

BOUNDEDNESS QUESTIONS FOR CALABI-YAU'S

- ① GENTLE INTRO TO THE MMP
- ② BOUNDEDNESS
- ③ - ④ BOUNDEDNESS for CY's (elliptic)
+ log CY pairs

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY
IS TO PRODUCE A COMPLETE CLASSIFICATION OF
PROJECTIVE VARIETIES.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY IS TO PRODUCE A COMPLETE CLASSIFICATION OF PROJECTIVE VARIETIES. TO THIS END, WE CAN PARTITION VARIETIES INTO EQUIVALENCE CLASSES USING 2 POSSIBLE EQUIVALENCE RELATIONS :

① ISOMORPHISM : $X \xrightleftharpoons[f]{f^{-1}} Y$

② BIRATIONAL : $X \xrightleftharpoons[g]{g^{-1}} Y$
 ISOMORPHISM g, g^{-1} are rational maps -

$\mathbb{C}(X) =$ field of rat'l functions on X

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY / \mathbb{C}

CANONICAL BUNDLE: $\det(\Omega_X^1)$

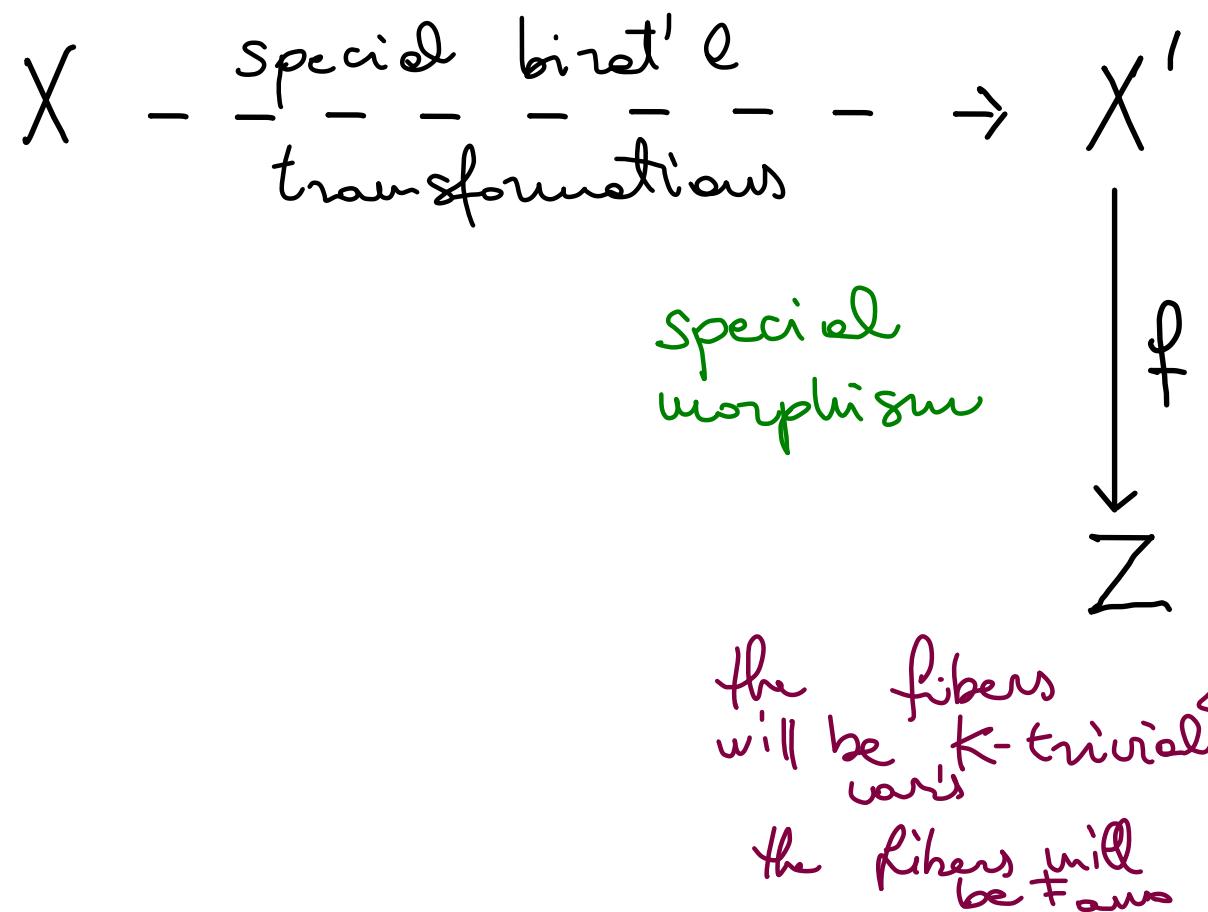
CANONICAL DIVISOR: Any Weil divisor K_X s.t.
 $\mathcal{O}_X(K_X) \cong$ canonical bundle.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY / \mathbb{C}

CONJECTURE

STARTING FROM X , THERE EXISTS AN
ALGORITHMIC WAY OF CONSTRUCTING A
DIAGRAM OF THE FOLLOWING FORM



SUCH THAT f SATISFIES
1 OF THE FOLLOWING:

