eta = a\cos(\theta).$$

Evaluating L from the flat bed (now located at $y = -h_0$) to the linear free surface $y = 0$ we find the expression for the averaged Lagrangian

$$L(\omega,k,a) = -\frac{\omega^2 a^2}{2k} - \frac{\omega^2 a^2\left(\mathrm{e}^{-2kh_0} - 1\right)}{4k} + \frac{\left(a^2 - 2h_0^2\right)g}{4}.$$

There are two terms which depend on h_0: if we assume the depth is very large, the exponential term can be neglected, and we notice that the other term $-2h_0^2 g/4$ contributes nothing to the variation. Consequently, the averaged Lagrangian can be written

$$L(\omega,k,a) = \frac{a^2\left(gk - \omega^2\right)}{4k}.$$

We find that the variation with respect to a yields

$$\frac{\partial L}{\partial a} = \frac{a(gk - \omega^2)}{2k} = 0 \Longrightarrow \omega^2 = gk$$

the dispersion relation. With this the entire averaged Lagrangian is equal to zero, which agrees with the idea that the averaged Lagrangian represents the difference $\bar{K} - \bar{V}$ of averaged kinetic and potential energies, which vanishes according to the calculations above (see also Section 1.2.2).

A slightly more sophisticated example involves the substitution of the second order expressions

$$\begin{aligned}
\eta &= a\cos(\theta) + \frac{1}{2}ka^2\cos(2\theta),\\
\phi &= \frac{a\omega}{k}\sin(\theta)\exp(ky) + \frac{\omega a^2}{2}\sin(2\theta)\exp(2ky).
\end{aligned}$$

In order to form the averaged Lagrangian, we need to integrate these expressions from $-h_0$ to the free surface η, while maintaining terms up to order $O(a^2k^2)$. In order to evaluate the integral, we need to employ the following trick: first use additivity of the integral to rewrite

$$\int_{-h_0}^{\eta} f dy = \int_{-h_0}^{0} f dy + \underbrace{\int_{0}^{\eta} f dy}_{(*)},$$

then, since the free surface at η is assumed to be a small perturbation, expand f in a Taylor series

$$f(y) = f(0) + y\frac{\partial}{\partial y}f|_{y=0} + \ldots$$

to make the dependence on the vertical coordinate explicit. Finally insert this Taylor series expansion into $(*)$ to obtain

$$(*) = \int_0^{\eta} f(0) + yf'(0) + \ldots \mathrm{d}y = f(0)\eta + \frac{\eta^2}{2} f'(0) + \ldots.$$

In this case f stands for any of the functions appearing in Luke's Lagrangian. The procedure to be carried out is algebraically somewhat lengthy, but essentially involves inserting the above ansatz into

$$L = \int_{-h_0}^{\zeta(x,t)} \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \phi_t \mathrm{d}y + \frac{g\zeta^2}{2}$$

and averaging as before. Note that we have taken out the gy term from the Lagrangian and neglected the contribution at h_0 in advance.

The resulting averaged Lagrangian turns out to be

$$L(\omega, k, a) = \frac{a^4 g\, k^3 + 4a^2 gk - 4\omega^2 a^2}{16k},$$

and

$$\frac{\partial L}{\partial a} = \frac{a\left(a^2 g\, k^3 + 2gk - 2\omega^2\right)}{4k} = 0$$

yields the deep-water dispersion relation

$$\omega^2 = gk(1 + \frac{a^2k^2}{2})$$

obtained at third order (see (3.11)). We can further generalise this by allowing for slow variations in the amplitudes up to second order, allowing $a = a(\epsilon t, \epsilon x)$. The averaged Lagrangian can then be used to derive the nonlinear Schrödinger equation for the envelope a, as shown by [YL75].

