The structure of algebraic varieties: birational classification and boundedness questions

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Geometries

Klein's & Grothendieck's idea

A geometry is the datum of:

- A class $\mathfrak{C}$ of **objects** that we would like to classify, and
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Topology $\longleftrightarrow$ sets & topologies + C^0 functions & maps
Differential Geometry $\longleftrightarrow$ manifolds + C^∞ functions & maps
Analytic Geometry $\longleftrightarrow$ $\mathbb{C}$-manifolds + C^ω functions & maps
Algebraic Geometry $\longleftrightarrow$ alg. varieties + polynomial functions & maps

Algebraic geometry: rigid as there are fewer functions;
powerful as we can use algebra & geometry together.

Algebraic Varieties

Let k be a field, e.g., $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

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$$X = \{(a_1, \ldots, a_n) \in k^n \mid f_1(a_1, \ldots, a_n) = \cdots = f_k(a_1, \ldots, a_n) = 0\}$$

for a finite collection $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$.

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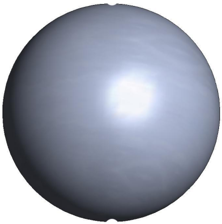
We will consider **irreducible** varieties, i.e. those that cannot further decomposed into unions of affine varieties.

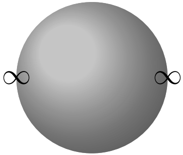
Algebraic Varieties, II

Question

What does X look like? For example is $X \neq \emptyset$?

Consider $f(x_1, x_2) = x_1^2 + x_2^2 + 1 \in k[x_1, x_2]$. $X := \{f = 0\}$
If $k = \mathbb{R}$ then X is empty, while if $k = \mathbb{C}$ then X contains infintely many points – it is a sphere minus 2 points.





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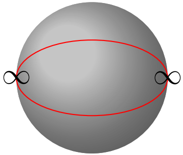

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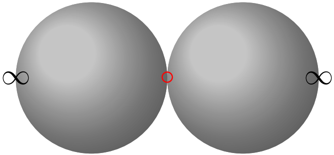

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In general, it is a hard problem to decide whether a variety X is non-empty. For example, consider $X = \{t_1^m + t_2^m - 1 = 0\}, m \in \mathbb{N}$. When $k = \mathbb{Q}$, deciding this is equivalent to Fermat Last Theorem.

This problem becomes much simpler if k is algebraically closed, e.g. $k = \mathbb{C}$. So, from now on we will assume that $k = \mathbb{C}$.

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Rational maps that are defined at every point of X are called **morphisms**.

Examples

$$i\colon \mathbb{C} \to \mathbb{C}$$
$$t \mapsto \frac{t^2+1}{t}$$

i is rational but not a morphism!

$$\phi\colon \mathbb{C} \to C$$
$$t \mapsto (t^3, t^2)$$

$C := \{x_1^2 - x_2^3 = 0\} \subset \mathbb{C}^2$

ϕ is a morphism!

$$c\colon \mathbb{C}^2 \to \mathbb{C}^2$$
$$(x_1, x_2) \mapsto (\tfrac{1}{x_1}, \tfrac{1}{x_2})$$

c is rational but not a morphism!

$c \circ c = Id$

Projective varieties

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A **projective variety** X in $\mathbb{P}^n$ is defined as the set of simultaneous zeros

$$X = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid g_1(a_0, \ldots, a_n) = \cdots = g_k(a_0, \ldots, a_n) = 0\}$$

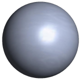
of homogenous polynomial equations $g_1, \ldots, g_k \in k[x_0, \ldots, x_n]$

Projective varieties, II

$\mathbb{P}^n$ and projective varieties are compact and moreover

$$\begin{array}{cccc} \mathbb{P}^n = & k^n & \cup & \mathbb{P}^{n-1} \\ & (a_0 : \cdots : a_{n-1} : 1) & & (b_0 : \cdots : b_{n-1} : 0) \end{array}$$

For example, $x_0^2 + x_1^2 + x_2^2 \in k[x_0, x_1, x_2]$ defines a quadric, which is the compactification of the affine variety $X = (t_1^2 + t_2^2 + 1 = 0)$ that we saw earlier.



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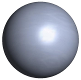

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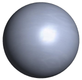

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We will consider from now on only projective varieties.