- ① f is BIRATIONAL
 K_2 is ample Z is a can model.
- ② f is a FIBRATION & $K_{X'} = f^* H$ ample on Z
- ③ f is a FIBR. & $-K_{X'}|_{f^{-1}(F)}$ is ample on $f^{-1}(F)$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

HENCE IN THE BIRATIONAL CLASSIFICATION
OF ALGEBRAIC VARIETIES WE HAVE 3
IMPORTANT CLASSES OF VARIETIES THAT
ACT AS BUILDING BLOCKS

- ① CANONICAL MODELS (K is ample)
- ② K -TRIVIAL VAR'S ($K \equiv 0$)
- ③ FANO VAR'S ($-K$ is ample)

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THE 2nd PART OF THE CLASSIFICATION AIMS
THEN TO CLASSIFY THOSE VARIETIES THAT BELONG
TO ONE OF THE 3 BUILDING BLOCKS

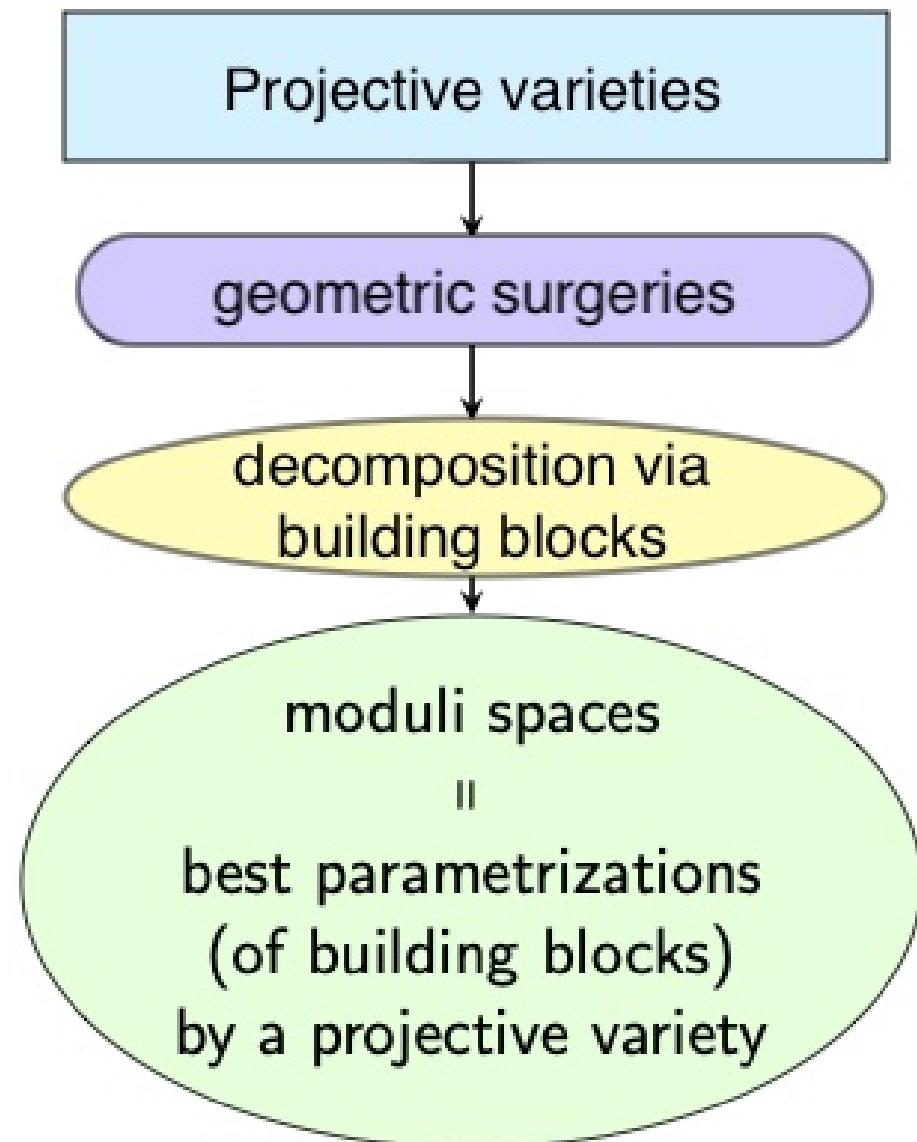
① CANONICAL MODELS (K IS AMPLE)

② K -TRIVIAL VAR'S ($K \equiv 0$)

