# Tutorial Literature Review & Model Setup

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November 28, 2024

## 1 Literature Review

- **AER Team Contests with Multiple Pairwise Battles (Qiang Fu, Jingfeng Lu, Yue Pan):**
  In this paper, Fu, Lu, and Pan analyze different temporal structures' effects on players' strategic behavior and their overall performance in contests. They consider the model that consists of two teams with odd risk-neutral players competing for the public good that is equally valued by all players in the winning team. Overall there are $2n+1$ distinct battles, and whichever team that secures the majority of battles is considered to be the winner and receives the prize. Besides the contest prize, each player can earn the same fixed private prize for the battle they participated in and exert a constant marginal cost. While the distribution of marginal cost is common knowledge, each player's realized cost remains private. The winning probability is determined by the palyer's effort level with homogenous degree zero.

  Based on the model setup above, they prove three neutrality properties: "history independence", "sequence independence", and "temporal-structure independence". "History independence" suggests that given the positive prize spread, the result of each battle depends on the player's inherent abilities regardless of the outcomes of previous battles. "Sequence independence" illustrates that no matter how player pairings are reshuffled, the winning probability of each team, the expected effort in each battle, and player's expected payoff all remain the same. Last, "temporal-structure independence" indicates that no matter how the battles are conducted, either simultaneously or sequentially, even the combinations of these two, the overall contest result and ex-ante expected effort stay constant.

  To derive these three results, they show each player in each round faces the common effective prize spread. Due to homogenous of degree zero, the unique equilibrium bids are linear with the scaling factor, which proves the "history independence" if the prize spread is positive. By backward induction, they analyze the contest and characterize the unique subgame perfect equilibrium. From sequentially setting to extensions to any temporal structure, as long as the pairing is fixed, they suggest the expected effort, expected payoff, and the winning probability are separated from the ordering and the structure of the contest.

- **Equilibrium player choices in team contests with multiple pairwise battles (Hideo Konishi, Chen-yu Pan, Dimitar Simeono):**
  This paper deals with how to choose which player to send in at which point in the competition by extending the analysis of multi-battle contest by Fu, Lu, and Pan. Konishi, Pan, and Simeono show that winning probability of each team at the equilibrium and ex ante expected effort of each player are independent of whether the game is one-shot or sequential, as long as the number of players on each team is consistent with the number of battles.

  They first reproduce the result derived by Hamilton and Romano for one-shot choice given exogenously battle winning probabilities. At the mixed strategy equilibrium, randomized matching would give both teams the equal chance of winning. Then they release the constraint and extend the equivalence to Fu, Lu, and Pan's work where the winning probabilities are determined by players' effort. The paper shows that this equivalence also applies to sequential battle. By backward induction, they find that at every stage of the game, the equilibrium strategy for each manager is to uniformly randomize over all of their players when selecting next player to send in. This is because that if team *A* uniformly randomizes over all players at every step, then team *B*'s expected winning probabilities would be the same regardless the strategy choice. Thus, team *B* becomes indifferent between all choices at every stage, and would like to randomize all of their remaining players. Consequently, the ex ante winning probabilities in each subgame are the same under randomization as in the one-shot game.

- **How to split the pie: Optimal rewards in dynamic multi-battle components (Xing Feng, Jingfeng Lu):**
  Feng and Lu explore the optimal design of rewards in sequential games with two risk-neutral players exerting marginal effort cost. They consider the battle to consist of three rounds where the organizer with fixed prize budget can allocate rewards to players based on the number of battles they win. The key finding is that the optimal design depends on the discriminatory power for which higher value rep-

resents more effectiveness in incentivizing effort. When the discriminatory power is in the low range, the winner-take-all prize allocation is optimal; however, as *r* increases, the optimal allocation gradually transfers from the winner-take-all to the proportional division rule, which is caused by the growing discouragement effect. When one player has a long lead or early advantage, the opponent will be discouraged from exerting effort in further competition. Because of this, the author conducts robustness check to verify the importance of mitigating the discouragement effect to derive the effort-maximizing prize design, ranging from relaxing constraints on the prize allocation, allowing for history-dependent prizes, considering asymmetric players, and extending the model to five-battle contests.

- **Multi-battle contests (Kai A. Konrad, Dan Kovenock):**

  Konrad and Kovenock characterize the "multi-battle contests" as the sequential battles in which winner is the first to win the predetermined number of victories. Assuming winner in each round receives the intermediate prize, they study the analytical solution for the unique subgame perfect equilibrium between two players who compete in simultaneous move all-pay auctions with complete information. The equilibrium specifically exhibits "endogenous uncertainty" and "pervasiveness" because of the positive equilibrium probability of any state to a certain outcome. This implies that all players will participate throughout the contest even if there is a large lead by other players.