4.4 Hamiltonian formulation of the water wave problem

Unfortunately the Hamiltonian formulation of the water wave problem, due to Zakharov [Zak68] is somewhat more involved than the Lagrangian. In particular, it needs careful reformulation of the problem. Since this reformulation is itself useful, we provide it here and sketch roughly how it relates to the Hamiltonian for water waves.

We start with the full water wave problem for inviscid, incompressible flow

$$u_t + uu_x + vu_y + wu_z = -P_x \tag{4.14}$$
$$v_t + uv_x + vv_y + wv_z = -P_y \tag{4.15}$$
$$w_t + uw_x + vw_y + ww_z = -P_z - g \tag{4.16}$$
$$u_x + v_y + w_z = 0 \tag{4.17}$$
$$\nabla \times \mathbf{u} = 0 \tag{4.18}$$
$$P = P_{atm} \text{ on } z = \zeta(x, y, t) \tag{4.19}$$
$$w_t = \zeta_t + u\zeta_x + v\zeta_y \text{ on } z = \zeta(x, y, t) \tag{4.20}$$
$$w = 0 \text{ on } z = -h \tag{4.21}$$

The condition of irrotationality implies $\exists \phi : \nabla\phi = \mathbf{u}$, reducing the field equation to Laplace's equation. Integrating the momentum equations with the help of the potential, this system can be rewritten as

$$\Delta\phi = 0 \text{ on } -h < z < \zeta \tag{4.22}$$
$$\zeta_t + \nabla_x\phi \cdot \nabla_x\zeta = \phi_z \text{ on } z = \zeta \tag{4.23}$$
$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\zeta = 0 \text{ on } z = -h \tag{4.24}$$
$$\phi_z = 0 \text{ on } z = -h \tag{4.25}$$

The next step relies on reducing the system (4.22)–(4.25) to an equivalent formulation on the free surface only, and subsequently applying a spatial Fourier transformation.

Akin to the approach undertaken by Zakharov [Zak68], a new variable is introduced for the potential at the free surface

$$\psi(x, y, t) = \phi(x, y, \zeta(x, y, t), t). \tag{4.26}$$

Hence

$$\psi_t = \phi_t + \phi_z\zeta_t \tag{4.27}$$
$$\psi_x = \phi_x + \phi_z\zeta_x \tag{4.28}$$
$$\psi_y = \phi_y + \phi_z\zeta_y \tag{4.29}$$

where ϕ_z is the vertical velocity at the free surface. Using this new notation in (4.23) yields:

$$\phi_z = \zeta_t + \zeta_x(\psi_x - \phi_z\zeta_x) + \zeta_y(\psi_y - \phi_z\zeta_y), \tag{4.30}$$

and using the notation $\nabla := \nabla_{(x,y)}$ for the horizontal gradient only, and $\nu = \phi_z(x, y, \zeta, t)$ this may be reformulated as

$$\zeta_t - \nu(1 + (\nabla\zeta)^2) + \nabla\psi \cdot \nabla\zeta = 0. \tag{4.31}$$

Using the same relations in (4.24), substituting from (4.30) for ζ_t, and collecting terms yields

$$\psi_t + g\zeta + \frac{1}{2}\left(\psi_x^2 + \psi_y^2\right) - \frac{1}{2}\phi_z^2(1 + \zeta_x^2 + \zeta_y^2) = 0, \tag{4.32}$$

or employing a more compact notation

$$\psi_t + g\zeta + \frac{1}{2}(\nabla\psi)^2 - \frac{1}{2}\nu^2\left(1 + (\nabla\zeta)^2\right) = 0. \tag{4.33}$$

(See [MSY18]). The philosophy behind these choices should be as follows: if ζ is known at some time $t = t_0$, the Laplace equation (4.22) can be solved uniquely if boundary values are specified. If at the same time $\psi(x, y, t_0) = \phi(x, y, \zeta(x, y, t_0), t_0)$ is known, along with bottom boundary condition (4.25), this is sufficient to specify a unique solution for the potential ϕ. From this solution one may calculate $\nu = \phi_z$ at the specified time, and subsequently the time evolution of ζ and ψ is simply given by the equations (4.31) and (4.33). Using Fourier transforms of these two conditions, and truncating at a fixed order in nonlinearity is precisely the idea behind the high-order spectral method (HOS) developed by Dommermuth and Yue [DY87] and West et al [WBJ+87], and widely used since then in a variety of applications.