Equivalence of varieties

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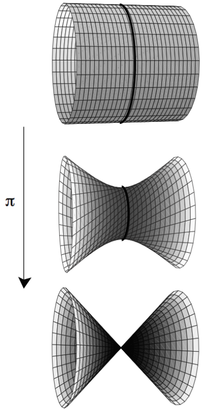
Being birational is an **equivalence relation**. Hence it partions algebraic varieties into equivalence classes.

Cone vs. Cilinder



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New Goal

Classify projective algebraic varieties up to birational equivalence.

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What does **simple geometry** mean for X?
We would like to have some way of measuring these improvements.

Plane Curves

In the projective plane $\mathbb{P}^2$, consider the projective variety X_f given by a homogenous polynomial of degree d, $f(X_0, X_1, X_2) = \sum a_{ijk} X_0^i X_1^j X_2^k$, $i + j + k = d$.

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How does the structure of X_f vary with d?

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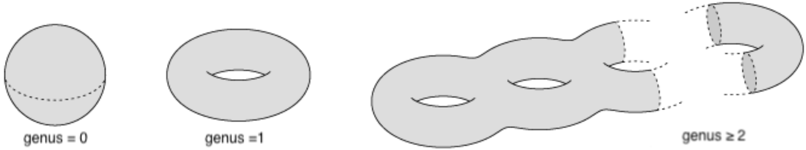

deg(f)	g(X)	Variety	Universal cover	Curvature
1,2	0	$\mathbb{P}^1$	S^2	> 0
3	1	Elliptic curves	$\mathbb{R}^2$	$= 0$
> 3	≥ 2	Hyperbolic curves	$\lvert x\rvert \leq 1 \subset \mathbb{R}^2$	< 0

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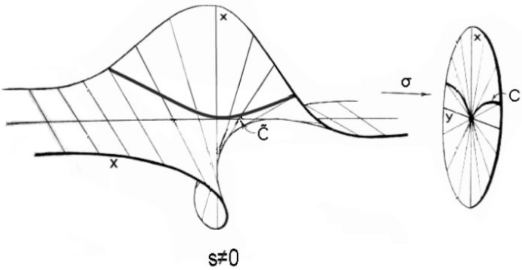

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Roughly, an algebraic variety X is smooth at a point $x \in X$, if locally around x, X is close to being isomorphic to $\mathbb{C}^n$. This is similar to the notion of smoothness in differential geometry. In fact, a smooth algebraic variety has a natural structure of complex manifold (implicit function theorem).

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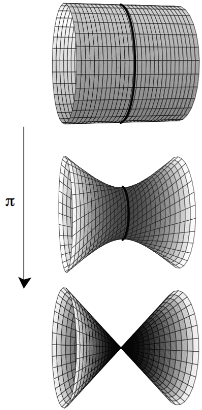
Cone vs. Cilinder: desingularization



Building blocks

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$$X \text{ is } \begin{cases} \textbf{Fano} \;\; \text{if it is positively curved} \\ \textbf{Calabi–Yau} \;\; \text{if it is flat} \\ \text{a } \textbf{canonical model} \;\; \text{if it is negatively curved} \end{cases}$$

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In $\mathbb{P}^n$, if $X = \{f = 0\}$ for a homogenous polynomial of degree d (and X is smooth) then

$$X \text{ is } \begin{cases} \textbf{Fano} \quad \text{if} & \deg(f) \leq n-1 \\ \textbf{Calabi–Yau} \quad \text{if} & \deg(f) = n \\ \text{a } \textbf{canonical model} \quad \text{if} & \deg(f) \geq n+1 \end{cases}$$

Minimal Model Program

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Given a smooth projective variety X then

(1) $X \dashrightarrow X'$ pass to a birational model X' by making the curvature as negative as possible using a combination of pre-determined birational maps.

(2) $X' \longrightarrow Z$ X' is expected to have a morphism to another projective variety whose fibers are in one of the 3 building block. Then repeat this process starting with Z.

The MMP is not only a classification algorithm but it is a much more versatile tool.
Given a question for X, it can be adapted to helps us to find a birational variety X' best suitable to the problem.

The MMP is known to work for projective varieties of dimension at most 3 by work of Mori (Fields Medal 1992), Kollár (Shaw Prize 2017) and many others. It is now know also to work in any dimension for varieties which are birational to canonical models by work of Shokurov, Birkar-Cascini-Hacon-McKernan and many others.

Figure: Mori

Figure: Kollár

Figure: Shokurov

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We can parametrize these simply by looking at their coefficients $(a_1 : a_2 : \cdots : a_{10})$.

