13. ABUNDANCE FOR THREEFOLDS,

CASE $\nu(X) = 1$

János Kollár, Kenji Matsuki, and James McKernan

This chapter treats the proof of the following result, proved in [Miyaoka88b] (see (11.1.2) for definitions):

13.1 Theorem. *Let X be a minimal threefold. If the numerical dimension $\nu(X)$ is one, then $|mK_X|$ is base point free for some $m > 0$.*

The main steps in the proof are almost identical to those of Chapter 11, but of course some steps are harder. Here is a generalization of (11.3.2) to dimension three.

13.2 Lemma. *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair (2.13.5), $\dim X = 3$. Suppose there is a nef divisor $D \in |m(K_X + \Delta)|$ such that $X \setminus D$ has terminal singularities. Let $B = D_{\mathrm{red}}$. Then there is a threefold $\hat{X}$, with boundary $\hat{\Delta} + \hat{B}$, where $\hat{B}$ is reduced, such that:*

(1) *The pair $(\hat{X}, \hat{\Delta} + \hat{B})$ has $\mathbb{Q}$-factorial log canonical singularities, $\hat{X} \setminus \hat{B}$ is isomorphic to $X \setminus B$ and there is a divisor $\hat{D} \in |m(K_{\hat{X}} + \hat{\Delta} + \hat{B})|$. Moreover $\hat{D}_{\mathrm{red}} = \hat{B}$.*

(2) *$K_{\hat{X}} + \hat{\Delta} + \hat{B}$ is nef.*

(3) *$\nu(X, K_X + \Delta) = \nu(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$ and $\kappa(X, K_X + \Delta) = \kappa(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$.*

Proof. By (6.16.1) or (20.9) there is a projective partial resolution of singularities $\mu : X_0 \to X$ such that

(1) the divisor $B_0 = (\mu^* B)_{\mathrm{red}}$ is a normal crossing divisor,

(2) $\mu : (X_0 \setminus B_0) \to (X \setminus B)$ is an isomorphism.

As (X, Δ) has klt singularities, $m(K_{X_0} + \mu_*^{-1}\Delta + E) = \mu^* D + \Gamma = \tilde{D}$, where E is the union of the μ-exceptional divisors and Γ is effective and supported on the exceptional locus. In particular, $\operatorname{Supp} \tilde{D} = \operatorname{Supp} \mu^* D$. Now we replace

The first identification is easy; given any open neighbourhood of F, we can always find a smaller one which retracts to F. For the second we use (12.2.8). In fact if we push down the short exact sequence

$$0 \longrightarrow \mathcal{O}_T(-F) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_F \longrightarrow 0$$

by f, we obtain a sequence

$$0 \longrightarrow R^1 f_* \mathcal{O}_T(-F) \longrightarrow R^1 f_* \mathcal{O}_T \longrightarrow R^1 f_* \mathcal{O}_F = H^1(F, \mathcal{O}_F) \longrightarrow 0.$$

The two spectral sequences give rise to the following commutative diagram of cohomology groups, with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(S, \mathbb{C}_S) & \longrightarrow & H^1(T, \mathbb{C}_T) & \longrightarrow & H^1(F, \mathbb{C}_F) \\ & & {\scriptstyle i_1}\downarrow & & {\scriptstyle j_1}\downarrow & & {\scriptstyle k_1}\downarrow \\ 0 & \longrightarrow & H^1(S, \mathcal{O}_S) & \longrightarrow & H^1(T, \mathcal{O}_T) & \longrightarrow & H^1(F, \mathcal{O}_F). \end{array} \tag{12.5.7}$$

We apply (12.5.1). We need to find a compatible splitting for k_1. Let F_j be the connected compnents of F. By (12.2.8) these come in three types. If F_j is a tree of rational curves then $H^1(F_j, \mathbb{C}) = 0$. If F_j is a cycle of rational curves then $H^1(F_j, \mathbb{C}) \to H^1(F, \mathcal{O}_F)$ is an isomorphism. Finally if F_j is a smooth elliptic curve then $H^1(T, \mathbb{C}_T) \to H^1(F, \mathbb{C}_F)$ factors through $H^1(S^\mu, \mathbb{C})$ hence the splitting of $k_1|F_j$ provided by Hodge decomposition works.

i_0 is automatically surjective, and there is a similar commutative diagram

$$\begin{array}{ccccccc} H^1(F, \mathbb{C}_S) & \longrightarrow & H^2(S, \mathbb{C}_T) & \longrightarrow & H^2(T, \mathbb{C}_F) & \longrightarrow & 0 \\ {\scriptstyle k_1}\downarrow & & {\scriptstyle i_2}\downarrow & & {\scriptstyle j_2}\downarrow & & \\ H^1(F, \mathcal{O}_S) & \longrightarrow & H^2(S, \mathcal{O}_T) & \longrightarrow & H^2(T, \mathcal{O}_F) & \longrightarrow & 0 \end{array} \tag{12.5.8}$$

(12.5.1.1) is vacuously satisfied, hence i_2 is surjective. □

12.5.8 Remark. One can see that the kernel of i_p in (12.1.2) is precisely $F^1 H^p(S, \mathbb{C}_S)$ (given by the natural mixed Hodge structure, cf. [Griffiths-Schmid73]). The proof given above could have been shortened by using more difficult Hodge theoretic methods.

$\mu_*^{-1}\Delta$ with Δ_0 where we only include those components of $\mu_*^{-1}\Delta$ which are not components of B_0. With this choice of Δ_0, $\Delta_0 + B_0$ is a boundary, and there is a divisor $D_0 \in |m(K_{X_0} + \Delta_0 + B_0)|$

We now apply the log minimal model program to $(X_0, \Delta_0 + B_0)$. We construct X_i, Δ_i, B_i and D_i satisfying (1) inductively. If $K_{X_i} + \Delta_i + B_i$ is not nef, there exists an elementary contraction $\phi_i : X_i \to Z_i$ [KMM87, 4-2-1 and 3-2-1] associated to a log extremal ray with respect to $K_{X_i} + \Delta_i + B_i$. Clearly the map ϕ_i is birational.

If ϕ_i is a divisorial contraction, we set $X_{i+1} = Z_i$, $\Delta_{i+1} = \phi_{i*}(\Delta_i)$, $B_{i+1} = \phi_{i*}B_i$ and $D_i = \phi_{i*}D_i$. (By [KMM87, 5-1-6], the image $(X_{i+1}, \Delta_{i+1}+B_{i+1})$ is $\mathbb{Q}$-factorial log terminal, and (13.2.4) implies that B_{i+1} and D_{i+1} are divisors.)

Otherwise there is a log flip, i.e., a small birational morphism $\phi_i^+ : X_{i+1} \to Z_i$. We take Δ_{i+1}, B_{i+1} and D_{i+1} to be the birational transforms of B_i and D_i under ϕ_i. (By [KMM87, 5-1-11] the log flip $(X_{i+1}, \Delta_{i+1} + B_{i+1})$ is log canonical and $\mathbb{Q}$-factorial in a neighborhood of B_{i+1}. The pluricanonical class pushes across the flip, because it may be defined using differential forms on a complement of any codimension 2 locus.)

$K_{X_i} + \Delta_i + B_i$ is negative relative to the morphism ϕ_i. Since $D_i \in |m(K_{X_i} + \Delta_i + B_i)|$ is supported on B_i, the exceptional locus of ϕ_i is contained in B_i. By (7.1) the process we have just described must terminate at some i, and we set $\hat{X} = X_i$, $\hat{\Delta} = \Delta_i$, $\hat{B} = B_i$ and $\hat{D} = D_i$.

Conditions (1) and (2) are automatic from the construction. (3) follows from (11.3.3) and (13.2.4) applied to the pullbacks of the divisors $m(K_X + \Delta)$ and D_i to a common resolution. □

13.2.4 Lemma. *The set theoretic image of an effective nef divisor under a birational morphism is divisorial.*

Proof. Let $f : X \to Y$ be a birational morphism, and let L be an effective nef $\mathbb{Q}$-Cartier divisor on X. We may assume that L is Cartier. Let $M = f_*L$ be the cycle theoretic push forward, and let $M_0 = f(\operatorname{Supp} L)$ be the set theoretic image. Write $M_0 - \operatorname{Supp} M = C_0 \cup \ldots \cup C_i$ where C_i are distinct irreducible components. By taking generic hyperplane sections of Y, we may assume that $\min\{\dim C_i\} = 0$. Using generic hyperplane sections on X we may assume $\dim X = 2$. Choosing a resolution of singularities for X and pulling back L, we may assume that X is smooth. But by the Hodge index theorem the intersection matrix of divisors supported on the exceptional locus of f over C_i is negative definite, and L is supported on this locus near C_i, a contradiction. □

13.3 Conclusion of Proof of (13.1).

Let X be a minimal threefold, i.e. a threefold with $\mathbb{Q}$-factorial terminal singularities such that K_X is nef. Suppose that $D \in |mK_X|$ (9.0.6) and let

$B = D_{\text{red}}$.

13.3.1 Lemma. *There is a threefold $\hat{X}$, birational to X, with reduced boundary $\hat{B}$ such that:*

(1) *The pair $(\hat{X}, \hat{B})$ is $\mathbb{Q}$-factorial and log canonical, $\hat{X} \setminus \hat{B}$ has terminal singularities, and there is a divisor $\hat{D} \in |mK_{\hat{X}}|$. Moreover $\hat{D}_{\text{red}} = \hat{B}$.*

(2) *$K_{\hat{X}} + \hat{B}$ is nef.*

(3) *$\nu(X) = \nu(\hat{X}, K_{\hat{X}} + \hat{B})$ and $\kappa(X) = \kappa(\hat{X}, K_{\hat{X}} + \hat{B})$.*

Proof. This is an immediate consequence of (13.2.1). □

13.3.2 Lemma. *There is a threefold X', birational to X, with a reduced boundary B' satisfying conditions (1–3) of (13.2.1) and*

(4) *every connected component of B' is irreducible.*

Proof. It remains to modify $\hat{X}$ further to achieve (4). Suppose S is a prime component of $\hat{B}$ which is not isolated in $\hat{B}$. We will apply the log minimal model program to $K_{\hat{X}} + \hat{B} - S$.

Suppose we have constructed a sequence of pairs (X_j, B_j) $((X_0, B_0) = (\hat{X}, \hat{B}))$ and birational morphisms $\phi_j : X_j \dashrightarrow X_{j+1}$, with respect to $K_{X_j} + B_j - S_j$ for $j \leq i-1$, where B_{j+1} and S_{j+1} are respectively either ${\phi_j}_*(B_j)$ and ${\phi_j}_*(S_j)$, if ϕ_j is a divisorial contraction, or the birational transforms of B_j and S_j under ϕ_j, if ϕ_j is a log flip. As in (13.2.1), (X_j, B_j) satisfies properties (1–3).

Suppose S_i is still not isolated in B_i. Then there is another component S' of B_i which meets S_i in a curve C (recall X_i is $\mathbb{Q}$-factorial). Let H be an ample divisor and set $C' = H \cap S'$. Let L_i be the line bundle $\mathcal{O}_{X_i}(m(K_{X_i} + B_i))$. It is automatic that $\nu(S, L_i|_S) = 0$ and so deg $L_i|_{C'} = 0$, as the curve C' lies in S'. On the other hand, as H is ample, $S \cdot C' = H \cdot C > 0$, and so

$$(K_{X_i} + B_i - S_i) \cdot C' < 0.$$

As L_i is nef, the Theorem on the Cone [KMM87, 4.2.1] implies there is a log extremal ray R such that

$$(K_{X_i} + B_i - S_i) \cdot R < 0.$$

with $L \cdot R = 0$. Now $S \cdot R > 0$ and the support of the base locus of K_{X_i} is a subset of supp (B_i) and so $R \subset$ supp $(B_i - S)$. By (8.1), there is a log flip of R with respect to L and by (7.1) this sequence of log flips terminates. Thus at some stage S_i is isolated in B_i.

However if T is another prime component of B' and T is isolated in B', then T_i (the component of B_i corresponding to T) is isolated in B_i (as each

ϕ_j only modifies points of S_j). In this way we isolate every component of B', one by one. □

Proof of (13.1.1). Pick a component S of B', and put $\Delta = \operatorname{Diff}_S(0)$. By (16.9.1), the pair (S, Δ) is semi log canonical and so (12.1.1) implies $K_S + \Delta$ is torsion. Just as before, by (11.3.6), we may find a finite Galois cover $\pi : \tilde{U} \longrightarrow U_1$, étale in codimension one, such that

$$\tilde{S} \sim m'\tilde{G}, \qquad K_{\tilde{U}} + \tilde{S} \sim d'\tilde{G}, \qquad \text{and} \qquad \omega_{\tilde{S}} = \mathcal{O}_{\tilde{U}}(\tilde{S})|_{\tilde{S}} = \mathcal{O}_{\tilde{S}}.$$

where $\tilde{S} = \pi^* S$. Now if we can apply (11.3.7), then we may conclude just as in Chapter 11. Conditions (1) and (2) of (11.3.7) are automatic.

