Shallow Water Equations Report

Richard Suhendra Carter Koehler

November 18, 2020

1 Introduction

The shallow water equations describe the flow of a thin layer of fluid, bounded below by a rigid topography, and bounded above by a free surface. They generally arise from situations where the length scale in the vertical dimension is much smaller than the typical length scale in the horizontal dimension. In this case, it's implied that the horizontal velocities remain constant with depth so that we can integrate out the vertical component.

The shallow water equations are widely applicable, in particular to situations where the wavelength is much larger than the depth of the fluid, which encompasses a large number of tidal phenomena, including tides, ocean currents, tsunamis, etc.

This report aims to provide a brief description as well as derivation of the shallow water equations. Then, it aims to provide an overview of methods of finite difference in the numerical approximation of these equations in their various forms.

The module containing these methods and equations is contained here: `https://github.com/rsuhendra/shallow_water`

2 Derivation

Much of the following derivation is based off of [1] and adapted to fit our notation and depth of explanation. Any steps taken from elsewhere are notated explicitly.

Let $\mathbf{u} = (u, v, w)$ be the 3-D velocity of the fluid, p be the pressure, and $H(x, y)$ describe the topography of the bottom of the pool.

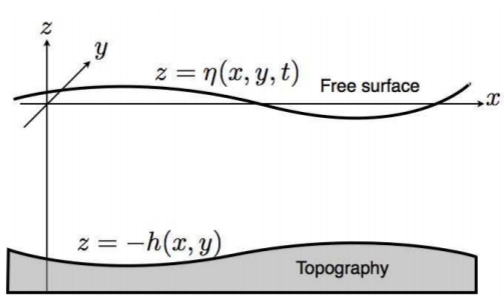


Figure 1: Taken from [1]. Our variables are named h instead of η and H instead of h.

We start out with a few conservation laws:

$$p = 0,\ \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h = w;\ \text{when } z = h(x, y, t) \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p + g\hat{z} = 0 \tag{2}$$

$$\nabla \cdot u = 0 \tag{3}$$

$$u \cdot \nabla (z + H(x, y)) = 0;\ \text{when } z = -H(x, y) \tag{4}$$

We integrate the incompressibility condition over the total depth of the water to get a conservation of mass condition.

$$0 = \int_{-H}^{h} (\nabla \cdot u) dz \tag{5}$$

Expanding and applying Leibniz gives us

$$\begin{aligned} =&\frac{\partial}{\partial x} \int_{-H}^{h} u\, dz - u(x, y, h) \frac{\partial h}{\partial x} - u(x, y, -H) \frac{\partial H}{\partial x} \\ &+ \frac{\partial}{\partial x} \int_{-H}^{h} v\, dz - v(x, y, h) \frac{\partial h}{\partial y} - v(x, y, -H) \frac{\partial H}{\partial y} \\ &+ w(x, y, h) - w(x, y, -H) \end{aligned} \tag{6}$$

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_{-H}^{h} u\, dz + \frac{\partial}{\partial y} \int_{-H}^{h} v\, dz \tag{7}$$

By (2) we have three equations [2]:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \tag{8}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + z \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \tag{9}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = 0 \tag{10}$$

Here we will assume that the medium is shallow and that acceleration in the vertical direction is negligible. Next, we will consider (10) and eliminate the vertical acceleration. This lets us integrate rather simply.

$$\int_{z}^{h} \frac{\partial p}{\partial z} dz = \int_{z}^{h} \rho g\, dz \tag{11}$$

Recalling that pressure is 0 at the surface,

$$p(x, y, z, t) = \rho g (h(x, y, t) - z) \tag{12}$$

We can plug this back into (8) and (9) to get

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial h}{\partial x} = 0 \tag{13}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial h}{\partial y} = 0 \tag{14}$$

And similarly, as a result of decoupling u and v from the z direction, (7) becomes the following.

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}((h+H)u) + \frac{\partial}{\partial y}((h+H)v) \tag{15}$$

Together, (13), (14), and (15) give us the Shallow Water Equations.

3 Equations

The shallow water equations come in many forms, owing to the inclusion or exclusion of certain physical effects. We present a few relevant versions in this section.

3.1 Full Shallow Water Equations

Often the effects of drag, viscosity, and rotation, and many others, are considered in fluid dynamical systems. We can amend the equations we derived in the previous section by including these some of these.

