Ordinary Differential Equations - 104131

Homework Page No. 3

1. The functions $e^x,\ xe^x,\ e^{2x}$ are solutions of a certain linear differential equation. Verify that they are linearly independent for every x.

2. (a) Show (without using the Wronskian) that the two functions $x,\ x^2$ are linearly independent for every x.

   (b) On the other hand, show that the Wronskian of these two function is zero for some x. Does this contradict their linear independence?

3. (a) Verify that $y(t) = t$ is a solution of the differential equation

   $$t^2y'' - t(t+2)y' + (t+2)y = 0.$$

   (b) Find another, independent solution of the same equation.

4. (a) Verify that $y(x) = \dfrac{1}{x+1}$ is a solution of the differential equation

   $$x(x+1)y'' - 2y' - 2y = 0.$$

   (b) Find another, independent solution of the same equation.

   (c) Find a solution which satisfies the initial value conditions $y(1) = 0,\ y'(1) = 3/2$.

5. Solve the following differential equations:

   (a) $y'' - 2y' + 6y = 0$.

   (b) $y^{(4)} - 5y'' + 4y = 0$.

   (c) $y^{(4)} - 4y''' + 4y'' = 0,\ \ y(1) = -1,\ y'(1) = 2,\ y''(1) = 0,\ y'''(1) = 0$.

6. It is given that $5x + \sin x$ is one of the solutions of a linear differential equation with constant coefficients,

   $$y^{(4)} + ay^{(3)} + by'' + cy' + dy = 0.$$

   Find the four real valued constants a, b, c, d.

Solution: 1. As solutions to some differential equation with constant coefficients (whose characteristic polynomial is $(r-1)^2(r-2)$ for example), it is enough to check the Wronskian.

2. a. If they are linearly dependent on $\mathbb{R}$ then there are constants c_1, c_2 not both zero such that $c_1x + c_2x^2 = 0$ for all x. But $x = 1$ gives us $c_1 = -c_2$ and $x = -1$ gives $c_1 = c_2$ and so $c_1 = c_2 = 0$.
b. The Wronskian is zero at $x = 0$. There is no contradiction because the theorems about the wronskian are for solutions for normalized equations. this means that x, x^2 are not solutions to a normalized euqation (the equation will have a problem at $x = 0$).

3. The normalized equation is $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$. We apply abel's formula for a second solutions and obtain

$$y_2(t) = y_1(t)\int \frac{W(t)}{y_1(t)^2}dt = t\int \frac{\exp(+t + 2\ln|t|)}{t^2} = t\int e^t = te^t.$$

4.b. The normalized equation is $y'' - \frac{2}{x(x+1)}y' - \frac{2}{x(x-1)}y = 0$. We apply abel's formula for a second solutions and obtain

$$y_2(x) = y_1(x)\int \frac{W(x)}{y_1(x)^2}dx = \frac{1}{x+1}\int (x+1)^2 \exp(2\ln|\frac{x}{x+1}|) =$$
$$\frac{1}{x+1}\int x^2 = \frac{x^3}{3(x+1)}.$$

The general solution is $y(x) = c_1(x+1)^{-1} + c_2\frac{x^3}{x+1}$.
c. $y'(x) = -c_1(x+1)^{-2} + c_2\frac{3x^2(x+1)-x^3}{(x+1)^2} = -c_1(x+1)^{-2} + c_2\frac{2x^3+3x^2}{(x+1)^2}$.

$$0 = y(1) = \frac{c_1}{2} + \frac{c_2}{2}$$
$$\frac{3}{2} = y'(1) = -\frac{c_1}{4} + \frac{5c_2}{4}.$$
$$c_1 = -1 \quad c_2 = 1.$$

$y(x) = -(x+1)^{-1} + \frac{x^3}{x+1}$.

5.a. $\ell(r) = r^2 - 2r + 6$. The roots are $r_{1,2} = \frac{2\pm\sqrt{4-24}}{2} = 1 \pm \sqrt{5}i$. Hence $y(x) = c_1\exp(x)\cos(\sqrt{5}x) + c_2\exp(x)\sin(\sqrt{5}x)$.
b. $\ell(r) = r^4 - 5r^2 + 4 = (r^2-1)(r^2-4)$. The roots are $\pm1, \pm2$. Hence $y(x) = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$.

c. $\ell(r) = r^4 - 4r^3 + 4r^2 = r^2(r^2 - 4r + 4) = r^2(r-2)^2$.
$y(x) = c_1 + c_2 x + c_3 e^{2x} + c_4 x e^{2x}$.

6. $y^{(4)} + ay^{(3)} + by'' + cy' + dy = 0$.

$$
\begin{aligned}
&y(x) = 5x + \sin x \\
&y'(x) = 5 + \cos x \\
&y''(x) = -\sin x \\
&y'''(x) = -\cos x \\
&y''''(x) = \sin x \\
&\sin x - a\cos x - b\sin x + c(5 + \cos x) + d(5x + \sin x) = 0 \\
&5c + 5dx + (1 - b + d)\sin x + (-a + c)\cos x = 0 \\
&5c = 0 \quad 5d = 0 \quad 1 - b + d = 0 \quad -a + c = 0 \\
&c = d = 0 \quad b = 1 \quad a = c = 0 \\
&y'''' + y'' = 0.
\end{aligned}
$$

An alternative solution is seeing that $x, \sin x$ are solutions which means that 0 is a root of multiplicity two and $\pm i$ are roots and so the characteristic polynomial is $r^2(r^2+1) = r^4 + r^2$ which is the characteristic polynomial of $y'''' + y'' = 0$.