Consider the commutative square

$$\begin{array}{ccc}
H^p(\tilde{S}_n, \mathbb{C}) & \longrightarrow & H^p(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \\
\big\downarrow & & \rho \big\downarrow \\
H^p(\tilde{S}, \mathbb{C}) & \xrightarrow{i_p} & H^p(\tilde{S}, \mathcal{O}_{\tilde{S}}).
\end{array}$$

where $\tilde{S}_n$ is defined as in Chapter 11. As the first vertical map is an isomorphism (the support of $\tilde{S}$ and $\tilde{S}_n$ are the same), and the map i_p is surjective (this is (12.1.2)), the map ρ is surjective as well, which is condition (3) of (11.3.7). □

14. ABUNDANCE FOR THREEFOLDS,

$\nu(X) = 2$ IMPLIES $\kappa(X) \geq 1$

DAN ABRAMOVICH, LUNG-YING FONG, and KENJI MATSUKI

We continue our treatment of Miyaoka's and Kawamata's proof of the abundance conjecture for threefolds. In this chapter, we look at the case $\nu(X) = 2$. The method we use in sections (14.3-4) is due to Kollár.

14.1 A special case

Let us first consider the following very special case, which gives some idea about the line of proof in the general case.

Assume X is a *smooth* minimal model with $\nu(X) = 2$, and assume the existence of a smooth member $D \in |mK_X|$. As $\nu(D, K_D) = 1$, by abundance for surfaces (11.3.1), $\kappa(D) = 1$ and D is an elliptic surface over some curve.

Let H be a hyperplane section of X. Kodaira vanishing on H gives

$$H^i(X, mK_X + lH) \simeq H^i(X, mK_X + (l+1)H)$$

for $i \geq 2$, $m > 1$ and $l \geq 0$. But since H is ample, this group vanishes for large l. Therefore we get

$$H^i(X, mK_X) = 0 \quad \text{for} \quad i \geq 2.$$

We now use Riemann–Roch. Since $K_X^3 = 0$, the coefficient of the leading (linear) term in $\chi(nK_X)$ is $K \cdot c_2(X)$, which is proportional to $c_2(D)$, which is nonnegative [BPV84, p. 188]. Hence $\chi(nK_X) \geq C$ for some constant C, and $h^0(X, nK_X) \geq C + h^1(X, nK_X)$. From the exact sequence

$$0 \to \mathcal{O}_X((n-m)K_X) \to \mathcal{O}_X(nK_X) \to \mathcal{O}_D(nK_X|_D) \to 0,$$

we get $h^2(D, nK_X|_D) = 0$ and

$$H^1(X, nK_X) \to H^1(D, nK_X|_D) \to 0.$$

Riemann–Roch on D implies that $\chi(D, nK_X|_D)$ is constant. Since $\nu(K_X|_D) = 1$, the abundance theorem on surfaces implies that both $h^0(D, nK_X|_D)$ and $h^1(D, nK_X|_D)$ grow with n. Hence $h^1(X, nK_X)$ grows with n. This proves that $\kappa(X) > 0$.

We begin with a construction similar to that of (13.2).

14.2 Lemma. *There is a normal threefold X', birational to X, with reduced boundary B', and such that*

(1) *(X', B') has $\mathbb{Q}$-factorial log canonical singularities and $(X', 0)$ has only log terminal singularities. There exists a divisor $D' \in |mK_{X'}|$. Moreover $B' = D'_{\mathrm{red}}$.*
(2) *$K_{X'} + B'$ is nef.*
(3) *$\nu(X) = \nu(X', K_{X'} + B')$ and $\kappa(X) = \kappa(X', K_{X'} + B')$.*
(4) *$L' = m(K_{X'} + B')$ is Cartier.*
(5) *If C is a curve in X' with $C \cdot (K_{X'} + B') = 0$, then $C \cdot K_{X'} \geq 0$. If D is a curve in X' with $D \cdot (K_{X'} + B') > 0$, then (X', B') is log terminal along the generic point of D.*

Proof. As in (13.2), we can apply the log minimal model program to construct a threefold $\tilde{X}$, with boundary $\tilde{B}$ satisfying (1), (2) and (3).

Replacing m by a multiple, we can assume that $L = m(K_{\tilde{X}} + \tilde{B})$ is a Cartier divisor.

We construct X' inductively. Take $X'_0 = \tilde{X}$, $B'_0 = \tilde{B}$ and $D'_0 = \tilde{D}$. If there exists a curve C in X'_i such that $C \cdot (K_{X'_i} + B'_i) = 0$ and $C \cdot K_{X'_i} < 0$, then there is some $K_{X'_i}$ extremal ray R_i lying on the hyperplane $\{\Gamma \mid \Gamma \cdot (K_{X'_i} + B'_i) = 0\}$. We have a divisorial contraction or a log flip $\phi_i : X'_i \dashrightarrow X'_{i+1}$ associated with R_i. Put $B'_{i+1} = \phi_{i*}(B'_i)$ and $D'_{i+1} = \phi_{i*}(D'_i)$. Then (1–4) are clearly satisfied. Since $(\tilde{X}, 0)$ has log terminal singularities, this process will stop and gives X'.

It remains to check inductively that if we contract a divisor by ϕ_i, we still have log terminal singularities generically along curves having positive intersection with $K_{X'_{i+1}} + B'_{i+1}$. Since $\nu(B'_i, L'_i|_{B'_i}) = 1$, (12.1.1) implies that $|m'L'|_{B'}|$ defines a morphism f from B'_i to some curve. Let S be a component of B'_i on which L'_i is not numerically trivial, and let $\lambda : S^\lambda \to S$ be the normalization. Consider the different Θ defined by $\lambda^*(K_{X'_i} + B'_i) = K_{S^\lambda} + \Theta$ (cf. (16.6)). Θ lies over the nonnormal locus of B'_i and the singular locus of X'_i. Let Θ_h be the horizontal part of Θ. If the generic fibre F of $f \circ \lambda$ is a smooth elliptic curve, then $\Theta_h = 0$. Otherwise $F \cong \mathbb{P}^1$, and $\Theta_h \cdot F = 2$. Decompose Θ_h as $\sum c_k \Gamma_k + \sum d_l \Delta_l$ in a neighborhood of F, where the Γ_k map under λ to the singular locus of X'_i and the Δ_l to the nonnormal locus of B'_i. Then

$$\sum c_k + \sum d_l = 2. \tag{14.2.1}$$

By the inductive assumption, (X'_i, B'_i) has log terminal singularities along $\lambda(\Gamma_k)$ and $\lambda(\Delta_l)$. In particular, X'_i is smooth along $\lambda(\Delta_l)$ and $d_l = 1$, while along $\lambda(\Gamma_k)$, X'_i has index m_k quotient singularities ($m_k \geq 2$) and $c_k = 1 - \frac{1}{m_k}$

(See (16.6)). The set of solutions $(d_1, \cdots; m_1, m_2, \cdots)$ of (14.2.1) can be easily enumerated:

Case 1 (1,1;),

Case 2 (1;2,2), and

Case 3 (;2,4,4), (;3,3,3), (;2,3,6), and (;2,2,2,2).

Suppose that ϕ_i contracts a $K_{X'_i}$ extremal curve C on X'_i. $C \cdot D'_i < 0$ implies that $C \subset B'_i$. $C \cdot (K_{X'_i} + B'_i) = 0$, and hence C is contained in a fibre of f. $C \cdot B'_i > 0$ implies that C has to intersect a component of B'_i positively. We see at once that ϕ_i can never contract components that are in Case 3. In Case 1, S intersects two other components S_1 and S_2 of B'_i. X'_i is smooth in a neighborhood of $\lambda(F)$, hence X'_{i+1} is generically smooth along the intersection of $\phi_i(S_1)$ and $\phi_i(S_2)$. In Case 2, X'_i has only two curves of A_1-singularities in a neighborhood of $\lambda(F)$, hence it is canonical. Therefore X'_{i+1} has terminal singularities in a neighborhood of $\phi_i(\lambda(F))$ (2.28.3), thus X'_{i+1} is generically smooth along $\phi_i(S)$. (In fact one can see that in this case $\lambda(F) \cdot K \geq 0$, and therefore we never have to contract $\lambda(F)$.) $\square$

14.3 Computing the second Todd class.

We now proceed with the proof of the abundance conjecture and establish an inequality involving the second Todd class on a resolution of X'. This is used in the final step when we apply Riemann–Roch.

14.3.1 Lemma. *X', B' and L' as in section 14.2. Let $\mu : V \to X'$ be a resolution of singularities. Then we have*

$$\mu^* L' \cdot \left(K_V^2 + c_2(V)\right) \geq L' \cdot \left(K_{X'}^2 + \hat{c}_2(\hat{\Omega}^1_{X'})\right).$$

Proof. Let

$$\mu_* \left(K_V^2 + c_2(V)\right) - \left(K_{X'}^2 + \hat{c}_2(\hat{\Omega}^1_{X'})\right) = \sum a_i C_i.$$

Then all the 1-cycles C_i are supported on the singular locus of X', and in particular they lie in B'. (By (13.2.4) X' has isolated singularities outside B'.) Because we are interested in the intersection of $\sum a_i C_i$ with $L' = m(K_{X'}+B')$, we only need to consider 1-cycles on components S of B' on which $\nu(L') \neq 0$, and focus on cycles C_i 'horizontal' to the map f defined in (14.2). They are contained in the Θ_h considered in (14.2) and we have a complete list of possible singularities there.

We can compute the numbers a_i by taking a transversal slice Γ_i at a general point P_i on C_i, and reduce the computation to the surface case. Let $\mu : \tilde{\Gamma}_i = \mu^{-1}(\Gamma_i) \to \Gamma_i$ be the resolution induced by μ. Notice that the number $c_1^2 + c_2$

does not change on blowing up a smooth point of a surface. We may assume that μ_i is the minimal resolution. If P_i is an index m_i point, then (10.8) (with $B = 0$) tells us that

(14.3.1.2)

$$a_i = \mu_* \left(K^2_{\tilde{\Gamma}_i} + c_2(\tilde{\Gamma}_i) \right) - \left(K^2_{\Gamma_i} + \hat{c}_2(\Gamma_i) \right)$$
$$= \left(K_{\tilde{\Gamma}_i} - \mu^* K_{\Gamma_i} \right)^2 + e_{\text{top}}(\mu^{-1}(P_i)) - \frac{1}{m_i}.$$

If the singularity is a Du Val singularity, then we can work out by explicit computation that a_i is $\frac{3}{2}$, $\frac{8}{3}$, $\frac{15}{4}$ and $\frac{35}{6}$, when m_i is 2, 3, 4 and 6 repectively. Otherwise, a_i is $\frac{4}{3}$, $\frac{3}{4}$ and $-\frac{5}{6}$, when m_i is 3, 4 and 6 respectively. Now an index 6 point is always accompanied by an index 2 point and an index 3 point, hence the sum of the corresponding a_i is at least 2. This completes the proof of the lemma. □

14.3.2 Lemma. $L' \cdot \hat{c}_2(\hat{\Omega}^1_{X'}) \geq L' \cdot \hat{c}_2 \left(\hat{\Omega}^1_{X'}(\log B') \right) - L' \cdot (K_{X'} + B') \cdot B'$.

Proof. By (10.8.8), the difference $\hat{c}_2(\hat{\Omega}^1_{X'}) - \hat{c}_2 \left(\hat{\Omega}^1_{X'}(\log B') \right) - (K_{X'} + B') \cdot B'$ is an effective 1–cycle supported on the singular locus of X'. □

14.3.3 Lemma. *Let $\mu : V \to X'$ be a resolution of singularities. Then we have*

$$\mu^* L' \cdot \left(K_V^2 + c_2(V) \right) \geq 0.$$

Proof. By (14.3.1) and (14.3.2), we have

$$\mu^* L' \cdot (K_V^2 + c_2(V)) \geq L' \cdot K_X^2 + L' \cdot \hat{c}_2 \left(\hat{\Omega}^1_{X'}(\log B') \right) - L' \cdot (K_{X'} + B') \cdot B'.$$

It follows from (10.13) that $L' \cdot \hat{c}_2 \left(\hat{\Omega}^1_{X'}(\log B') \right) \geq 0$. $L' = m(K_{X'} + B')$ and $\nu(B', L'|_{B'}) = 1$, so that $L' \cdot (K_{X'} + B') \cdot B' = 0$. Write $K_{X'}$ as $\sum b_i S_i$, where $b_i \geq 0$ and S_i are components of B'. Moreover $S_i \cdot L'$ is equivalent to an effective sum of curves having zero intersection with $(K_{X'} + B')$. By condition (5) of (14.2), this implies $S_i \cdot L' \cdot K_{X'} \geq 0$. Hence $L' \cdot K^2_{X'} \geq 0$. This completes the proof of the lemma. □

14.3.4 Remarks.

(i) From the proof we see that the inequality in (14.3.3) is strict, unless the map f_0 has smooth elliptic fibres on all the components of B where $\nu(L') = 1$.

(ii) The above proof works in any dimension.

