Ordinary Differential Equations - 104131

Homework No. 1

1. Solve the following differential equations:

   (a) $y' - \dfrac{2}{x^3}y = 0.$

   (b) $y' + \dfrac{2}{x^3}y = 0,\ y(2) = 3.$

   (c) $y' + \dfrac{2}{x^3}y = 0,\ y(0) = 3.$ Try to explain what is the difference between this initial value problem and the previous one.

   (d) $y' + \dfrac{2}{x}y = \dfrac{\cos x}{x^2}.$

   (e) $x^2y' + 3xy = \frac{\sin x}{x}$.

   (f) $y' + \tan(x)y = x\sin(2x)$.

   (g) $t\dfrac{dy}{dt} + (t+1)y = t, \quad y(\ln 2) = 1$.

   (h) $y' - 3x^2y = -x^2, \quad y(0) = 1$.

   solutions:

   a. $p(x) = -2/x^3$ and so $P(x) = 1/x^2$ and therefore $y(x) = c\exp(-\frac{1}{x^2})$.

   b. $p(x) = 2/x^3$ and so $P(x) = -1/x^2$ and therefore the general solution is $y(x) = c\exp(\frac{1}{x^2})$. The initial condition means $3 = y(2) = ce^{\frac{1}{4}}$ and so $c = 3e^{-\frac{1}{4}}$ and so $y(x) = 3e^{-\frac{1}{4}}\exp(\frac{1}{x^2})$ where $x > 0$.

   c. The initial value problem is not valid and so there is no solution.

   d. We calculate two integrals:

$$\int 2/x\ dx = 2\ln|x| + c = \ln x^2 + c$$

   and then

$$\int \exp(\ln x^2)\frac{\cos x}{x^2}dx = \int \cos x dx = \sin x + c$$

   and the general solution is $y(x) = \frac{c}{x^2} + \frac{\sin x}{x^2}$

   e. We solve $y' + \frac{3}{x}y = \frac{\sin x}{x^3}$.

$$\int \frac{3}{x}dx = 3\ln|x| + c = \ln|x|^3 + c.$$

in this case we get an integrating factor which is $\exp(\ln|x|^3) = |x|^3$ but it is easy to see that x^3 is also an integrating factor. Therefore the second integral we solve is

$$\int x^3 \frac{\sin x}{x^3} dx = -\cos x.$$

Then the general solution is $y = \frac{c}{x^3} - \frac{\cos x}{x^3}$.

f. the first integral is

$$\int \tan x dx = -\ln|\cos x| + c$$

and so the integrating factor is $\frac{1}{|\cos x|}$ but again, we can easily check that $\frac{1}{\cos x}$ is an integrating factor. and so the second integral is

$$\int \frac{x\sin(2x)}{\cos x} dx = \int \frac{2x\sin x\cos x}{\cos x} dx = \int 2x\sin x dx = -2x\cos x + 2\sin x + c$$

and so the general solution is $y = c\cos x - 2x\cos^2 x + 2\sin x\cos x = c\cos x - 2x\cos^2 x + \sin 2x$.

g. We solve $y' + \frac{t+1}{t} y = 1$.

$$\int \frac{t+1}{t} dt = t + \ln|t| + c$$

The integrating factor should be $|t|e^t$ but it is easy to check te^t is also an integrating factor. Then the second integral is

$$\int te^t dt = te^t - e^t + c$$

and the general solution is $y = c\frac{e^{-t}}{t} + \frac{e^{-t}}{t}(te^t - e^t) = c\frac{e^{-t}}{t} + 1 - \frac{1}{t}$.

It is easy to see that none of these solutions can be defined at $x = 0$ and so these are also the solutions to the original equation.

h. $\int -3x^2 dx = -x^3 + c$ and

$$\int -x^2 \exp(-x^3) dx = \frac{1}{3}\exp(-x^3)$$

and so the general solution is $y = c\exp(x^3) + \frac{1}{3}$. the initial condition gives $c = \frac{2}{3}$.

