Boundedness of fibered K-trivial varieties

Roberto Svaldi
with Philip Engel, Stefano Filipazzi, Francois Greer, & Mirko Mauri

Università degli Studi di Milano

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Introduction: Boundedness

Definition

A collection $\{X_i\}_{i \in I}$ of projective varieties is *bounded* if there exists a family of projective varieties $\mathcal{X} \to \mathcal{T}$, over a quasiprojective base $\mathcal{T}$, such that for all $i \in I$, there exists $t_i \in \mathcal{T}$ for which $X_i \simeq \mathcal{X}_{t_i}$.

In particular, one central property here is the fact that T can only finitely many irreducible/connected components.

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Variants

1. Birational boundedness
2. Bounded in codimension one
3. Boundedness of pairs
4. Finiteness of deformation classes
5. Topological boundedness

Introduction: Boundedness

Examples of bounded collections

1. Subschemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial
2. Smooth Fano varieties of a fixed dimension
3. K3 and abelian surfaces: Algebraically unbounded, but finite number of deformation classes (moduli spaces $\mathcal{F}_{2d}$ for any degree $2d > 0$)

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10 second summary: Minimal Model Program

The MMP provides an algorithmic that aims to build all birational equivalence classes of algebraic varieties using 3 fundamental classes:

- Fano varieties,
- K-trivial varieties,
- canonical models.

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Question

Under what conditions the fundamental classes of projective varieties are bounded?

Boundedness in 2 out 3 cases

Theorem (Hacon-M^{c}Kernan-Xu)

Fix $d \in \mathbb{Z}_{>0}, v \in \mathbb{Q}_{>0}$. The collection of varieties X such that

1. $\dim X = d$ *and X is semi-log-canonical,*
2. K_X *is ample and* $K_X^d = v$

is bounded

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Theorem (Birkar's solution to BAB)

Fix $d \in \mathbb{Z}_{>0}, \epsilon \in \mathbb{R}_{>0}$. The collection of varieties X such that

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A *K-trivial variety* is a normal projective variety X with canonical singularities such that $K_X \sim 0$. We say that X is

(CY) *(irreducible) Calabi-Yau* if $H^0(X, \Omega^{[k]}) = 0$ for all $0 < k < \dim X \geq 3$ and the same holds for all quasi-étale covers of X;

(PS) *(primitive) symplectic* if $H^0(X, \Omega^{[1]}) = 0$, $H^0(X, \Omega^{[2]}) = \mathbb{C}\sigma$ and σ is symplectic on the smooth locus of X;

(AV) *abelian* if $H^0(X, \Omega^{[1]}) = \dim X$.

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Theorem (Beauville-Bogomolov decomposition (Greb-Kebekus-Peternell, Druel, Höring-Peternell, . . .))

Every variety with numerically trivial canonical bundle and klt singularities admits a quasi-étale cover which is a product of CY varieties, symplectic varieties, and an abelian variety.

Introduction: K-trivial varieties

Examples

- Smooth hypersurfaces of degree $d+1$ in $\mathbb{P}^d$, $d > 2$, are Calabi-Yau.
- IHS: $\mathrm{K3}^{[n]}$, $\mathrm{Kum}^n(A)$, OG6, OG10.
- Enriques, Log Enriques.

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Example (Alexeev)

Let G be a finite group acting on a lattice L and suppose $L_{\mathbb{C}}$ is an irreducible representation of G. For any abelian surface A,

$$X := G \backslash L \otimes A$$

is symplectic; though X has a quasi-étale cover by $L \otimes A \simeq A^{\oplus \mathrm{rk}\, L}$.

Introduction: Motivating Question

Today's motivating question [Reid, Yau, . . .]

Is the collection of Calabi-Yau varieties in a fixed dimension bounded? Are K-trivial varieties topologically bounded, i.e., do K-trivial varieties have a finite number of deformation types?

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Is the collection of Calabi-Yau varieties in a fixed dimension bounded? Are K-trivial varieties topologically bounded, i.e., do K-trivial varieties have a finite number of deformation types?