14.4 Proving that $\kappa(X) > 0$. We now can prove the main theorem along the lines of the smooth case as in (14.1).

14.4.1 Theorem. [Kawamata91b] *Let X be a minimal 3-fold over $\mathbb{C}$. Suppose that $\nu(X) = 2$. Then $\kappa(X) > 0$.*

Proof. Construct X' and L' as in (14.2). Let $\mu : V \to X'$ be a desingularization of X'. Since X' is log terminal, X' has only rational singularities. Therefore

(14.4.1.1)

$$\begin{aligned}\chi(X', nL') =& \chi(V, n\mu^* L') \\ =& \frac{n}{12}\left(K_V^2 + c_2(V)\right) \cdot \mu^* L' + \chi(\mathcal{O}_V).\end{aligned}$$

(14.3.3) shows that the linear term in (14.4.1.1) is nonnegative. Therefore

(14.4.1.2) $$\chi(X', nL') \geq C \qquad \text{for some constant } C.$$

Now look at the exact sequence:

(14.4.1.3) $$0 \to \mathcal{O}_{X'}(nL'(-B')) \to \mathcal{O}_{X'}(nL') \to \mathcal{O}_{B'}(nL'|_{B'}) \to 0.$$

Recall that $L' = m(K_{X'} + B')$, thus $nL'(-B') \equiv K_{X'} + (nm-1)M'$ where $M' = K_{X'} + B'$. Take a general ample hyperplane section H' of X'. Using the restriction exact sequence and the Kawamata–Viehweg vanishing theorem [KMM87, 1-2-5] we see that

$$H^i\left(X', nL'(-B') + lH'\right) \simeq H^i\left(X', nL'(-B') + (l+1)H'\right)$$

for $i \geq 2$ and $l \geq 0$. The last group vanishes when l is large, thus $H^2\left(X', nL'(-B')\right) = 0$. Moreover, since B' is Cohen-Macaulay,

$$h^2\left(B', nL'|_{B'}\right) = h^0\left(B', \omega_{B'}(-nL'|_{B'})\right) = 0$$

for n large. Therefore we have $H^2(X', nL') = 0$ for large n. Combined with (14.4.1.2), this shows that

$$h^0(X', nL') \geq h^1(X', nL') + C.$$

Thus it is sufficient to prove that $h^1(X', nL')$ grows linearly with n. Note that $\chi(X', K_{X'} + (n-1)L') = -\chi(X', (1-n)L')$ has the same linear term as in (14.4.1.1). Hence it follows from (14.4.1.3) that $\chi(B', nL'|_{B'})$ is actually a constant. Then (12.1.1) together with the vanishing of $H^2(B', nL'|_{B'})$ implies that both $h^0(B', nL'|_{B'})$ and $h^1(B', nL'|_{B'})$ grow with n. We have

$$H^1(X', nL') \to H^1(B', nL'|_{B'}) \to H^2(X', nL'(-B')) = 0.$$

This shows that $h^1(X, nL')$ grows with n and completes the proof of the theorem. □

15. LOG ELLIPTIC FIBER SPACES

JÁNOS KOLLÁR

The aim of this chapter is to complete the proof of abundance for threefolds. Instead of the cohomological approach of [Kawamata85] we present a rather geometric one. Many of the arguments work for an arbitrary nef divisor B such that $\nu(B) = 2$ and $\kappa(B) \geq 1$. The underlying variety can have arbitrary dimension or even positive characteristic. We however formulate everything for a klt divisor $K_X + \Delta_X$ in characteristic zero, where the necessary flips are known to exist.

15.1 Definition. (15.1.1) A *log elliptic fiber space* is a proper morphism $g : (V, \Delta_V) \to W$ such that $g_*\mathcal{O}_V = \mathcal{O}_W$, the generic fiber E_g is an irreducible curve and $(K_V + \Delta_V) \cdot E_g = 0$.

(15.1.2) Let (X, Δ_X) be a log variety. A *log elliptic structure* on X is a diagram

$$\begin{array}{ccc} (X, \Delta_X) & \xleftarrow{\ h\ } & (V, \Delta_V) \\ & & \Big\downarrow g \\ & & W \end{array}$$

where h is a birational morphism, $g : (V, \Delta_V) \to W$ is a log elliptic fiber space and $K_V + \Delta_V = h^*(K_X + \Delta_X) + F$, where F is effective and $\operatorname{Supp} F$ contains every h-exceptional divisor.

15.1.3 Comments. The second definition is motivated by two examples. First, assume that (V, Δ_V) is a log elliptic fiber space and assume that (X, Δ_X) is obtained from (V, Δ_V) by $(K_V + \Delta_V)$-extremal contractions and flips. Then (X, Δ_X) has a log elliptic structure (we may have to blow up V a little).

Second, if X has terminal singularities, $\Delta_X = 0$ and X is birational to an elliptic fiber space $(V, 0)$ then X has an elliptic structure.

15.2 Proposition. *Let (X, Δ_X) be a proper klt variety. Assume that $K_X + \Delta_X$ is nef and that X has a log elliptic structure. Then there is an open set $U \subset X$ and a proper morphism $f_U : U \to Z$ which is a log elliptic fiber space.*

Proof. Let $E_g \subset V$ be the generic fiber of g. Then

$$0 = E_g \cdot (K_V + \Delta_V) = E_g \cdot h^*(K_X + \Delta_X) + E_g \cdot F \geq E_g \cdot F.$$

Thus F is disjoint from E_g and h is an isomorphism in a neighborhood of E_g. $\square$

For higher dimensional fibers the situation is more complicated. The following result (which is not used in the sequel) generalizes [Grassi91, 1.8].

15.3 Theorem. *Let X be a proper variety with $\mathbb{Q}$-factorial terminal singularities. Assume that $mK_X = 0$ for some $m > 0$ and $\rho(X) = 1$. Let $p : X \dashrightarrow Z$ be a dominant rational map with connected fibers. Then $p^{-1}(z)$ is of general type for every general $z \in Z$.*

Proof. Let $g : Y \to X$ be a proper birational morphism such that $f = p \circ g : Y \to Z$ is a morphism. Let $E \subset Y$ be the exceptional divisor of g. We may assume that Y is smooth. Let $H \subset Z$ be a divisor. Then $g(f^*(H))$ is an effective divisor on X, hence ample.

$$g^*(g(f^*(H))) = f^*(H) + F_1 \quad \text{where } \operatorname{Supp} F_1 \subset E.$$

Let $z \in Z - H$ be a point such that $f^{-1}(z)$ is smooth and $g|f^{-1}(z)$ is birational. Then

$$g^*(g(f^*(H)))|f^{-1}(z) = F_1|f^{-1}(z)$$

is the pull back of an ample divisor by the birational morphism $g|f^{-1}(z)$. In particular it is big. On the other hand, $K_Y = g^*K_X + F_2$ where $\operatorname{Supp} F_2 = E$. Thus

$$mK_{f^{-1}(z)} = mK_Y|f^{-1}(z) = mF_2|f^{-1}(z).$$

Since $\operatorname{Supp} F_1 \subset \operatorname{Supp} F_2$ this implies that $K_{f^{-1}(z)}$ is big. $\square$

15.4 Theorem. *Let (X, Δ_X) be a projective $\mathbb{Q}$-factorial threefold such that $K_X + \Delta_X$ is klt. Assume that*

(15.4.1) $K_X + \Delta_X$ is nef;

(15.4.2) $\dim |m(K_X + \Delta_X)| \geq 1$ for some $m > 0$;

(15.4.3) there is an open set $U \subset X$ and a proper morphism $f_U : U \to Z$ which is a log elliptic fiber space.

Then $K_X + \Delta_X$ is eventually free.

15.4.4 Remark. If $p : X \to Y$ is a morphism such that $K_X + \Delta_X = p^*(K_Y + \Delta_Y)$ then $K_X + \Delta_X$ is eventually free iff $K_Y + \Delta_Y$ is. Similarly, if $p : X \dashrightarrow Y$

is a $(K_X+\Delta)$-flop then $K_X+\Delta_X$ is eventually free iff $K_Y+p_*(\Delta_X)$ is. We use these observations to change X.

15.5 End of the proof of (11.1.1). Let X be a minimal threefold. The only case still open is when $\nu(X)=2$. We woud like to check the conditions of (15.4) in case $\Delta_X=0$. (15.4.1) is assumed and (14.4.1) shows (15.4.2). (15.4.3) requires a little work.

Lt (X', B') be as in (14.2). B' is a semi log canonical surface with $\nu(B')=1$. Thus by (11.3.1) it has an irreducible component which is birational to either a ruled or to an elliptic surface. We already know that $\dim|m(K_X+\Delta_X)|\geq 1$. Assume that we can construct (X', B') such that B' moves in a pencil. We obtain that X' contains a pencil of ruled or elliptic surfaces. X' is not uniruled, thus it has a pencil of elliptic surfaces. Therefore X is birational to an elliptic threefold, hence (15.2) implies (15.4.3).

Let us go back to the construction in (14.2) which was started in (13.2). (We use the notation employed there.) If $D\in|m(K_X+\Delta_X)|$ moves in a pencil then we can choose $\mu: X_0\to X$ such that $\tilde{D}$ still moves in a pencil. This pencil survives in all the contractions and flips. At the end we obtain (X', B') as in (14.2) such that B' moves in a pencil $\{B'_t\}$ and (X', B'_t) is log canonical for general t. At least one of the moving components of B'_t has $\nu(B'_t)=1$. Thus the above argument applies and (15.4) completes the proof of the abundance theorem for threefolds. $\square$

15.6 Definition. We say that an effective divisor $D\subset X$ is $(K_X+\Delta_X)$-*trivially connected* if for any two points $x_1, x_2\in D$ there is a connected curve $x_1, x_2\in C\subset D$ such that $K_X+\Delta_X$ is numerically trivial on every irreducible component of C.

15.7 Lemma. *Assume (15.4.1 and 2). Let $D\subset X$ be $(K_X+\Delta_X)$-trivially connected. Then one of the following holds:*

(15.7.1) $K_X+\Delta_X$ is eventually free and is composed of a pencil,

(15.7.2) there is an effective divisor D' and natural numbers d, m such that $dD+D'\in|m(K_X+\Delta_X)|$, $\operatorname{Supp} D\not\subset \operatorname{Supp} D'$ and $D\cap D'\neq\emptyset$.

Proof. Let $|m(K_X+\Delta_X)|=F+|M|$ where F is the fixed-part. Assume first that $|M|$ is composed of a free pencil. Let $p: X\to C$ be the corresponding morphism with connected fibers. Assume that we can not find $dD+D'$ as required. Then $\operatorname{Supp} D$ is a fiber of p, hence F is contained in a union of fibers. Since F is nef, F is the sum of rational multiples of fibers, hence some multiple of $K_X+\Delta_X$ is the pull-back of an ample divisor from C.

Otherwise there is a pencil $F'+|N_t|\subset|m(K_X+\Delta_X)|$ such that every N_t is connected and $|N_t|$ has a base point $b\in X$. $D\subset X$ is $(K_X+\Delta_X)$-trivially connected, thus if $B\in|m(K_X+\Delta_X)|$ intersects D then D is an irreducible component of B. If $b \in D$ then $D \subset F$ and any general N_t intersects D but i different from it. If $b \notin D$ then there is a t_0 such that N_{t_0} intersects D. N_t is connected and also contains b, thus we are again done. □

15.8 Lemma. *Assumptions as in (15.7) and assume that (15.7.2) holds Then $K_X + \Delta_X + \epsilon D$ is not nef for $\epsilon > 0$.*

Proof. By assumption there is an irreducible curve $C \subset D$ such that $C\cdot(K_X + \Delta_X) = 0$ satisfying $C \cap D' \neq \emptyset$ and $C \not\subset D'$. Thus

$$0 = C \cdot (dD + D') = dC \cdot D + C \cdot D', \quad \text{hence} \quad C \cdot D < 0.$$

Therefore $C \cdot ((K_X + \Delta_X) + \epsilon D) = \epsilon C \cdot D < 0$. □

15.9 Corollary. *Assumptions as in (15.4). Then one of the following holds:*

(15.9.1) $|n(K_X + \Delta_X)|$ is composed of a free pencil for some $n > 0$; or

(15.9.2) there is a log variety $(X', \Delta_{X'})$ which is log birational to (X, Δ_X) and satisfies all the assumptions of (15.4) and such that X' does not contain any $(K_{X'} + \Delta_{X'})$-trivially connected divisors.