In contrast to the Lagrangian variational principle, which in 1D produces a second-order equation of motion akin to $F = m\ddot{x}$, the Hamiltonian produces the associated system of first order equations $\dot{x} = p/m$, $\dot{p} = F$ in terms of a (generalised) position and (generalised) momentum. In our setting, this means that we will have Hamiltonian equations

$$\frac{\partial\zeta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\zeta}, \tag{4.34}$$

which are exactly the boundary formulation of the water wave problem (4.33) and (4.31). The associated Hamiltonian is

$$H = \int_{-h}^{\zeta} \frac{1}{2}|\nabla\phi|^2 dz + \frac{1}{2}g\eta^2, \tag{4.35}$$

which should be varied as

$$\delta \iint H \mathrm{d}x\mathrm{d}y.$$

The principle difficulty compared to Luke's variational principle is that we need to consider the variation of H with respect to the potential evaluated at the free surface ψ, while the Hamiltonian itself depends on the potential in the entire fluid bulk ϕ. The resolution of this problem requires some careful algebra and a few tricks, which we will not cover here.

Appendix A

Basics of vector calculus and the governing equations

A.1 Some mathematical preliminaries

Before we proceed, we recall a few important definitions from vector calculus.

Definition A.1.1 (Path) *A path is a map* $c : I \subset \mathbb{R} \to \mathbb{R}^n$.

For example, the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is the image of the path

$$c(t) = (\sin t, \cos t), \quad 0 \le t \le 2\pi.$$

The parabola is the image of the path

$$c(t) = (t, t^2), \quad t \in \mathbb{R}.$$

In case the mapping is differentiable, we can speak of a velocity:

Definition A.1.2 (Velocity vector) *The velocity vector of a differentiable path* $c(t)$ *is defined as the derivative with respect to the parametrizing variable. If* $c(t) = (x(t), y(t))$ *then*

$$c'(t) = (x'(t), y'(t)).$$

$c'(t)$ *is the vector tangent to the path* $c(t)$ *at time* t.

We also recall

Definition A.1.3 (Chain Rule) *Let* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *be open sets,* $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$, *with* $g(U) \subset V$. *If* g *is differentiable at* x_0 *and* f *is differentiable at* $y_0 = g(x_0)$, *then* $f \circ g$ *is differentiable at* x_0 *and*

$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0).$$

Recall that for $f : \mathbb{R}^n \to \mathbb{R}^m$, $Df(x_0)$ is the $m \times n$ matrix with entries $t_{ij} = \partial f_i / \partial x_j$.

Let us apply the chain rule to a differentiable path $c : \mathbb{R} \to \mathbb{R}^3$ and a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, with $c(t) = (x(t), y(t), z(t))$. Then

$$\begin{aligned} D(f \circ c)(t_0) = Df(c(t_0))Dc(t_0) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}\right)\Big|_{c(t_0)} \cdot \left(x'(t), y'(t), z'(t)\right)^T\Big|_{t_0} \\ &= \nabla f(c(t_0)) \cdot c'(t_0) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\Big|_{t=t_0}. \end{aligned}$$

Definition A.1.4 (Acceleration) *The acceleration along a path is denoted by*

$$a = \frac{dv}{dt} = c''(t).$$

For example, given a particle moving in a circular trajectory on a path of radius r_0

$$r(t) = (r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0})$$

we compute the speed $\|r'(t)\|$ and acceleration.

$$\|r'(t)\| = s,\ r'' = -\omega^2 r(t),$$

where we write $\omega = s/r_0$ for the frequency of the rotation.

