Hyperbolicity and log pairs

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July 27 2015

Hyperbolicity

We will work over the field of complex numbers.

Definition (Kobayashi, Brody)

Let X be a complex manifold. We say that X is hyperbolic if there exists no holomorphic map $f : \mathbb{C} \to X$.

The original definition of hyperbolicity, due to Kobayashi, is slightly different. For time reason, we will use the equivalent definition above, due to Brody.

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Three classical conjectures

Conjecture (Green-Griffiths)

Let X be a projective manifold of general type. Then, no entire curve $f: \mathbb{C} \to X$ is Zariski dense.

Conjecture (Lang)

Let X be a projective manifold. Then, X is hyperbolic if and only if all subvarieties of X are general type.

Conjecture (Kobayashi)

Let X be a projective manifold. If X is hyperbolic then K_X is ample.

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Hyperbolicity and positivity

Moral of the conjectures

Hyperbolicity should correspond to (or at least imply) positivity properties of X (Ω^1_X, K_X).

From the point of view of birational geometry, we can see a first realization of this principle in the Cone Theorem.

Cone Theorem (Mori, Kawamata, Kollár, Shokurov, Ambro, Fujino)

Let X be a smooth projective variety.
There exist (countably many) K_X-negative rational curves R_i on X such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{>0}[R_i].$$

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One can immediately recover the following corollary.

Corollary

Let X be a smooth projective variety. Assume that X contains no rational curves.
Then K_X is nef, i.e. $K_X \cdot C \geq 0$ for any irreducible curve $C \subset X$.

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A quest for weaker conditions

Hyperbolicity is a very strong property for a complex manifold.
In fact, on a complex manifold X, hyperbolicity forbids the presence of rational curves and complex tori, for example.

Question 1

What are weaker conditions of hyperbolic type that one can impose on a complex manifold that still implies positivity properties of the cotangent bundle (or a suitable modification)?

Question 2

What can we way in the case of quasi-projective varieties?

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Log smooth pairs and quasi projective varieties

A pair (X, D) is said to be log smooth if X is a smooth proper variety and $D = \sum_i D_i$ is a snc divisor on X.

Strategy

We will use log smooth pairs to give some answers to the previous questions.

If U is a smooth quasi-projective variety, by Hironaka's desingularization theorem, we can always compactify U to a smooth projective variety X and the pair $(X, X \setminus U)$ is log smooth.

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Log smooth pairs and quasi projective varieties, II

Given a log smooth pair (X, D), we can induce a natural stratification of the variety X.

The closed strata will be X itself, the components of D, D_i, $i = 1, \ldots, k$, and then the irreducible components of the intersections of the D_i.

The open strata will be the closed strata minus the intersections with all other strata not containing them.That is, $X \setminus D$, $D_i \setminus (\cup_{j \neq i} D_j)$ and more in general

$$\cap_{i \in I} D_i \setminus (\cup_{j \in \{1,\ldots,k\} \setminus I} D_j) \text{ for } I \subset \{1, \ldots, k\}$$

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A first theorem

From now on, we will focus on log smooth pair (X, D).

Theorem (Lu-Zhang, -)

Let (X, D) be a projective log smooth pair.
If $K_X + D$ is not nef, there exists a non-constant morphism $f\colon \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the strata of (X, D).

Definition (L-Z)

A pair (X, D) is said to be Mori hyperbolic if there exists no non-constant morphism $f\colon \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the strata of (X, D).

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Ingredients for a proof

- Adjunction: we can restrict to the strata of (X, D) and work inductively. Hence we can assume that $K_X + D$ is nef when restricted to the components of D.
- Cone Theorem: If $K_X + D$ is not nef along a stratum W, by the Cone Thm there exists a K_X-negative rational curve R. We will use R to obtain a morphism $f : \mathbb{A}^1 \to X$ intersecting D in at most one point. We can actually assume even more, i.e., that there exists a morphism

$$\pi : X \to Y$$

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Ingredients for a proof, II

- KV-vanishing: Fibers for π are either 0- or 1-dimensional. When they are 1-dimensional, Kawamata-Viehweg vanishing implies that the generic one is a smoothly embedded copy of $\mathbb{P}^1$.
- For a rational curve R in the (positive-dimensional) fibers of π, it is enough to show now that the intersection with D is supported in just one point.