2. Solve the initial value problem $y' + \left(\frac{\ln^2 x}{\sin^2 x}\right) y = 0, \quad y(5) = 0$.

solution: The solution is $y = c\exp\left(\int \frac{\ln^2 x}{\sin^2 x}dx\right)$. The initial condition gives $0 = y(5) = c\exp(\textit{some number})$ and since the exponent is always nonzero this means $c = 0$ which gives us $y \equiv 0$ as the solution. the interval of definition for the solution is an interval of definition of $\frac{\ln^2 x}{\sin^2 x}$ which contains 5 and the interval of definition for $\frac{\ln^2 x}{\sin^2 x}$ is $x > 0$ and $x \neq k\pi$ which means $(\pi, 2\pi)$ because it contains 5.

another way to solve this is to guess $y \equiv 0$ which is always a solution for a homogeneous equation. it also satisfies the initial condition and so is the unique solution according the the existence and uniqueness theorem for linear equations and the interval of definition is as said above.

3. Solve the following differential equations:

   (a) $(\sin^2 u + v\tan u)\, du - dv = 0$.

   (b) $(2uv - e^{-2v})\, dv + v\, du = 0$.

   What is preferable: $u = u(v)$ or $v = v(u)$?

   Solutions: (a)$(\sin^2 u + v\tan u)\, du - dv = 0$. Since $(\sin^2 u + v\tan u)'_v = \tan(u)$ and $(-1)'_u = 0$ then the equation isnt exact. so we look for an integrating factor which is a function of u alone:

$$\frac{(\sin^2 u + v\tan u)'_v - (-1)'_u}{-1} = -\tan u.$$

   And so we have an integrating factor which is $\mu(u) = \exp\int -\tan u du = \exp(\ln|\cos x|) = |\cos x|$. We guess $\cos x$ will work as well.

$$(\cos u \sin^2 u + v\sin u)du - \cos u dv = 0$$

   is the equation we obtain. We check whether it is exact: $(\cos u \sin^2 u + v\sin u)'_v = \sin u$ and $(-\cos(u))'_u = \sin u$ and so the new equation is exact. We find the potential function:

$$F(x,y) = \int F'_u du = \int \cos u \sin^2 u + v\sin u du = \frac{1}{3}\sin^3 u - v\cos u + g(v).$$

$$F(x,y) = \int F'_v dv = \int -\cos u dv = -v\cos u + h(u).$$

and so $F(x,y) = \frac{1}{3}\sin^3 u - v\cos u (+c)$. The general solution in implicit form is $\frac{1}{3}\sin^3 u - v\cos u = c$ or $v = \frac{c}{\cos u} + \frac{1}{3\cos u}\sin^3 u$ in explicit form where v is a function of u. Having multiplied the equation by $\cos u$ which is zero for $u = \frac{\pi}{2} + k\pi$ then we need to check whether we added these solutions to the original equation $(\sin^2 u + v\tan u)\, du - dv = 0$ meaning to $(\sin^2 u + v\tan u)\frac{du}{dv} - 1 = 0$ and we see that no constant solution $u(v)$ is a solution and so the general solution in implicit form is $\frac{1}{3}\sin^3 u - v\cos u = c$ excluding the solutions $u(v) \equiv \frac{\pi}{2} + k\pi$.
another way to solve this equation is to look at the original equation where v is a function of u. Which means $(\sin^2 u + v\tan u) - \frac{dv}{du} = 0$ or $v' - (\tan u)v = -\sin^2 u$ whose solution is $v = \frac{c}{\cos u} + \frac{1}{3\cos u}\sin^3 u$.
(b) $(2uv - e^{-2v})\, dv + v\, du = 0$. This can be solved using integrating factor. We will solve it using another method. Consider the equation where u is a function of v. $(2uv - e^{-2v}) + v\frac{du}{dv} = 0$ or $u' + 2u = \frac{e^{-2v}}{v}$. An integrating factor is e^{2v} and then $(e^{2v}u)' = \frac{1}{v}$ and so $e^{2v}u = \ln|v| + c$ and $u = ce^{-2v} + e^{-2v}\ln|v|$

4. Solve the following equations:

   (a) $y' + y^2 \sin x = 0$.