Two further ideas will come in quite useful:

Differentiation of an integral We occasionally find ourselves in situations where we need to differentiate integral expressions where the boundaries vary. For example, the generic integral

$$I(t) = \int_{a(t)}^{b(t)} f(x,t)dx$$

can be differentiated with respect to t as follows:

$$\frac{dI}{dt} = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx. \tag{A.1}$$

This follows from the standard chain rule when the integral is viewed (correctly) as a function of the boundaries: $I(t) = I(a(t), b(t), f(x,t))$.

The following important theorem (due to Gauss) allows us to exchange volume integrals over the divergence of a vector field by surface integrals.

Theorem A.1.1 (Divergence theorem) *If $\mathcal{D}$ is a closed region whose boundary – denoted by $\partial\mathcal{D}$ – is simple and piecewise smooth (continuously differentiable), and if the vector field $\mathbf{u}$ has continuous partial derivatives in $\mathcal{D}$ and on $\partial\mathcal{D}$, then*

$$\iiint_{\mathcal{D}} \nabla \cdot \mathbf{u}\, \mathrm{d}x \mathrm{d}y \mathrm{d}z = \iint_{\partial\mathcal{D}} \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S.$$

Here $\mathrm{d}S$ denotes the surface integral over $\partial\mathcal{D}$, and $\mathbf{n}$ is the outward pointing normal vector to the surface.

The foregoing theorem, which is used in deriving the differential form of the equations of motion, can also be used to generalise integration by parts. If $\mathbf{u} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(x,y,z) \mapsto (u(x,y,z), v(x,y,z), w(x,y,z))$ is a vector field, and $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a scalar field, the following generalisation of the product rule holds:

$$\nabla \cdot (\mathbf{u}\psi) = \mathbf{u} \cdot (\nabla\psi) + \psi(\nabla \cdot \mathbf{u}) = u\psi_x + v\psi_y + w\psi_z + \psi(u_x + v_y + w_z).$$

Inserting the vector field $\mathbf{u}\psi$ into the divergence theorem yields

Theorem A.1.2 (Green's theorem)

$$\iiint_{\mathcal{D}} \psi\,(u_x + v_y + w_z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{\partial\mathcal{D}} (\mathbf{u}\psi) \cdot \mathbf{n}\,\mathrm{d}S - \iiint_{\mathcal{D}} u\psi_x + v\psi_y + w\psi_z \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

which is the three dimensional integration-by-parts "formula".

A.2 Basic ideas of fluid flow

Fluid dynamics is based on classical mechanics, that is, on Newton's laws of motion.

In simple classical scenarios, we can keep track of the motion of individual particles. For a simple pendulum, consisting of a mass on a rod of fixed length l, we only need to keep track of the angle θ of the mass to the vertical, and can write the governing equation as

$$\theta'' + \frac{g}{l}\sin(\theta) = 0.$$

Here g is the acceleration of gravity, and the prime $'$ denotes differentiation with respect to time.

As more bodies become involved, the problem becomes increasingly complex, such as the set of differential equations for bodies under the force of gravity

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j\neq i} \frac{G m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3}.$$

Here m_i are the individual masses, q_i the positions, G the gravitational constant, and t is time. We must deal with a coupled system of nonlinear ordinary differential equations (ODEs).

To describe the large scale motions of a fluid means finding a way to describe the motions of millions or billions of molecules of that fluid (whether air, water, or otherwise). Rather than trying to keep track of each individual fluid particle[1], it is useful to assume a description in terms of vector fields.

We use the following notation:

1. The velocity of the fluid is denoted by $\mathbf{u}(\mathbf{x}, t)$, where $\mathbf{x}$ is a position vector, t is time.
2. The density (mass/unit volume) of the fluid is $\rho(\mathbf{x}, t)$.
3. The pressure at any point in the fluid is $P(\mathbf{x}, t)$.