Proof. Assume that X contains a $(K_X + \Delta_X)$-trivially connected divisor D. Then either (15.7.1) holds or $K_X + \Delta_X + \epsilon D$ is not nef. After a sequence of D-flops (with respect to $K_X + \Delta_X$) the birational transform of D becomes contractible. For this it is sufficient to observe that the birational transform of D under a sequence of flops stays $(K + \Delta)$-trivially connected. The general fiber of the elliptic fibration is disjoint from D, thus (15.4.3) is preserved under flops and $(K + \Delta)$-trivial contractions. Repeating this procedure, we eventually stop at X'. □

15.10 Theorem. *Assumptions as in (15.4). Assume furthermore that X does not contain any $(K_X+\Delta_X)$-trivially connected divisors. Then f_U extends to a morphism $f : X \to \bar{Z}$ with 1-dimensional fibers.*

Proof. By shrinking Z we may assume that f_U is flat. Thus we get a morphism $Z \to \mathrm{Chow}(X)$. (See [Hodge-Pedoe52, X.6-8] for basic results about Chow varieties.) Let $\bar{Z}$ be the normalization of the closure of the image and let $g : \bar{U} \to \bar{Z}$ be the universal family. Let $u : \bar{U} \to X$ be the natural morphism. We prove that u is an isomorphism.

u is an isomorphism over $g^{-1}(Z)$. Assume that $F \subset \bar{U}$ is a divisor contracted by u. Then $g(F)$ is at most one dimensional. Since g has one dimensional fibers, $g(F)$ is one dimensional. Let $E = g^{-1}(g(F))$. $\dim u(E) = 2$ since a 1-dimensional subvariety of X supports only countably many different cycles in $\mathrm{Chow}(X)$. (This is the point where we need Chow instead of Hilb.) Thus there are divisors $E_1, E_2 \subset E$ such that

$$\dim u(E_1) = 2; \quad \dim u(E_2) \leq 1 \quad \text{and} \quad E_1 \cap E_2 \text{ dominates } g(F).$$

We claim that $u(E_1)$ is $(K_X+\Delta_X)$-trivially connected. Indeed, $u(E_1\cap E_2)\subset u(E_2)$ and every curve in $u(E_2)$ has zero intersection with $K_X+\Delta_X$. Any two points of $u(E_1)$ can be connected by images of fibers of $E_1\to g(F)$ and by $u(E_1\cap E_2)$.

This contradiction shows that u does not contract any divisors. Since X is $\mathbb{Q}$-factorial, u can not contract curves, and thus u is an isomorphism. $\square$

15.11 Lemma. *Let X be a variety with log terminal singularities. Let $f: X\to Z$ be a proper morphism onto a normal variety Z such that every fiber has dimension k for some fixed k. If $\dim Z>2$ then assume that K_Z is $\mathbb{Q}$-Cartier. Then Z has only log terminal singularities.*

Proof. Choose a projective embedding of X. Fix $z\in Z$. Let $H\subset X$ be a complete intersection of k general hyperplanes. $H\to Z$ is dominant and we may assume that $H\to Z$ is finite over z. H has a log terminal singularities (cf. [Reid80, 1.13]) thus by (20.3.1) Z has a log terminal singularity at z. $\square$

15.11.1 Remark. Shokurov pointed out that under the assumptions of (15.11) if X is $\mathbb{Q}$-factorial then so is Z.

15.12 Proposition. *Let $f:(X,\Delta_X)\to Z$ be a log elliptic fiber space with 1-dimensional fibers. Assume that (X,Δ_X) is lc and nef. Then there is a line bundle L on Z such that*

$$n(K_X+\Delta_X)\sim f^*L \quad \text{for some } n>0.$$

Proof. A general fiber E_g of f is either an elliptic curve (which is disjoint from Δ_X) or is a rational curve. In either case a multiple of $K_X+\Delta_X$ is linearly equivalent to zero on the generic fiber. Thus there is a (not necessarily effective) divisor D which is disjoint from E_g and is linearly equivalent to $n_0(K_X+\Delta_X)$ for some $n_0>0$. Let $C_i\subset Z$ be the irreducible components of $f(\operatorname{Supp} D)$. We can write $D=\sum D_i$ where the D_i are those components that map onto C_i. Let z_i be a general point of C_i. Then D_i is nef on $f^{-1}(z_i)$, thus D_i is a rational multiple of $f^*(C_i)$. Hence $n_iD_i=f^*(m_iC_i)$ for some $n_i>0$ (possibly $m_i<0$). Choose M such that

$$M\sum\frac{m_i}{n_i}C_i$$

is Cartier. Then

$$Mn_0(K_X+\Delta_X)\sim f^*\mathcal{O}_Z\left(M\sum\frac{m_i}{n_i}C_i\right). \quad \square$$

(15.13) Proof of (15.4). If $|n(K_X + \Delta_X)|$ is composed of a base point free pencil then we are done. Otherwise $\nu(K_X + \Delta_X) \geq 2$.

By (15.9) there is a series of flops and $(K + \Delta)$-trivial contractions $X \dashrightarrow X'$ such that X' does not contain $(K + \Delta)$-trivially connected surfaces. By (15.4.4) it is sufficient to show that $K_{X'} + \Delta_{X'}$ is eventually free. (15.10) gives a proper morphism $f : X' \to \bar{Z}$ and by (15.12) there is a line bundle L on $\bar{Z}$ such that $n(K_{X'} + \Delta_{X'}) \sim f^*L$.

I claim that L is ample. This is proved using the Nakai–Moishezon criterion. Let H be ample on X and let E_g be a general fiber of f. Then

$$(E_g \cdot H)(L \cdot L) = H \cdot f^*L \cdot f^*L = n^2 H \cdot (K_{X'} + \Delta_{X'}) \cdot (K_{X'} + \Delta_{X'}) > 0.$$

If $C \subset Z$ is an irreducible curve such that $C \cdot L = 0$ then $K_{X'} + \Delta_{X'}$ is numerically trivial on $f^{-1}(C)$, a contradiction. Thus L is ample, and hence a suitable multiple of L is generated by global sections. □

16. ADJUNCTION OF LOG DIVISORS

ALESSIO CORTI

In this chapter we discuss several matters connected with the adjunction formula for a Weil divisor $S \subset X$ inside a normal space X. The first goal is to define a *different* Diff which is a $\mathbb{Q}$-divisor on S so that the following adjunction formula holds:

$$K_S + \mathrm{Diff} = K_X + S\,|_S\,.$$

[Shokurov91,Ch.3] defines the different as a divisor on the normalization S^ν of S, and uses the notion to establish some elementary properties of log terminal singularities. However, it is desirable to deal with the reduced part of the boundary of a log divisor without normalizing it. For this reason we define the different directly on S.

Once the different is defined, we use it to relate properties of (X, S) to (S, Diff).

We begin with some preliminaries on Weil divisors on nonnormal varieties. In the following, X is a pure dimensional reduced scheme. After (16.7) we always assume that X is defined over an algebraically closed field of characteristic zero. X may be reducible and not necessarily S_2. $K(X)$ denotes the sheaf of total quotient rings (see e.g. [Hartshorne77, II.6]).

16.1 Definition.

(16.1.1) A *Weil divisorial subsheaf* is a coherent $\mathcal{O}_X$-module $\mathcal{L}$, which is principal in codimension one and saturated, *together with the choice of an embedding* $\mathcal{L} \subset K(X)$. The condition that $\mathcal{L}$ is free in codimension one implies $\mathcal{L} \cong \mathcal{L}^{**}$, provided X is S_2. The embedding $\mathcal{L} \subset K(X)$ is very important, although, following common useage in the literature, I will occasionally be sloppy about it (see 16.3.3).

(16.1.2) Define the product $\mathcal{L} \cdot \mathcal{L}' \subset K(X)$ in the natural way (i.e. $\mathcal{L} \cdot \mathcal{L}'$ is the saturation of the product of sheaves $\mathcal{L}\mathcal{L}' \subset K(X)$). Note that in general the natural homomorphism $\mathcal{L} \otimes \mathcal{L}' \to \mathcal{L} \cdot \mathcal{L}'$ is neither injective nor surjective (it is, however, an isomorphism, whenever $\mathcal{L}$ or $\mathcal{L}'$ is locally $\mathcal{O}_X$-free). We also write $\mathcal{L}^{[n]}$ for the product of $\mathcal{L}$ with itself n-times. With these laws, the set of Weil divisorial subsheaves is a group which we denote by $\mathrm{WSh}(X)$. In a natural way $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^{-1} = \{x \in K(X) \mid x \cdot \mathcal{L} \subset \mathcal{O}_X\} \subset K(X)$.

Equivalently, let $\mathrm{CDiv}(U)$ be the group of Cartier divisors on a scheme U. Then

$$\mathrm{WSh}(X) = \operatorname{proj\,lim} \mathrm{CDiv}(X \setminus S)$$

where the limit is over all closed subschemes $S \subset X$ such that $\operatorname{codim}_X S \geq 2$.

If X is normal then this is the usual definition. However for nonnormal schemes unexpected things can happen. Let for instance

$$X = \operatorname{Spec} \mathbb{C}[x, y, z, z^{-1}]/(x^2 - zy^2).$$

The ideals (x) and (y) define different Weil divisorial subsheaves such that $(x)^{[2]} = (y)^{[2]}$.

(16.1.3) The group of $\mathbb{Q}$-Weil divisorial sheaves is defined as $\mathrm{WSh}(X)_{\mathbb{Q}} = \mathrm{WSh}(X) \otimes \mathbb{Q}$.

(16.1.4) To each unit $x \in K(X)^*$ there is a naturally associated Weil divisorial subsheaf $(x) = x \cdot \mathcal{O}_X \subset K(X)$. We say that two Weil divisorial subsheaves $\mathcal{L}$ and $\mathcal{L}'$ are *linearly equivalent* and write $\mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L}^{-1} \cdot \mathcal{L}' = (x)$ for some $x \in K(X)^*$.

(16.1.5) If $\mathcal{L}$ is a Weil divisorial subsheaf, we define the *support* of $\mathcal{L}$ to be the Zariski closed subset $\mathrm{Supp}(\mathcal{L}) \subset X$ of points where $\mathcal{L} \neq \mathcal{O}_X$.

(16.1.6) $\mathcal{L} \subset K(X)$ is effective if $\mathcal{O}_X \subset \mathcal{L} \subset K(X)$.

16.2 Definition.

(16.2.1) A Weil divisor on X is a formal linear combination:

$$D = \sum n_\Gamma \Gamma,$$

where the sum extends over all points of codimension one $\Gamma \subset X$ such that $\mathcal{O}_{X,\Gamma}$ is a DVR, and n_Γ are integers, only finitely many of which are nonzero. The group of all Weil divisors is denoted by $\mathrm{WDiv}(X)$. As in (16.1.3), $\mathrm{WDiv}(X)_{\mathbb{Q}} = \mathrm{WDiv}(X) \otimes \mathbb{Q}$

(16.2.2) There is a natural injective group homomorphism $\mathrm{WDiv}(X) \ni D \mapsto \mathcal{O}(D) \in \mathrm{WSh}(X)$. Let $\Gamma \subset X$ be a codimension one prime of X, then $\mathcal{O}(D)$ is uniquely determined by $\mathcal{O}(D)_\Gamma = \mathcal{O}_{X,\Gamma}$ if X is not regular at Γ, and $\mathcal{O}(D)_\Gamma = t^{n_\Gamma} \cdot \mathcal{O}_{X,\Gamma}$ if $\mathcal{O}_{X,\Gamma}$ is a DVR.

If $\mathcal{L}$ is a Weil divisorial subsheaf, $\mathcal{L}(D)$ as usual denotes $\mathcal{L} \cdot \mathcal{O}(D)$.

We say that D and D' are linearly equivalent if the corresponding sheaves are.

Also, perhaps inappropriately, we say that a Weil divisorial subsheaf $\mathcal{L} \subset K(X)$ is a Weil divisor if $\mathcal{L} = \mathcal{O}(D)$ for some Weil divisor D. Of course, this is equivalent to saying that no codimension one component of the support of $\mathcal{L}$ is contained in the singular locus of X.

16.3 Remarks and more definitions.

(16.3.1) The inclusion $\mathrm{WDiv}(X) \subset \mathrm{WSh}(X)$ induces an isomorphism

$$\mathrm{WDiv}(X)/\sim \,\cong \mathrm{WSh}(X)/\sim,$$

and we denote any of these two groups by $\mathrm{Weil}(X)$.

(16.3.2) $\mathcal{O}_X \subset K(X)$ is a Weil divisorial subsheaf precisely when X is S_2.

(16.3.3) The dualizing sheaf ω_X (as in [Hartshorne77, III.7], that is, $\omega_X = H^{-d}(\omega_X^{\cdot})$ if $\omega_X^{\cdot}$ is the normalized dualizing complex) is torsion free of rank one, and admits therefore an embedding $\omega_X \subset K(X)$. Since we also know that ω_X is saturated (see e.g. [Reid80, App. to §1]), ω_X is a Weil divisorial subsheaf precisely when X is Gorenstein in codimension one. This is why later (16.5) we shall assume this condition (which is satisfied for example if X has normal crossings in codimension one). If this is the case then with an appropriate choice of embedding $\omega_X \subset K(X)$, ω_X is actually a Weil divisor, whose linear equivalence class is denoted by K_X.

(16.3.4) Weil divisors and sheaves are codimension one constructions. This means that X may always be replaced with any open subset $U \subset X$ such that $\mathrm{codim}_X(X \setminus U) \geq 2$. This principle is used in many natural constructions like pullbacks and restrictions, as well as in many proofs (sometimes without explicit mention).

(16.3.5) Let $p : X' \to X$ be a finite dominant morphism. There is a natural pullback

$$p^w : \mathrm{WSh}(X) \to \mathrm{WSh}(X').$$

This is defined on $\mathcal{L}$ by taking $U \subset X$ open with $\mathrm{codim}_X(X \setminus U) \geq 2$, and such that $\mathcal{L}$ is locally free on U. Then on $V = p^{-1}(U)$, $p^w(\mathcal{L}) = p^*(\mathcal{L})$ is a locally free subsheaf of $K(V)$, and defines a Weil divisorial subsheaf on X' (16.3.4).