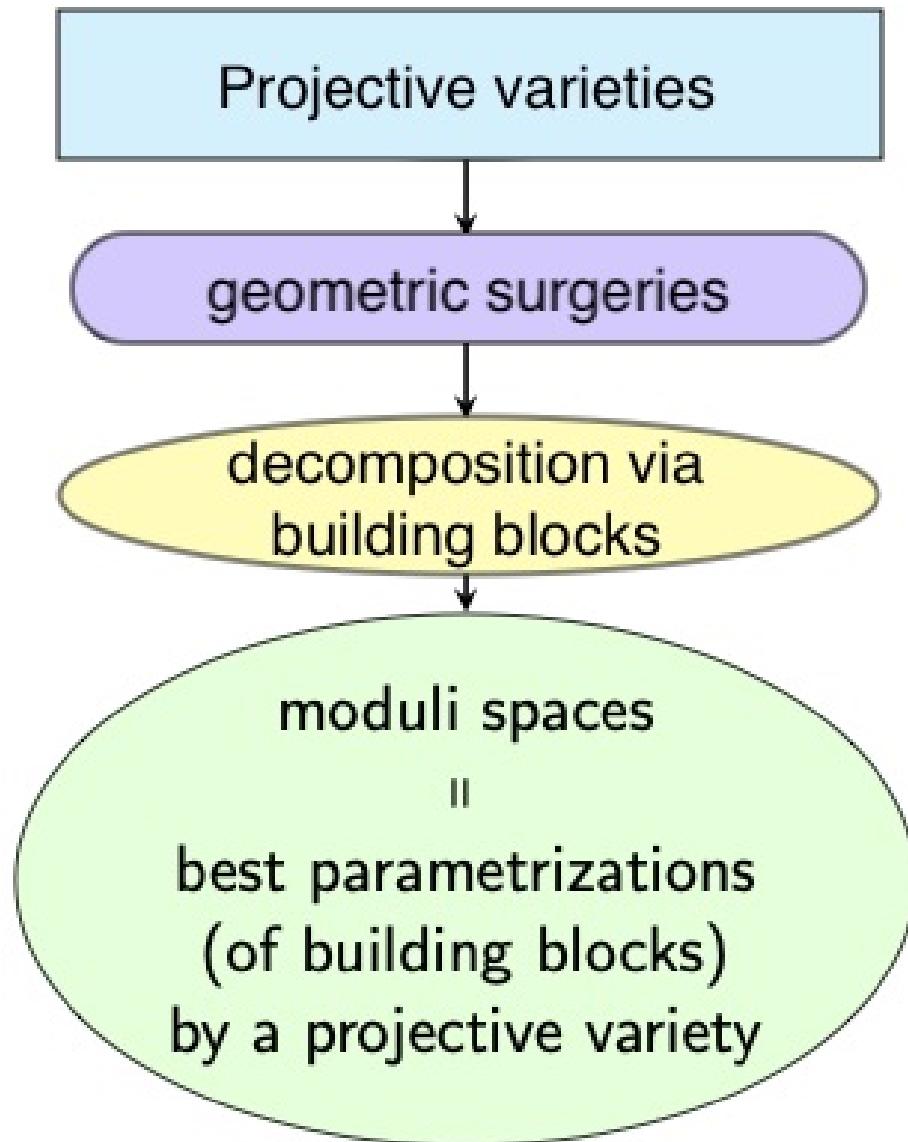
③ FANO VAR'S (- K IS AMPLE)

MODULI SPACES :

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION



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IN ORDER TO COMPLETE SUCH CLASSIFICATION SCHEME, WE NEED ALSO TO WORK WITH SINGULAR VARIETIES.

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X NORMAL PROJECTIVE VARIETY / \mathbb{C}

CANONICAL BUNDLE: $i_* \det(\Omega^1_{X^{sm}})$ rank 1 - sheaf

$$i: X^{sm} \hookrightarrow X$$

CANONICAL DIVISOR: Zer. closure of any

$$K_{X^{sm}}$$

K_X will be a Weil divisor
(not. nec. Cartier)

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION

- ① A PAIR IS THE DATUM OF :
- (i) X a normal variety
 - (ii) B an effective Weil divisor with coefficients $\in (0, 1]$
- ② A LOG PAIR IS A PAIR (X, B) SUCH THAT
 $K_X + B$ IS \mathbb{R} -CARTIER
(can be written as a \mathbb{R} -sum)
of Cartier divisors

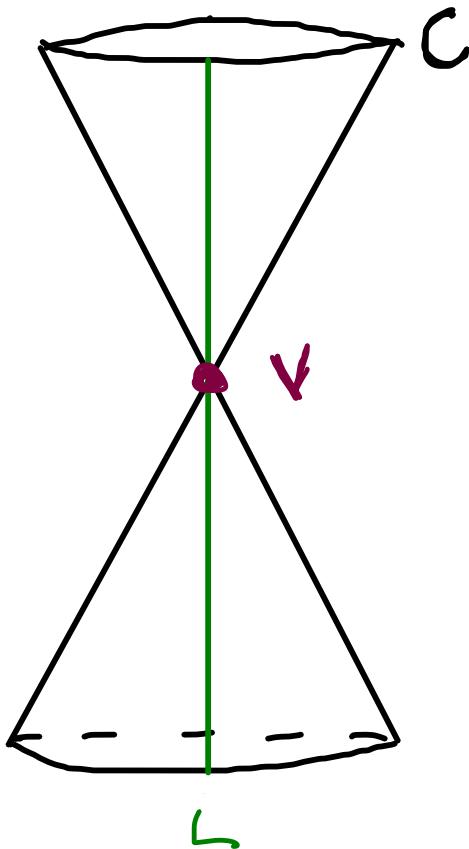
MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

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CONSIDER FOR EXAMPLE



$$V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^3$$

C is SING. at V

L is a Weil divisor

L is not a Cartier divisor at V.

