PROBLEM SOLVING SESSIONS

All varieties are assumed to be normal.

1. Line bundles and linear series

Exercise 1.1. *Let A and B be two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. Show that*

- *$A + B$ is ample when A is ample and B is nef,*
- *$A + B$ is nef when both A and B are nef,*
- *$A + B$ is base point free when A and B are base point free.*

For a Cartier divisor D on a projective variety X, we define $\kappa(D)$ to be the dimension of $\phi_{|mD|}(X)$ for divisible enough $m \gg 0$, where $\phi_{|mD|}$ is the map associated to $|mD|$. One can show that $\kappa(D) = r$ if and only if $h^0(X, \mathcal{O}_X(mD)) \sim Cm^r$ for some $C > 0$. We say that D is big if $\kappa(D) = \dim X$.

Exercise 1.2. *Let L be a nef Cartier divisor on a (smooth) projective surface S such that $L^2 > 0$. Show that L is big. Hint: show that $H^2(X, mL) = 0$ for $m \gg 0$ and use the Riemann-Roch formula.*

Exercise 1.3. *Let X be a projective variety and let L be any Cartier divisor. Show that $\kappa(L) \leqslant \dim X$. Hint: find an ample divisor A such that $A - L$ is effective, and show the statement for A by looking at its Hilbert polynomial.*

Exercise 1.4. *Let X be a projective variety and let L be a Cartier divisor on X.*

- *Assume that L is big and let F be any effective divisor. Show that $H^0(X, mL - F) \neq 0$ for some $m \gg 0$. Hint: consider the short exact sequence*
$$0 \to \mathcal{O}_X(mL - F) \to \mathcal{O}_X(mL) \to \mathcal{O}_F(mL|_F) \to 0$$
.
- *Show that L is big if and only if $L \sim_{\mathbb{Q}} A + E$ for some ample $\mathbb{Q}$-divisor A and effective $\mathbb{Q}$-divisor E.*
- *Show that $L + N$ is big for any nef $\mathbb{Q}$-divisor N when L is big.*

Exercise 1.5. *Let α be any real number and let L be an ample Cartier divisor on a projective variety X of dimension d such that $L^d > \alpha$. Show that for any point $x \in X$ there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ such that $\operatorname{mult}_x D > \alpha$. Hint: Consider the exact sequence*
$$0 \to \mathcal{O}_X(mL) \otimes m_x^k \to \mathcal{O}_X(mL) \to \mathcal{O}_X/m_x^k \to 0$$
.

Exercise 1.6. *Let X be a smooth projective surface, and let L be an ample Cartier divisor on X. Show that if $K_X + L$ is ample and the base locus of $K_X + L$ is of dimension zero, then $3(K_X + L)$ is base point free (*).*

2. Singularities of the MMP

Exercise 2.1. *Let $C \subseteq \mathbb{P}^n$ be a smooth projective curve embedded into some $\mathbb{P}^n$ and let $S \subseteq \mathbb{A}^{n+1}$ be its cone. Show that*

- *S is klt if $g(C) = 0$,*
- *S is lc but not klt if $g(C) = 1$,*
- *S is not lc if $g(C) > 1$.*

Exercise 2.2. *Let (X, Δ) be a klt pair and let A be a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Show that there exists an effective $\mathbb{Q}$-divisor $E \sim_{\mathbb{Q}} A$ such that $(X, \Delta + E)$ is klt.*

Exercise 2.3. *Show that $\operatorname{lct}(\mathbb{P}^2; C)$ is equal to 1 when C is a nodal rational curve, and $\frac{5}{6}$ when C is a rational cuspidal curve.*

Exercise 2.4. *Let X be a (non-necessarily) smooth toric variety, and let D be the union of all toric divisors. Show that (X, D) is log canonical.*

Exercise 2.5. *Let $f : X \to Y$ be a projective fibration.*

- *Assume that X is ϵ-klt. Show that a general fibre of f is ϵ-klt as well.*
- *Using the canonical bundle formula, show that if X is klt and $\rho(X/Y) = 1$, then Y is klt.*

