FOLIATION EXERCISES

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Exercise 1. *Consider the foliation $\mathcal{F}$ on $\mathbb{P}^n$ given by a pencil of degree d hypersurfaces. What is the general log leaf of $\mathcal{F}$? What is $K_{\mathcal{F}}$?*

Exercise 2.
Let $\mathcal{F}$ be a del Pezzo foliation on a quadric $Q \subset \mathbb{P}^{n+1}$. Show that the general log leaf $(F, \Delta) = (Q', H)$ where $H \in \mathcal{O}_Q(1)$.

Exercise 3. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^{n+1}$ and let X be a smooth degree d hypersurface. Show that $\mathcal{F}$ induces a foliaton on X, call it $\mathcal{F}_X$.*

What is the relation between $K_{\mathcal{F}}$ and $K_{\mathcal{F}_X}$?

Hint: Consider the cases where X is $\mathcal{F}$-invariant and not $\mathcal{F}$-invariant separately.

Exercise 4. *Let $\mathcal{F}$ be an algebraically integrable foliation on $\mathbb{P}^{n+1}$. Let X be a general hypersurface. Is $\mathcal{F}_X$ algebraically integrable? What is the relation between the general log leaf of $\mathcal{F}$ and the general log leaf (if it exists) of $\mathcal{F}_X$?*

For the next few exercises we will make use of the following definition:

Definition 1. *Let $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ be a birational morphism between foliated varieties, this means that π is birational and $d\pi(\mathcal{F}') = \mathcal{F}$. Suppose that $\mathcal{F}$ is $\mathbb{Q}$-Gorenstein.*

We can write $K_{\mathcal{F}'} = \pi^*(K_{\mathcal{F}}) + \sum a_i E_i$.

The a_i's are the foliated discrepancies. We say that $\mathcal{F}$ is terminal if every discrepancy is > 0, we say that $\mathcal{F}$ is canonical if every discrepancy is ≥ 0.

Exercise 5. *Let $\mathcal{F}$ be a smooth foliation on a smooth variety X.*

1. *Blow up $\mathcal{F}$ at a closed point. What is the discrepancy of this blow up? Is the transformed foliation still smooth?*
2. *Can you think of an example of a smooth foliation which is not terminal? What about on a surface?*

Exercise 6. *Blow up the following vector fields at the origin and compute the discrepancy of the blow up:*

1. $x\partial_x + y\partial_y$
2. $x\partial_x - y\partial_y$
3. $x^5y^7\partial_x + (x+y)\partial_y$
4. $ax\partial_x + by\partial_y$, $a, b \in \mathbb{Z}$

In the last example, suppose a, b are positive. Is the foliation canonical?

Exercise 7. *Let X be a surface and $\mathcal{F}$ a rank 1 foliation on X.*

1. *Show that we have an exact sequence $0 \to \mathcal{F} \to T_X \to I_Z \cdot N_{\mathcal{F}} \to 0$ where I_Z is the ideal sheaf of $sing(\mathcal{F})$*
2. *If $K_{\mathcal{F}}$ is the canonical divisor and $N^*_{\mathcal{F}}$ is the co-normal divisor show that $K_X = K_{\mathcal{F}} + N^*_{\mathcal{F}}$.*

(3) *Let $X \to B$ be a smooth fibration over a curve, let $\mathcal{F}$ be the corresponding foliation. Let $X_1 \to X$ be the blow up at a point p, and let $\mathcal{F}_1$ be the foliation associated to $X_1 \to B$. What is the foliated discrepancy of this blow up? Show that $\mathcal{F}_1$ has exactly one singular point.*
(4) *Let $X_2 \to X_1$ be the blow up at the unique singular point of $\mathcal{F}_1$, and let $\mathcal{F}_2$ be the foliation associated to $X_2 \to B$. What is the discrepancy of this blow up?*
(5) *Compare $K_{\mathcal{F}_2}$ and $K_{X_2/B}$. Are they equal?*
(6) *Let $f : X \to B$ be any fibration between smooth surfaces and $\mathcal{F}$ be the corresponding foliation. What is $N^*_{\mathcal{F}}$? Keep in mind that $N^*_{\mathcal{F}}$ is saturated in Ω^1_X. Use this to deduce the general relation between $K_{X/B}$ and $K_{\mathcal{F}}$.*

