Exercise 2.2.6

Suppose we are given a relation R on a set S. Define the relation R' as follows

$$R' = R \cup \{(s,s) \mid s \in S\}$$

Show that R' is the reflexive closure of R.

Proof. By construction, R' is reflexive, so all we need to show is that R' is the smallest reflexive relation containing R. Suppose that T was another reflexive relation such that $R \subseteq T$. We will show that $R' \subseteq T$. Let $z = (x,y) \in R'$. By the defintion of R' it suffices to consider two cases:

Case 1: $z \in R$ Then clearly, $z \in T$ since $R \subseteq T$.

Case 2: $z \notin R$ Then, $z = (s,s)$ for some $s \in S$. Because T is reflexive, $z = (s,s) \in T$.

In either case we have show $z \in T$, hence $R' \subseteq T$. So, R' is the smallest reflexive relation containing R i.e. is it the transitive closure. □

Exercise 2.2.7

A more constructive definition of transitive closure is

$$\begin{aligned}
R_0 &= R \\
R_{i+1} &= R_i \cup \{(s,u) \mid \text{for some } t,\ (s,t) \in R_i \text{ and } (t,u) \in R_i\} \\
R^+ &= \bigcup_i R_i
\end{aligned}$$

Show that R^+ is the transitive closure of R.

Proof. First, we show that R^+ is transitive. Suppose that $t \in S$ is such that $(s,t) \in R^+$ and $(t,u) \in R^+$ for some $s, u \in S$. By the definition of R^+ and the fact that the R_i's form an increasing sequence, there is a $k \in \mathbb{N}$ such that $(s,t) \in R_k$ and $(t,u) \in R_k$. It follows that $(s,u) \in R_{k+1} \subseteq R^+$ which proves that R^+ is transitive.

Next, we show that R^+ is the smallest transitive relation containing R. Suppose that T was another transitive relation containing R. We show by induction that $R_i \subseteq T$ for each $i \in \mathbb{N}$ so that $R^+ = \bigcup_i R_i \subseteq T$.

Base case: $R_0 = R \subseteq T$ by assumption.

Inductive step: Assume that we have shown $R_i \subseteq T$. We show that $R_{i+1} \subseteq T$. Let $z = (s,u) \in R_{i+1}$. By the definition of R_{i+1}, either $z \in R_i$ or there is a $t \in S$ such that $(s,t) \in R_i$ and $(t,u) \in R_i$. In the former case, $z \in T$ by the IH. In the later case, $(s,t) \in T$ and $(t,u) \in T$ by the IH. Because T is transitive, it follows that $z = (s,u) \in T$. Hence, $R_{i+1} \subseteq T$.

By induction, $R_i \subseteq T$ and it follows that $R^+ \subseteq T$ which proves R^+ is the transitive closure. □

Exercise 2.2.8

Suppose R is a binary relation on a set S and P is a predicate on S that is preserved by R. Show that P is also preserved by $R*$.

Proof. Note that R^*, the reflexive and transitive closure of R can be obtained as $R^* = (R')$. Hence, R^* can be formed as

$$\begin{aligned}
R_0 &= R' \\
R_{i+1} &= R_i \cup \{(s,u) \mid \text{for some } t,\ (s,t) \in R_i \text{ and } (t,u) \in R_i\} \\
R^* &= \bigcup_i R_i
\end{aligned}$$

We will show that P is preserved by R_i for all $i \in bN$ by induction.

Base case: Let $z = (s,u) \in R_0 = R'$ and suppose that $P(s)$ holds. Then, either $z \in R$ or $u = s$. If $z \in R$ then $P(u)$ holds because R preserved P. If $u = s$, then $P(u)$ holds vacuously.

Inductive step: Assume that P is preserved by R_i. We will show that R_{i+1} preserves P. Let $z = (s,u) \in R_{i+1}$ and suppose that $P(s)$ holds. By the definition of R_{i+1} either $z \in R_i$ or there is a $t \in S$ such that $(s,t) \in R_i$ and $(t,u) \in R_i$. In the former case, $P(u)$ holds by the IH. In the latter case, because R_i preserves P and $P(s)$ holds we first have $P(t)$ holds which implies that $P(u)$ holds. Hence, P is preserved by R_{i+1}.

By induction, P is preserved by R_i for all $i \in \mathbb{N}$. Because $R^* = \bigcup_i R_i$ it follows that P is preserved by R^*. □