(16.3.6) Similarly, let $i : S \hookrightarrow X$ be a subscheme of pure codimension one. Denote by $\mathrm{WSh}_S(X)$ the subgroup of sheaves $\mathcal{L}$ which are $\mathbb{Q}$-Cartier at all points $P \subset S$ of codimension one, and such that S and $\mathrm{Supp}(\mathcal{L})$ have no common irreducible components (if these conditions are satisfied we say that $\mathcal{L}$ has good support on S). Then we have a natural restriction homomorphism:

$$i^w : \mathrm{WSh}_S(X) \to \mathrm{WSh}(S)_{\mathbb{Q}}.$$

This is defined as follows. If $\mathcal{L}$ is Cartier at points $P \subset S$ of codimension one, let $U \subset X$ be an open subset such that $\mathrm{codim}_X(X\setminus U) \geq 2$, $\mathrm{codim}_S(S\setminus U) \geq 2$ and $\mathcal{L}$ is Cartier on U. Then on $V = S \cap U$, $i^w\mathcal{L}$ is the usual restriction of a Cartier divisor (and, because $\mathcal{L}$ has good support on S, $\mathcal{L} \subset K(X)$ induces $i^w\mathcal{L} \subset K(S)$). This determines $i^w\mathcal{L}$ on S. If $\mathcal{L} \in \mathrm{WSh}_S(X)$, then $\mathcal{L}^{[n]}$ is Cartier at points $P \subset S$ of codimension one for some $n > 0$. $i^w\mathcal{L}$ is defined to be $\frac{1}{n} i^w\mathcal{L}^{[n]}$. This is independent of the choice of n. We also write $\mathcal{L}|S$ instead of $i^w(\mathcal{L})$. The whole point of this construction is of course that we want to define i^w in such a way that it is functorial and a group homomorphism.

Next we state the adjunction formula for a divisor $i : S \hookrightarrow X$. If $\mathcal{F}$ is a sheaf on X, we write $i^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_S$ and

$$i^{[*]}\mathcal{F} \stackrel{\text{def}}{=\!=} \text{saturation of } (i^*\mathcal{F}/\mathrm{Torsion}_{\mathcal{O}_S}(i^*\mathcal{F})).$$

16.4 Proposition. *Let X be a normal scheme (actually it is enough that X is S_2), and $i : S \hookrightarrow X$ a reduced subscheme of pure codimension one. Then there is a canonical isomorphism:*

$$\omega_S = i^{[*]}\omega_X(S).$$

In particular:

(16.4.1) If X is S_3 and S is a Cartier divisor, then $\omega_S = \omega_X \otimes \mathcal{O}_X(S) \otimes \mathcal{O}_S$.

(16.4.2) If $\omega_X(S)$ is locally free and S is S_2, then $\omega_S = \omega_X(S) \otimes \mathcal{O}_S$. In particular ω_S is locally free and S is Gorenstein if it is CM (=Cohen-Macaulay).

(16.4.3) If $\omega_X(S)$ is Cartier at every codimension one point $P \in S$, then $\omega_S = i^w\omega_X(S)$. In particular then S is Gorenstein in codimension one, and choosing suitable embeddings we may write the above isomorphism in the form $K_S = K_X + S|S$.

Proof. By assumption X is CM outside a set Z of codimension three; by considering $X \setminus Z$ we may assume that X is CM.

Along the lines of [Hartshorne77, III.7] it is easy to check that $\omega_S = \mathcal{E}xt^1(\mathcal{O}_S, \omega_X)$ is a dualizing sheaf for S. Applying $\mathcal{H}om_{\mathcal{O}_X}(\,\cdot\,, \omega_X)$ to the exact sequence:

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

(since S is a Weil divisor, $\mathcal{I}_S = \mathcal{O}_X(-S) \subset \mathcal{O}_X$, with the notation of (16.2.2)), we obtain an exact sequence:

$$0 \to \omega_X \to \omega_X(S) \to \omega_S \to 0,$$

which fits into a commutative diagram (with exact rows):

$$\begin{tikzcd}
 & \omega_X(S) \otimes \mathcal{O}_X(-S) \arrow[r] \arrow[d] & \omega_X(S) \arrow[r] \arrow[d, equal] & \omega_X(S) \otimes \mathcal{O}_S \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & \omega_X \arrow[r] & \omega_X(S) \arrow[r] & \omega_S \arrow[r] & 0.
\end{tikzcd}$$

This shows that $\alpha : \omega_X(S) \otimes \mathcal{O}_S \to \omega_S$ is *surjective*. S is a Weil divisor, and hence X is smooth at every generic point of S. Therefore α is an isomorphism at generic points of S, so $\omega_X(S) \otimes \mathcal{O}_S / \mathrm{Torsion}_{\mathcal{O}_S}(\omega_X(S) \otimes \mathcal{O}_S) \cong \omega_S$, which is what we want. $\square$

16.4.4 Example. Let $A \subset \mathbb{P}^{n-1}$ be a smooth, projectively normal, Abelian surface and let $X \subset \mathbb{P}^n$ be the cone over A with vertex $x \in X$. Then X is normal, lc and $\omega_X \cong \mathcal{O}_X(-1)$. However X is not S_3. Let $x \in H \subset X$ be a hyperplane section, smooth outside x. H is not normal; let $p : \bar{H} \to H$ be the normalisation. Then

$$\omega_H = p_*(\mathcal{O}_{\bar{H}}) \neq \mathcal{O}_H = Ext^1(\mathcal{O}_H, \omega_X).$$

The aim is to generalize the adjunction formula (16.4.3) to the case where $\omega_X(S)$ is only $\mathbb{Q}$-Cartier at codimension one points $P \subset S$. This is accomplished in the following:

16.5 Proposition - Definition. *Let X be a normal scheme, $i : S \hookrightarrow X$ a reduced subscheme of pure codimension one. Assume that S is Gorenstein in codimension one and that $\omega_X(S) \in \mathrm{WSh}_S(X)$. Then there is a naturally defined effective different $\mathcal{D}iff(0) \in \mathrm{WSh}(S)_{\mathbb{Q}}$ so that:*

$$\omega_S \cdot \mathcal{D}iff(0) = i^w \omega_X(S).$$

If $B \in \mathrm{WSh}_S(X)_{\mathbb{Q}}$, we also define the different of B by $\mathcal{D}iff(B) = \mathcal{D}iff(0) \cdot i^w B$.

Proof. We systematically remove codimension 2 subsets $Z \subset S$, whenever needed, without warning.

From the adjunction formula (16.4) we know that $\omega_S = \omega_X(S)_S$. Suppose that $\omega_X(S)^{[n]}$ is Cartier at every codimension one point $P \in S$. Consider the sequence of maps

$$\left(\omega_X(S) \otimes \mathcal{O}_S\right)^{\otimes n} \cong \omega_X(S)^{\otimes n} \otimes \mathcal{O}_S^{\otimes n} \to \omega_X(S)^{\otimes n} \otimes \mathcal{O}_S \to \omega_X(S)^{[n]} \otimes \mathcal{O}_S.$$

Taking the quotient by the torsion submodules we obtain

$$\omega_S^{[n]} \xrightarrow{\delta} \omega_X(S)^{[n]} \otimes \mathcal{O}_S$$

which is an isomorphism at the generic points of S because X is normal. δ defines a Weil divisorial subsheaf $\mathcal{D}$ on S so that $\omega_S^{[n]} \cdot \mathcal{D} = i^w \omega_X(S)^{[n]}$. Since the isomorphism of the adjunction formula is natural, $\mathcal{D}$ is well defined (i.e., it does not depend on the embedding $\omega_X(S) \subset K(X)$). Set $\mathcal{D}iff(0) = \frac{1}{n}\mathcal{D}$. $\square$

We now apply the different to study log canonical and log terminal singularities. The nice fact is that if $K_X + S$ is log canonical in codimension two, the different is actually a Weil divisor (i.e., no codimension one component of the support of $\mathcal{D}iff$ is contained in the singular locus of S). Also, under the same assumptions, we compute the different.

16.6 Proposition. *Let X be a normal space, $S \subset X$ a reduced subscheme of pure codimension one and B a $\mathbb{Q}$-Weil divisor. Assume that K_X+S+B is log canonical in codimension two. Then S has normal crossings in codimension one, so the assumptions of (16.5) are satisfied. Moreover the different $\mathcal{D}iff(B)$ is a $\mathbb{Q}$-Weil divisor (that is, no codimension one component of the support of $\mathcal{D}iff(B)$ is contained in the singular locus of S), which is denoted by* $\mathrm{Diff}(B)$.

Let $P \subset S$ be a codimension one point of S. The following computes the coefficient p of the different $\mathrm{Diff}(0)$ *at P:*

(16.6.1) If S has two branches at P then $P \notin \mathrm{Supp}\, B$ and $p = 0$.
This follows from the more precise result that one of the following holds:

(16.6.1.1) $K + S$ is lt at P, X is smooth at P, and S is a normal crossing divisor at P.

(16.6.1.2) $K + S$ is lc but not lt at P. Then $K + S$ is Cartier at P. More precisely, locally analytically at P, $S \subset X$ is isomorphic to $(C \subset T) \times \mathbb{C}^{d-2}$, where $(C \subset T) \cong ((xy = 0) \subset \mathbb{C}^2/\mathbb{Z}_m)$ and $\mathbb{Z}_m$ acts with weights $(1, q)$ with $(q, m) = 1$.

(16.6.2) If S has one branch at P, and $K + S$ is lc but not lt at P, then $p = 1$.
More precisely $K + S$ has index two at P. Let $\pi : X' \to X$ be the index one cover, and $S' = \pi^{-1}(S)$. Then $S' \subset X'$ is as in (16.6.1.2).

(16.6.3) If S has one branch at P and $K + S$ is lt at P, then, locally analytically at P, $S \subset X$ is isomorphic to $(C \subset T) \times \mathbb{C}^{d-2}$, where $(C \subset T) \cong ((x = 0) \subset \mathbb{C}^2/\mathbb{Z}_m)$ and $\mathbb{Z}_m$ acts with weights $(1, q)$ with $(q, m) = 1$. Also, the local class group $\mathrm{Weil}(\mathcal{O}_{X,P}) \cong \mathbb{Z}_m$, *and X is smooth at P iff $m = 1$. In particular:*

$$p = \frac{m-1}{m},$$

where m is characterized by any of the following properties:

(16.6.3.1) m is the index of $K + S$ at P;

(16.6.3.2) m is the index of S at P;

(16.6.3.3) m is the order of the cyclic group $\mathrm{Weil}(\mathcal{O}_{X,P})$.

Proof. I may assume that X is a surface. All the statements then follow from the classification of log canonical surface singularities in Chapter 3. That $\mathcal{D}iff(B)$ is a Weil divisor also follows from the classification, more specifically from (16.6.1) above. In (16.6.2), it is easy to check that $K_{S'} = (\pi|S')^w(K_S + P)$. □

16.7 Corollary. *Assumptions as in (16.6). Let $B = \sum b_i B_i$. The coefficient of $[P]$ in* $\mathrm{Diff}(B)$ *is*

$$\begin{array}{ll} 0 & \textit{in case (16.6.1);} \\ 1 & \textit{in case (16.6.2);} \\ 1 - \dfrac{1}{m} + \sum \dfrac{r_i b_i}{m} & \textit{in case (16.6.3), for suitable } r_i \in \mathbb{N}. \end{array}$$

Proof. In the first two cases $P \notin \operatorname{Supp} B$, so (16.6) applies directly. In the last case the local class group has order m. Thus mB_i is Cartier at P hence $i^w(\mathcal{O}_X(B_i)) = (r_i/m)\mathcal{O}_S(P)$ for some $r_i \geq 0$. □

16.8 Remark. The different is used in the following situation. Let X be a normal variety, and $K_X + S + B$ a log divisor with S reduced and $\llcorner B \lrcorner = 0$. Then if $K_X + S + B$ is lt, it should be true that $K_S + \mathrm{Diff}(B)$ is lt (and conversely) in some suitable sense. Now in general S is a variety with double normal crossings in codimension one and we need to use the appropriate notions of semi log terminal etc. introduced in (12.2).

Unfortunately we encounter the following technical problem:

The birational transform of $S \subset X$ in a log resolution of $(X, S + B)$ is in general not a semi resolution of S since different components may get separated. Also, the exceptional role of higher normal crossing points complicates the formulation of the result (cf. (16.9.2)). (Recent results of Szabó seem to have settled this problem.)

In dimension three one can overcome some of these problems. The results become somewhat cumbersome, mostly due to our choice of definition of log terminal.

16.9 Proposition. *Let X be a normal threefold, $K + S + B$ a log divisor with S reduced. Then:*

(16.9.1) If $K + S + B$ is lc then $K_S + \mathrm{Diff}(B)$ is slc.

(16.9.2) Let $K + S + B$ be dlt, and $f : Y \to X$ a good divisorial resolution. Assume that $\llcorner B \lrcorner = \emptyset$. Then, outside a number of triple normal crossing points at which f is an isomorphism, $K_S + \mathrm{Diff}(B)$ is semi lt. Moreover, S has a semiresolution without pinch points.