  Additionally, the study defines "separating states" where one player has positive continuation value equal to the final prize, while the other players have zero continuation value. The player's effort, the aggregate effort, and the winning probabilities do not consistently increase or decrease. Instead, effort levels are non-monotonic in the closeness of the race depending on the structure of the contest. In states where both players have an equal number of wins, efforts are highest as they compete for both the intermediate and final prizes; however, players only compete for the intermediate prize when sufficient asymmetry is exhibited. Moreover, they find that the total effort expended by a player throughout the game can substantially exceed the final prize value, as the previous efforts are considered as sunk costs and are independent of future decisions.

- **Winner-pay contests (Andrew J. Yates):**

  Yates verifies a pure strategy Nash equilibrium exists under winner-pay contexts in which two players select bids. Under six weak assumptions of the contest success function, the uniqueness of Nash equilibrium can be proved, which also holds for the Tullock contest. For some specific function forms of the contest success function under symmetric and asymmetric cases, they obtain the explicit formula for the equilibrium bid. In addition, they also discuss how private information affects the bidding strategies compared to the assumption of complete information. With specific contest success functions, the paper derives a Bayesian Nash equilibrium with two types and shows the ex ante expected bid is lower under private information.

# 2 Model and Analysis

## 2.1 One-shot Game

Consider two teams $A$ and $B$ competing in a simplified one-round contest. The manager writes a limited liability contract for each agent on their team. Players on each team exert effort $x_a$ and $x_b$ in participating in the game with constant marginal cost $c_a$ and $c_b$. Since the effort level is not contractible, managers assign optimal rewards(wages), $w_a$ and $w_b$ for players on their team contingent upon their player winning the contest, and 0 if they lose. Extending from the basic general Tullock contest, we incorporate the general discriminatory power $r \in (0,1]$, the representation of effectiveness of effort, into our model. The winning probability of team $A$ is given by

$$\mathsf{P}_A(x_A, x_B) = \frac{x_A^r}{x_A^r + x_B^r},$$

and similarly for team $B$ such that

$$\mathsf{P}_B(x_A, x_B) = \frac{x_B^r}{x_A^r + x_B^r}.$$

Each manager values the competition prize as $V$ and maximizes the expected payoff from winning the contest. They solve the optimization problem as follows,

$$\max_{w_A}(V - w_A)\frac{x_A^r}{x_A^r + x_B^r},$$
$$\max_{w_B}(V - w_B)\frac{x_B^r}{x_A^r + x_B^r}.$$

For agents on the team, to maximize their payoff given the wages offered and marginal cost of efforts, they adjust their efforts

$$\max_{w_A} w_A \frac{x_A^r}{x_A^r + x_B^r} - c_A x_A,$$
$$\max_{w_B} w_B \frac{x_B^r}{x_A^r + x_B^r} - c_B x_B.$$

By FOC, for player on team $A$

$$c_A = \frac{rx_A^{r-1}w_A(x_A^r + x_B^r) - w_A x_A^r r x_A^{r-1}}{(x_A^r + x_B^r)^2}$$
$$c_A = w_A \frac{rx_A^{r-1}x_B^r}{(x_A^r + x_B^r)^2}$$

By symmetry, agent on team $B$ has

$$c_B = w_B \frac{rx_A^r x_B^{r-1}}{(x_A^r + x_B^r)^2}$$

Notice that

$$\frac{rw_B x_A^r x_B^{r-1}}{c_B} = (x_A^r + x_B^r)^2 = \frac{rw_A x_A^{r-1}x_B^r}{c_A} \Rightarrow \frac{w_B x_A^r x_B^{r-1}}{c_B} = \frac{w_A x_A^{r-1}x_B^r}{c_A}$$
$$\Rightarrow w_B x_A^r x_B^{r-1} = \frac{w_A x_A^{r-1} x_B^r c_B}{c_A} \text{ (multiply each side by } c_B)$$
$$\Rightarrow w_B x_A = \frac{w_A x_B c_B}{c_A}$$
$$\Rightarrow x_B = \frac{c_A}{c_B}\frac{w_B x_A}{w_A}$$