3. MMP

Exercise 3.1. *Let X be a klt projective variety such that K_X is big and nef. Using the base point free theorem show that K_X is semiample.*

Exercise 3.2. *Let X be a smooth projective surface.*

- *Let C be a curve such that $C^2 < 0$ and $K_X \cdot C < 0$. Show that C is an exceptional curve (i.e. $C \simeq \mathbb{P}^1$ and $C^2 = -1$).*
- *Assume that X is not uniruled (covered by rational curves). Using the cone theorem and the contraction theorem show that K_X is nef if and only if there are no exceptional curves on X.*

Exercise 3.3. *Let $f : X \to Y$ be a morphism from a $\mathbb{Q}$-Gorenstein variety X such that $\operatorname{codim} \operatorname{Exc} f \geqslant 2$ and $-K_X$ is f-ample. Show that K_Y is not $\mathbb{Q}$-Cartier.*

Exercise 3.4. *Let (X, B) be a log Fano variety. Show that you can run the MMP for any divisor D.*

Exercise 3.5. *Let X be a normal projective variety.*

- *Let $f : X \to Y$ be a birational morphisms. Show that $H^0(X, f^*D + E) \simeq H^0(Y, D)$ for any Cartier divisor D on Y and an effective Cartier divisor E on X.*
- *Let $f : X \dashrightarrow Y$ be a K_X-negative divisorial contraction or a flip between $\mathbb{Q}$-Gorenstein varieties. Show that $H^0(X, mK_X) \simeq H^0(Y, mK_Y)$ for divisible enough $m \in \mathbb{N}$.*

Assume that X is terminal and the full MMP and the abundance conjecture (i.e. K_X nef implies semiample) holds. Show that X is either uniruled or $\kappa(X) \geqslant 0$.

Exercise 3.6. *Let X and Y be two birational terminal projective varieties such that both K_X and K_Y are nef (*).*

- *Show that X and Y are isomorphic in codimension one.*
- *Let A be an ample $\mathbb{Q}$-Cartier divisor on Y, and let A_X be its strict transform on X. Show that if $K_X + A_X$ is nef, then $X \simeq Y$.*
- *Show that X and Y are connected by a sequence of log flips.*
- *Show that X and Y are connected by a sequence of flops (**).*

We say that $f \colon X \dashrightarrow Y$ is a log flip*, if there exists a klt pair (X, Δ) s.t. f is a $(K_X + \Delta)$-flip. We call f a* flop *if it is a K_X-crepant log flip.*

4. Boundedness and effective results

Exercise 4.1. *Show that polarised varieties (X, L) with L very ample and the Hilbert polynomial $\chi(tL)$ fixed, belong to a bounded family.*

Exercise 4.2. *Let n be a natural number. Show (using the boundedness of Fano varieties) that there exists $b \in \mathbb{N}$ depending only on n such that $h^0(X, -K_X) \leqslant b$ for any smooth projective Fano variety X of dimension n.*

Exercise 4.3. *Let $\mathcal{X} \to T$ be a bounded family of klt varieties. Show that there exists $\epsilon > 0$ such that every element of the family is ϵ-klt.*

Exercise 4.4. *Let m be a natural number.*

- *Using the BAB theorem, show that klt del Pezzo surfaces X with mK_X Cartier are bounded.*
- *Show the same statement without using the BAB theorem (**). Hint: show that the volume is bounded and then use Kollár's effective base point freeness.*

Exercise 4.5. *Let L be a Cartier divisor such that $L - K_X$ is ample. Show that $H^0(X, iL) \neq 0$ for some $1 \leqslant i \leqslant \dim X + 1$. (*)*

Exercise 4.6. *Fix a natural number n and $\epsilon > 0$. Let (X, Δ) be an ϵ-lc n-dimensional pair, let Δ be big, and let $K_X + \Delta \sim_{\mathbb{Q}} 0$. Using the general version of BAB conjecture and the boundedness of lc-thresholds, show that there exists $0 < \delta = \delta(n, \epsilon)$ such that if (X, Φ) is a log pair with $\Phi > (1-\delta)\Delta$ and $K_X + \Phi \sim_{\mathbb{Q}} 0$, then (X, Φ) is klt (*).*