Families of varieties

It is important to understand how varieties behave not only as standalone objects, but also in families. Consider the homogenous polynomials of degree 3 in 3 variables,

$$\begin{aligned} f(X_0, X_1, X_2) = \quad & a_1 X_0^3 + a_2 X_1^3 + a_3 X_2^3 + a_4 X_0 X_1^2 + a_5 X_0 X_2^2 + a_6 X_1 X_0^2 + \\ & + a_7 X_1 X_2^2 + a_8 X_2 X_0^2 + a_9 X_2 X_1^2 + a_{10} X_0 X_1 X_2. \end{aligned}$$

We can parametrize these simply by looking at their coefficients $(a_1 : a_2 : \cdots : a_{10})$.
To each such f, there corresponds an algebraic curve $X_f = \{f = 0\}$, which we saw is an elliptic curve when it is smooth.

Boundedness of curves

The space of coefficients $(a_1 : \cdots : a_{10}) = \mathbb{P}^9$ parametrizes these elliptic curves (and their degenerations).

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Moreover, if $W \to Z$ is a morphism whose fibers are elliptic curves, one should expect to get a map $m\colon Z \dashrightarrow \mathbb{P}^9$.
One then could try to classify W by classifying Z and understanding the map $Z \dashrightarrow \mathbb{P}^9$.

Boundedness

As we are interested not only in understanding the behavior of algebraic varieties as standalone objects, but also in families, we need a suitable notion to indicate that a set of varieties can be described by many families.

Boundedness

An infinite number of algebraic varieties can be defined by a finite number of parameters.

How to show boundedness?

Given a class $\mathfrak{D}$ of projective algebraic varieties, we need to show that they can all be embedded in a fixed projective space $\mathbb{P}^l$ and the volume of these embeddings is bounded.

State of the art

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Canonical models [Hacon-McKernan-Xu, 2009-15]:

N-dimensional canonical models with mild singularities with fixed volume v are bounded.

Fano varieties [Birkar, 2016]:

N-dimensional Fano varieties with bounded singularities are bounded.

These results were one of 2 main achievements that earned Hacon-McKernan the 2017 Breakthrough Prize and Birkar the 2018 Fields Medal.

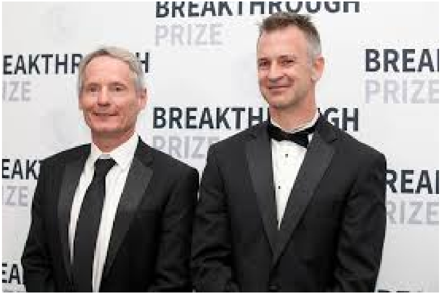

Figure: McKernan and Hacon

Figure: Birkar

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Fix a positive integer $n \geq 3$. Are n-dimensional projective Calabi–Yau varieties bounded?

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Conjecture

Fix a positive integer $n \geq 3$. Are n-dimensional projective Calabi–Yau varieties bounded?

This is a very obscure problem. Only known result:

Elliptic CY 3-folds [Gross, 1994]:

Elliptic Calabi–Yau threefolds $f : X \to Y$ are bounded up to birational equivalence, assuming that f is non-isotrivial.

My work

In a recent project, joint with G. Di Cerbo, we focus on Calabi-Yau varieties with an elliptic fibration. They have been intensively studied in mathematics and they seem to be the most relevant in physics (string theory and F-theory).

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Theorem

Let $n \leq 5$ be an integer. Then there are finitely many families of elliptic Calabi-Yau varieties $f : X \to Y$, up to birational isomorphism, such that $\dim(X) = n$, X is not a product and there exists a section $s : Y \to X$.

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A section $s \colon Y \to X$ is simply a map such that $f \circ s = Id_Y$. In particular, it gives a point of reference on every fiber.

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(1) We first show that bases of elliptic CYs are bounded. Using the MMP, they decompose birationally into a tower of morphisms. We show that at each step in the tower boundedness holds and it propagates upwards.

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(2) Since the elliptic fibration has a section, we can deduce boundedness of the total space from boundedness of the bases.

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- It would be interesting to remove the assumption on the existence of the section. The result of M. Gross in dimension 3 does not have this assumption. In higher dimension, there are subtle problems related to the type of singular fibers that appear in the fibration and how to deal with those.
- Finally, it should be enough to assume that the base of the fibration is covered by many copies of $\mathbb{P}^1$. At the moment we can prove this only in dimension 4 and we are working with Birkar to remove this obstruction.

Thank you!