Ordinary Differential Equations - 104131

Homework Page No. 6

1. Draw the phase portrait of the following systems:

   (a) $\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix} \vec{x}$

   (b) $\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \vec{x}$

   (c) $\vec{x}'(t) = \begin{pmatrix} -4 & -2 \\ 5 & 3 \end{pmatrix} \vec{x}$

   (d) $\vec{x}'(t) = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} \vec{x}$

   (e) $\vec{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

2. Solve the following equations using power series, meaning find a recursive formula for the coefficients.

   (a) $(1-x^2)y'' + 2y = 0$

   (b) $y'' - xy' + y = 0$

   (c) $(1-x)y'' - xy' + y = 0$

Solutions: 2.a.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$
\begin{aligned}
&(1-x^2)y'' + 2y = 0 \\
&y'' - x^2y'' + 2y = 0 \\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x^2\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2} + 2\sum_{n=0}^{\infty}a_nx^n = 0 \\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}-n(n-1)a_nx^n + \sum_{n=0}^{\infty}2a_nx^n = 0 \\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\big)x^n = 0 \\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - (n^2-n-2)a_n\big)x^n = 0 \\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - (n-2)(n+1)a_n\big)x^n = 0 \\
&(n+2)(n+1)a_{n+2} - (n-2)(n+1)a_n = 0 \qquad n \geq 0 \\
&a_{n+2} = \frac{(n-2)(n+1)}{(n+2)(n+1)}a_n \qquad n \geq 0 \\
&a_{n+2} = \frac{n-2}{n+2}a_n \qquad n \geq 0
\end{aligned}
$$

b.

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}a_nx^n \\
y'(x) &= \sum_{n=0}^{\infty}na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
y''(x) &= \sum_{n=0}^{\infty}n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^{n-1} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n
\end{aligned}
$$

$$
\begin{aligned}
&y'' - xy' + y = 0\\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0\\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0\\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - na_n + a_n\big)x^n = 0\\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - (n-1)a_n\big)x^n = 0\\
&(n+2)(n+1)a_{n+2} - (n-1)a_n = 0 \quad\quad n \geq 0\\
&a_{n+2} = \frac{n-1}{(n+2)(n+1)}a_n \quad\quad n \geq 0
\end{aligned}
$$

c.

$$
\begin{aligned}
&y(x) = \sum_{n=0}^{\infty} a_n x^n\\
&y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\\
&y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^{n-1} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n
\end{aligned}
$$

$$
\begin{aligned}
&(1-x)y'' - xy' + y = 0\\
&y'' - xy'' - xy' + y = 0\\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)na_{n+1}x^{n-1} - x\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0\\
&\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n - \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0\\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} - na_n + a_n\big)x^n = 0\\
&\sum_{n=0}^{\infty}\big((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} - (n-1)a_n\big)x^n = 0\\
&(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} - (n-1)a_n = 0 \quad\quad n \geq 0\\
&a_{n+2} = \frac{(n+1)na_{n+1} + (n-1)a_n}{(n+2)(n+1)} \quad\quad n \geq 0
\end{aligned}
$$

Remarks:

1. An initial condition is just a_0, a_1 since $y(0) = a_0$ and $y'(0) = a_1$. This means that a_0, a_1 determine the solution.

2. In 2.a we see that for $n = 2$ we get $a_4 = 0$ and therefore $a_{2n} = 0$ for $n \geq 2$. We also see that if $a_1 = 0$ then $a_{2n+1} = 0$ for $n \geq 0$ and in this case we have a solution which is a polynomial which is $a_0 + a_2 x^2 = a_0 - a_0 x^2$.

3. In 2.b we see that for $n = 1$ we get $a_3 = 0$ and so $a_{2n+1} = 0$ for $n \geq 1$. And if $a_0 = 0$ then $a_{2n} = 0$ for $n \geq 0$ and in this case we have a solution which is a polynomial $a_1 x$.