Substituting the expression of $x_B$ back to the expression of $c_A$, we have

$$c_A = \frac{rw_A x_A^{r-1}\left(\frac{c_A^r}{c_B^r}\frac{w_B^r x_A^r}{w_A^r}\right)}{\left(x_A^r + \frac{c_A^r}{c_B^r}\frac{w_B^r x_A^r}{w_A^r}\right)^2} \Rightarrow c_A = \frac{r\frac{c_A^r}{c_B^r}\frac{w_B^r x_A^{2r-1}}{w_A^{r-1}}}{\left(x_A^r + \frac{c_A^r}{c_B^r}\frac{w_B^r x_A^r}{w_A^r}\right)^2}$$
$$\Rightarrow c_A = \frac{r\frac{c_A^r}{c_B^r}\frac{w_B^r}{w_A^{r-1}x_A}}{\left(1 + \frac{c_A^r}{c_B^r}\frac{w_B^r}{w_A^r}\right)^2} \text{ (simplify the righthand side expression by dividing } x_A^{2r})$$
$$\Rightarrow x_A = \frac{r\frac{c_A^r}{c_B^r}\frac{w_B^r}{w_A^{r-1}}}{c_A\left(1 + \frac{c_A^r}{c_B^r}\frac{w_B^r}{w_A^r}\right)^2}$$
$$\Rightarrow x_A = \frac{r\frac{c_A^r}{c_B^r}\frac{w_B^r}{w_A^r}w_A}{c_A\left(\frac{w_A^r c_B^r + w_B^r c_A^r}{w_A^r c_B^r}\right)^2}$$
$$\Rightarrow x_A = \frac{w_A}{c_A}\frac{rc_A^r c_B^r w_B^r w_A^r}{(w_A^r c_B^r + w_B^r c_A^r)^2}$$

By symmetry,

$$x_B = \frac{w_B}{c_B}\frac{rc_A^r c_B^r w_B^r w_A^r}{(w_A^r c_B^r + w_B^r c_A^r)^2}$$

Thus, under which a Nash equilibrium exists, the winning probability for player on team $A$ is

$$\begin{aligned}
\frac{x_A^r}{x_A^r+x_B^r} &= \frac{\frac{w_A^r}{c_A^r}\frac{r^r(c_A^r)^2(c_B^r)^2(w_B^r)^2(w_A^r)^2}{(w_A^r c_B^r+w_B^r c_A^r)^2}}{\frac{w_A^r}{c_A^r}\frac{r^r(c_A^r)^2(c_B^r)^2(w_B^r)^2(w_A^r)^2}{(w_A^r c_B^r+w_B^r c_A^r)^2}+\frac{w_B^r}{c_B^r}\frac{r^r(c_A^r)^2(c_B^r)^2(w_B^r)^2(w_A^r)^2}{(w_A^r c_B^r+w_B^r c_A^r)^2}} \\
&= \frac{\frac{w_A^r}{c_A^r}}{\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r}}
\end{aligned}$$

Since players would like to maximize their win chances based on the wages provided, we have

$$\begin{aligned}
\frac{\partial}{\partial w_A}\left(\frac{x_A^r}{x_A^r+x_B^r}\right) &= \frac{\frac{1}{c_A^r}rw_A^{r-1}(\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r})-\frac{w_A^r}{c_A^r}\frac{1}{c_A^r}rw_A^{r-1}}{(\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r})^2} \\
&= \frac{\frac{1}{c_A^r}rw_A^{r-1}\frac{w_B^r}{c_B^r}}{(\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r})^2}
\end{aligned}$$

Similarly, for team $B$,

$$\begin{aligned}
\frac{x_B^r}{x_A^r+x_B^r} &= \frac{\frac{w_B^r}{c_B^r}}{\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r}}, \\
\frac{\partial}{\partial w_B}\left(\frac{x_B^r}{x_A^r+x_B^r}\right) &= \frac{\frac{1}{c_B^r}rw_B^{r-1}\frac{w_A^r}{c_A^r}}{(\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r})^2}
\end{aligned}$$

**Nash Equilibrium**

For manager $A$, given the winning probability in terms of $w_A$, $w_B$, $c_A$, and $c_B$, they choose the optimal wage $w_A$ to maximize their expected payoff

$$\max_{w_A}(V-w_A)\frac{x_A^r}{x_A^r+x_B^r}.$$

To find the optimal wage, we derive the FOC from the maximization problem such that

$$\frac{\partial}{\partial w_A}\left[(V-w_A)\frac{x_A^r}{x_A^r+x_B^r}\right]=0.$$

Expanding and simplifying the equation, we have

$$
\begin{aligned}
(-1)\frac{x_A^r}{x_A^r+x_B^r}+(V-w_A)\frac{\partial}{\partial w_A}\left(\frac{x_A^r}{x_A^r+x_B^r}\right)=0 &\Rightarrow (-1)\frac{\frac{w_A^r}{c_A^r}}{\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r}}+(V-w_A)\frac{\frac{1}{c_A^r}rw_A^{r-1}\frac{w_B^r}{c_B^r}}{(\frac{w_A^r}{c_A^r}+\frac{w_B^r}{c_B^r})^2}=0\\
&\Rightarrow (-1)w_A^r+(V-w_A)rw_A^{r-1}\frac{w_B^r}{c_B^r}=0\\
&\Rightarrow (-1)w_A+(V-w_A)r\frac{w_B^r}{c_B^r}=0 \text{ (divide by } w_A^{r-1})\\
&\Rightarrow w_A=(V-w_A)r\frac{w_B^r}{c_B^r} \text{ (implicit solution)}\\
&\Rightarrow w_A=\frac{Vr\frac{w_B^r}{c_B^r}}{r\frac{w_B^r}{c_B^r}+1} \text{ (explicit solution)}
\end{aligned}
$$

