Ordinary Differential Equations - 104131

Homework Page No. 5

Solve the following differential systems and initial value problems:

1. $\vec{x}'(t) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \vec{x}$

2. $\vec{x}'(t) = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \vec{x}$

3. $\vec{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$

4. $\vec{x}'(t) = \begin{pmatrix} 1789 & 1848 \\ 1914 & 1939 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

5. $\vec{x}'(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$

6. $\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix} \vec{x}$

7. $\vec{x}'(t) = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & -2 \\ 2 & 8 & -2 \end{pmatrix} \vec{x}$

8. $\vec{x}'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix}$

9. It is given that $\vec{x}^{(1)}(t) = \vec{w}e^t$ and $\vec{x}^{(2)}(t) = \vec{u}e^t + \vec{v}te^t$ are solutions of a 2×2 differential system $\vec{x}' = A\vec{x}$, (where $\vec{u}, \vec{v}, \vec{w}$ are constant vectors and A is a constant matrix).

   (a) What are the eigenvalues of A?

   (b) What are the eigenvectors of A?

   (c) What are $det(A)$ and $trace(A)$?

10. Find a differential system $\vec{x}\,' = A\vec{x}$, where A is a 2×2 matrix, such that

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}, \qquad \vec{x}^{(2)} = \begin{pmatrix} t \\ t-1 \end{pmatrix} e^{-2t}$$

are it solutions. What are the eigenvalues and the eigenvectors of A?

Solutions: 2.

$$p(r) = \det \begin{pmatrix} 3-r & 1 & 1 \\ 1 & -r & 2 \\ 1 & 2 & -r \end{pmatrix} = (r+2)(r-1)(r-4).$$

For $r = -2$

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $r = 1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $r = 4$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

and the general solution is

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{4t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

4. $\vec{x}(t) \equiv 0$ solves the equation and satisfies the initial condition and so by existence qand uniqueness it is the solution.

6.

$$p(r) = \det\begin{pmatrix} 3-r & 2 \\ -5 & 1-r \end{pmatrix} = r^2 - 4r + 13.$$

For $r = 2 + 3i$

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \to \quad a\begin{pmatrix} 2 \\ -1+3i \end{pmatrix}$$

$$e^{(2+3i)t}\begin{pmatrix} 2 \\ -1+3i \end{pmatrix} = e^{2t}(\cos(3t) + i\sin(3t))\begin{pmatrix} 2 \\ -1+3i \end{pmatrix} =$$
$$\begin{pmatrix} 2e^{2t}\cos(3t) \\ -e^{2t}\cos(3t) - 3e^{2t}\sin(3t) \end{pmatrix} + i\begin{pmatrix} 2e^{2t}\sin(3t) \\ -e^{2t}\sin(3t) + 3e^{2t}\cos(3t) \end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1\begin{pmatrix} 2e^{2t}\cos(3t) \\ -e^{2t}\cos(3t) - 3e^{2t}\sin(3t) \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t}\sin(3t) \\ -e^{2t}\sin(3t) + 3e^{2t}\cos(3t) \end{pmatrix}$$

7.

$$p(r) = \det\begin{pmatrix} -r & 8 & 0 \\ 0 & -r & -2 \\ 2 & 8 & -2-r \end{pmatrix} = (r+2)(-r^2 - 16).$$

For $r = -2$

$$\begin{pmatrix} 2 & 8 & 0 \\ 0 & 2 & -2 \\ 2 & 8 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad c\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}.$$

For $r = 4i$

$$\begin{pmatrix} -4i & 8 & 0 \\ 0 & -4i & -2 \\ 2 & 8 & -2-4i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad b \begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix}.$$

$$e^{4it} \begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix} = (\cos(4t) + i\sin(4t)) \begin{pmatrix} -2i \\ 1 \\ -2i \end{pmatrix} =$$

$$\begin{pmatrix} 2\sin(4t) \\ \cos(4t) \\ 2\sin(4t) \end{pmatrix} + i \begin{pmatrix} -2\cos(4t) \\ \sin(4t) \\ -2\cos(4t) \end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(4t) \\ \cos(4t) \\ 2\sin(4t) \end{pmatrix} + c_3 \begin{pmatrix} -2\cos(4t) \\ \sin(4t) \\ -2\cos(4t) \end{pmatrix}$$

8.

$$p(r) = \det \begin{pmatrix} -r & 1 & 0 \\ 0 & -r & 1 \\ 4 & -4 & 1-r \end{pmatrix} = (r-1)(-r^2-4).$$

For $r = 1$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $r = 2i$

$$\begin{pmatrix} -2i & 1 & 0 \\ 0 & -2i & 1 \\ 4 & -4 & 1-2i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \to \quad b \begin{pmatrix} 1 \\ 2i \\ -4 \end{pmatrix}.$$

$$e^{2it}\begin{pmatrix}1\\2i\\-4\end{pmatrix} = (\cos(2t)+i\sin(2t))\begin{pmatrix}1\\2i\\-4\end{pmatrix} =$$

$$\begin{pmatrix}\cos(2t)\\-2\sin(2t)\\-4\cos(2t)\end{pmatrix} + i\begin{pmatrix}\sin(2t)\\2\cos(2t)\\-4\sin(2t)\end{pmatrix}$$

and so the general solution is

$$\vec{x}(t) = c_1e^t\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}\cos(2t)\\-2\sin(2t)\\-4\cos(2t)\end{pmatrix} + c_3\begin{pmatrix}\sin(2t)\\2\cos(2t)\\-4\sin(2t)\end{pmatrix}$$

Now for the initial condition

$$\begin{pmatrix}2\\-2\\-8\end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\-4\end{pmatrix} + c_3\begin{pmatrix}0\\2\\0\end{pmatrix} \rightarrow \begin{matrix}c_1=0\\c_2=2\\c_3=-1\end{matrix}$$

$$\vec{x}(t) = 2\begin{pmatrix}\cos(2t)\\-2\sin(2t)\\-4\cos(2t)\end{pmatrix} - \begin{pmatrix}\sin(2t)\\2\cos(2t)\\-4\sin(2t)\end{pmatrix}$$