Exercise 8. (1) *Suppose that $f : X \to B$ is a smooth fibration of a surface over a curve, let $\mathcal{F} = T_{X/B}$. Let A be an ample divisor. Suppose that $0 \le A' \sim_{\mathbb{Q}} A$ is general so that (X, A') is klt and that $K_{\mathcal{F}} + A$ is f-trivial. Then $K_{\mathcal{F}} + A = f^*J$ where J is nef. Hint: Kodaira's canonical bundle formula*
(2) *Let $\mathcal{F}$ be as above. Show that $\mathcal{F}$ cannot be Fano. Hint: Look at $A = -K_{\mathcal{F}} - \epsilon f^*P$.*
(3) *See if you can generalize this when X, B, f aren't necessarily smooth and in higher dimensions. What if there is a boundary? A good place to start might be by looking at a generalization of Kodaira's canonical bundle formula.*

Exercise 9. (1) *Show that there are no smooth foliations on $\mathbb{P}^2$.*
(2) *Let A be an abelian variety with $\rho(A) = 1$. Are there smooth foliations on A?*
(3) *Let X be a smooth surface of general type with $\rho(X) = 1$. Show that there are no smooth foliations on X. (Bogomolov Sommese Vanishing is handy here)*

The goal of the next few exercises is to decide when a Fano foliation exists on a hypersurface of degree d. (N.b., exercises 10 and 11 are just general statements about the cohomology of the exterior powers of the cotangent bundle of hypersurfaces in $\mathbb{P}^n$ and don't contain any explicitly foliated content. You might want to skip them here and think about them some other time.)

Exercise 10. *Bott's formulae: Let p, k, n be integers with $n \ge 1$.*
(1) *if $0 \le p \le n$ and $k > p$ then $h^0(\mathbb{P}^n, \Omega^p(k)) = \binom{k+n-p}{k}\binom{k-1}{p}$*
(2) *if $k = 0$ and $p = 0$ $h^0(\mathbb{P}^n, \Omega^p(k) = 1$*
(3) *for any other choice of p, q, n we have $h^0(\mathbb{P}^n, \Omega^p(k)) = 0$.*
(4) *$h^1(\mathbb{P}^n, \Omega^p(k)) = 0$*

For the next exercise it might help to assume the following: If X is a smooth hypersurface in $\mathbb{P}^{n+1}$ then
(1) $h^0(X, \Omega^r_X(s) = 0$ for $s < r \le n-1$,
(2) $h^1(X, \Omega^r_X(s)) = 0$ for $0 \le r \le n-2$ and $s \le r-2$.

Exercise 11. *Let $n \ge 3$ and let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \ge 3$. Let k, q be such that $k \le q \le n-2$ and $q \ge 1$.*
Then $H^0(X, \Omega^q_X(k)) = 0$.
Hint: We have an exact sequence of sheaves on X

$$0 \to \Omega^{q-1}_X(q-d) \to \Omega^q_{\mathbb{P}^{n+1}}(q)|_X \to \Omega^q_X(q) \to 0$$

Exercise 12. *(Proposition 4.7 in On Fano Foliations):*

Let $n \geq 3$ and let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 3$. Let r, ι be positive integers such that $2 \leq r \leq n-1$. Then there exists a Fano foliation of rank r and index ι on X if and only if $d + \iota \leq r + 1$.

Hint: The existence of such a foliation would give some section of $H^0(X, \Omega^a(b))$ for some a, b. In the converse direction, a section of $H^0(X, \Omega^a(b))$ defines a distribution, but not necessarily a foliation. Can you choose ω so that it is integrable?

Exercise 13. *Let $\mathcal{E}$ be an ample rank 2 vector bundle on $\mathbb{P}^\ell$ and let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^\ell$. Let $\mathcal{O}_X(1)$ be the tautological bundle.*

Let $\mathcal{F}$ be a rank r foliation with $-K_\mathcal{F} = \mathcal{O}_X(r-1)$.

1. *Show that $X \to \mathbb{P}^\ell$ is tangent to the foliation and hence $\mathcal{F}$ is pulled back from a foliation $\mathcal{G}$ on $\mathbb{P}^\ell$. Deduce from this that $r = 3$.*
2. *Is $\mathcal{G}$ a Fano foliation? What if $\mathcal{E}$ isn't ample?*
3. *Now suppose that $\mathcal{E}$ is rank 3 and that a general fibre of π is transverse to $\mathcal{F}$. Show that the rank of $\mathcal{F}$ is 2.*

See On Fano Foliations Proposition 7.10 and Theorem 9.6 for more on this theme.