Similarly, for manager $B$, they would set wages at

$$
\begin{aligned}
w_B &= (V-w_B)r\frac{w_A^r}{c_A^r} \text{ (implicit solution)}\\
&= \frac{Vr\frac{w_A^r}{c_A^r}}{r\frac{w_A^r}{c_A^r}+1} \text{ (explicit solution)}
\end{aligned}
$$

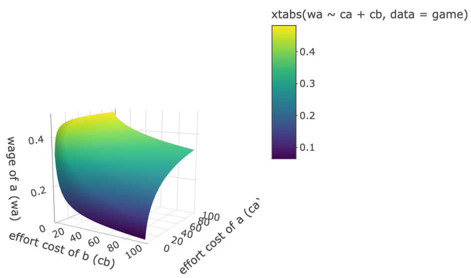
Therefore, at the equilibrium where both managers have no incentive to deviate from their choice of wages, the equilibrium wage of this one-shot game is

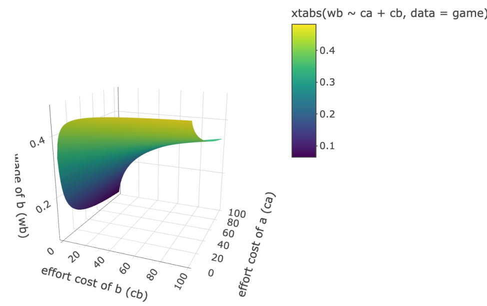
$$
\begin{aligned}
w_A &= (V-w_A)r\frac{w_B^r}{c_B^r},\\
w_B &= (V-w_B)r\frac{w_A^r}{c_A^r}.
\end{aligned}
$$

**Simulations**

After deriving the general expression, we conduct the simulation to visualize the relationship between different variables.

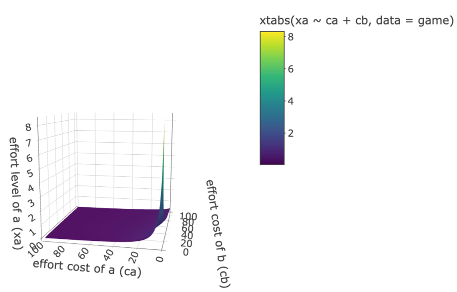
**The Relationship between Cost of Effort and Wages:**

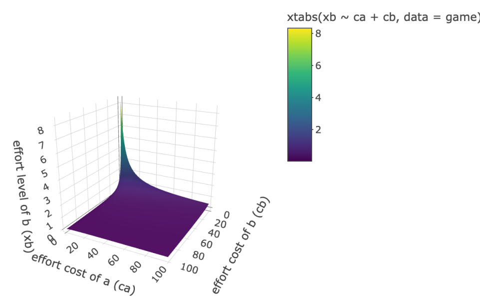





The graphs given above depict how wages are varied by changing in cost of efforts $c_A$ and $c_B$. The graph on the left shows that fixing $c_B$, as the effort cost of team $A$ $c_A$ increases, wages $w_A$ increases as well. This is because as cost of participating in the contest raises, higher wages are needed to incentivize players to ensure participation, as long as the compensation is balanced between the team's performance relative to cost. Additionally, for team $B$, as the cost of effort increases, it becomes more expensive for team $B$ to exert high levels of effort. Perceiving this, manager $A$ don't need to excessively incentivize their player to win the prize. Thus, to economize the expense on wages, manager $A$ decrease their wages to players on team $A$ when team $B$ is experiencing high cost of effort. This also holds for the graph on the right, from which as $c_A$ goes up, the $w_B$ decreases accordingly.

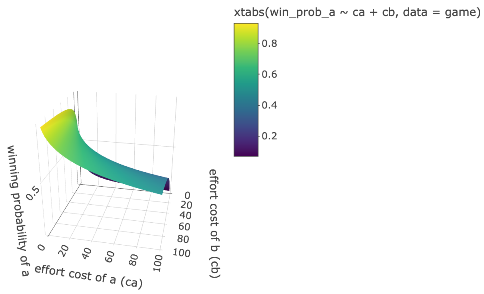
**The Relationship between Cost of Effort and Effort Level:**

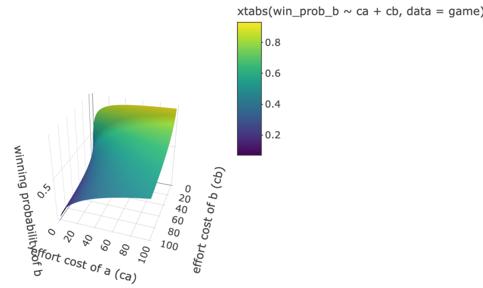





Graphs above illustrate how the cost of efforts influences players' choice of effort level. From the graph on the left, as $c_a$ increases, the effort level $x_A$ sharply decreases, especially when $c_A$ is in low range. This could be interpreted as higher personal costs lead to significantly reduced effort, as players are discouraged by exerting high cost relative to the prize. Such relationship also exists between $c_B$ and $x_A$, but not as dramatically as with $c_A$. As $c_b$ increases, team $A$ perceives lower threat from team $B$, thus reducing their effort. A similar argument will be applied to the graph on the right.