   (b) $y' = \dfrac{x^2}{y(1+x^3)}$.

   (c) $y' = 2(1+x)(1+y^2), \quad y(0) = 0$.

Solutions: (a) $y' + y^2 \sin x = 0$ or $y' = -y^2\sin x$ which is a separable equation as well as a bernouli equation. We will solve as a separable equation. Then $y \equiv 0$ is a constant solution which we will check if it is singular. Then we calculate the integrals

$$\int -\sin x dx = \cos x (+c)$$

$$\int y^{-2} dy = -\frac{1}{y}(+c)$$

and so the general solution in implicit form is $-\frac{1}{y} = \cos x + c$ and explicitly $y = \frac{1}{-\cos x + c}$. We now check if $y \equiv 0$ is singular. For that we must find a constant c such that $0 = \frac{1}{-\cos x + c}$ for all x. Clearly there isnt and therefore the solution $y \equiv 0$ is a singular solution.

(b) $y' = \frac{x^2}{y(1+x^3)}$. Again a separable equation but no constant solutions and so we turn to the integrals.

$$\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\ln|1+x^3|(+c)$$

$$\int ydy = \frac{y^2}{2}(+c)$$

and so the general solution is $\frac{y^2}{2} = \frac{1}{3}\ln|1+x^3|+c$ or $y = \pm\sqrt{\frac{2}{3}\ln|1+x^3|+c}$.
(c) $y' = 2(1+x)(1+y^2), \quad y(0) = 0$. A separable equation. there are no constant solutions and so we solve the integrals

$$\int 2(1+x)dx = 2x + x^2(+c)$$

$$\int \frac{1}{1+y^2}dy = \arctan y(+c)$$

and so the general solution is $\arctan y = 2x+x^2+c$ or $y = \tan(2x+x^2+c)$. The initial condition gives us $0 = \tan(c)$ and so $c = k\pi$. However, $y = \tan(2x+x^2+k\pi) = \tan(2x+x^2)$ because the period of $\tan x$ is π. Alternatively, we use $\arctan y = 2x+x^2+c$ to find c and obtain $c = 0$.

5. Solve the equations

   (a) $y' - 2y = y^2 - 3$
   (b) $t^2\frac{dy}{dt} + 2ty - y^3 = 0$.
   (c) $y' = \frac{2}{x}y + \frac{x}{y^2}$.

   solutions: (a) $y' - 2y = y^2 - 3$ or $y' = y^2 + 2y - 3 = (y-1)(y+3)$. A separable equation and so we look for constant solutions and $y \equiv 1$ and $y \equiv -3$) are constant solution. We will check later whether they are singular solutions. We now calculate the integrals:

$$\int dx = x(+c)$$

$$\int \frac{1}{(y-1)(y+3)}dy = \int \frac{1}{4}\left(\frac{1}{y-1} - \frac{1}{y+3}\right) = \frac{1}{4}\ln\left|\frac{y-1}{y-3}\right|(+c)$$

and so the general solution is $\frac{1}{4}\ln\left|\frac{y-1}{y+3}\right| = x + c$ or $\ln\left|\frac{y-1}{y+3}\right| = 4x + c$ or $\left|\frac{y-1}{y+3}\right| = e^c e^{4x}$ or $\frac{y-1}{y+3} = \pm e^c e^{4x}$ or $\frac{y-1}{y+3} = ce^{4x}$ where $c \neq 0$ or $y = \frac{1+3ce^{4x}}{1-ce^{4x}}$

where $c \neq 0$. We now find the singular solutions: Let us check whether $y \equiv 1$ is singular: $1 = \frac{1+3ce^{4x}}{1-ce^{4x}}$ and we get $-1 = 3$ since $c \neq 0$. Let us check whether $y \equiv -3$ is singular: $-3 = \frac{1+3ce^{4x}}{1-ce^{4x}}$ and we get $-3 = 1$. So both solutions are singular solutions.

