Ordinary Differential Equations - 104131

Homework Page No. 4

1. Solve the following non-homogeneous linear differential equations by the method of "Variation of parameters":

   (a) $y'' + y = \tan t$
   Hint: $\sec t = 1/\cos t$, $\int \sec t\, dt = \ln|\sec t + \tan t|$.

   (b) $y'' + y = \dfrac{5}{\cos^2 t}$

   (c) $y'' - 2y' + y = \dfrac{e^t}{t^2+1}$

2. (a) The homogeneous equation $ty'' - (1+t)y' + y = 0$, $t > 0$, has a soltion $y_1(t) = 1 + t$. Find the general solution of this equation.

   (b) Find the general solution of the non-homogeneous equation

$$ty'' - (1+t)y' + y = t^2 e^{2t}$$

3. Solve the following equations by comparison of coefficients:

   (a) $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1,\ y'(0) = 0$

   (b) $y'' + y' - 2y = 2t, \quad y(0) = 0,\ y'(0) = 1$

   (c) $y''' - 2y'' + y' = t^3$

   (d) $y''' + 2y'' + y' = t^3 e^t$

   (e) $y^{(4)} + 2y^{(2)} + y = 3t + 3$

   (f) $y^{(6)} + y^{(3)} = t$

4. Solve

   (a) $x^2y'' - 2xy' + 4y = 2x^2$.

   (b) $x^2y'' - 2xy' + 2y = 4x^3$.

   (c) $x^2y'' + 9xy' + 25y = 50$.

(d) $x^3y''' + 3x^2y'' + xy' + 8y = 65\cos\ln x$.

(e) $y''' = \frac{6y}{x^3}$.

Solutions: 1.(a) $r^2 + 1 = 0$ is the characteristic equation with solutions $\pm i$. Therefore $y_h(t) = c_1\cos t + c_2\sin t$. We now solve the nonhomogeneous equation using variation of parameters. $y_p(t) = c_1(t)\cos t + c_1(t)\sin t$. Then

$$\begin{aligned} c_1'\cos t + c_2'\sin t &= 0 \\ -c_1'\sin t + c_2'\cos t &= \tan t. \end{aligned}$$

Then

$$\begin{aligned}
&c_1' = -c_2'\frac{\sin t}{\cos t} \\
&c_2'\frac{\sin^2 t}{\cos t} + c_2'\cos t = \tan t \\
&c_2'\frac{\sin^2 t + \cos^2 t}{\cos t} = \frac{\sin t}{\cos t} \\
&c_2' = \sin t \\
&c_1' = -\frac{\sin^2 t}{\cos t} = -\frac{1-\cos^2 t}{\cos t} = -\frac{1}{\cos t} + \cos t \\
&c_1(t) = -\ln|\sec t + \tan t| + \sin t + c_1 \\
&c_2(t) = -\cos t + c_2 \\
&y(t) = y_h(t) + y_p(t) = \\
&c_1\cos t + c_2\sin t + (-\ln|\sec t + \tan t| + \sin t)\cos t - \cos t\sin t
\end{aligned}$$

(b) The homogeneous equation is the same as in previous case. Therefore $y_h(t) = c_1\cos t + c_2\sin t$ and $y_p(t) = c_1(t)\cos t + c_1(t)\sin t$.

$$\begin{aligned} c_1'\cos t + c_2'\sin t &= 0 \\ -c_1'\sin t + c_2'\cos t &= \frac{5}{\cos^2 t}. \end{aligned}$$

Then

$$\begin{aligned}
&c_1' = -c_2' \frac{\sin t}{\cos t} \\
&c_2' \frac{\sin^2 t}{\cos t} + c_2' \cos t = \frac{5}{\cos^2 t} \\
&c_2' \frac{\sin^2 t + \cos^2 t}{\cos t} = \frac{5}{\cos^2 t} \\
&c_2' = \frac{5}{\cos t} \\
&c_1' = -\frac{5 \sin t}{\cos^2 t} \\
&c_2(t) = 5 \ln|\sec t + \tan t| + c_1 \\
&c_1(t) = -\frac{5}{\cos t} + c_2 \\
&y(t) = y_h(t) + y_p(t) = \\
&c_1 \cos t + c_2 \sin t + (-\frac{5}{\cos t}) \cos t + (5 \ln|\sec t + \tan t|) \sin t
\end{aligned}$$