Connectedness Theorem (Shokurov, Kollár)

Let $\pi\colon X \to Y$ a morphism. Assume that $-(K_X + D)$ is ample along every fiber of π.
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Cone Theorem and hyperbolicity

Theorem (-)

Let (X, D) be a log smooth pair. There exist countalby many $(K_X + D)$-negative curves C_i s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta \geq 0} + \sum_{i \in I} \mathbb{R}_{\geq 0}[C_i].$$

Moreover, one of the two following conditions holds:

- $C_i \cap (X \setminus D)$ *contains the image of a non-constant map* $f \colon \mathbb{A}^1 \to X$*;*
- *there exists an open stratum* W *of* D *such that* $C_i \cap W$ *contains the image of a non-constant map* $f \colon \mathbb{A}^1 \to W$*.*

Log-hyperbolicity and ampleness

Theorem (Kleiman, Nakai, Moishezon)

Let X be a proper variety and D a Cartier divisor on X.
D is ample iff $D^{\dim V} \cdot V > 0$, $\forall V \subset X$ irreducible subvariety.

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Log canonical pairs

Let (X, Δ) be a pair of a normal variety X and en effective $\mathbb{R}$-divisor $\Delta = \sum d_i \Delta_i,\ d_i \in (0, 1]$ s.t. $K_X + \Delta$ is $\mathbb{R}$-Cartier.
Given a log-resolution $\psi\colon Z \to X$, we can write

$$K_Z + \widetilde{\Delta} = \psi^*(K_X + \Delta) + \sum a_i E_i$$

Definition

The pair (X, Δ) is a log canonical pair if $a_i \geq -1$, $\forall i$.

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Non-klt locus

In the previous equation, the components E_i of coefficient -1 are very important.

Their image via ψ on X is called the non-klt locus of the pair (X, Δ), $\mathrm{Nklt}(X, \Delta)$.

$\mathrm{Nklt}(X, \Delta)$ comes equipped with a stratification analogous to the one defined in the log smooth case and in fact induced by the one on $(Z, \lfloor \widetilde{\Delta} - \sum a_i E_i \rfloor)$.

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The main theorems

Theorem (-)

Let (X, Δ) be a projective log canonical pair.
If $K_X + \Delta$ is not nef, there exists a non-constant morphism $f\colon \mathbb{A}^1 \to X$ and $f(\mathbb{A}^1)$ is contained in one of the strata of the non-klt locus of (X, Δ).

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Differences with the log smooth case

- Adjunction: log canonical pairs are much more general and singular then log smooth ones.

  Hence, one needs a better theory of adjunction in order to apply the previous strategy of proof.

Theorem (-)

Let (X, Δ) be log canonical pair. Let W be a lc center and let W^ν be the normalization.
On W^ν there exists a divisor Δ_{W^ν} such that (W^ν, Δ_{W^ν}) is a log pair,

$$K_{W^\nu} + \Delta_{W^\nu} \sim (K_X + \Delta)|_{W^\nu}$$

and $\mathrm{Nklt}(W^n u, \Delta_{W^\nu}) = \mathrm{Nklt}(X, \Delta)|_{W^\nu}$

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The ample case

Theorem (-)

Let (X, Δ) be a dlt pair.

Assume that there is no non-constant morphism $f : \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the open strata of (X, Δ).

The following are equivalent:

- $K_X + \Delta$ *is ample*
- $(K_X + \Delta)^{\dim W} \cdot W > 0$, $\forall W \subset X$ *stratum of* (X, D)
- $K_X + \Delta$ *is big and its restriction along* $\lfloor \Delta \rfloor$ *is ample.*

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Thanks for your attention!!