(b) $t^2\frac{dy}{dt} + 2ty - y^3 = 0$. We first solve $\frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3$. This is a Bernouli equation. Since $n = 3$ then $y \equiv 0$ is a solution. We now substitute $z = y^{1-3} = y^{-2}$. Then $z' = -2y^{-3}y'$. We rewrite the equation as $y^{-3}y' + \frac{2}{t}y^{-2} = \frac{1}{t^2}$. Then $-\frac{1}{2}z' + \frac{2}{t}z = \frac{1}{t^2}$ or $z' - \frac{4}{t}z = -\frac{2}{t^2}$. The integrating factor is t^{-4} and so $(t^{-4}z)' = -\frac{2}{t^6}$ hence $t^{-4}z = \frac{2}{5}t^{-5} + c$ and then $z = ct^4 + \frac{2}{5t}$. Then $y = \pm\sqrt{\frac{1}{ct^4+\frac{2}{5t}}}$. We now find out whether $y \equiv 0$ is singular: $0 = \pm\sqrt{\frac{1}{ct^4+\frac{2}{5t}}}$ and clearly no number c gives us a constant function and therefore $y \equiv 0$ is singular.

(c) $y' = \frac{2}{x}y + \frac{x}{y^2}$. Rewrite as $y' - \frac{2}{x}y = \frac{x}{y^2}$ and it is a bernouli with $n = -2$. Then $z = y^{1-(-2)} = y^3$ and so $z' = 3y^2y'$. rewrite the equation as $y^2y' - \frac{2}{x}y^3 = x$ and then $\frac{1}{3}z' - \frac{2}{x}z = x$ or $z' - \frac{6}{x}z = 3x$. The integrating factor is x^{-6} which means $(x^{-6}z)' = 3x^{-5}$. Then $x^{-6}z = -\frac{3}{4}x^{-4} + c$ hence $z = cx^6 - \frac{3}{4}x^2$. Then $y = \left(cx^6 - \frac{3}{4}x^2\right)^{\frac{1}{3}}$.

6. Solve $y' - y = y^2$ by two different methods.

solution: $y' - y = y^2$. first we solve as separable. $y' = y^2 + y = y(y+1)$ and so $y \equiv 0$ and $y \equiv -1$ are constant solutions. The integrals are

$$\int dx = x(+c)$$

$$\int \frac{1}{y(y+1)}dy = \int \frac{1}{y} - \frac{1}{y+1}dy = \ln\left|\frac{y}{y+1}\right| + c$$

and therefore the general solution is $\ln\left|\frac{y}{y+1}\right| = x + c$ or $\left|\frac{y}{y+1}\right| = e^c e^x$ or $\frac{y}{y+1} = \pm e^c e^x$ or $\frac{y}{y+1} = ce^x$ where $c \neq 0$. Then $y = \frac{ce^x}{1-ce^x} = \frac{1}{ce^{-x}-1}$ where $c \neq 0$. clearly $y \equiv 0$ is a singular solution but note that for $c = 0$ we get $y \equiv -1$. however, $c = 0$ is not allowed and so both constant solutions are singular.

We now solve as bernouli: then $n = 2$ and $y \equiv 0$ is a solution. then $z = y^{1-2} = y^{-1}$ and $z' = -y^{-2}y'$. Rewrite the equation $y^{-2}y' - y^{-1} = 1$ and then $-z' - z = 1$ or $z' + z = -1$. The general solution to this is $z = ce^{-x} - 1$. Then $y = \frac{1}{ce^{-x}-1}$. Clearly $y \equiv 0$ is a singular solution but

$y \equiv -1$ is NOT singular since $c = 0$ gives it and is legal in this way of solution. This means that a solution may be singular when solving one way and not singular when solving another way.

7. Find all solutions of

   (a) $y' = \dfrac{x+2y}{x}$,

   (b) $y' = \dfrac{4y-3x}{2x-y}$ with $y(1) = 3$ and with $y(1) = -3$.