Proof. Let us prove (16.9.2) first. Let $S' = f_*^{-1}(S)$. Since $K + S + B$ is lc in codimension 2, S is semismooth outside a finite set. We have by definition:

$$K_Y + S' = f^*(K_X + S + B) + \sum a_i E_i \tag{16.9.3}$$

with all $a_i > -1$ ($\llcorner B \lrcorner = 0$). In particular, f is generically an isomorphism above the normal crossing locus of S. Also, because X is divisorial lt, no component of the double curve of S' is mapped to a point. All this says that $S' \to S$ is a good semiresolution outside the triple points. By our definitions, S' has no pinch points. Note that since f is divisorial, it sends a neighbourhood of the triple normal crossing locus of S' isomorphically to a neighbourhood of the triple normal crossing locus of S. Now from (16.9.3) and (16.5) we get that

$$(16.9.4) \qquad K_{S'} = (f|S')^*(K_S + \mathrm{Diff}(B)) + \sum a_i E_i|S'.$$

We see later in (17.5) that S is S_2 and seminormal. This however is not important for the rest of the chapter.

(16.9.1) is similar but easier: it is not true that S' is a semiresolution of S, but this does not affect the slc property (cf. [KSB88, 4.30]). □

16.10 Corollary. *Let $(x \in X)$ be a three dimensional germ, $S \subset X$ a reduced boundary. If $K_X + S$ is divisorial log terminal and S has at least three components at x, $(x \in S \subset X)$ is analytically isomorphic to $(0 \in (xyz = 0) \subset \mathbb{C}^3)$.*

Proof. By (12.2.7) an slt point cannot have three or more components. □

16.11 Example. The assumption dlt is necessary in (16.9.2) and (16.10). Indeed, let $S \subset X$ be $(xw = 0) \subset ((xy + zw = 0) \subset \mathbb{C}^4)$. Then $K_X + S$ is lt, as can be seen on any of the two standard small resolutions. $K_X + S$ however is not dlt. Here $K_S = K_S + \mathrm{Diff}(0)$ and S has a log canonical quadruple point at the origin.

The rest of the chapter is devoted to the classification of log terminal singularities (X, D) in dimension three where $\llcorner D \lrcorner$ is "large". These results will not be used later. It gives however a good flavour of how to work with log terminal singularities and with the different.

The presence of a reduced boundary imposes strong restrictions on log terminal singularities; an example is (16.10). A key tool in classifying terminal and log terminal singularities are standard coverings of various kinds (cf. [CKM88,6.7]):

16.12 Lemma. *Let $0 \in X$ be a germ of a normal variety, $D \subset X$ a $\mathbb{Q}$-Cartier integral Weil divisor. There is a cyclic covering $p : X' \to X$, which is uniquely determined by the following properties:*

(16.12.1) $p^*D = D' \subset X'$ is a Cartier divisor.

(16.12.2) p is étale in codimension one and is (totally) ramified precisely along the locus where D is not Cartier.

X' can also be characterized as the smallest covering of X such that D' is Cartier. X' is called the index one cover relative to D. □

To a log divisor $K_X + B$ as above one can associate two index one covers $X' \to X$, relatively to $K_X + B$ or B. It is useful to be able to relate the log terminal property of X and X'.

16.13 Lemma. *Let X be a normal variety, $p : X' \to X$ any finite morphism which is étale in codimension one. Then:*

(16.13.1) If X has canonical (terminal) singularities, so does X'.

(16.13.2) Let $B \subset X$ be a boundary (possibly empty) and let $B' = p^*B$.

(16.13.2.1) $K_X + B$ is lc iff $K_{X'} + B'$ is lc.

(16.13.2.2) $K_X + B$ is plt iff $K_{X'} + B'$ is plt

(16.13.2.3) If p is a cyclic cover, X is a threefold and $K_X + B$ is dlt (resp. lt), then so is $K_{X'} + B'$. Furthermore,

$$(B' \subset X') \cong ((xyz = 0) \subset \mathbb{C}^3) \quad \Leftrightarrow \quad (B \subset X) \cong ((xyz = 0) \subset \mathbb{C}^3).$$

Proof. (16.13.1) is [CKM88, 6.7.(ii)]. (16.13.2.1-2) is proved in (20.3). We only prove (16.13.2.3) for dlt here, the lt case is the same. This also illustrates pretty well the difficulties involved in working with the notion of log terminal.

Let $f : Y \to X$ be a good divisorial resolution such that $K_Y + f_*^{-1}B + E = f^*(K_X + B) + \sum a_i E_i$ with all $a_i > 0$ where $E = \sum E_i$ is the f-exceptional divisor. Let $Y' = (Y \times_X X')^\nu$ be the normalized pull back, so that we have a diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{p} & X \end{array}$$

Let E' be the f'-exceptional set. The crux of the argument is to be able to construct a good divisorial resolution $\varphi : \tilde{Y} \to Y'$, with the property that the image of the φ-exceptional locus is entirely contained in E'. The point is that since p is étale in codimension one, p' can only be ramified along E, and since E is a normal crossing divisor, Y' has toroidal singularities. Set $B' = p^*(B)$.

Pick a point $q \in f_*^{-1}B$. Choose local coordinates (x, y, z) near $q \in Y$ such that the components of $E \cup f_*^{-1}B$ are the coordinate planes. Locally, the covering is the normalization of

$$(t^d = x^a y^b z^c) \subset \mathbb{C}^1 \times \mathbb{C}^3.$$

The local equation of $f_*^{-1}B$ is one of the following: $(xyz = 0)$, $(xy = 0)$, $(x = 0)$ or $(1 = 0)$. In the first case p' is unramified along the coordinate planes, thus $a = b = c = 0$ and p' is étale above q. In the second case p' is unramified along two of the the coordinate planes, thus $a = b = 0$ and $(f')_*^{-1}B' \subset Y'$ is a (double) normal crossing point. In the third case p' is unramified along one of the the coordinate planes, thus $a = 0$. Let T be the normalization of the surface singularity $(t^d = y^b z^c)$. Then

$$[(f')_*^{-1}B' \subset Y'] \cong [T \times \{0\} \subset T \times \mathbb{C}].$$

Therefore Y' is smooth along $(f')_*^{-1}B'$, except possibly for some curves $C_i \subset Y'$ of cyclic quotient singularities that meet $(f')_*^{-1}B'$ transversally. We begin constructing a resolution by resolving Y' along C_i. (We care only about a neighborhood of $(f')_*^{-1}B'$ in this step.) This gives $\varphi' : Y'' \to Y'$. Y'' is smooth in a neighborhood of $(\varphi')_*^{-1}(f')_*^{-1}B'$, and $(\varphi')_*^{-1}(f')_*^{-1}B' + E''$ is a global normal crossing divisor in a neighborhood of $(\varphi')_*^{-1}(f')_*^{-1}B'$. It is clear that a good divisorial resolution can now be achieved by blowing up centers contained in E'' only (and not intersecting $(\varphi')_*^{-1}(f')_*^{-1}B'$).

The rest is an easy consequence of the log ramification formula (20.2). The situation now is the following:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{p}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{p} & X. \end{array}$$

Here $\tilde{f} : \tilde{Y} \to X'$ is a good divisorial resolution, $\tilde{p}$ is generically finite, and F_j being any $\tilde{f}$-exceptional component, $\tilde{p}(F_j) \subset E_i$ for some f-exceptional component E_i. Write

$$K_{\tilde{Y}} + \tilde{f}_*^{-1}B' = \tilde{f}^*(K_{X'} + B') + \sum b_j F_j.$$

Then if $\tilde{p}^* E_i = \sum_j e_{ij} F_j$, we have $b_j = \sum_i e_{ij} a_i + r_j$ with $r_j \geq 0$, by the log ramification formula. Since $\tilde{p}(F_j) \subset E_i$ for some i, we see that $b_j > 0$.

Finally, if $f : Y \to X$ is not the identity then by our construction $\tilde{f} : \tilde{Y} \to X'$ is not the identity, thus $(B' \subset X')$ is different from $((xyz = 0) \subset \mathbb{C}^3)$. □

16.14 Remark. The converse to (16.13.2.3) is probably also true. Here, however, the problem is to find a suitable resolution of X, without blowing up the double locus of B. This does not follow directly from Hironaka. (Recently Szabó settled this question.)

From now on X is a threefold, and $B \subset X$ a $\mathbb{Q}$-Cartier reduced boundary such that $K_X + B$ is divisorial log terminal. We begin by classifying these singularities.

16.15 Theorem. *Let $x \in B \subset X$ be a three dimensional germ, assume $K + B$ dlt and B $\mathbb{Q}$-Cartier. Then:*

(16.15.1) If B has three components, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (xyz = 0) \subset \mathbb{C}^3.$$

(16.15.2) If B has two components, both of which are $\mathbb{Q}$-Cartier, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_m(q_1, q_2, 1) \quad \text{where} \quad (q_1, q_2, m) = 1.$$

(16.15.3) If B has two components, neither of which is $\mathbb{Q}$-Cartier, then $x \in B \subset X$ is analytically isomorphic to

$$0 \in (z = 0) \subset (xy + zf(z,t) = 0) \subset \mathbb{C}^4/\mathbb{Z}_m(q_1, -q_2, 1, a)$$
$$\text{where} \quad (q_i, a, m) = (q_1, q_2, m) = 1.$$

Proof. (16.15.1) is a special case of (16.10), so let's prove (16.15.2–3).

Let $p : X' \to X$ be the index one cover relative to $K_X + B$, and set $B' = p^*B$. By (16.13.2.3), $K_{X'} + B'$ is dlt. Note that $K_B + \mathrm{Diff}(0) = K_B + \sum \frac{m_i - 1}{m_i} P_i$, where $P_i \subset B \subset X$ are codimension two singular points on X as in (16.6.3). Also by (16.6.3), B' is smooth at $p^{-1}(P_i)$, and $p|B'$ is ramified in codimension one precisely at $\sum m_i P_i$. It follows then from (16.13.2.3) that B' has two components. Also then $K_{B'} = (p|B')^*\big(K_B + \mathrm{Diff}(0)\big)$ is semi log terminal of index one, by (16.9). Then by [KSB88,4.21] $B' = B_1' + B_2'$, where B_1' and B_2' are smooth and cross normally.

In case (16.15.2), each component of B' is $\mathbb{Q}$-Cartier and Cartier in codimension two. It is easy then to show that X' must be smooth along B'. Indeed let $p' : X'' \to X'$ be the index one cover relative to B_1'. Then, since B_1' is Cartier in codimension two, $p|B_1'' : B_1'' \to B_1'$ is unramified in codimension one. It follows that B_1'' is regular in codimension one. But X'' has rational singularities (it is log terminal), hence CM, so B_1'' is also CM, and normal by the Serre criterion. But then $p|B_1'' : B_1'' \to B_1'$ is a split cover, since it is unramified in codimension one and B_1' is smooth. This means that $p' = \mathrm{id}$, and since B_1' is smooth and Cartier, X' is smooth. Now (16.15.2) follows at once: $B' \subset X' \cong (xy = 0) \subset \mathbb{C}^3$, and $x \in B \subset X$ is analytically isomorphic to $0 \in (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_m(q_1, q_2, q_3)$. We may assume $q_3 = 1$, because p is unramified along $B_1 \cap B_2$, and $(q_1, q_2, m) = 1$ because p is unramified in codimension one.

In case (16.15.3), let $p' : X'' \to X'$ be the index one cover relative to B'. Then it is clear that, as for B', $B'' = B_1'' + B_2''$, where B_1'' and B_2'' are smooth and cross normally. Then, $p'|B'' : B'' \to B'$ is a split covering, since it is unramified in codimension one. It follows that $p' = \mathrm{id}$ and B' is also Cartier. Then X' has cDV singularities and the result follows at once. $\square$

16.16 Remark. It should be possible to check directly (although I did not do it) that the singularities in (16.15.2–3) are dlt.

If B has only one component, it is not possible to give a compact description as above. Even if B is Cartier, we know from inversion of adjunction (16.9) that any $\mathbb{Q}$-Gorenstein deformation (in particular the trivial deformation) of a surface quotient singularity is log terminal. However, under further restrictions, it is possible to come up with a short list:

16.17 Proposition. *[KSB88] Let $x \in B \subset X$ be a three dimensional germ, assume $K + B$ is dlt and B is Cartier. Also assume that X is cDV outside B. Then $x \in B \subset X$ is analytically isomorphic to one of the following:*

(16.17.1) $0 \in (xyz = 0) \subset \mathbb{C}^3$;

(16.17.2) $0 \in (t = 0) \subset (x^2 + f(y,z,t) = 0) \subset \mathbb{C}^4$ *where* $(x^2 + f(y,z,0) = 0)$ *defines a Du Val singularity;*

(16.17.3) $0 \in (t = 0) \subset (xy + f(z^r, t) = 0) \subset \mathbb{C}^3/\mathbb{Z}_r(a, -a, 1, 0)$. $\square$

17. ADJUNCTION AND DISCREPANCIES

JÁNOS KOLLÁR

The aim of this chapter is to investigate the problem posed in Chapter 16 of comparing the discrepancies of $(X, S+B)$ and $(S, \mathrm{Diff}(B))$. Before formulating the first result, we need to define some other variants of discrep(X).