2. We find a second solution to the homogeneous equation using abel's formula.

$$y_2(t) = y_1(t) \int \frac{W(t)}{y_1(t)^2} dt = (1+t) \int \frac{\exp(\int \frac{1+t}{t})}{(1+t)^2} dt = (1+t) \int \frac{te^t}{(1+t)^2} dt = (1+t) \frac{e^t}{1+t} = e^t.$$

We now use variation of parameters $y_p(t) = c_1(t)(1+t) + c_2(t)e^t$. Recall variation of parameter needs a normalized equation $y'' - \frac{1+t}{t} y' + \frac{1}{t} y = te^{2t}$. Then

$$\begin{aligned}
&c_1'(1+t) + c_2' e^t = 0 \\
&c_1' + c_2' e^t = te^{2t}
\end{aligned}$$

Then

$$
\begin{aligned}
c_1' &= -c_2' \frac{e^t}{1+t} \\
-c_2' \frac{e^t}{1+t} &+ c_2' e^t = te^{2t} \\
c_2'(1 - \frac{1}{1+t}) &= te^t \\
c_2' &= (1+t)e^t \\
c_1' &= -e^{2t} \\
c_1(t) &= -\frac{1}{2}e^{2t} \\
c_2(t) &= te^t.
\end{aligned}
$$

which means

$$
\begin{aligned}
y_p(t) &= -\frac{1}{2}e^{2t}(1+t) + te^t e^t = \frac{t-1}{2}e^{2t} \\
y(x) &= c_1(1+t) + c_2 e^t + \frac{t-1}{2}e^{2t}.
\end{aligned}
$$

3.(a) $r^2 - 2r - 3 = (r-3)(r+1)$. $y_h(t) = c_1 e^{3t} + c_2 e^{-t}$. Using comparison of coefficients we see that the multiplicity of 2 as a root of $r^2 - 2r - 3 = (r-3)(r+1)$ is zero. Then $y_p(t) = R_1(t)e^{2t} = (c_0 + c_1 t)e^{2t}$.

(b) $r^2 + r - 2 = (r-1)(r+2)$. $y_h(t) = c_1 e^t + c_2 e^{-2t}$. Using comparison of coefficients we see that the multiplicity of 0 as a root of $r^2 + r - 2 = (r-1)(r+2)$ is zero. Then $y_p(t) = R_1(t) = c_0 + c_1 t$.

(c) $r^3 - 2r^2 + r = r(r-1)^2$. $y_h(t) = c_1 + c_2 e^t + c_3 te^t$. Using comparison of coefficients we see that the multiplicity of 0 as a root of $r^2 + r - 2 = (r-1)(r+2)$ is 1. Then $y_p(t) = tR_3(t) = c_0 t + c_1 t^2 + c_2 t^3 + c_3 t^4$.

(d) $r^3 + 2r^2 + r = r(r+1)^2$. $y_h(t) = c_1 + c_2 e^{-t} + c_3 te^{-t}$. Using comparison of coefficients we see that the multiplicity of 1 as a root of $r^3 + 2r^2 + r = r(r+1)^2$ is 0. Then $y_p(t) = R_3(t)e^t = (c_0 + c_1 t + c_2 t^2 + c_3 t^3)e^t$.

(e) $r^4 + 2r^2 + 1 = (r^2+1)^2$. $y_h(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$. Using comparison of coefficients we see that the multiplicity of 0 as a root of $r^4 + 2r^2 + 1 = (r^2+1)^2$ is 0. Then $y_p(t) = R_1(t) = c_0 + c_1 t$.

(f) $r^6 + r^3 = r^3(r^3+1)$. $y_h(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + c_5 e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + c_6 e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$.

Using comparison of coefficients we see that the multiplicity of 0 as a root of $r^6 + r^3 = r^3(r^3+1)$ is 3. Then $y_p(t) = t^3 R_1(t) = c_0 t^3 + c_1 t^4$.

4.(a) $x^2y'' - 2xy' + 4y = 2x^2$. $x = e^t$, $t = \ln|x|$ $Y(t) = y(e^t)$. Then $\ell(r) = r(r-1) - 2r + 4 = r^2 - 3r + 4$ and $r_{1,2} = \frac{3+\sqrt{9-16}}{2}$ which gives us $Y_h(t) = c_1 e^{\frac{3\pm\sqrt{9-16}}{2}t} + c_2 e^{\frac{3-\sqrt{9-16}}{2}t}$.