   (c) $y' = \dfrac{x^2+3y^2}{2xy}$

   (d) $y' = \dfrac{x^2+3xy+y^2}{x^2}$ with $y(1) = 1$ and with $y(1) = -1$.

   solutions: (a) $y' = \frac{x+2y}{x} = 1 + 2\frac{y}{x}$. Then $v = \frac{y}{x}$ or $y = xv$ and then $y = v + xv' = 1 + 2v$ or $v' = \frac{1+v}{x}$. This is a separable equation and $v \equiv -1$ is a constant solution. The integrals are

$$\int \frac{1}{x} dx = \ln|x|(+c)$$

$$\int \frac{1}{1+v} dv = \ln|1+v|(+c)$$

   and the general solution is $\ln|1+v| = \ln|x| + c$. Then $|1+v| = |x|e^c$ or $1 + v = \pm e^c x$ or $1 + v = cx$ where $c \neq 0$. this means that $v = cx - 1$ where $c \neq 0$. Then $\frac{y}{x} = cx - 1$ where $c \neq 0$. Then $y = cx^2 - x$ where $c \neq 0$. Recall $v \equiv -1$ was a solution as well which means $y = -x$ is a solution and it is indeed singular since $c \neq 0$.

   Note that the equation is also a linear equation whose solution is $y = cx^2 - x$ where $c = 0$ is allowed.

   (b) $y' = \frac{4y-3x}{2x-y} = \frac{4\frac{y}{x}-3}{2-\frac{y}{x}}$. Use $v = \frac{y}{x}$ or $y = xv$ and then $y' = v + xv' = \frac{4v-3}{2-v}$ or $xv' = \frac{4v-3}{2-v} - v = \frac{v^2+2v-3}{2-v} = \frac{(v-1)(v+3)}{2-v}$. This is a separable equation and so the constant solutions are $y \equiv 1$ and $y \equiv -3$. $y \equiv -3$ solves the equation with the initial condition $y(1) = -3$ and since it is easy to check the existence and uniqueness holds, it is the only such solution. so we are left to find the solution to the other initial condition. We integrate

$$\int \frac{1}{x} dx = \ln|x|(+c)$$

$$\int \frac{2-v}{(v-1)(v+3)}dv = \int \frac{1}{4}\left(\frac{1}{v-1} - \frac{1}{v+3}\right)dv = \frac{1}{4}\ln\left|\frac{v-1}{v+3}\right|(+c)$$

and so the general solution is $\frac{1}{4}\ln\left|\frac{v-1}{v+3}\right| = \ln|x|+c$ or $\ln\left|\frac{v-1}{v+3}\right| = \ln x^4+c$ or $\left|\frac{v-1}{v+3}\right| = x^4e^c$ or $\frac{v-1}{v+3} = \pm e^cx^4$ or $\frac{v-1}{v+3} = cx^4$ where $c \neq 0$. and so $y = \frac{1+3cx^4}{1-cx^4}$ where $c \neq 0$. It is easy to verify that the constant solutions $y \equiv 1, -3$ are singular. We not proceed to solve the initial condition problem $y(1) = 3$. then $3 = \frac{1+3c}{1-c}$ which gives $c = \frac{1}{3}$.

(c) $y' = \frac{x^2+3y^2}{2xy} = \frac{1}{2}\frac{x}{y} + \frac{3}{2}\frac{y}{x}$. Use $v = \frac{y}{x}$ or $y = xv$ and then $y' = v+xv' = \frac{1}{2v} + \frac{3v}{2} = \frac{1+3v^2}{2v}$ or $xv' = \frac{1+3v^2}{2v} - v = \frac{v^2+1}{2v}$. This is a separable equation with no constant solutions. The integrals are

$$\int \frac{1}{x}dx = \ln|x|(+c)$$

$$\int \frac{2v}{v^2+1}dv = \ln|v^2+1|(+c)$$

and then the general solution is $\ln|v^2+1| = \ln|x|+c$. We get $v^2+1 = cx$ where $c \neq 0$ or $v = \pm\sqrt{cx-1}$ where $c \neq 0$. this means $y = \pm x\sqrt{cx-1}$ where $c \neq 0$.