17.1 Definition. Let X be a normal scheme, $D = \sum d_i D_i$ a boundary and let $Z \subset S \subset X$ be closed subschemes. (More generally, we may allow X to be nonnormal as long as the conditions of (2.6) are satisfied.) We use the following refinements of (1.6):

$$\begin{aligned}
&\operatorname{discrep}(X, D) \\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq \mathrm{Center}_X(E)\}; \\
&\operatorname{discrep}(\mathrm{Center} \subset Z, X, D) \\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq \mathrm{Center}_X(E) \subset Z\}; \\
&\operatorname{discrep}(S \cap \mathrm{Center} \subset Z, X, D) \\
&\qquad = \inf_E \{a(E, X, D) | E \text{ is exceptional, } \emptyset \neq S \cap \mathrm{Center}_X(E) \subset Z\};
\end{aligned}$$

One can also define versions where we allow E to be nonexceptional as well. These are denoted by totaldiscrep. Of course, totaldiscrep = discrep if Z has codimension at least two. We write discrep$(S \cap \mathrm{Center} \neq \emptyset, X, D)$ instead of discrep$(S \cap \mathrm{Center} \subset S, X, D)$ which is misleading in appearance.

17.1.1 Proposition. *(17.1.1.1) Any of the discrepancies defined above is either $-\infty$ or ≥ -1 and the infimum is a minimum.*

(17.1.1.2) For any $Z \subset S \subset X$

$$\begin{aligned}
\operatorname{discrep}(\mathrm{Center} \subset Z, X, D) &\geq \operatorname{discrep}(S \cap \mathrm{Center} \subset Z, X, D) \\
&\geq \operatorname{totaldiscrep}(X, D);
\end{aligned}$$

(17.1.1.3) If discrep(Center $\subset Z, X, D) \geq -1$ *then there is an open neighborhood* $Z \subset U \subset X$ *such that* totaldiscrep$(U, D) \geq -1$.

Proof. (17.1.1.2) is clear from the definition.

In order to see the other two claims, take a log resolution $f : Y \to (X, D)$. If $a(E, X, D) \geq -1$ for every divisor $E \subset Y$ then

$$\text{totaldiscrep}(X, D) = \min_E \{a(E, X, D) | E \subset Y\}$$

by (4.12.1.2). Similarly, (4.12.1.1) implies (17.1.1.1) for the other versions.

Assume now that there is a divisor $E \subset X$ such that $a(E, X, D) = -1 - c$ for some $c > 0$. Let $p \in E$ be any point. Choose a general codimension one subvariety $p \in W \subset E$. Let $g_1 : Y_1 \to Y$ be the blow up of W and let $E_1 \subset Y_1$ be the exceptional divisor. If $g_i : Y_i \to Y$ and $E_i \subset Y_i$ are already defined then let $g_{i+1} : Y_{i+1} \to Y_i \to Y$ be the blow up of $E_i \cap (g_i)_*^{-1}(E)$ and let E_{i+1} be the exceptional divisor of $Y_{i+1} \to Y_i$. By an easy computation $a(E_j, X, D) = -jc$. Let $p_j \in E_j$ be a point such that $g_j(p_j) = p$ and let F_j be the divisor obtained by blowing up p_j. Then

$$a(F_j, X, D) \leq -jc + \text{const.} \quad \text{hence} \quad \text{discrep}(\text{Center} \subset f(p), X, D) = -\infty.$$

Choosing p such that $f(p) \in Z$ completes the proof. $\square$

An upper bound is harder to find:

17.1.2 Conjecture. *[Shokurov88] Let $0 \in (X, D)$ be an n-dimensional normal singularity. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then*

$$\text{discrep}(\text{Center} \subset 0, X, D) \leq \dim X - 1,$$

and equality holds only if X is smooth and $0 \notin D$. (cf. (1.8)).

17.1.3 Remark. Assume that the conjecture fails for $0 \in X$. Then (X, D) is terminal. Thus if a list of terminal singularities is known, the conjecture can be verified. Therefore (17.1.2) is trivial if $\dim X \leq 2$. For $\dim X = 3$ it was checked by Markushevich (unpublished).

The following is the easy direction in comparing discrepancies:

17.2 Theorem. *Let X be a variety and let $S + B$ be a Weil divisor. Assume that S is reduced and $K + S + B$ is lc in codimension two. Assume furthermore that $K + S + B$ is $\mathbb{Q}$-Cartier. Let $Z \subset S$ be a closed subscheme. Then*

(17.2.1)

$$\begin{aligned} \text{totaldiscrep}(\text{Center} \subset Z, S, \text{Diff}(B)) &\geq \text{discrep}(\text{Center} \subset Z, X, S + B) \\ &\geq \text{discrep}(S \cap \text{Center} \subset Z, X, S + B). \end{aligned}$$

In particular,

$$\text{(17.2.2)} \qquad \operatorname{totaldiscrep}(S, \operatorname{Diff}(B)) \geq \operatorname{discrep}(X, S+B).$$

Proof. Set $Z = S$ in (17.2.1) to obtain (17.2.2). Also, the second inequality of (17.2.1) is obvious. For the rest we need a simple lemma which we state in a general setup:

17.2.3 Lemma. *Let $f : Y \to X$ be a proper birational morphism with exceptional divisors E_j. Assume that Y is normal. Let $S + B$ be a $\mathbb{Q}$-divisor on X and let S' be the birational transform of S on Y. Assume that (X, S) and (Y, S') are lc in codimension two. Let $D \subset S$ be the union of all codimension one points of S above which $S' \to S$ is not an isomorphism and let $D' \subset S'$ be the preimage of D. Finally let*

$$K_Y + f_*^{-1}(S+B) \equiv f^*(K_X + S + B) + \sum a(E_j, S+B)E_j.$$

Then

(17.2.4)

$$(f|S')_* \operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S+B)E_j\right) = \operatorname{Diff}_S(B) + 2[D]; \quad \textit{and}$$

$$K_{S'} + \operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S+B)E_j\right) \equiv (f|S')^*\left(K_S + \operatorname{Diff}_S(B)\right).$$

Proof. The left hand side of the second eqality is $f^*(K+S+B)|S'$ and the right hand side is $f^*(K+S+B|S)$. Thus the second equality is clear.

The first is a codimension one question on S, so that by shrinking X, we may assume that S is semismooth and $f : S' \to S$ is finite. Assume that $m(K_X + S + B)$ is Cartier. Then

$$\begin{aligned} mK_{S'} &+ m(f|S')_*^{-1}(\operatorname{Diff}(B)) + mD' \\ &= (f|S')^*\left(m(K_S + \operatorname{Diff}(B))\right) \\ &= f^*\left(m(K_X + S + B)|S'\right) \\ &= mK_{S'} + m\operatorname{Diff}_{S'}\left(f_*^{-1}B - \sum a(E_j, S+B)E_j\right), \end{aligned}$$

where all the equalities are equalities of divisors. Pushing this down to S gives the first equality. □

In order to see (17.2) let $f : Y \to X$ be a log resolution of $(X, S+B)$ with exceptional divisors E_j. Let $E_j \cap S' = \sum C_{jk} + \sum D_{jk}$ where the C_{jk} are the $(f|S')$-exceptional components of the intersection and $f|D_{jk}$ is birational. For simplicity assume that S' is disjoint from $f_*^{-1}(B)$.

Restricting (17.2.4) to S' we obtain:

$$K_{S'} + (f|S')_*^{-1}(\mathrm{Diff}(B)) + D' \equiv f^*(K_S + \mathrm{Diff}(B)) + \sum_{j,k} a(E_j, S+B)C_{jk}.$$

Therefore

$$(17.2.5) \qquad \begin{aligned} a\,(C_{jk}, S, \mathrm{Diff}(B)) &= a(E_j, X, S+B), \quad \text{and} \\ a\,(D_{jk}, S, \mathrm{Diff}(B)) &= a(E_j, X, S+B). \end{aligned}$$

Every exceptional divisor over S appears as an irreducible component of $E_j \cap S'$ for a suitable choice of f. The only problem is that $f(C_{jk}) \subset Z$ does not imply $f(E_j) \subset Z$. However if we blow up C_{jk} then we obtain a new exceptional divisor E_{jk} such that

$$f(E_{jk}) = f(C_{jk}) \subset Z \quad \text{and} \quad a(E_{jk}, X, S+B) = a(E_j, X, S+B).$$

This proves (17.2.1). □

The following conjecture asserts that the inequalities in (17.2) are equalities. Special cases were discussed earlier in [KSB88,Chapter 6; Stevens88; Shokurov91,3.3]. The conjecture (or similar results and conjectures) will be frequently referred to as *adjunction* (if we assume something about X and obtain conclusions about S) or *inversion of adjunction* (if we assume something about S and obtain conclusions about X).

17.3 Conjecture. *Notation as in (17.2). Then*
(17.3.1)

$$\begin{aligned} \mathrm{totaldiscrep}(\mathrm{Center} \subset Z, S, \mathrm{Diff}(B)) &= \mathrm{discrep}(\mathrm{Center} \subset Z, X, S+B) \\ &= \mathrm{discrep}(S \cap \mathrm{Center} \subset Z, X, S+B). \end{aligned}$$

In particular,

$$(17.3.2) \qquad \mathrm{totaldiscrep}\,(S, \mathrm{Diff}\,(B)) = \mathrm{discrep}\,(\mathrm{Center} \cap S \neq \emptyset, X, S+B).$$

Unfortunately, I do not know how to prove these in full generality. The rest of the chapter is devoted to proving some important special cases.

The following technical result is crucial in (17.6–7). It was proved by [Shokurov91,5.7] for surfaces.

17.4 Theorem. *Let X, Z be normal varieties (or analytic spaces) and let $h : X \to Z$ be a proper morphism with connected fibers. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X. Assume that*

(17.4.1) if $d_i < 0$ then $h(D_i)$ has codimension at least two in Z; and

(17.4.2) $-(K_X + D)$ is h-nef and h-big. (If h is birational then h-big is automatic.)

Let

$$f : Y \xrightarrow{g} X \xrightarrow{h} Z$$

be a resolution of singularities such that $\operatorname{Supp} g^{-1}(D)$ *is a divisor with normal crossings. Let*

$$K_Y = g^*(K_X + D) + \sum e_i E_i.$$

Further let

$$A = \sum_{i:e_i > -1} e_i E_i \quad \text{and} \quad F = -\sum_{i:e_i \leq -1} e_i E_i.$$

Then $\operatorname{Supp} F = \operatorname{Supp} \llcorner F \lrcorner$ *is connected in a neighborhood of any fiber of f.*

Proof. By definition

$$\ulcorner A \urcorner - \llcorner F \lrcorner = K_Y + (-g^*(K_X + D)) + \{-A\} + \{F\},$$

and therefore by [KMM87,1-2-3]

$$R^1 f_* \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) = 0.$$

Applying f_* to the exact sequence

$$0 \to \mathcal{O}_Y(\ulcorner A \urcorner - \llcorner F \lrcorner) \to \mathcal{O}_Y(\ulcorner A \urcorner) \to \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner) \to 0$$

we obtain that

$$f_* \mathcal{O}_Y(\ulcorner A \urcorner) \to f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner) \tag{17.4.3}$$

is surjective. Let E_i be an irreducible component of $\ulcorner A \urcorner$. Then either E_i is g-exceptional or E_i is the birational transform of some D_i and $d_i = -e_i < 0$.

Thus $g_*(\ulcorner A \urcorner)$ is h-exceptional and

$$f_* \mathcal{O}_Y(\ulcorner A \urcorner) = h_*(\mathcal{O}_X(g_*(\ulcorner A \urcorner))) = \mathcal{O}_Z.$$

Assume that $\llcorner F \lrcorner$ has at least two connected components $\llcorner F \lrcorner = F_1 \cup F_2$ in a neighborhood of $f^{-1}(z)$ for some $z \in Z$. Then

$$f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(z)} \cong f_* \mathcal{O}_{F_1}(\ulcorner A \urcorner)_{(z)} + f_* \mathcal{O}_{F_2}(\ulcorner A \urcorner)_{(z)},$$

and neither of these summands is zero. Thus $f_* \mathcal{O}_{\llcorner F \lrcorner}(\ulcorner A \urcorner)_{(z)}$ cannot be the quotient of the cyclic module $\mathcal{O}_{z,Z} \cong f_* \mathcal{O}_Y(\ulcorner A \urcorner)_{(z)}$. $\square$

17.5 Corollary. *If (X, D) is lt then $\llcorner D \lrcorner$ is seminormal and it has a semiresolution with normal crossing points only. If (X, D) is dlt then $\llcorner D \lrcorner$ is seminormal and S_2. If (X, D) is dlt and every irreducible component of $\llcorner D \lrcorner$ is $\mathbb{Q}$-Cartier then every irreducible component of $\llcorner D \lrcorner$ is normal.*

Proof. We apply (17.4) to $h : X \cong Z$. Let $g : Y \to X$ be a log resolution. Then $F = \llcorner F \lrcorner$ is the birational transform of $\llcorner D \lrcorner$. By assumption F has only normal crossing points. In particular, F is seminormal and S_2. We can successively blow up the normal crossing points of multiplicity at least 3 starting with the highest multiplicity locus to obtain a semiresolution of $\llcorner D \lrcorner$ with normal crossing points only.