2L is a Cartier divisor,

ADJ. FORM:

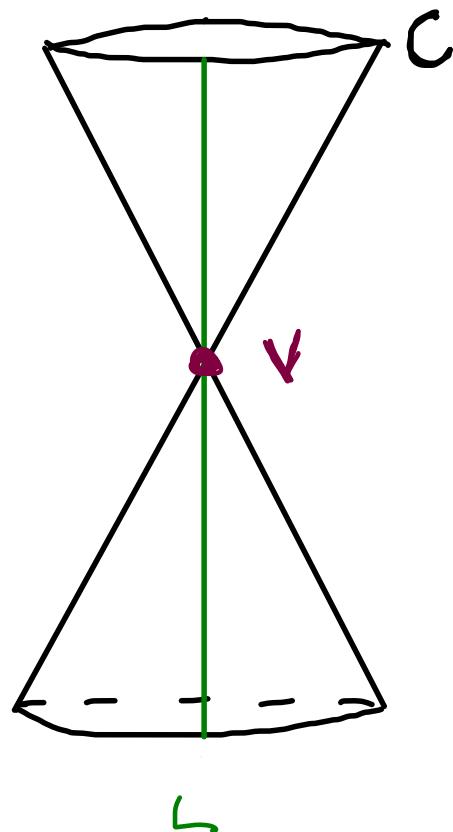
X SMOOTH
 D SMOOTH
codim 1

$$(K_X + D)|_D = K_D$$

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CONSIDER FOR EXAMPLE



$$V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^3$$

ADJ. FORM:

$X_{D'|}$ SMOOTH
 D' SMOOTH
codim 1

$$(K_X + D)|_D = K_D$$

$$(K_{C \cup L} + L|_{C \cup L})|_L = K_{L|_{C \cup L}}$$

$$(K_C + L) \cdot L = -\frac{3}{2} \longrightarrow (K_C + L)|_L = K_L + \frac{1}{2} \vee$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

CONSIDER A FIBRATION IN K-TRIVIAL VARIETIES

$$f : Y \longrightarrow X$$

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WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.
TO DO THAT, WE NEED TO MEASURE THEIR SINGULARITIES.
WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

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DEFINITION LET (X, B) BE A PAIR. A BIRAT'L MORPHISM

$$\pi: X' \longrightarrow X$$

IS A LOG RESOLUTION WHEN :

① π IS A PROPER RESOLUTION OF X

② $\tilde{B} + \text{Exc}(\pi)$ is

strict
transform
of B on X'

SIMPLE
NORMAL
CROSSING

$\forall x' \in X$
locally analytically
 $(X', (\tilde{B} + \text{Exc}(\pi)))$
 $(\mathbb{C}^n, \sum_{i=1}^k \{z_i = 0\})$

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IS A LOG RESOLUTION WHEN :

- ① X' IS SMOOTH;
- ② $(X', \tilde{B} + \text{Exc}(\pi))$ IS SNC.

THEOREM [HIRONAKA] LOG RESOLUTION EXIST IN CHAR 0.