If it is not further specified, the usual rectangular Cartesian coordinates are used, with components

$$\mathbf{x} = (x, y, z) \text{ and } \mathbf{u} = (u, v, w).$$

We assume without further comment that all functions are suitably smooth.

A.3 Governing equations

In standard Euclidean coordinates, let us imagine the path of a particle of fluid, denoted by $\mathbf{x}(t) = (x(t), y(t), z(t))$. A schematic depiction is given in Figure A.1. In a fluid, we assume a continuum of particles, each with their own paths, in a motion such that two particles never occupy the same location. We wish to concentrate on the velocity vector field associated with this motion.

On the one hand, if we are measuring this velocity field at a certain point $(x(t_0), y(t_0), z(t_0))$ at a certain instant t_0 we would expect the value of the velocity to coincide exactly with the velocity of *that* particle whose path takes it through said point at precisely the measurement time, so

$$\mathbf{u}(x(t_0), y(t_0), z(t_0), t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$

[1]It is possible to formulate the laws of fluid motion by keeping track of each particle, called the *Lagrangian description* of fluid motion. This can be very useful for specialized applications, but the majority of work on fluids uses the *Eulerian description* which we shall pursue.

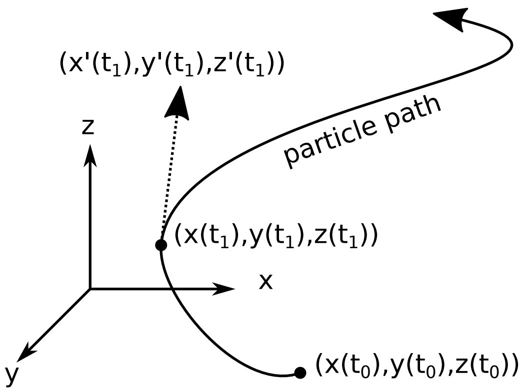


Figure A.1: Path of a particle parametrised by t in xyz-space. Also indicated is the velocity vector (tangent to the path) at time t_1.

However, we can keep measuring at the same point at a *different* time, at which point different particles will pass through the measurement point and the velocity field will change as a function of time. This is why the velocity field $\mathbf{u}(x, y, z, t)$ is allowed to depend explicitly on the time coordinate t.

Of course, we can follow the change in the velocity *as we follow a particle through the flow.* This change of velocity is the acceleration, and we write $a = \frac{d^2\mathbf{x}(t)}{dt^2} = \frac{d}{dt}\mathbf{u}(x(t), y(t), z(t), t)$, where d/dt stands for the total derivative. This may be expanded using the chain rule as follows:

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{u}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{u}}{\partial t}$$

which results in the expression

$$\mathbf{a} = u\mathbf{u}_x + v\mathbf{u}_y + w\mathbf{u}_z + \mathbf{u}_t.$$

This is written compactly and defined as follows:

Definition A.3.1 (Material derivative) *The differential operator* $\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla$ *is called the material derivative. This depicts the rate of change of a quantity with the flow of a fluid (following a particle through the flow).*

We are now in a position to formulate Newton's second law for a fluid, namely (for a unit volume)

$$F = ma \Leftrightarrow \rho\frac{D\mathbf{u}}{Dt} = F.$$

It remains to specify the forces contained in the term F on the right hand side. We generally have the following forces (per unit volume):

1. **Pressure**: $F = -\nabla P(x, y, z, t)$
2. **Gravity**: $F = -\nabla(\rho g z)$

which yield the equation of conservation of momentum in the following form

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla P - (0, 0, \rho g),} \tag{A.2}$$

called the **Euler equation**, or in Cartesian coordinates

$$u_t + uu_x + vu_y + wu_z = -P_x/\rho \tag{A.3}$$
$$v_t + uv_x + vv_y + wv_z = -P_y/\rho \tag{A.4}$$
$$w_t + uw_x + vw_y + ww_z = -P_z/\rho - g \tag{A.5}$$

We now define:

Definition A.3.2 (Incompressible flow) *A fluid flow is termed incompressible if the density of a fluid particle does not change as it moves through the fluid, i.e.*

$$\frac{D\rho}{Dt} = 0.$$

Using a standard argument based on volumes of fluid, it can be seen that conservation of mass for a fluid is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.$$

Thus, for an incompressible flow, mass conservation is equivalent to

$$\boxed{\nabla \cdot \mathbf{u} = 0.} \tag{A.6}$$

When it is common to work with incompressible flows (e.g. if the density of the fluid can be assumed constant) equation (A.6) is simply called the equation of **mass conservation.** We shall employ this equation throughout these notes.

A.3.1 Examples of simple flows

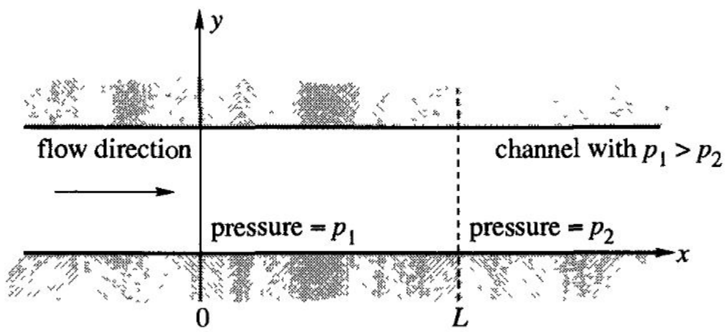
As a simple example, we shall use the Euler equation (A.2) to derive the form of **hydrostatic pressure**. Assume a stationary fluid $\mathbf{u} = 0$ with $\rho = 1$. The fluid is to be acted upon by the forces of gravity, given by $(0, 0, -g)$ with g constant, and pressure. Assume furthermore that $z = h_0$ denotes the surface of the fluid (in a glass, say), where the pressure is constant – what is the pressure throughout the fluid?

Clearly $\mathbf{u} = 0$ is divergence free, satisfying (A.6). Inserting into the Euler equations (A.3)–(A.5), we find only equation (A.5) in the z-direction is nontrivial:

$$0 = -P_z - g.$$

Integrating once yields $P = -gz + C$. By adjusting $C = gz_0 + P_{atm}$ where P_{atm} denotes atmospheric pressure, we have the hydrostatic solution, where the pressure at the top of the vessel is equal to the constant atmospheric pressure of the air. This yields the result that, in a still fluid, the pressure increases linearly with depth from a reference surface.

We consider a second example, depicted by the figure below.



Neglecting gravity, assume the pressure p_1 at $x = 0$ is larger than the pressure p_2 at $x = L$ in a 2D channel, so fluid is pushed from left to right. Find a solution to the Euler equations for incompressible flow in the form

$$\mathbf{u}(x, y, t) = (u(x, t), 0, 0), \quad P(x, y, t) = p(x).$$

We insert into the Euler equation with constant density to get $u_t + uu_x = -p_x$. Mass conservation yields $u_x = 0$, so that the Euler equation simplifies to $u_t = -p_x$. Consequently, $p_{xx} = 0$, so that p is linear in x :

$$p(x) = p_1 - \left(\frac{p_1 - p_2}{L}\right) x$$

in order to satisfy the boundary conditions at $x = 0$ and $x = L$. Substituting back into the Euler equation and integrating yields

$$u = \left(\frac{p_1 - p_2}{L}\right) t + C.$$

Thus the flow in this channel with constant pressure gradient increases indefinitely as a function of time – clearly an unrealistic scenario. The remedy for this problem in pipe flow is to be found in viscosity, which requires the Navier-Stokes equations.

A.4 Properties of fluid flow

We start with a number of useful definitions:

Definition A.4.1 (Particle path) *A particle path or trajectory is the curve traced out by a particle of the fluid as time progresses, and is thus a solution to the differential equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t)$$

with suitable initial conditions.