This is a famous and very difficult problem; with few techniques to approach the general question. Hence, for today we'll focus on a special class of K-trivial varieties.

Definition

A *fibration* $f : X \to Y$ is a surjective, proper morphism of normal varieties with connected fibers and $0 < \dim Y < \dim X$.

If $K_X \sim 0$, by adjunction, the general fiber of f is also K-trivial.

Main Results

Theorem A (Engel-Filipazzi-Greer-Mauri-S.)

Irreducible Calabi-Yau varieties X of fixed dimension, fibered in either

1. *abelian varieties, or*
2. *symplectic varieties of a fixed deformation class*

are birationally bounded.

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Theorem B (EFGMS)

Lagrangian fibrations, of fixed dimension, lie in finitely many deformation classes of symplectic variety.

Main results: Some consequences

Conjecture (HyperKähler SYZ/Generalized Abundance)

If X is a symplectic variety, and $L \to X$ is a nef line bundle with $L^{2d} = 0$, then L is semiample.

Corollary (After Theorem B)

If the above holds, there are only finitely many deformation classes of primitive symplectic variety X, of a fixed dimension $2d$, with $b_2(X) \geq 5$.

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Moral of the story

That abelian varieties deform to each other is classical.
The corollary provides strong evidence that the symplectic factors in the BB decomposition only come in finitely many flavors.
The boundedness (or lack thereof) of CY factors is still rather mysterious.

Proof: Key steps

We consider fibered Calabi–Yau varieties $f\colon X \to Y$ in a fixed dimension $d > 2$:

0 Canonical bundle formula: there exists a special (generalized) log Calabi–Yau structure $(Y, B, \mathbf{M})$ on the base.

1 Bound the integer $c > 0$ for which $c\mathbf{M}$ is (b-)free.

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4 Bound the *Tate–Shafarevich group*—birational classes of abelian fibrations $f\colon X \to Y$ with a specified Albanese f^{Alb} (and B divisor).

This strategy was devised by Gross, and Dolgachev-Gross in the early 1990's to prove the birationally boundedness elliptic CY 3-folds.

Canonical bundle formula

Let $f : X \to Y$ be a fibration in K-trivial varieties ($K_X \sim_f 0$). The base admits the structure of a *generalized pair* $(Y, B, \mathbf{M})$:

1. An effective $\mathbb{Q}$-divisor (the *boundary divisor*)

$$B := \sum_{\substack{P \subset Y \text{ prime} \\ \text{divisors}}} a_P P$$

   measuring singularities of X over the codimension 1 points of Y;

2. The *moduli divisor* (or *Hodge bundle*)

$$\mathbf{M} := c_1(\mathcal{F}^g),$$

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$$\boxed{K_X \sim f^*(K_Y + B + \mathbf{M})}$$

Canonical bundle formula: Example

Let $f : X \to Y$ be a minimal elliptic surface. Then

$$K_X \sim f^*(K_Y + B + j^*\mathcal{O}(\tfrac{1}{12}))$$

where $j : Y \to \mathbb{P}^1$ is the j-invariant, and $B = \sum a_P P$ and $a_P \in [0,1)$ depends on the Kodaira classification of the fiber:

$f^{-1}(P)$	$I_n(m)$	II	III	IV	I_n^*	II^*	III^*	IV^*
a_P	$\frac{m-1}{m}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$

Proof: Step 1 (effective *b*-semiampleness)

Let $f : X \to Y$ be a fibration in K-trivial varieties. Then $\mathbf{M}$ is b-nef, and it is b-semiample – the famous "b-semiampleness conjecture" of Prokhorov and Shokurov, recently solved by Bakker-Filipazzi-Mauri-Tsimerman.

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For abelian and symplectic varieties, the moduli space $\Gamma\backslash\mathbb{D}$ is locally Hermitian symmetric, and the moduli divisor is the Hodge bundle λ. A (projective) Baily-Borel compactification $\overline{\Gamma\backslash\mathbb{D}}$ (on which λ is ample) is known to exist, by the Baily-Borel theorem.

