Ordinary Differential Equations - 104131

Homework Page No. 5

Solve the following differential systems and initial value problems:

1. $\vec{x}'(t) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \vec{x}$

2. $\vec{x}'(t) = \begin{pmatrix} 1789 & 1848 \\ 1914 & 1939 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

3. $\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix} \vec{x}$

4. $\vec{x}'(t) = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & -2 \\ 2 & 8 & -2 \end{pmatrix} \vec{x}$

5. $\vec{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix}$

6. It is given that
$$\vec{x}(t) = \begin{pmatrix} 2e^t + 8e^{-t} \\ -2e^t + 12e^{-t} \end{pmatrix}$$
is a solution of a 2×2 differential system $\vec{x}' = A\vec{x}$, where A is a constant matrix.

   (a) What are the eigenvalues of A?

   (b) What are the eigenvectors of A?

   (c) What are $det(A)$ and $trace(A)$?

7. solve $\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} t \\ 1 \end{pmatrix}$

Solutions: 1.

$$p(r) = \det \begin{pmatrix} 3-r & 1 & 1 \\ 1 & -r & 2 \\ 1 & 2 & -r \end{pmatrix} = (r+2)(r-1)(r-4).$$

For $r = -2$

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $r = 1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $r = 4$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

and the general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

2. $\vec{x}(t) \equiv 0$ solves the equation and satisfies the initial condition and so by existence qand uniqueness it is the solution.

3.

$$p(r) = \det \begin{pmatrix} 3-r & 2 \\ -5 & 1-r \end{pmatrix} = r^2 - 4r + 13.$$

For $r = 2 + 3i$

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \to \quad a \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$$

$$e^{(2+3i)t}\begin{pmatrix} 2 \\ -1+3i \end{pmatrix} = e^{2t}(\cos(3t)+i\sin(3t))\begin{pmatrix} 2 \\ -1+3i \end{pmatrix} =$$

$$\begin{pmatrix} 2e^{2t}\cos(3t) \\ -e^{2t}\cos(3t)-3e^{2t}\sin(3t) \end{pmatrix} + i\begin{pmatrix} 2e^{2t}\sin(3t) \\ -e^{2t}\sin(3t)+3e^{2t}\cos(3t) \end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1\begin{pmatrix} 2e^{2t}\cos(3t) \\ -e^{2t}\cos(3t)-3e^{2t}\sin(3t) \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t}\sin(3t) \\ -e^{2t}\sin(3t)+3e^{2t}\cos(3t) \end{pmatrix}$$

4.

$$p(r) = \det\begin{pmatrix} -r & 8 & 0 \\ 0 & -r & -2 \\ 2 & 8 & -2-r \end{pmatrix} = (r+2)(-r^2-16).$$

For $r=-2$

$$\begin{pmatrix} 2 & 8 & 0 \\ 0 & 2 & -2 \\ 2 & 8 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}.$$

For $r=4i$

$$\begin{pmatrix} -4i & 8 & 0 \\ 0 & -4i & -2 \\ 2 & 8 & -2-4i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad b\begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix}.$$

$$e^{4it}\begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix} = (\cos(4t)+i\sin(4t))\begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix} =$$

$$\begin{pmatrix} 2\sin(4t) \\ \cos(4t) \\ 2\sin(4t) \end{pmatrix} + i\begin{pmatrix} -2\cos(4t) \\ \sin(4t) \\ -2\cos(4t) \end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(4t) \\ \cos(4t) \\ 2\sin(4t) \end{pmatrix} + c_3 \begin{pmatrix} -2\cos(4t) \\ \sin(4t) \\ -2\cos(4t) \end{pmatrix}$$

5.

$$p(r) = \det \begin{pmatrix} -r & 1 & 0 \\ 0 & -r & 1 \\ 4 & -4 & 1-r \end{pmatrix} = (r-1)(-r^2-4).$$

For $r = 1$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \rightarrow \quad a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $r = 2i$

$$\begin{pmatrix} -2i & 1 & 0 \\ 0 & -2i & 1 \\ 4 & -4 & 1-2i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \rightarrow \quad b \begin{pmatrix} 1 \\ 2i \\ -4 \end{pmatrix}.$$

$$e^{2it} \begin{pmatrix} 1 \\ 2i \\ -4 \end{pmatrix} = (\cos(2t) + i\sin(2t)) \begin{pmatrix} 1 \\ 2i \\ -4 \end{pmatrix} =$$

$$\begin{pmatrix} \cos(2t) \\ -2\sin(2t) \\ -4\cos(2t) \end{pmatrix} + i \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \\ -4\sin(2t) \end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \\ -4\cos(2t) \end{pmatrix} + c_3 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \\ -4\sin(2t) \end{pmatrix}$$

Now for the initial condition

$$\begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 2 \\ c_3 = -1 \end{matrix}$$

$$\vec{x}(t) = 2 \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \\ -4\cos(2t) \end{pmatrix} - \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \\ -4\sin(2t) \end{pmatrix}$$

6.

$$\vec{x}(t) = \begin{pmatrix} 2e^t + 8e^{-t} \\ -2e^t + 12e^{-t} \end{pmatrix} = e^t \begin{pmatrix} 2 \\ -2 \end{pmatrix} + e^{-t} \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

and so we see that 1 is an eigenvalue with eigenvectors $a \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and -1 is an eigenvalue with eigenvectors $a \cdot \begin{pmatrix} 8 \\ 12 \end{pmatrix}$. Finally, since $\det(A)$ is the multiplication of all eigenvalues and $trace(A)$ is the sum of all eigenvalues, then $\det(A) = -1$ and $trace(A) = 0$.

7.

$$p(r) = \det \begin{pmatrix} -r & -1 \\ 2 & 3-r \end{pmatrix} = (r-1)(r-2)$$

For $r = 1$

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightarrow \quad a \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $r = 2$

$$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightarrow \quad a \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and therefore

$$\vec{x}_h(t) = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

We now use variation of parameters.

$$\vec{x}_p(t) = c_1(t)e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(t)e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$c_1'(t)e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2'(t)e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$\begin{aligned} c_1'(t)e^t + c_2'(t)e^{2t} &= t \\ -c_1'(t)e^t - 2c_2'(t)e^{2t} &= 1 \end{aligned}$$

$$\begin{aligned} c_2'(t) &= -(t+1)e^{-2t} \\ c_1'(t) &= te^{-t} + (1+t)e^{-t} = (1+2t)e^{-t} \end{aligned}$$

$$\begin{aligned} c_2(t) &= \tfrac{(3+2t)e^{-2t}}{4} \\ c_1(t) &= -(3+2t)e^{-t} \end{aligned}$$

$$\vec{x}_p(t) = c_1(t)e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(t)e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} =$$
$$-(3+2t)e^{-t}e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{(3+2t)e^{-2t}}{4} e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} =$$
$$\begin{pmatrix} -(3+2t) + \frac{3+2t}{4} \\ 3+2t - \frac{3+2t}{2} \end{pmatrix} = \begin{pmatrix} -\frac{9+6t}{4} \\ \frac{3+2t}{2} \end{pmatrix}$$

and the general solution is

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -\frac{9+6t}{4} \\ \frac{3+2t}{2} \end{pmatrix}$$