The full shallow water equations can be described by the parabolic partial differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} - bu + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} - bv + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
&\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\big((h+H)u\big) + \frac{\partial}{\partial y}\big((h+H)v\big) = 0
\end{aligned} \tag{16}$$

where:
$u(x, y, t)$ is velocity in the x-direction
$v(x, y, t)$ is velocity in the y-direction
$h(x, y, t)$ describes the vertical displacement of the surface relative to $z = 0$
$H(x, y)$ describes the bottom topography relative to $z = 0$
g is the coefficient of gravitational acceleration
f is the coefficient describing the Coriolis effect
b is the drag coefficient
ν is the viscosity coefficient

The bottom topography variable $H(x, y)$ can also be seen as the mean depth of the fluid and can be taken to be a constant value if we want a flat bottom.

3.2 Shallow Water Equations

It is important to note that viscosity (and even drag) are often not considered when studying or simulating the shallow water equations. Indeed, we only derived the simpler versions of the equations.

If we throw out viscosity effects, we get the hyperbolic partial differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} - bu \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} - bv \\
&\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\big((h+H)u\big) + \frac{\partial}{\partial y}\big((h+H)v\big) = 0
\end{aligned} \tag{17}$$

3.3 Linearized Shallow Water Equations

The linearized equations come from assuming we have an equilibrium state (in this case a flat surface), with small perturbations in each variable, which we write as $u = 0 + \delta_u$, $v = 0 + \delta_v$, $h = 0 + \delta_h$, respectively. Specifically, we would say $\delta_u \ll 1$. Accordingly, $\frac{\partial \delta_u}{\partial x} \ll 1$, and so on. This allows us to ignore nonlinear terms, as, for example $\delta_u \frac{\partial \delta_u}{\partial x} \ll \delta_u$. This gives us the following.

$$\begin{aligned} \frac{\partial \delta_u}{\partial t} + g\frac{\partial \delta_h}{\partial x} &= 0 \\ \frac{\partial \delta_v}{\partial t} + g\frac{\partial \delta_h}{\partial y} &= 0 \\ \frac{\partial \delta_h}{\partial t} + \frac{\partial}{\partial x}(H\delta_u) + \frac{\partial}{\partial y}(H\delta_v) &= 0 \end{aligned} \tag{18}$$

Removing the deltas, we get the linearized shallow water equations.

$$\begin{aligned} &\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0 \\ &\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} = 0 \\ &\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0 \end{aligned} \tag{19}$$

We can also include the other linear effects and take H to be a flat constant such that this just becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} - bu \\ &\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y} - bv \\ &\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} + H\frac{\partial v}{\partial y} = 0 \end{aligned} \tag{20}$$

4 Numerical Methods

4.1 Necessary Conditions

For the shallow water equations, it is critically important that

$$h(x, y, t) + H(x, y) \geq 0 \quad \forall\ x, y, t \tag{21}$$

since this quantity describes the amount of mass (or fluid volume if you will) at any one point and must be non-negative.

While we cannot necessarily ensure this will hold for our numerical simulation, it would be good practice to set an initial condition that's greater than some threshold to ensure that (21) holds

$$h(x, y, 0) + H(x, y) > a \quad \forall\ x, y \text{ and } a > 0 \tag{22}$$

4.2 Explicit Methods

Since most versions of the equation are hyperbolic, they can be very easily time-stepped using explicit methods. In particular, we look to use Predictor Corrector Method

$$\begin{aligned} X^{n+\frac{1}{2}} &= X^n + \Delta t F(X^n) \\ X^{n+1} &= X^n + \frac{\Delta t}{2}\big(F(X^n) + F(X^{n+\frac{1}{2}})\big) \end{aligned} \tag{23}$$

In our case, Forward Euler leads to instability, which we will discuss in following sections

$$X^{n+1} = X^n + \Delta t F(X^n) \tag{24}$$

We will be timestepping the regular (17) and linearized (20) versions of the shallow water equations with explicit methods.

4.3 Implicit Methods

Since stability analysis showed that Forward Euler on (18) was unstable, we instead showed that Backward Euler would give us a stable result. This will be shown in following sections.

Since our equation is in two dimensions, implicit methods become expensive unless we employ operator splitting. Here, we outline an example of Backward Euler with operator splitting

$$\begin{aligned} X^{n+\frac{1}{4}} &= \left(M_x + \frac{\Delta t}{2} L_x \right)^{-1} X^n \\ X^{n+\frac{3}{4}} &= \left(M_y + \Delta t L_y \right)^{-1} X^{n+\frac{1}{4}} \\ X^{n+1} &= \left(M_x + \frac{\Delta t}{2} L_x \right)^{-1} X^{n+\frac{3}{4}} \end{aligned} \tag{25}$$

4.4 Implicit-Explicit Methods

Our full shallow water equations includes both non-linear and Laplacian terms. Because of this, the natural conclusion would be to employ implicit methods to deal with the Laplacian terms, and explicit methods to deal with the non-linear terms.