(d) $y' = \frac{x^2+3xy+y^2}{x^2} = 1 + 3\frac{y}{x} + \frac{y^2}{x^2}$. Use $v = \frac{y}{x}$ or $y = xv$ and then $y' = v + xv' = 1 + 3v + v^2$ or $xv' = 1 + 2v + v^2 = (v+1)^2$. This is a separable equation with $y \equiv -1$ as a constant solutions. The integrals are

$$\int \frac{1}{x}dx = \ln|x|(+c)$$

$$\int \frac{1}{(v+1)^2}dv = -\frac{1}{v+1}(+c)$$

and therefore the general solution is $-\frac{1}{v+1} = \ln|x| + c$ or $v = -1 + \frac{1}{c-\ln|x|}$. Since no c yields a constant solution then the solution $v \equiv -1$ is singular. the general solution is $y = -x + \frac{x}{c-\ln|x|}$ with $y = -x$ as a singular solution.

now to solve the initial condition problems: for $y(1) = 1$ we see that the singular solution doesnt satisfy it so we look in the general solution: $1 = -1 + \frac{1}{c-\ln|1|}$ which gives $c = \frac{1}{2}$.

for $y(1) = -1$ we see that the singular solution $y \equiv -1$ satisfies the initial condition and so is the unique solution (why is it unique?).

8. For each of the following first order differential equations decide whether it is linear and homogeneous, linear and non-homogeneous, separable, or of homogeneous type, is it written in a normalized form or not. Note that an equation may belong to several types.

   (a) $y' \sin x = 8x^2 y$
   (b) $x^2 y' = y^2$
   (c) $y' + y \tan x = \dfrac{1}{\cos x}$
   (d) $xy' = y$
   (e) $y' = \dfrac{y^2 - 2xy}{x^2 - 2xy}$

9. (a) $\frac{y}{x}\, dx + (y^3 + \ln x)\, dy = 0$.
   (b) $e^{-y}\, dx - (2y + xe^{-y})\, dy = 0$.
   (c) $(2xy^2 + 2y) + (2x^2 y + 2x)\, y' = 0$.

   solutions: (a) $\left(\frac{y}{x}\right)'_y = \frac{1}{x}$ and $((y^3 + \ln x))'_x = \frac{1}{x}$ and so the equation is exact.

$$F(x, y) = \int \frac{y}{x} dx = y \ln x + g(y)$$
$$F(x, y) = \int y^3 + \ln x dy = \frac{y^4}{4} + y \ln x + h(x).$$

   hence $F(x, y) = \frac{y^4}{4} + y \ln x + c$ and the general solution is $\frac{y^4}{4} + y \ln x = c$.
   (b) $\left(e^{-y}\right)'_y = -e^{-y}$ and $(-(2y + xe^{-y}))'_x = -e^{-y}$ and so the equation is exact.

$$F(x, y) = \int e^{-y} dx = xe^{-y} + g(y)$$
$$F(x, y) = \int -2y - xe^{-y} dy = -y^2 + xe^{-y} + h(x).$$

   hence $F(x, y) = -y^2 + xe^{-y} + c$ and the general solution is $-y^2 + xe^{-y} = c$.
   (c) $(2xy^2 + 2y)dx + (2x^2 y + 2x)\, dy = 0$. since $(2xy^2 + 2y)'_y = 4xy + 2$ and $(2x^2 y + 2x)'_x = -4xy + 2$ then the equation is exact.

$$F(x, y) = \int 2xy^2 + 2y dx = x^2 y^2 + 2xy + g(y)$$
$$F(x, y) = \int 2x^2 y + 2x dy = x^2 y^2 + 2xy + h(x).$$

   hence $F(x, y) = x^2 y^2 + 2xy + c$ and the general solution is $x^2 y^2 + 2xy = c$.