Exercise 14. *Let X be a smooth variety and let $\mathcal{E}$ be a rank $r$vector bundle on X. Let $\mathbb{P} = \mathbb{P}_X(E)$*

1. *Suppose that $\mathcal{E}$ is equipped with a connection $\nabla : \mathcal{E} \to \Omega^1 \otimes \mathcal{E}$. Show that we can associate a distribution $\mathcal{D}(\nabla)$ on $\mathbb{P}$ to ∇. Hint: ∇ defines a splitting of $T_\mathbb{P}$.*
2. *If you know something about flatness of connections you might want to try and show that if ∇ is flat then $\mathcal{D}(\nabla)$ is a foliation. Otherwise just assume that ∇ is flat and that $\mathcal{D}(\nabla)$ is a foliation.*
3. *Show that ∇ defines a representation of $\pi_1(X)$ into PGL_r and that if this representation is finite then $\mathcal{D}(\nabla)$ is algebraically integrable.*

Exercise 15. *Basic facts on the Harder-Narasimhan filtration:*

1. *if $\mathcal{E}, \mathcal{F}$ are two semi-stable vector bundles with $\mu(\mathcal{E}) > \mu(\mathcal{F})$ then $Hom(\mathcal{E}, \mathcal{F}) = 0$.*
2. *if $\mathcal{E}, \mathcal{F}$ are two vector bundles with $\mu_{min}(\mathcal{E}) > \mu_{max}(\mathcal{F})$ then $Hom(\mathcal{E}, \mathcal{F}) = 0$.*
3. *Compute $\mu(\mathcal{E} \otimes \mathcal{F})$ and $\mu(\bigwedge^2 \mathcal{E})$ in terms of $\mu(\mathcal{E}), \mu(\mathcal{F})$.*

Exercise 16. *Let $\mathcal{E}$ be a vector bundle on a smooth curve C. Suppose $\mu_{min}(\mathcal{E}) > 0$. Show that $\mathcal{E}$ is ample.*

Hint: Maybe try the case where $\mathcal{E}$ is semi-stable first.

We recall the following theorem due to Bogomolov and McQuillan:

Theorem 0.1 (Bogomolov-McQuillan theorem)**.** *Let $(X, \mathcal{F})$ be a smooth foliated projective variety with $\mathcal{F}$ locally free near $C \subset X$ a smooth curve. Suppose that $\mathcal{F}|_C$ is ample. Then the general leaf through C is a rationally connected variety.*

Exercise 17. *Let $(X, \mathcal{F})$ be a smooth variety with a foliation (not necessarily smooth). Let H be an ample divisor and suppose that $K_\mathcal{F} \cdot H^{n-1} < 0$.*

Show that there is a foliation $\mathcal{G} \subset \mathcal{F}$ whose general leaf is a rationally connected variety.

Exercise 18. *Let X be a smooth variety with $\rho(X) = 1$. Suppose that $\mathcal{F}$ is a Fano foliation. Show that $\mathcal{F}$ isn't smooth.*

Exercise 19. *Let $(X, \mathcal{F})$ be a foliated smooth variety. Let $f : \mathbb{P}^1 \to X$ be a free rational curve on X transverse to $\mathcal{F}$, i.e., f^*T_X is a nef vector bundle. Let M be the component of $Mor(\mathbb{P}^1, X)$ containing $[f]$.*

Let $ev : \mathbb{P}^1 \times M \to X$ be the evaluation map and let $\pi : \mathbb{P}^1 \times M \to M$ be the projection.

Denote by $\mathcal{F}_{tang}$ the saturation of $\pi_* ev^* \mathcal{F}$ in T_M

Show that $\mathcal{F}_{tang}$ is a foliation. What is the moduli interpretation of this foliation?

Exercise 20. *The goal of this exercise is to look at the minimal model of a smooth foliation $\mathcal{F}$ on a smooth surface X.*

(1) *Show that if $K_{\mathcal{F}}$ is not psef then X is a $\mathbb{P}^1$ bundle over a curve B and $\mathcal{F} = T_{X/B}$.*
(2) *Let C be an irreducible curve transverse to $\mathcal{F}$. Show that $(K_{\mathcal{F}} + C) \cdot C \geq 0$. Conclude that if $K_{\mathcal{F}} \cdot C < 0$ that $K_{\mathcal{F}}$ isn't psef*
(3) *Show that if $K_{\mathcal{F}}$ is psef and $K_{\mathcal{F}} \cdot C < 0$ then C is a rational curve tangent to the foliation.*
(4) *Let L be a leaf of a rank 1 foliation on a surface. Show that the foliation gives rise to a connection on $N_{L/X}$.*
(5) *Conclude that either $\mathcal{F}$ is a $\mathbb{P}^1$ fibration or $K_{\mathcal{F}}$ is nef. Hint: Having a connection on a line bundle (especially a line bundle on $\mathbb{P}^1$) is a strong condition!*