The nonhomogeneous equation in t is $Y'' - 3Y' + 4Y = 2e^{2t}$. We solve by comparison of coefficients: The multiplicity of 2 as a root of the characteristic polynomial is 0 and so $Y_p(t) = ce^{2t}$. Then $4ce^{2t} - 6e^{2t} + 4ce^{2t} = 2e^{2t}$. Then $c = 1$. Then

$$Y(t) = c_1 e^{\frac{3\pm\sqrt{9-16}}{2}t} + c_2 e^{\frac{3-\sqrt{9-16}}{2}t} + e^{2t}$$

or

$$y(x) = c_1 x^{\frac{3\pm\sqrt{9-16}}{2}} + c_2 x^{\frac{3-\sqrt{9-16}}{2}} + x^2$$

(b) $x^2y'' - 2xy' + 2y = 4x^3$. $x = e^t$, $t = \ln|x|$ $Y(t) = y(e^t)$. Then $\ell(r) = r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-1)(r-2)$ which gives us $Y_h(t) = c_1 e^t + c_2 e^{2t}$.

The nonhomogeneous equation in t is $Y'' - 3Y' + 2Y = 2e^{3t}$. We solve by comparison of coefficients: The multiplicity of 3 as a root of the characteristic polynomial is 0 and so $Y_p(t) = ce^{3t}$. Then $9ce^{3t} - 9e^{3t} + 2ce^{3t} = 4e^{3t}$. Then $c = 2$. Then

$$Y(t) = c_1 e^t + c_2 e^{2t} + 2e^{3t}$$

or

$$y(x) = c_1 x + c_2 x^2 + 2x^3$$

(c) $x^2y'' + 9xy' + 25y = 50$. $x = e^t$, $t = \ln|x|$ $Y(t) = y(e^t)$. Then $\ell(r) = r(r-1) + 9r + 25 = r^2 + 8r + 25$ and $r_{1,2} = \frac{-8\pm\sqrt{64-100}}{2} = -4 \pm 3i$ which gives us $Y_h(t) = c_1 e^{-4t}\cos 3t + c_2 e^{-4t}\sin 3t$.

The nonhomogeneous equation in t is $Y'' + 8Y' + 25Y = 50$. We solve by comparison of coefficients: The multiplicity of 0 as a root of the characteristic polynomial is 0 and so $Y_p(t) = c$ and $c = 2$. Then

$$Y(t) = c_1 e^{-4t}\cos 3t + c_2 e^{-4t}\sin 3t + 2$$

or

$$y(x) = c_1 x^{-4}\cos(3\ln|x|) + c_2 x^{-4}\sin(3\ln|x|) + 2$$

(d) $x^3y''' + 3x^2y'' + xy' + 8y = 65\cos\ln x$. $x = e^t,\ t = \ln|x|\ \ Y(t) = y(e^t)$. Then $\ell(r) = r(r-1)+3r+1 = r^2+2r+1 = (r+1)^2$ which gives us $Y_h(t) = c_1e^{-t}+c_2te^{-t}$. The nonhomogeneous equation in t is $Y'' + 2Y' + Y = 65\cos t$. We solve by comparison of coefficients: The multiplicity of i as a root of the characteristic polynomial is 0 and so $Y_p(t) = c_1\cos t + c_2\sin t$. Then

$$-c_1\cos t - c_2\sin t - 2c_1\sin t + 2c_2\cos t + c_1\cos t + c_2\sin t = 65\cos t$$

Then $c_2 = \frac{65}{2}\ \ c_1 = 0$. Then

$$Y(t) = c_1e^{-t} + c_2te^{-t} + \frac{65}{2}\sin t$$

or

$$y(x) = c_1x^{-1} + c_2x^{-1}\ln|x| + \frac{65}{2}\sin(\ln|x|).$$

(e) $y''' = \frac{6y}{x^3}$ or $x^3y''' - 6y = 0$. $x = e^t,\ t = \ln|x|\ \ Y(t) = y(e^t)$. Then $\ell(r) = r(r-1)(r-2) - 6 = r^3 - 3r^2 + 2r - 6 = (r-3)(r^2+2)$ which gives us $Y_h(t) = c_1e^{3t} + c_2\cos\sqrt{2}t + c_3\sin\sqrt{2}t$ or $y_h(x) = c_1x^3 + c_2\cos(\sqrt{2}\ln|x|) + c_3\sin(\sqrt{2}\ln|x|)$