10. For the equation $(x^2+y^2+2x)\,dx+2y\,dy=0$ find an integrating factor which depends only on one variable. Solve the differential equation.

solution: $P'_y=2y$ and $Q'_x=0$. Not exact. We look for an integration factor which is a function of x alone: If there is one, then the expression $\frac{P'_y-Q'_x}{Q} = \frac{2y}{2y} =$ needs to be a function of x alone and it is. Then $\mu(x)=e^x$ and the equation $e^x(x^2+y^2+2x)\,dx+e^x 2y\,dy=0$ is exact (verify this!). Then

$$F(x,y)=\int e^x(x^2+y^2+2x)dx = x^2e^x+y^2e^x+g(y)$$
$$F(x,y)=\int e^x 2ydy = y^2e^x+h(x).$$

hence $F(x,y)=x^2e^x+y^2e^x+c$ and the general solution is $x^2e^x+y^2e^x=c$.

11. Solve the equation $(3xy+y^2)\,dx+(x^2+xy)\,dy=0$ by two methods: As an equation of homogeneous type and by an integration factor.

solution: integration factor: $P'_y=3x+2y$ and $Q'_x=2x+y$. We look for an integration factor: as a function of x alone: $\frac{3x+2y-2x-y}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$ and therefore the integration factor is $\mu(x)=\exp(\int \frac{1}{x}dx = x$ and so the equation $(3x^2y+xy^2)\,dx+(x^3+x^2y)\,dy=0$ is exact (verify!).

$$F(x,y)=\int 3x^2y+xy^2dx = x^3y+\frac{1}{2}x^2y^2+g(y)$$
$$F(x,y)=\int x^3+x^2ydy = x^3y+\frac{1}{2}x^2y^2+h(x).$$

hence $F(x,y)=x^3y+\frac{1}{2}x^2y^2+c$ and the general solution is $x^3y+\frac{1}{2}x^2y^2=c$. Note however, that we multiplied the original equation by x which is zero when $x=0$ and so we may have added the solution $x(y)\equiv 0$ and must check whether this is a solution to the original equation $(3xy+y^2)\,dx+(x^2+xy)\,dy=0$ and since x is a function of y then we find out whether $x\equiv 0$ is a solution to $(3xy+y^2)\,\frac{dx}{dy}+(x^2+xy)=0$. Substitution shows it is a solution and so we did not add solutions.

We now solve as a homogeneous equation: $y' = -\frac{3xy+y^2}{x^2+xy} = -\frac{3\frac{y}{x}+\frac{y^2}{x^2}}{1+\frac{y}{x}}$. Then $v=\frac{y}{x}$ or $y=xv$ and then $y=v+xv' = -\frac{3v+v^2}{1+v}$ or $xv' = -\frac{3v+v^2}{1+v}-v = \frac{-2v^2-4v}{1+v} = \frac{2v(-v-2)}{1+v}$. This is a separable equation and

$v \equiv 0, -2$ are a constant solutions. The integrals are

$$\int \frac{1}{x} dx = \ln |x| (+c)$$

$$\int -\frac{1}{2} \frac{1+v}{v(v+2)} dv = -\frac{1}{2} \int \frac{1}{2} \left(\frac{1}{v} + \frac{1}{v+2} \right) dv = -\frac{1}{4} \ln |v(v+2)| (+c)$$

and the general solution is $-\frac{1}{4} \ln |v(v+2)| = \ln |x| + c$. Then $v(v+2) = cx^{-4}$ where $c \neq 0$. This shows us the the constant solutions are singular. We return to x, y and obtain $\frac{y}{x}(\frac{y}{x} + 2) = cx^{-4}$ where $c \neq 0$. Then $x^2y^2 + 2yx^3 = c$ where $c \neq 0$. The singular solutions are $y \equiv 0$ and $y \equiv -2x$. How does this match the solution we got using integrating factor? The implicit form is the same but c cannot be zero in the second form and substituting $c = 0$ we get $x^2y^2 + 2yx^3 = 0$ which defines implicitly two functions: $y \equiv 0$ and $y = -2x$.