Since our equations are in two spatial dimensions, it is also wise to employ operator splitting. In particular, we employ Strang-Splitting for second order accuracy.

We outline an example of Forward-Backward Euler Method with operator splitting

$$\begin{aligned} X^{n+\frac{1}{6}} &= \left(M_x + \frac{\Delta t}{2} L_x \right)^{-1} X^n \\ X^{n+\frac{2}{6}} &= \left(M_y + \frac{\Delta t}{2} L_y \right)^{-1} X^{n+\frac{1}{6}} \\ X^{n+\frac{4}{6}} &= X^{n+\frac{2}{6}} + \Delta t F(X^{n+\frac{2}{6}}) \\ X^{n+\frac{5}{6}} &= \left(M_y + \frac{\Delta t}{2} L_y \right)^{-1} X^{n+\frac{4}{6}} \\ X^{n+1} &= \left(M_x + \frac{\Delta t}{2} L_x \right)^{-1} X^{n+\frac{5}{6}} \end{aligned} \tag{26}$$

where F describes the non-linear term and M_x, L_x and M_y, L_y describe diffusion in the x and y direction respectively.

For our purposes, Forward-Backward Euler is not stable. However, we can default to Crank-Nicolson-Predictor-Corrector with operator splitting for the purposes of our simulation. For stability purposes, it is likely that we can change the implicit timestepper, but not the explicit one as stability fails in Forward Euler.

We will be timestepping the full shallow water equations (16) with IMEX methods.

4.5 Boundary Conditions

In our simulations, we implemented the boundary conditions

$$u(x_l) = u(x_r) = 0$$
$$v(y_l) = v(y_r) = 0$$

The aim is to simulate a rectangular(in our case square) basin with rigid walls. We implement these boundary conditions for the three equations above (16)-(17)-(20), though careful care should be taken when changing boundary conditions for these problems, in particular for the full shallow water equations case (16).

4.6 Von Neumann Analysis

We also need to make sure that our method is stable. We employ Von Neumann Analysis to find the correct relationship between Δx and Δt. We write each of our variables as follows (where I is the imaginary unit). We also assume that $H = 1$ as a constant for this instance.

$$u^n_{i,j} = z^n u^0_{i,j} Exp(Iki\Delta x)\, Exp(Ikj\Delta x)$$
$$v^n_{i,j} = z^n v^0_{i,j} Exp(Iki\Delta x)\, Exp(Ikj\Delta x)$$
$$h^n_{i,j} = z^n h^0_{i,j} Exp(Iki\Delta x)\, Exp(Ikj\Delta x)$$

4.6.1 Forward Euler

We then apply a Forward Euler differencing scheme and collect the z terms on one side, we get the following.

$$z h^0_{i,j} = h^0_{i,j} - \frac{\Delta t}{\Delta x} I \sin k\Delta x u^0_{i,j} - \frac{\Delta t}{\Delta x} I \sin k\Delta x v^0_{i,j}$$
$$z u^0_{i,j} = -\frac{g\Delta t}{\Delta x} I \sin k\Delta x h^0_{i,j} + u^0_{i,j}$$
$$z v^0_{i,j} = -\frac{g\Delta t}{\Delta x} I \sin k\Delta x h^0_{i,j} + v^0_{i,j}$$

If we consider this as an eigenvalue problem, where the gain z is the eigenvalue and the initial state (h^0, u^0, v^0) is the eigenvector, we can solve for possible values of z, which gives:

$$z = \left\{1, 1 - \frac{\sqrt{-1+I}\Delta t\sqrt{g}\sin(\Delta x k)}{\Delta x}, 1 + \frac{\sqrt{-1+I}\Delta t\sqrt{g}\sin(\Delta x k)}{\Delta x}\right\}$$

Notably, the third value for z achieves values greater than 1 regardless of how we choose Δt, so we will not use Forward Euler, and we will instead go with an implicit method for the linearized equations.