Proof: Step 1 (effective *b*-semiampleness)

Given an abelian (or symplectic) fibration $f : X \to Y$, there is a *period map*

$$\Phi : Y \dashrightarrow \overline{\Gamma \backslash \mathbb{D}}.$$

Thus, $c\mathbf{M}$, $c > 0$ is (b-)free once the universal moduli divisor $c\lambda$ is free.

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Issue

: moduli spaces of abelian g-folds are not finite in number. For all sequences $\mathbf{d}$ of integers $d_1 \mid \cdots \mid d_g$ we have a DM stack $\mathcal{A}_{g,\mathbf{d}}$ of $\mathbf{d}$-polarized abelian g-folds.

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Question

Why is c uniform for all $\mathbf{d}$?

Proof: Completion of Step 1 (effective b-semiampleness)

Answer

The "Zarhin trick." All the Baily-Borel compactifications map into a single moduli space

$$\overline{A}_{g,\mathbf{d}} \to \overline{A}_{8g}$$

of PPAVs, and the Hodge bundle λ_{8g} pulls back to $8\lambda_g$ on each $\overline{A}_{g,\mathbf{d}}$ (critically, independent of $\mathbf{d}$).

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Proof.

The Zarhin trick sends $Zar : A \mapsto A^{\oplus 4} \oplus (A^*)^{\oplus 4}$ and so
$$H^{8g,0}(Zar(A)) \simeq H^{g,0}(A)^{\otimes 8}.$$
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(For families of symplectic varieties, apply Kuga-Satake, then Zarhin.)

Proof: Step 2 (bounding the base)

STEP 1 COMPLETE! (We bounded c for which $c\mathbf{M}$ is free.)

Theorem (Birkar–Di Cerbo–S.)

The bases $(Y, B, \mathbf{M})$ *are bounded in codimension* 1*, if* Y *is rationally connected.*

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The bases are RC when X is CY or primitive symplectic.
So there is a finite type family

$$(\mathcal{Y}, \mathcal{B}, \mathcal{M}) \to \mathcal{T}$$

which contain all possible bases (up to small modification) of an abelian fibration $f : X \to Y$ of a CY/symplectic variety.

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STEP 2 COMPLETE! The base $(Y, B, \mathbf{M})$ is bounded (in codim 1).

Proof: Step 3 (bounding the Albanese fibration)

Next goal (crucial): Bound the polarization type $\mathbf{d}$ of the fibers, and the classifying morphism

$$\begin{aligned} \Phi : Y &\dashrightarrow \mathcal{A}_{g,\mathbf{d}} \\ y &\mapsto \mathrm{Aut}^0(X_y) \end{aligned}$$

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X is birational to X^{Alb} iff X admits a rational section.

Proof: Step 3, bounding the Albanese fibration

Again use Zarhin: By a volume argument, rational maps

$$Zar \circ \Phi : Y \dashrightarrow \overline{A}_{8g}$$

for which $(Zar \circ \Phi)^*(\lambda_{8g}) \equiv \mathbf{M}$ are bounded. So the space of all possible "Zarhin-tricked" period maps $(Y, Zar \circ \Phi)$ is bounded.

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Key Question

Can we undo the Zarhin trick and in turn bound the original period map $\Phi : Y \dashrightarrow \overline{A}_{g,\mathbf{d}}$?

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Short answer: *In general, no!*

Proof: Step 3 (bounding the Albanese fibration)

For all $A \in \mathcal{A}_{g,\mathbf{d}}$ we have a (very stupid) abelian fibration $A \to *$. After the Zarhin trick, the period maps $* \to \overline{A}_{8g}$ lie in a bounded family. But the original maps are not bounded, since $\mathbf{d}$ is arbitrary.

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Lemma

If $H^2(X, \mathcal{O}) = 0$ or $f : X \to Y$ is Lagrangian, the Zar $\circ\, \Phi$ pullback of the universal $\mathbb{Z}^{16g}$-local system on $\mathcal{A}_{8g}$ to the smooth locus $Y^o \subset Y$ of f recovers a polarization type $\mathbf{d}$.