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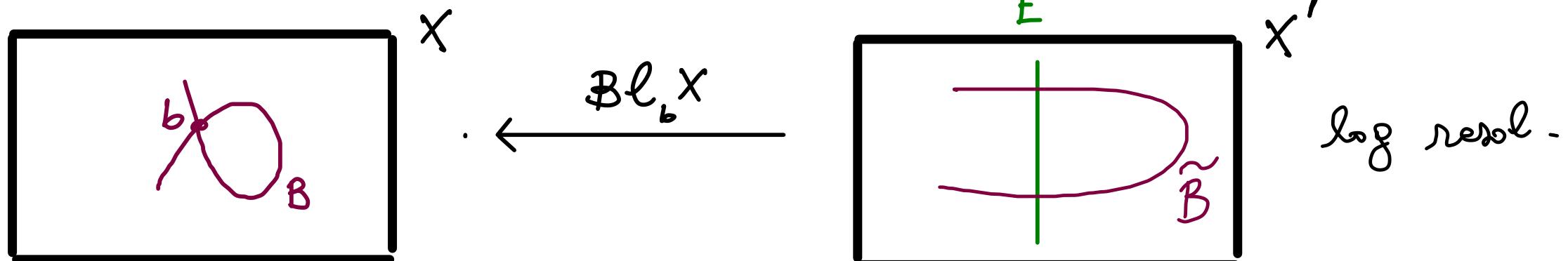
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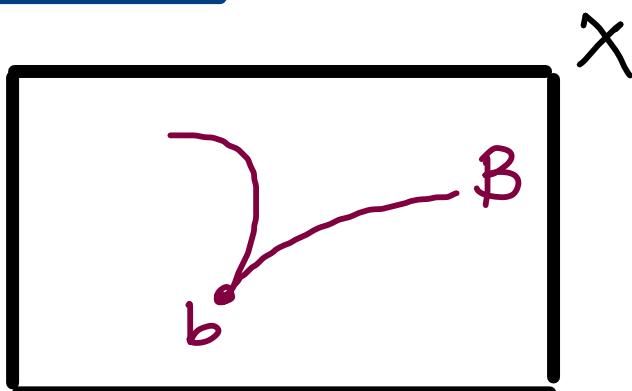
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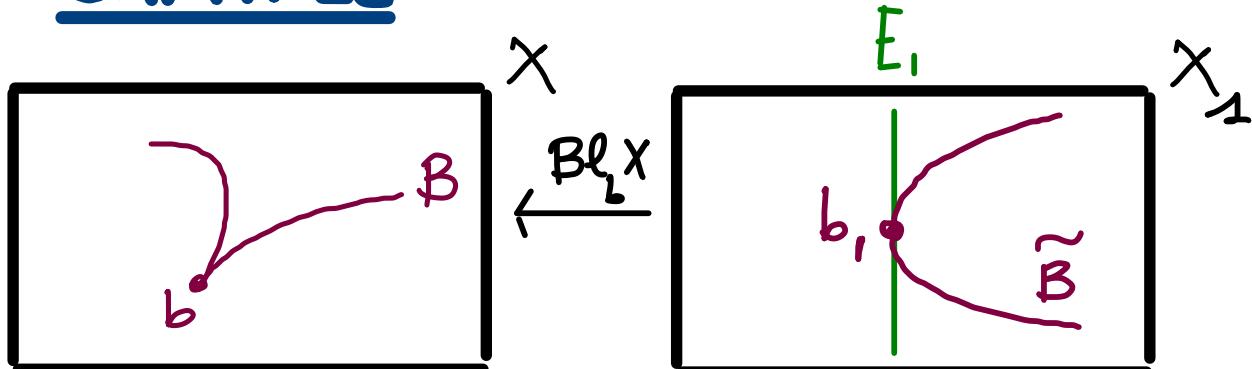
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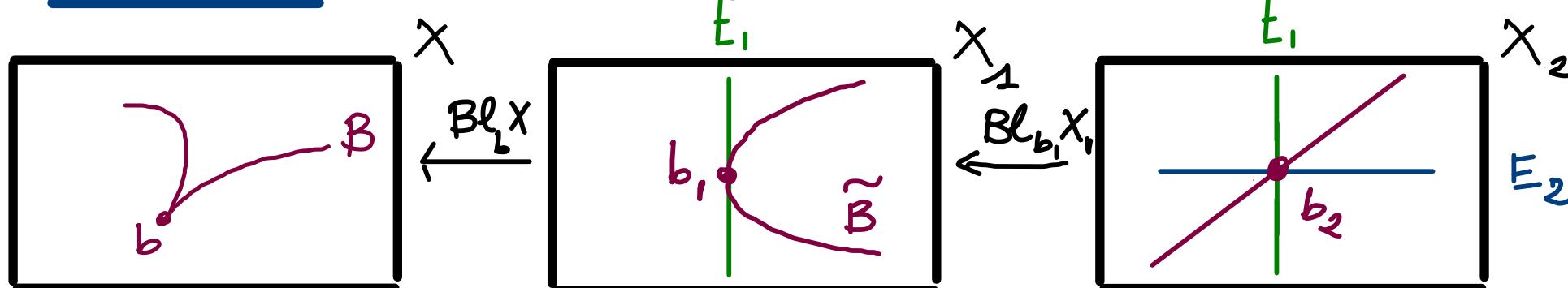
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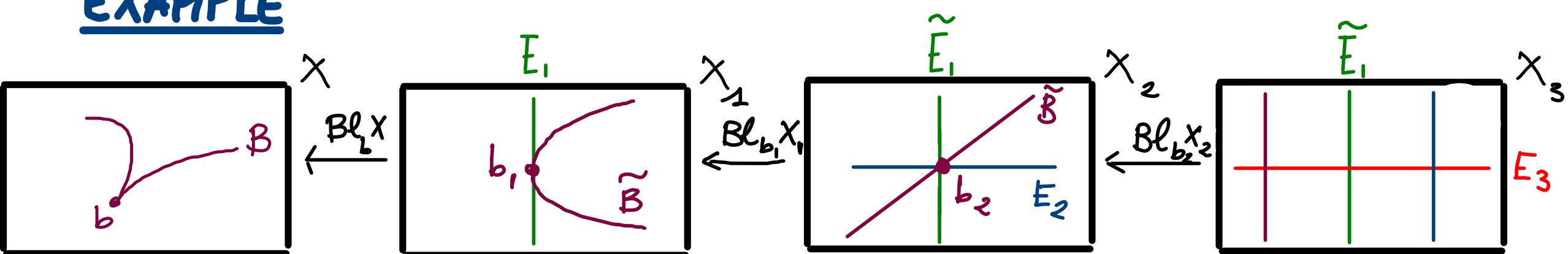
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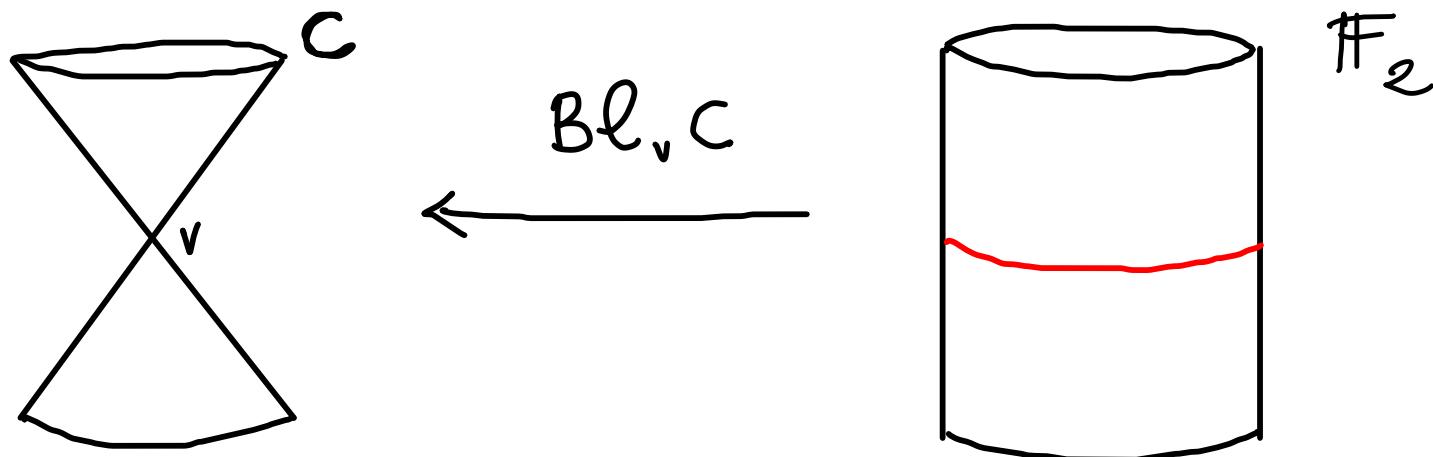
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WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.
TO DO THAT, WE NEED TO MEASURE THEIR SINGULARITIES.
WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

