Power Spectrum

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1 Definition

Power spectrum is defined as the fourier transform of the autocorrelation function. The autocorrelation of a function $\delta(x)$ is

$$a(\tau) = \int_{-\infty}^{\infty} \delta(x)\delta(x+\tau)dx??$$

And the Power Spectrum is,

$$\begin{aligned}
P(k) &= \int_{-\infty}^{\infty} \exp(-ik\tau)a(\tau)d\tau \\
&= \int_{-\infty}^{\infty} \exp(-ik\tau) \int_{-\infty}^{\infty} \delta(x)\delta(x+\tau)dxd\tau \\
&= \int_{-\infty}^{\infty} \exp(-ik(\tau+x)) \int_{-\infty}^{\infty} \exp(ikx)\delta(x)\delta(x+\tau)dxd\tau \\
&= \int_{-\infty}^{\infty} \exp(ikx)\delta(x) \int_{-\infty}^{\infty} \exp(-ik(x+\tau))\delta(x+\tau)d\tau dx \\
&= \tilde{\delta}^*(k)\tilde{\delta}(k)
\end{aligned}$$

2 Discrete and Continous

The power spectrum is defined as follows

$$(2\pi)^3\delta_D(\boldsymbol{k}-\boldsymbol{k'})P(\boldsymbol{k}) =< \delta(\boldsymbol{k})\delta^*(\boldsymbol{k'}) > \tag{1}$$

The DFT fourier space interval given by $\Delta k = \Delta k_x = \Delta k_y = \Delta k_z = 2\pi/L$,where L is the real space interval. Multiplying both sides by $(\Delta k)^3 = \Delta k_x \Delta k_y \Delta k_z$ we get

$$(2\pi)^3\delta_D(\boldsymbol{k}-\boldsymbol{k'})P(\boldsymbol{k})(\Delta k)^3 =< \delta(\boldsymbol{k})\delta^*(\boldsymbol{k'}) > (\Delta k)^3 \tag{2}$$

Now lets discretize $\boldsymbol{k}'$ by saying that it belongs to a set which is equally spaced by dk along all three axis. Either $\boldsymbol{k}'$ lies inside the interval $[\boldsymbol{k}-d\boldsymbol{k}/2, \boldsymbol{k}+d\boldsymbol{k}/2]$

or outside the interval. When it is outside the interval, then the RHS of eq[2] is identically zero. When $\boldsymbol{k}'$ is inside the interval, $\boldsymbol{k}' \approx \boldsymbol{k}$ on the RHS and the LHS can be modified as,

$$\sum_{\boldsymbol{k}'} (2\pi)^3 \delta_D(\boldsymbol{k} - \boldsymbol{k}') P(\boldsymbol{k}) (\Delta k)^3 = < \delta(\boldsymbol{k}) \delta^*(\boldsymbol{k}) > (\Delta k)^3 \tag{3}$$

The summation above is over the discrete set in which $\boldsymbol{k}'$ belongs. When Δk is very small we can replace the summation on the RHS above with integration.

$$\int (2\pi)^3 \delta_D(\boldsymbol{k} - \boldsymbol{k}') P(\boldsymbol{k}) d^3 k' = < \delta(\boldsymbol{k}) \delta^*(\boldsymbol{k}) > \Delta k_x \Delta k_y \Delta k_z \tag{4}$$

$$(2\pi)^3 P(\boldsymbol{k}) = < \delta(\boldsymbol{k}) \delta^*(\boldsymbol{k}) > (2\pi)^3 / L^3 \tag{5}$$

The continous can be replaced with discrete fourier transform as follows(See Math methods notes for more details.) $\delta(\boldsymbol{k}) = \delta_{\boldsymbol{k}} L^3$. After this substitution, eqn[5] becomes,

$$P(\boldsymbol{k}) / L^3 = < \delta^2_{\boldsymbol{k}} > \equiv P_{discrete} \tag{6}$$