Proof.

Undo Zarhin trick on level of local systems. Then, Deligne's theorem of the fixed part shows: Monodromy-invariant classes in H^2 of the fibers are necessarily of $(1,1)$-type. □

Proof: Step 3 (bounding the Albanese fibration)

Thus, we have bounded the polarization type $\mathbf{d}$ of a general fiber of $f : X \to Y$. Repeating the earlier volume argument, we bound the classifying morphism

$$\Phi^o : Y^o \to \mathcal{A}_{g,\mathbf{d}}$$

and in turn the birational class of the Albanese $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$. **STEP 3 COMPLETE!**

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(The role of the boundary divisor B is very subtle—it is critical to make a distinction between the period map to the coarse space $Y \dashrightarrow A_{g,\mathbf{d}}$ and the classifying map to the DM stack $Y^o \to \mathcal{A}_{g,\mathbf{d}}$. Lifts from the coarse space to the stack are controlled via $\operatorname{supp} B$.)

Proof: Step 4 (bounding the Tate–Shafarevich group)

This step is rather technical ($>$ 50 pages). Question: How to bound $f : X \to Y$ from the data of $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$?

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Key Tools

Kulikov models, semiabelian group schemes, étale-analytic comparisons, etc.

Definition

A *Kulikov model* is a semistable, relatively K-trivial morphism.

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Definition

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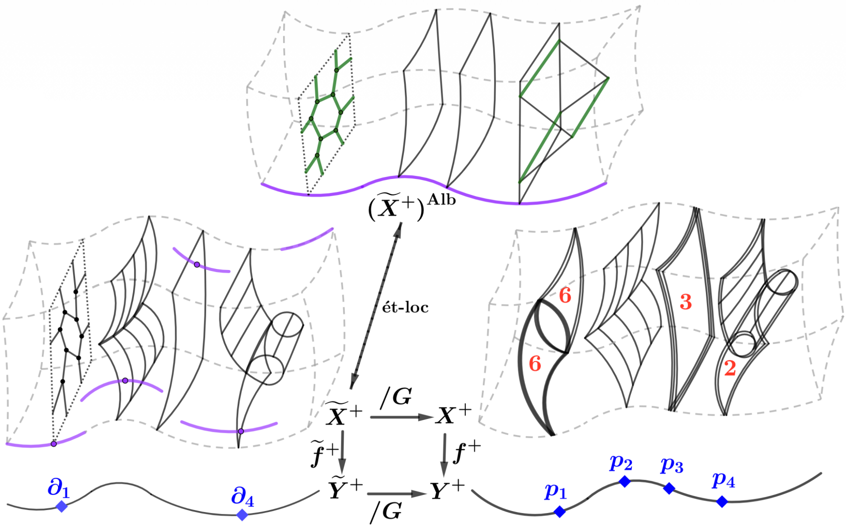
Question

In what context do they exist?

Answer

Unclear, but OK after a finite G-Galois base change $\widetilde{Y}^+ \xrightarrow{/G} Y^+$ of a **big** open subset $Y^+ \subset Y$.

Proof: Step 4 (bounding the Tate–Shafarevich group)



The sections of $(\widetilde{f}^+)^{\mathrm{Alb}}$ form a semiabelian group scheme $\mathcal{P} \to \widetilde{Y}^+$.

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Two important exact sequences: The component sequence

$$0 \to \mathcal{P}^0 \to \mathcal{P} \to \mu \to 0$$

where $\mu \to \widetilde{Y}^+$ is the relative component group of the Kulikov model, and the exponential exact sequence

$$0 \to \Gamma \to \mathfrak{p} \xrightarrow{\exp} \mathcal{P}^0 \to 0$$

associated to the sheaf $\mathfrak{p}$ of Lie algebras of $\mathcal{P}$. Note: Γ is a constructible sheaf of finitely generated $\mathbb{Z}$-modules.