GIVEN (X, B) A LOG PAIR AND A LOG RESOL.
 $\pi: X' \longrightarrow X$

WE CAN ALWAYS WRITE IN A UNIQUE WAY

$$K_{X'} + \tilde{B} = \pi^*(K_X + B) + \sum_{\text{exc div.}} a_i E_i$$

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WE CAN ALWAYS WRITE IN A UNIQUE WAY

$$K_{X'} + B' = \pi^*(K_X + B)$$

$$\text{WHERE } B' = \pi_*^{-1} B + \sum_{\substack{\text{exc.} \\ \text{div.}}} a_i E_i$$

DEFINITION LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$\alpha(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'$$

↑
log discrepancy

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[THIS IS CALLED THE LOG DISCREPANCY OF D wrt (X, B)]

② THE TOTAL LOG DISCREPANCY OF (X, B) is

$$a(X, B) := \inf a(D; X, B)$$

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② THE TOTAL LOG DISCREPANCY OF (X, B) is

$$a(X, B) := \inf a(D; X, B)$$

SPECIAL DICHOTOMY: $a(X, B) = -\infty$

OR

$$a(X, B) \geq 0 \quad \text{MMP} \quad \smiley$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION LET (X, B) BE A LOG PAIR.

THEN (X, B) IS

LOG CANONICAL

KLT
LOG TERM.