Definition A.4.2 (Streamline) *For a fluid flow with velocity field* $\mathbf{u}(\mathbf{x}, t)$, *a streamline at a fixed time is an integral curve of* $\mathbf{u}$. *Thus,* $\mathbf{x}(s)$ *is a streamline at time* t_0 *if it is a parametrized curve that satisfies*

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(s), t_0).$$

Streamlines and particle paths do not generally coincide, except in stationary or steady flow:

Definition A.4.3 (Steady flow) *A fluid flow is steady or stationary if it is independent of time* t, *i.e.*

$$\frac{\partial \mathbf{u}}{\partial t} = 0.$$

Note that this is not the same as $d\mathbf{u}/dt = 0$, and does not mean there is no acceleration – it simply means the velocity field does not change explicitly with time.

In unsteady flow, the streamlines give a picture of the *instantaneous* velocity directions, while the particle paths give the velocity directions of *one particle over time.* Both of these look drastically different if the reference frame is changed e.g. if the observer is at rest relative to fluid motions or not.

Definition A.4.4 (Vorticity) *The curl of a fluid velocity field is referred to as its vorticity,*

$$\omega = \nabla \times \mathbf{u}.$$

A vector field with vanishing vorticity is called irrotational.

In rectangular Cartesian coordinates, the components of the vorticity vector are

$$\omega = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{A.7}$$

A well-known theorem from vector calculus attests to the importance of irrotational flows:

Theorem A.4.1 *For any irrotational flow* $\mathbf{u}$ *there exists a special function, called a potential, and usually denoted* ϕ *with*

$$\mathbf{u} = \nabla\phi.$$

In fluid flows, when the vector field is a velocity field, the function ϕ is commonly called the **velocity potential.**

CAUTION: some authors define the velocity potential such that $\mathbf{u} = -\nabla\phi$. This should be checked carefully when reading other texts!

We make note of one important fact: for an incompressible, irrotational fluid flow, the potential is a harmonic function, i.e.

$$\boxed{\Delta\phi = 0.}$$

A.4.1 Bernoulli's equations

The Bernoulli equation is an integrated form of the Euler equations, and provides a relationship between the pressure field and the kinematics, which turns out to be quite useful. Such an equation can be derived under two assumptions:

1. Steady flow
2. Irrotational flow

These lead to similar looking, but distinct, equations, both of which are commonly called *the Bernoulli equation.* To simplify the algebra, we begin our derivation in 2D, first for irrotational flow. For two dimensional flows $\mathbf{u} = (u, 0, w)$, the irrotationality condition becomes

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}.$$

Write the Euler equations in the $xz-$plane:

$$\begin{aligned} u_t + uu_x + wu_z &= -\frac{1}{\rho}p_x \\ w_t + uw_x + ww_z &= -\frac{1}{\rho}p_z - g \end{aligned}$$

Using irrotationality (see (A.7)) $u_z = w_x$ these become

$$\begin{aligned} u_t + \frac{1}{2}\left((u^2)_x + (w^2)_x\right) &= -\frac{1}{\rho}p_x \\ w_t + \frac{1}{2}\left((u^2)_z + (w^2)_z\right) &= -\frac{1}{\rho}p_z - g \end{aligned}$$

Now introduce the velocity potential ϕ with $\phi_x = u$, $\phi_z = w$:

$$\frac{\partial}{\partial x}\left[\phi_t + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho}\right] = 0$$
$$\frac{\partial}{\partial x}\left[\phi_t + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho}\right] = -g$$

Integrating both equations leads to

$$\phi_t + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho} = C_1(z,t)$$
$$\phi_t + \frac{1}{2}\left(u^2 + w^2\right) + \frac{p}{\rho} = -gz + C_2(x,t)$$

and comparing both sides leads to the compact expression

$$\boxed{\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \frac{p}{\rho} + gz = C(t)} \tag{A.8}$$

which relates the fluid pressure, particle elevation, and velocity potential, and is the Bernoulli equation for irrotational flow. For three dimensional flows, there is an additional factor $(1/2)\phi_y^2$ on the left hand side.