Proof: Step 4 (bounding the Tate–Shafarevich group)

$$\begin{tikzcd}
X^+ \arrow[d, "f^+"] & \widetilde{X}^+ \arrow[l, "/G"'] \arrow[d, "\widetilde{f}^+"] \arrow[r, dashed, leftrightarrow, "\text{et-loc}"] & (\widetilde{X}^+)^{\mathrm{Alb}} \arrow[dl, "(\widetilde{f}^+)^{\mathrm{Alb}}"] \\
Y^+ & \widetilde{Y}^+ \arrow[l, "/G"] &
\end{tikzcd}$$

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Birational class of $f\colon X \to Y$ is encoded by an element in the G-equivariant sheaf cohomology group

$$t(f) \in Ш_{G,\,\text{ét}} := H^1_G(\widetilde{Y}^+, \mathcal{P}).$$

Thus, we are reduced to proving finiteness of $Ш_{G,\,\text{ét}}$.

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We must analyze the diagram

$$
\begin{tikzcd}
 & & H^1_{G,\,\text{ét}}(\widetilde{Y}^+, \mathcal{P}^0) \arrow[d, "\text{analytification}"] & \\
H^1_G(\widetilde{Y}^+, \Gamma) \arrow[r] & H^1_{G,\,\text{an}}(\widetilde{Y}^+, \mathfrak{p}) \arrow[r] & H^1_{G,\,\text{an}}(\widetilde{Y}^+, \mathcal{P}^0) \arrow[r] & H^2_G(\widetilde{Y}^+, \Gamma)
\end{tikzcd}
$$

coupled with a general theorem of Raynaud and basic group cohomology, that the upper group is torsion. [n.b. torsion ≠ finite, cf. $\mathbb{Q}/\mathbb{Z}$]

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$$\begin{tikzcd} & & H^1_{G,\mathrm{\acute{e}t}}(\widetilde{Y}^+,\mathcal{P}^0) \arrow[d, "\text{analytification}"] & \\ H^1_G(\widetilde{Y}^+,\Gamma) \arrow[r] & H^1_{G,\mathrm{an}}(\widetilde{Y}^+,\mathfrak{p}) \arrow[r] & H^1_{G,\mathrm{an}}(\widetilde{Y}^+,\mathcal{P}^0) \arrow[r] & H^2_G(\widetilde{Y}^+,\Gamma) \end{tikzcd}$$

coupled with a general theorem of Raynaud and basic group cohomology, that the upper group is torsion. [n.b. torsion $\neq$ finite, cf. $\mathbb{Q}/\mathbb{Z}$]

The image of $H^1_G(\widetilde{Y}^+,\Gamma_{\mathbb{C}}) \to H^1_{G,\mathrm{an}}(\widetilde{Y}^+,\mathfrak{p})$ is $H^2(X,\mathcal{O})/H^2(Y,\mathcal{O})$.
STEP 4 COMPLETE! (Tate–Shafarevich finiteness)

Proof: Step 4 (bounding the Tate–Shafarevich group)

Philosophical point (why it works): Once we control the Tate–Shafarevich twist on a big open set $Y^+ \subset Y$, we can apply Hartogs' type results to relate the analytic and étale worlds: $Ш_{G,\mathrm{ét}} = (Ш_{G,\mathrm{an}})_{\mathrm{tors}}$.

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This is far from true when $Y^+ \subset Y$ is not big.

Example

Consider an elliptic surface $S \to C$ and a logarithmic transform $S' \to C$ at some points $p_i \in C$. These are biholomorphic over $C \setminus \{p_i\}$ but not bimeromorphic over any neighborhood of p_i.

Birational type of the Albanese

Theorem (EFGMS)

If $f : X \to Y$ is an abelian fibration of a K-trivial variety, then so is $f^{\mathrm{Alb}} : X^{\mathrm{Alb}} \to Y$. Similarly, if f is Lagrangian fibration of a primitive symplectic variety, then so is f^{Alb}.

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Question

Can we construct new deformation classes of symplectic varieties, by passing to the Albanese of a Lagrangian fibration? For example: $OG10$ is the Albanese of a Lagrangian fibration on $K3^{[5]}$.

Thank you for your time!