By (17.4.3) the composite

$$g_* \mathcal{O}_Y(\ulcorner A \urcorner) \cong \mathcal{O}_X \to \mathcal{O}_{\llcorner D \lrcorner} \hookrightarrow g_* \mathcal{O}_F \hookrightarrow g_* \mathcal{O}_F(\ulcorner A \urcorner)$$

is surjective, and hence

$$\mathcal{O}_{\llcorner D \lrcorner} \cong g_* \mathcal{O}_{\llcorner F \lrcorner}. \tag{17.5.1}$$

Let $n : B \to \llcorner D \lrcorner$ be the seminormalizsation of $\llcorner D \lrcorner$. Then $B \times_n F \to F$ is a homeomorphism, thus an isomorphism. Therefore $F \to \llcorner D \lrcorner$ factors through n. Thus by (17.5.1) $n_* \mathcal{O}_B = \mathcal{O}_{\llcorner D \lrcorner}$, hence n is an isomorphism.

Assume now that (X, D) is dlt. Let $Z \subset \llcorner D \lrcorner$ be a closed subset of codimension ≥ 2. I claim that $Z' = \operatorname{Sing} F \cap g^{-1}(Z)$ has codimension ≥ 2 in F. Assume the contrary. Then there is an irreducible component $Z'' \subset Z'$ such that $Z'' \subset Y$ has codimension two and it is contained in the exceptional set of g. Therefore Z'' is contained in an exceptional divisor E of g. Since $\operatorname{Supp} g^{-1}(D)$ is a normal crossing divisor, there is at most one irreducible component of F containing Z''. This contradicts $Z'' \subset \operatorname{Sing} F$.

Let $n' : B' \to \llcorner D \lrcorner$ be the S_2-ization of $\llcorner D \lrcorner$ [EGA, IV.5.10.16-17]. Then $B' \times_{n'} F \to F$ is finite and birational on every irreducible component. Furthermore, by the above considerations, it is a homeomorphism in codimension one. Since F is seminormal and S_2, this implies that it is an isomorphism. Therefore $F \to \llcorner D \lrcorner$ factors through n'. Thus by (17.5.1) $n'_* \mathcal{O}_{B'} = \mathcal{O}_{\llcorner D \lrcorner}$, hence n' is an isomorphism.

Assume that every irreducible component of $\llcorner D \lrcorner$ is $\mathbb{Q}$-Cartier and let $D_1 \subset \llcorner D \lrcorner$ be an irreducible component. We can replace D by $D' = D - (1/2)(\llcorner D \lrcorner - D_1)$. Then (X, D') is dlt and $\llcorner D' \lrcorner = D_1$. Thus D_1 is seminormal and S_2. By the classification of Chapter 3, it is also smooth in codimension one, hence normal. □

17.5.2 Example. (cf. (16.11)) Let $X = (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2} \sum_{i=1}^{4} (x + 2^i u = y + 2^{-i} v = 0).$$

Then (X, D) is lt and $\llcorner D \lrcorner$ is two planes intersecting at a single point. Thus it is not S_2.

The most important application of the above connectedness result is to the problem of inversion of adjunction. The following theorem shows that in the notation of (17.3)

$$\operatorname{totaldiscrep}(S, \operatorname{Diff}(B)) > -1 \quad \Leftrightarrow \quad \operatorname{discrep}(\text{Center} \cap S \neq \emptyset, X, S+B) > -1.$$

17.6 Theorem. *Let X be normal and let $S \subset X$ be an irreducible divisor. Let B be an effective $\mathbb{Q}$-divisor such that $\llcorner B \lrcorner = \emptyset$ and assume that K_X+S+B is $\mathbb{Q}$-Cartier. Then K_X+S+B is plt in a neighborhood of S iff $K_S+\operatorname{Diff}(B)$ is klt.*

Proof. Let $g : Y \to X$ be a resolution of singularities and as in (17.4) let

$$K_Y = g^*(K_X + S + B) + A - F.$$

Let $S' \subset Y$ be the birational transform of S and let $F = S' \cup F'$. By adjunction

$$K_{S'} = g^*(K_S + \operatorname{Diff}(B)) + (A - F')|S'.$$

$K_X + S + B$ is plt iff $F' = \emptyset$ and $K_S + \operatorname{Diff} B$ is plt iff $F' \cap S' = \emptyset$. Let $h : X \to X$ be the identity. By (17.4) $S' \cup F'$ is connected, hence $F' = \emptyset$ iff $F' \cap S' = \emptyset$. $\square$

17.7 Theorem. *Let X be normal and let $S \subset X$ be an irreducible divisor. Let B and B' be effective $\mathbb{Q}$-divisors such that $\llcorner B \lrcorner = \emptyset$. Assume furthermore that*

(17.7.1) B' is $\mathbb{Q}$-Cartier, $K_X + S + B$ is $\mathbb{Q}$-Cartier, and

(17.7.2) $K_X + S + B$ is plt.

Then $K_X + S + B + B'$ is lc in a neighborhood of S iff $K_S + \operatorname{Diff}(B + B')$ is lc.

Proof. By (2.17.5) $K_X + S + B + B'$ (resp. $K_S + \operatorname{Diff}(B + B')$) is lc iff $K_X + S + B + tB'$ (resp. $K_S + \operatorname{Diff}(B + tB')$) is plt for every $0 \leq t < 1$. Thus (17.6) implies (17.7). $\square$

The following corollary is very important in Chapter 18. (See (18.3) for the definition of maximally lc.)

17.8 Corollary. *Let X be normal, $\mathbb{Q}$-factorial and let $S \subset X$ be an irreducible divisor. Let $\sum d_i D_i$ be an effective $\mathbb{Q}$-divisor. Assume that $K_X + S$ is plt. Set*

$$\Delta = \operatorname{Diff}_S(0) \quad \text{and} \quad B_i = i^w \mathcal{O}_X(D_i),$$

where $i: S \to X$ is the natural injection and i^w is defined in (16.3.6).

Then $K_X + S + \sum d_i D_i$ is maximally lc near a point $x \in S$ iff $K_S + \Delta + \sum d_i B_i$ is maximally lc near $x \in S$. $\square$

The rest of the chapter is devoted to showing that if the minimal model program works in dimension n then (17.3) holds for small discrepancies for $\dim X = n$. The precise assumptions are the following.

17.9 Assumption. For the rest of the chapter we use the following special case of the Log Minimal Model Program:

$(*_n)$. Let $f: Y \to X$ be a proper birational morphism. Assume that Y is normal, $\mathbb{Q}$-factorial and $\dim Y \le n$. Let D be a $\mathbb{Q}$-Weil divisor on Y such that (Y, D) is log terminal. Then the steps of the $(K_Y + D)$-MMP (as described in (2.26)) all exist and the process terminates with a relative minimal model $\bar{f}: (\bar{Y}, \bar{D}) \to X$.

We know that $(*_2)$ and $(*_3)$ hold.

We start with the following result which is of considerable interest in itself. It is a generalisation of (6.9.4).

17.10 Theorem. *Assume $(*_n)$. Let (X, B) be a log canonical pair, $\dim X \le n$. Let $f: Y \to X$ be a log resolution. Let $\mathcal{E}$ be a subset of the exceptional divisors $\{E_i\}$ such that*

(17.10.1.1) If $a(E_i, B) = -1$ then $E_i \subset \mathcal{E}$;

(17.10.1.2) If $E_j \subset \mathcal{E}$ then $a(E_j, B) \le 0$.

Then there is a factorization

$$f: Y \overset{h}{\dashrightarrow} X(\mathcal{E}) \overset{g}{\longrightarrow} X$$

with the following properties:

(17.10.2.1) h is a local isomorphism at every generic point of $\mathcal{E}$;

(17.10.2.2) h contracts every exceptional divisor not in $\mathcal{E}$;

(17.10.2.3)
$$\begin{aligned} &h_*\left(K_Y + f_*^{-1}(B) + \sum -a(E_i, B)E_i\right) \\ &= K_{X(\mathcal{E})} + g_*^{-1}(B) + \sum_{E_i \subset \mathcal{E}} -a(E_i, B) h_*(E_i) \\ &\equiv g^*(K_X + B) \quad \text{is log terminal.} \end{aligned}$$

Proof. For a small ϵ let

(17.10.3)
$$d(E_i) = \begin{cases} -a(E_i, B) & \text{if } E_i \subset \mathcal{E}; \\ \max\{-a(E_i, B) + \epsilon, 0\} & \text{if } E_i \not\subset \mathcal{E}. \end{cases}$$

Then

$$K_Y + f_*^{-1}(B) + \sum d(E_i)E_i \equiv f^*(K_X + B) + \sum_{E_j \notin \mathcal{E}} (d_j + a(E_j, B))E_j.$$

Apply the $(K_Y + f_*^{-1}(B) + \sum d(E_i)E_i)$-MMP to Y/X. Every extremal ray is supported in (the birational transform of) $h_*(\mathcal{E})$. Also, an effective exceptional divisor is never nef. Thus the MMP stops with a factorization

$$f : Y \overset{h}{\dashrightarrow} X(\mathcal{E}) \xrightarrow{g} X$$

such that $h_*(\mathcal{E}) = \emptyset$ and h is an isomorphism at every generic point of $\mathcal{E}$. □

17.11 Corollary. *Assume* $(*_n)$. *Let* $(X, S + B)$ *be as in (17.2) such that* $\dim X \le n$ *and* X *is* $\mathbb{Q}$*-factorial. Assume furthermore that either,*

(17.11.1) $(X, S + B)$ *is plt and* $d = \operatorname{discrep}(S \cap \operatorname{Center} \subset Z, X, S + B) \le 0$*;*
or

(17.11.2) $(X, S + B)$ *is lc and* $d = -1$.

Then the equalities (17.3.1) hold.

Proof. Let $f : Y \to X$ be a log resolution of $(X, S + B)$ such that $f^{-1}(Z)$ is a divisor with normal crossings. Let $S'' \subset Y$ be the birational transform of S.

Let $\mathcal{E}$ be the set of exceptional divisors with discrepancy d such that $S \cap \operatorname{Center}_X(E) \subset Z$. By assumption $\mathcal{E} \ne \emptyset$. We apply the

$$\left(K_Y + f_*^{-1}(S + B) + \sum d(E_i)E_i\right)\text{-MMP on } f : Y \to X.$$

At the end we obtain $h : Y \dashrightarrow X(\mathcal{E})$ and $g : X(\mathcal{E}) \to X$ such that

$$h_*\left(K_Y + f_*^{-1}(S + B) + \sum d(E_i)E_i\right) = g^*(K + S + B).$$

Let $S' \subset X(\mathcal{E})$ be the birational transform of S. Since X is $\mathbb{Q}$-factorial, the exceptional set of g is exactly $h_*(\mathcal{E})$, hence S' intersects the exceptional divisor $h_*(\mathcal{E})$. $f(S') \cap f(h_*(\mathcal{E})) \subset Z$, hence every irreducible component $C \subset S' \cap h_*(\mathcal{E})$ lies above Z.

By (16.7) the coefficient $p(C)$ of $[C]$ in $\operatorname{Diff}(g_*^{-1}B - dh_*(\mathcal{E}))$ is

$$p(C) = 1 - \frac{1}{m} + \sum \frac{r_i b_i}{m} + \frac{r_0(-d)}{m} \ge 1 - \frac{1+d}{m} \ge -d,$$

and by (17.2.3) $a(C, S, \operatorname{Diff}(B)) = -p(C)$. Combining with (17.2) we are done. □

17.12 Corollary. *Assume* $(*_n)$. *Let* $(X, S+B)$ *be as in (17.2) such tha* $\dim X \leq n$ *and* X *is* $\mathbb{Q}$*-factorial. Then*

$$\operatorname{totaldiscrep}(S, \operatorname{Diff}(B)) = \operatorname{discrep}(\operatorname{Center} \cap S \neq \emptyset, X, S+B).$$

Proof. Let $d = \operatorname{discrep}(\operatorname{Center} \cap S \neq \emptyset, X, S+B)$. By blowing up a codimension one smooth point of S we see that $d \leq 0$. If $d > -1$ then $(X, S+B)$ is plt, thus (17.11.1) implies the required equality.

If $d = -1$ then we can apply (17.11.2).

Finally assume that $d = -\infty$. We need to show that $(S, \operatorname{Diff}(B))$ cannot be lc. Let $f : (Y, f_*^{-1}(S+B)+E) \to X$ be a log terminal model of $(X, S+B)$ where E is the reduced exceptional divisor. Write

$$K_Y + f_*^{-1}(S+B) + E \equiv f^*(K_X + S + B) - F,$$

where by (2.19) F is effective and either $F = 0$ or $\operatorname{Supp} F = \operatorname{Supp} E$. In the former case $(X, S+B)$ is lc. In the latter case let $S' \subset Y$ denote the birational transform of S. Then S' and E intersect nontrivially and

$$K_{S'} + \operatorname{Diff}_{S'}(f_*^{-1}(B) + E + F) = f^*(K_S + \operatorname{Diff}_S(B))$$

contains a component with coefficient greater than 1 by (16.7).

Thus $(S, \operatorname{Diff}(B))$ is not lc. □