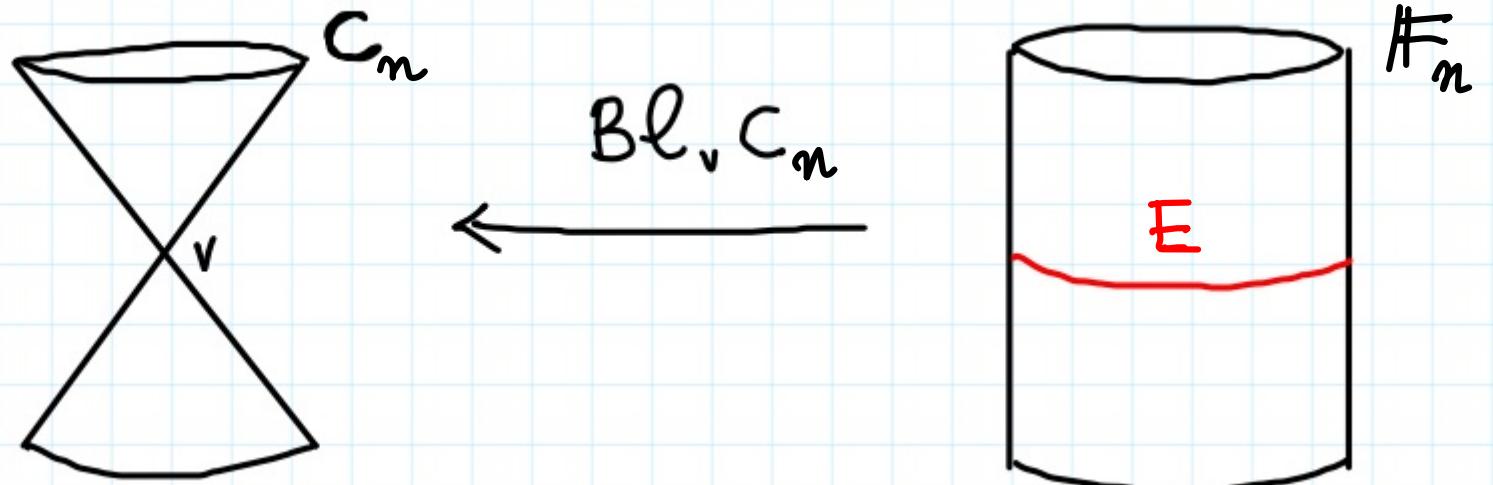
CANONICAL

TERMINAL

≥ 0
 > 0
 ≥ -1
 > -1

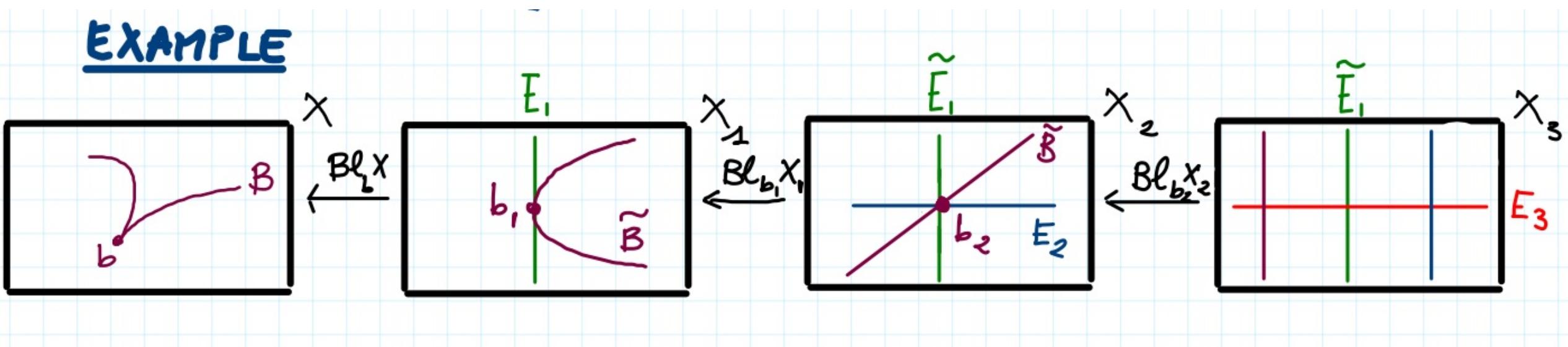
IF $a(X, B)$

EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF $\deg n$

EXAMPLE



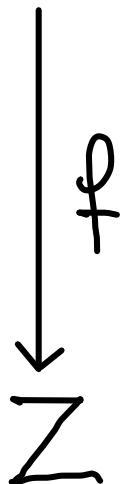
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~~X SMOOTH PROJECTIVE VARIETY / C~~ (X, B) LOG CANONICAL PAIR

CONJECTURE

STARTING FROM (X, B) , THERE EXISTS AN ALGORITHMIC WAY OF CONSTRUCTING A DIAGRAM OF THE FOLLOWING FORM

$X \xrightarrow{\text{composition of special birat'l maps}} (X', B')$



SUCH THAT f SATISFIES
1 OF THE FOLLOWING :

① f BIRAT'L & $K_{X'} + B'$ is AMPLE

② f FIBRATION &
 $K_X + B' = f^* H$ ample

③ f FIBRATION &
 $- (K_X + B')$ AMPLE on fibers

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

IN ORDER TO ACHIEVE SUCH RESULT , WE WILL TRY
TO CONTROL & TUNE THE POSITIVITY/NEGATIVITY OF
LOG DIVISORS .

(X, B) LOG PAIR

$K_X + B$ is \mathbb{R} -Cartier

$\forall C \subseteq X$ proper curve

$$(K_X + B) \cdot C$$

GOAL : either make $K_X + B$ nef after many
birational transformations

OR

die trying (but in finite time)

$$K_X + B \cdot C \geq 0$$

$\forall C \subseteq X$ proper curve

THE CONE THEOREM

THEOREM [Mori, Kollar, Shokurov, ...]

LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X + B \geq 0} + \overline{\text{NE}}(X)_{K_X + B < 0}$$

the closure the one of
effective
cones

THE CONE THEOREM

THEOREM

LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X+B \geq 0} + \overline{\text{NE}}(X)_{K_X+B < 0} =$$

$$= \overline{\text{NE}}(X)_{K_X+B \geq 0} + \sum_{i \in I} R_i + [C_i]$$

extremal rays
rat'l curves
 $-2\dim X \leq (K_X+B) \cdot C_i < 0$

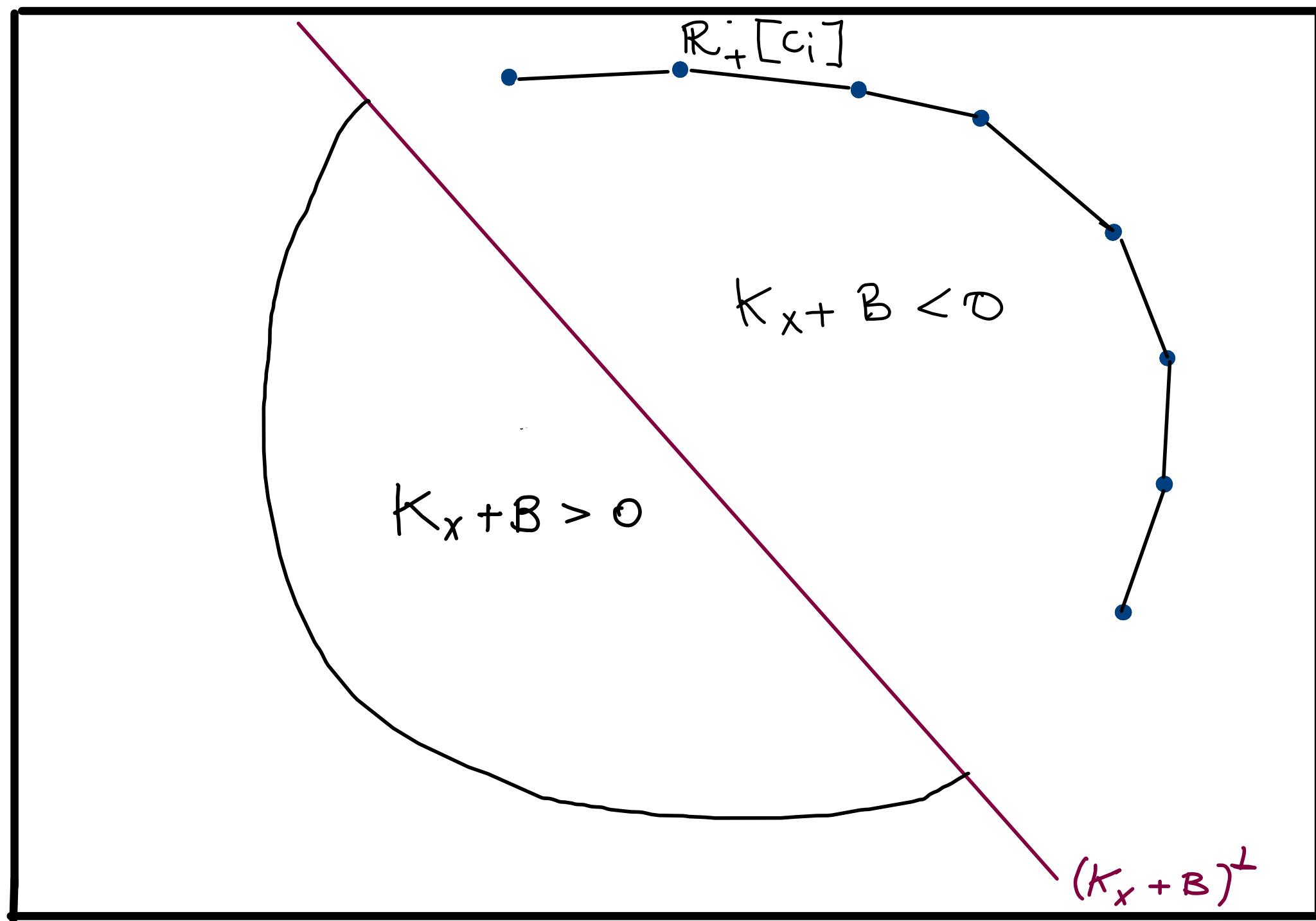
locally discrete away from $(K_X+B)^\perp$

countable set

HORIZONTAL SLICE OF $\overline{\text{NE}}(x)$

extremal rays $\stackrel{\text{def}}{=} v_1 + v_2 \in R_i$

$$\downarrow \\ v_1, v_2 \in R_i$$



THE CONE THEOREM

THEOREM

LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X+B \geq 0} + \overline{\text{NE}}(X)_{K_X+B < 0} =$$
$$= \overline{\text{NE}}(X)_{K_X+B \geq 0} + \sum_{i \in I} [R_i + [C_i]]$$

*R_i: extremal ray
↑ rat'l curves
at most countable set -2dim X ≤ (K_X+B) · C_i < 0*

MOREOVER, $\forall i \in I, \exists \text{cont}_{R_i}: X \rightarrow \mathbb{Z}_i$ SUCH THAT

cont_{R_i} HAS CONNECTED FIBERS & IF $\text{cont}_{R_i}(C) = \text{pt} \Rightarrow [C] \in R_i$.