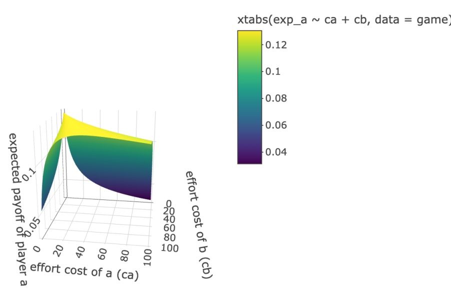
**The Relationship between Cost of Effort and Winning Probabilities:**

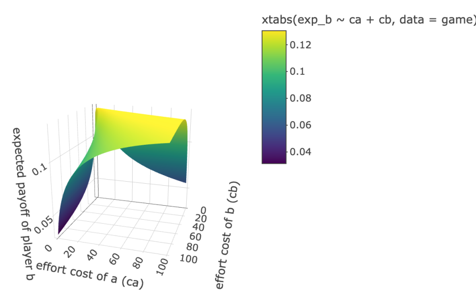



These two graphs generalize the relationship between the winning probabilities and the effort cost of both teams. Keeping $c_b$ constant, as $c_A$ increases, the win chances of team $A$ decrease dramatically, particularly when $c_A$ is at lower levels. This suggests that high cost level will reduce the team's ability to compete effectively. In contrast, when $c_B$ increases, the winning probabilities of team $A$ increase. This is because it becomes costly for team $B$ to participate in game, then team $A$ has better chance of winning. Interestingly, in scenarios where both $c_A$ and $c_B$ are high, the winning probabilities for team $A$ decreases, which shows that high cost of efforts can generally depress the capabilities of both team and leads to more equalized win chances. Similar relationships are illustrated in the graph on the right as well.

**The Relationship between Cost of Effort and Players' Expected Payoff:**

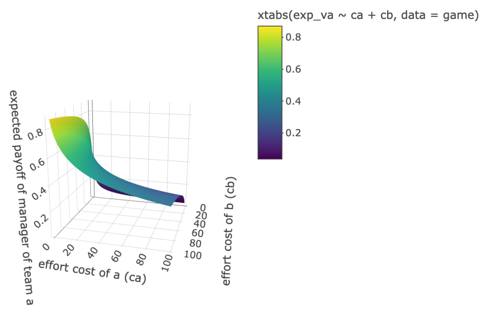


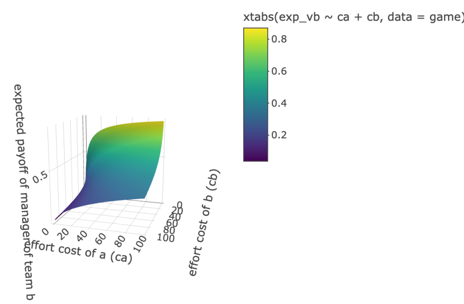




Here, when $c_a$ is low, the expected payoff of the agent is relatively high. The peak generally occurs when $c_B$ is at medium to high levels. Before subsequently decline, the player $A$'s expected payoff increases as $c_A$ increases. We interpret this as diminishing returns in player's expected payoff. Beyond the optimal point, the costs outweigh the benefits. As $c_A$ increases, player $A$'s expected payoff decreases. This intuitively makes sense as higher effort costs will lead to lower total benefits. Particularly, when $c_b$ is at lower level, team $A$ finds it significantly difficult to receive high payoffs as their cost of effort increases.

However, even if team $A$ has advantage in cost of efforts, the rising cost in team $B$ does not always result in straightforward increase in player $A$'s expected payoff. At first, when $c_A$ is low, the increase in $c_B$ increases player $A$'s expected payoff and then decreases. This is because initially team $A$ has relative advantage in participating in the contest; as $c_B$ increases, initial benefit of low $c_A$ might lead to overinvestment in effort, eventually reach a point of diminishing returns. As $c_A$ increases to the medium and high level, as $c_b$ increases, player on team $A$ has higher expected payoff. This indicates that player $A$ benefited from player $B$'s high cost pressure, thus having a relative advantage in the expected payoff.