In the case of steady but possibly rotational flows, we use the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2}\right) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

inserted into the Euler equation:

$$\mathbf{u}_t + \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{P}{\rho} + gz\right) = \mathbf{u} \times \omega$$

For steady flow, $\mathbf{u}_t = 0$. Note subsequently that the right-hand side $\mathbf{u} \times \omega$ is perpendicular to both $\mathbf{u}$ and ω. We now claim that the following equation holds on streamlines:

$$\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{P}{\rho} + gz = \text{const.}$$

Let $\mathbf{x}(s)$ be a streamline, and evaluate the expression:

$$\left.\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{P}{\rho} + gz\right|_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} = \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{P}{\rho} + gz\right) \cdot \mathbf{x}'(s)ds$$
$$= \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} (\mathbf{u} \times \omega) \cdot \mathbf{x}'(s)ds = 0,$$

since, by the definition of a streamline, $\mathbf{x}'(s) = \mathbf{u}(\mathbf{x}(s))$.

A.4.2 Two-dimensional flows

Let $\mathbf{u} = (u, v)$ be a two dimensional, irrotational and incompressible flow. It follows that $\mathbf{u} = \nabla\phi = (\phi_x, \phi_y)$ and from (A.6) that $\Delta\phi = 0$. Assuming the fluid domain is simply connected, we can find a harmonic conjugate for ϕ using the Cauchy-Riemann equations:

$$\phi_x = \psi_y$$
$$\phi_y = -\psi_x$$

We call this function $\psi(x, y)$ the stream-function of the flow, and it is immediate that it also satisfies the Laplace equation. The streamlines in 2D flow are the level lines of the function ψ. This is seen as follows: let $\mathbf{x}(s) = (x(s), y(s))$ be a streamline, so that $x'(s) = u(x(s), y(s), t)$ and $y'(s) = v(x(s), y(s), t)$. The claim is that ψ is constant on the curve $\mathbf{x}(s)$:

$$\frac{d}{ds}\psi(x(s), y(s)) = \frac{\partial\psi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' = -vu + uv = 0.$$

As a simple example, we determine the stream function for the following velocity potential

$$\phi(x, y, t) = (-3x + 5y)\cos\frac{2\pi t}{T}$$

The velocity components are

$$u = \phi_x = -3\cos\frac{2\pi t}{T}$$
$$v = \phi_y = 5\cos\frac{2\pi t}{T}$$

Using the Cauchy-Riemann equations and integrating, we find

$$\psi(x, y, t) = -3y\cos\frac{2\pi t}{T} + C_1(x, t)$$
$$\psi(x, y, t) = -5x\cos\frac{2\pi t}{T} + C_2(y, t)$$

and so deduce that the stream function must be

$$\psi(x, y, t) = -(5x + 3y)\cos\frac{2\pi t}{T} + C(t).$$

For a fixed time t, the streamlines are lines $\psi = C$, which we see to be families of lines with slopes of -5/3.

The stream-function also gives rise to a powerful reformulation of the water-wave problem in 2D. We may define the complex potential via

$$\Omega(z) = \phi(x, y) + i\psi(x, y),$$

where ϕ is the velocity potential of the flow and ψ the stream function. For example, a uniform flow with speed u_c and flow direction making an angle θ_c to the positive $x-$axis can be specified by the complex potential

$$\Omega(z) = u_c e^{-i\theta_c} z$$

where u_c, $\theta_c \in \mathbb{R}^+$. This is easy to see by remembering that $\nabla_{(x,y)}\phi = (u, v)$ and $z = x + iy$. It is then possible to bring to bear many elegant tools from complex analysis – the best-known examples are in calculating lift forces on airfoils and other objects in a flow.

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