4.6.2 Backward Euler

$$z h^0_{i,j} + z\frac{\Delta t}{\Delta x} I \sin k\Delta x u^0_{i,j} + z\frac{\Delta t}{\Delta x} I \sin k\Delta x v^0_{i,j} = h^0_{i,j}$$
$$z u^0_{i,j} + z\frac{g\Delta t}{\Delta x} I \sin k\Delta x h^0_{i,j} = u^0_{i,j}$$
$$z v^0_{i,j} + z\frac{g\Delta t}{\Delta x} I \sin k\Delta x h^0_{i,j} = v^0_{i,j}$$

We can play the same game here as above, except this time we are interested in the eigenvalues of the inverse of the left-hand side matrix, which are the below.

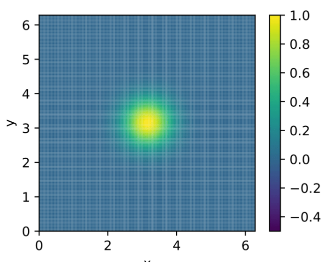
$$z = \left\{1, \frac{1}{1+\sqrt{-1+I}\,\frac{\Delta t}{\Delta x}\sqrt{g}\sin(\Delta x k)}, \frac{1}{1+\frac{\sqrt{2+2I}}{1-I}\,\frac{\Delta t}{\Delta x}\sqrt{g}\sin(\Delta x k)}\right\}$$

This scheme is stable when $\frac{\Delta t}{\Delta x} \leq 2^{\frac{1}{4}}\sqrt{g}$, as the magnitude of the denominators in the second and third cases will be at least 1.

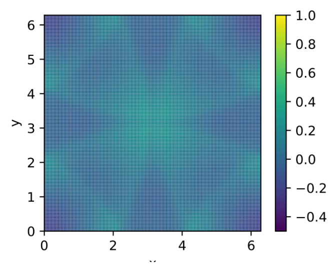
5 Simulations

Here are some pretty pictures (with descriptions of course): For all of the following we have

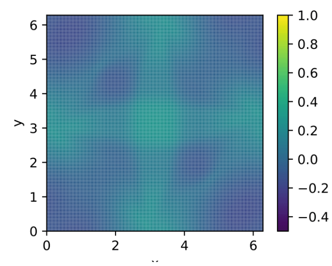
$$g = 9.81,\ f = 0.5,\ b = 0,\ \nu = 0.2 \text{ if included}$$



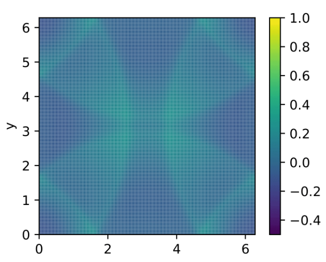
(a) Initial condition 2D Gaussian



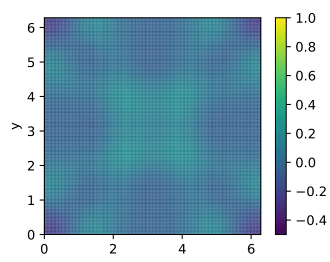
(b) Flat Bottom



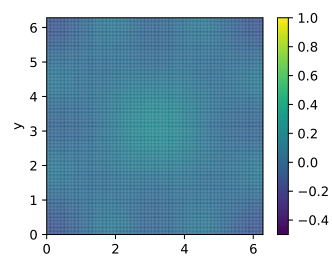
(c) Very rough bottom



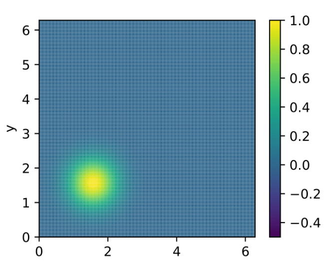
(d) Ridged Bottom



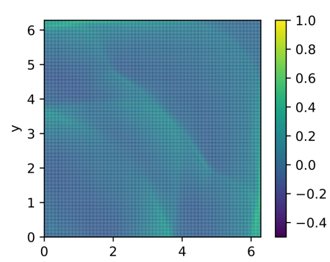
(e) Linear Shallow Water



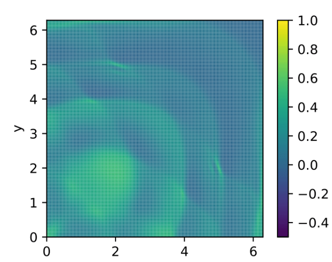
(f) Full Shallow Water



(g) Initial Condition for BC



(h) BC with 1 initial drop



(i) BC with 3 drops over time

References

[1] Harvey Segur. *Lecture 8: The Shallow-Water Equations*. 2009.

[2] Isla Simpson. *The Shallow water System*. 2011.