**The Relationship between Cost of Effort and Managers' Expected Payoff:**



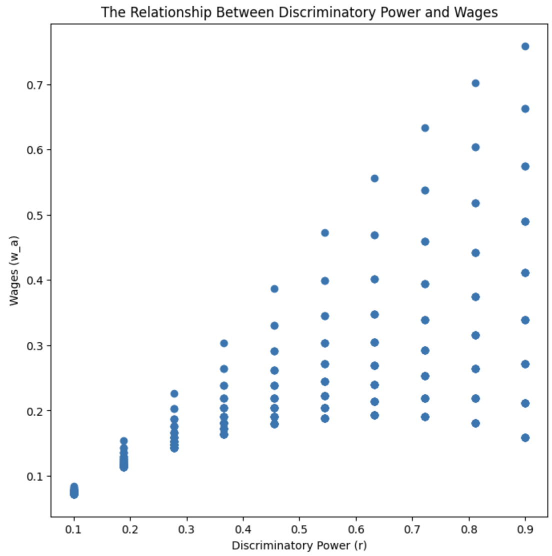




From the graph, we observe that as $c_a$ increases, manager $A$'s expected payoff decreases, since it is more costly for team $A$ to exert effort, leading to lower expected payoff. In contrast, the increase in $c_b$ raises manager $A$'s expected payoff, since team $A$ can potentially gain from opponent's high cost pressure. Thus, team $A$'s win chances could improve. The graph on the right offers us a similar insight.

**The Effect of Discriminatory Power:**

To focus on the effect of discriminatory power, we simplify our model to $c_A = c_B$, which suggests the equalized winning probabilities for each team regardless. Without loss of generality, we focus on the effect of discriminatory power on Team $A$'s wages and manager's payoff.

- **The Relationship between Discriminatory Power and Wages:**

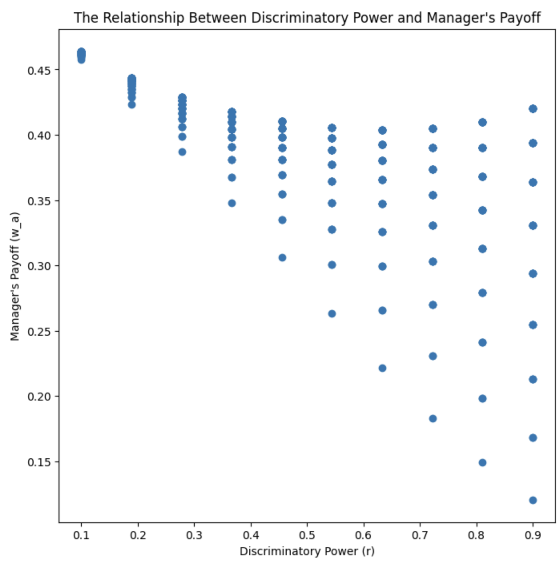



Each layer of points represents different levels of cost of effort level. Here, higher wages are associated with higher levels of cost. This is because when cost effot is high, without sufficient incentives, players are less likely to exert high effort under costly conditions. Thus, higher $c_A$ lead to higher wages.

Given the different cost levels, we notice that there is nonlinear relationship between discriminatory power and wages. When $r$ is in lower levels, the wages are relatively low. This is possibly because at lower levels of $r$ managers are unable to differentiate which players should be rewarded more given their effort level, and manager's ability to strategically assign wages will be improved along with the increase in the discriminatory power. Additionally, this shows that when discriminatory power is low, contest success function is less sensitive to the differences of effort level. Thus, players may be not strongly motivated to make high effort, as their marginal contribution to win chances is low. Subsequently, managers don't need to pay high wages to motivate players. As $r$ increases to the medium levels, the wages increase. At this level of discriminatory power, the contest becomes more competitive; thus players need more incentive to maintain their effort level. Notice that for those layers with relatively low cost, when $r$ approaches higher levels, the wages start to decrease again. This might be because, at high levels of discriminatory power, players are inherently motivated to exert high effort, since small differences in effot can substantially affect the winning probabilities. Therefore, managers can offer lower wages to players. In contrast, for layers with higher cost levels, the wages consistently increases as $r$ increases, since when the cost of effort is high, players require higher wages to compensate for increased cost of participating in the contest. Thus, managers need to pay higher wages.

Consider the graph cross-sectionally, at lower level of discriminatory power, there is less variation in wages across the players with different cost levels. As the discriminatory power increases, there is a wider spread in the wage distribution. This indicates that managers leverage discriminatory ability to assign wages and make more strategic decisions. However, this could lead to the potential for rent dissipation, where significant amount of budget is allocated to players to secure critical wins. While this might result in short-term gains, such wage distribution may be inefficient.

- **The Relationship between Discriminatory Power and Manager's Expected Payoff:**




There is a cluster of manager's expected payoff at the low range of $r$ from 0.1 to 0.5. Within such a range, as $r$ increases, the manager's payoff decreases. This suggests that at lower levels of discriminatory power, manager's ability to incentivize players is limited. And the contest success function is less sensitive to the effort levels exerted by the players. Consequently, players may not be sufficiently motivated to make efforts, leading to lower manager's expected payoff. As $r$ increases, the sensitivity of the contest success function to effort levels increases as well. In other words, at higher levels of $r$, small differences in effort can lead to significant changes in win chances. This means that as $r$ increases, players are more aggressively incentivized to exert effort, and managers can allocate prizes more effectively, thus having higher expected payoffs. The U-shape pattern is obvious across all cost layers, indicating that the relationship between discriminatory power and manager's expected payoff holds true regardless. However, the depth and steepness of the U-shape vary across the cost layers. When cost of effort is low, U-shape is more evident compared to the orange layer. Recall that as $c_A$ increases, manager $A$'s expected payoff would decrease. Thus, positive impact of higher discriminatory power on manager $A$'s payoff is more significant when $c_A$ is at lower level.

## 2.2 Best of three with fixed ordering

Extending from the one-shot game, here we consider a dynamic three-battle contest where the pairings and ordering of matches are fixed. Suppose $V = 1$. Let wage be $w_{A_{(i,j)}}$ and $w_{B_{(i,j)}}$, cost of efforts be $c_{A_{(i,j)}}$ and $c_{B_{(i,j)}}$, where $(i, j)$ indicates the stage where team $A$ has win $i$ games and team $B$ has win $j$ games. Let $V_{A_{(i,j)}}$ and $V_{B_{(i,j)}}$ be continuation values respectively, the gross expected payoff for winning the team prize at stage $(i, j)$. By solving the game backward, we aim to identify the equilibrium wages and continuation values at each stage given each player's cost of effort level.

**Stage** $(1,1)$

The equilibrium of stage $(1,1)$ coincides with the one from the one-shot game. Thus, given $V = 1$,

$$w_{A_{(1,1)}} = (1 - w_{A_{(1,1)}})r\frac{w^r_{B_{(1,1)}}}{c^r_{B_{(1,1)}}},$$
$$w_{B_{(1,1)}} = (1 - w_{B_{(1,1)}})r\frac{w^r_{A_{(1,1)}}}{c^r_{A_{(1,1)}}}.$$

Then the continuation value is determined by the assigned wages, which could be calculated as

$$V_{A_{(1,1)}} = (1 - w_{A_{(1,1)}})\frac{x^r_{A_{(1,1)}}}{x^r_{A_{(1,1)}} + x^r_{B_{(1,1)}}} = (1 - w_{A_{(1,1)}})\frac{\frac{w^r_{A_{(1,1)}}}{c^r_{A_{(1,1)}}}}{\frac{w^r_{A_{(1,1)}}}{c^r_{A_{(1,1)}}} + \frac{w^r_{B_{(1,1)}}}{c^r_{B_{(1,1)}}}},$$
$$V_{B_{(1,1)}} = (1 - w_{B_{(1,1)}})\frac{x^r_{B_{(1,1)}}}{x^r_{A_{(1,1)}} + x^r_{B_{(1,1)}}} = (1 - w_{B_{(1,1)}})\frac{\frac{w^r_{B_{(1,1)}}}{c^r_{B_{(1,1)}}}}{\frac{w^r_{A_{(1,1)}}}{c^r_{A_{(1,1)}}} + \frac{w^r_{B_{(1,1)}}}{c^r_{B_{(1,1)}}}}.$$

**Stage** $(1,0)$

At stage $(1,0)$, manager $A$'s expected payoff is given by

$$\begin{aligned}
&(1-w_{A_{(1,0)}})\frac{x^r_{A_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}}+V_{A_{(1,1)}}\frac{x^r_{B_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}}\\
&=V_{A_{(1,1)}}+(1-w_{A_{(1,0)}}-V_{A_{(1,1)}})\frac{x^r_{A_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}}
\end{aligned}$$

Thus, both managers, solve the maximization problems respectively

$$\begin{aligned}
&\text{Manager } A\text{: } \max_{w_A} V_{A_{(1,1)}}+(1-w_{A_{(1,0)}}-V_{A_{(1,1)}})\frac{x^r_{A_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}},\\
&\text{Manager } B\text{: } \max_{w_B}(V_{B_{(1,1)}}-w_{B_{(1,0)}})\frac{x^r_{B_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}}
\end{aligned}$$

Notice that note that the managers' maximization problems have the same form as in the state $(1,1)$. Therefore, the equilibrium wage at stage $(1,0)$ is

$$\begin{aligned}
w_{A_{(1,0)}}&=(1-w_{A_{(1,0)}}-V_{A_{(1,1)}})r\frac{w^r_{B_{(1,0)}}}{c^r_{B_{(1,0)}}},\\
w_{B_{(1,0)}}&=(V_{B_{(1,1)}}-w_{B_{(1,0)}})r\frac{w^r_{A_{(1,0)}}}{c^r_{A_{(1,0)}}}.
\end{aligned}$$

Here the continuation values for both teams are

$$\begin{aligned}
V_{A_{(1,0)}}&=(V_{A_{(1,1)}}-w_{A_{(1,0)}})\frac{x^r_{A_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}},\\
V_{B_{(1,0)}}&=(V_{B_{(1,1)}}-w_{B_{(1,0)}})\frac{x^r_{B_{(1,0)}}}{x^r_{A_{(1,0)}}+x^r_{B_{(1,0)}}}.
\end{aligned}$$

**Stage** $(0,1)$

By symmetry from the stage $(1,0)$, at stage $(0,1)$, managers face the maximization problem

$$\begin{aligned}
&\text{Manager } A\text{: } \max_{w_A}(V_{A_{(1,1)}}-w_{A_{(0,1)}})\frac{x^r_{A_{(0,1)}}}{x^r_{A_{(0,1)}}+x^r_{B_{(0,1)}}},\\
&\text{Manager } B\text{: } \max_{w_B}(1-w_{B_{(0,1)}})\frac{x^r_{B_{(0,1)}}}{x^r_{A_{(0,1)}}+x^r_{B_{(0,1)}}}+V_{B_{(1,1)}}\frac{x^r_{A_{(0,1)}}}{x^r_{A_{(0,1)}}+x^r_{B_{(0,1)}}}=V_{B_{(1,1)}}+(1-w_{B_{(0,1)}}-V_{B_{(1,1)}})\frac{x^r_{A_{(0,1)}}}{x^r_{A_{(0,1)}}+x^r_{B_{(0,1)}}},
\end{aligned}$$

which give us the equilibrium wage at $(0,1)$ where

$$w_{A_{(1,0)}} = (V_{A_{(1,1)}} - w_{A_{(0,1)}}) r \frac{w^r_{A_{(0,1)}}}{c^r_{A_{(0,1)}}},$$
$$w_{A_{(1,0)}} = (1 - w_{B_{(0,1)}} - V_{B_{(1,1)}}) r \frac{w^r_{B_{(0,1)}}}{c^r_{B_{(0,1)}}}.$$

And the continuation values are

$$V_{A_{(0,1)}} = (V_{A_{(1,1)}} - w_{A_{(0,1)}}) \frac{x^r_{A_{(0,1)}}}{x^r_{A_{(0,1)}} + x^r_{B_{(0,1)}}},$$
$$V_{B_{(0,1)}} = (V_{B_{(1,1)}} - w_{B_{(0,1)}}) \frac{x^r_{B_{(0,1)}}}{x^r_{A_{(0,1)}} + x^r_{B_{(0,1)}}}.$$

**Stage** $(0,0)$

Solving the game iteratively, at the initial stage of the game, managers' expected payoff maximization problems are given as

$$\text{Manager } A\text{: } \max_{w_A} (V_{A_{(1,0)}} - w_{A_{(0,0)}}) \frac{x^r_{A_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}} + V_{A_{(0,1)}} \frac{x^r_{B_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}} = V_{A_{(0,1)}} + (V_{A_{(1,0)}} - w_{A_{(0,0)}} - V_{A_{(0,1)}}) \frac{x^r_{A_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}},$$
$$\text{Manager } B\text{: } \max_{w_B} (V_{A_{(0,1)}} - w_{B_{(0,1)}}) \frac{x^r_{B_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}} + V_{B_{(1,0)}} \frac{x^r_{A_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}} = V_{B_{(1,0)}} + (V_{A_{(0,1)}} - w_{B_{(0,1)}} - V_{B_{(1,0)}}) \frac{x^r_{A_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}},$$

Thus, the equilibrium wages are solved as

$$w_{A_{(0,0)}} = (V_{A_{(1,0)}} - w_{A_{(0,0)}} - V_{A_{(0,1)}}) r \frac{w^r_{B_{(0,0)}}}{c^r_{B_{(0,0)}}},$$
$$w_{B_{(0,0)}} = (V_{B_{(0,1)}} - w_{B_{(0,0)}} - V_{B_{(1,0)}}) r \frac{w^r_{A_{(0,0)}}}{c^r_{A_{(0,0)}}}.$$

Lastly, the continuation values at the initial stage are determined by $w_{A_{(0,0)}}$ and $w_{B_{(0,0)}}$, in which

$$V_{A_{(0,0)}} = (V_{A_{(1,0)}} - w_{A_{(0,0)}}) \frac{x^r_{A_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}},$$
$$V_{B_{(0,0)}} = (V_{B_{(0,1)}} - w_{B_{(0,0)}}) \frac{x^r_{B_{(0,0)}}}{x^r_{A_{(0,0)}} + x^r_{B_{(0,0)}}}.$$

# 3 Working Results (verify as a group)