CHAPTER 13

Birational Relation Among Mori Fiber Spaces

The purpose of this chapter is to discuss the **birational relation among Mori fiber spaces**. Here we focus our attention on the most important subject, the **Sarkisov program**, due to Sarkisov [3], Reid [6], Corti [1], which gives an algorithm for factoring a given birational map between Mori fiber spaces into a sequence of certain elementary transformations called "**links**." While it is a higher-dimensional analogue of the Castelnuovo–Noether theorem (cf. Theorem 1-8-8), its true meaning becomes clearer in the framework of the logarithmic category, with the main machinery of the program working under the log MMP discussed in Chapter 11. Our presentation is mostly in dimension 3, where all the necessary ingredients are established (with the most subtle part of showing "termination of Sarkisov program" ingeniously settled by Corti [1], as discussed in Section 13-2), leaving the details of the higher-dimensional case to the reader, where the general mechanism goes almost verbatim but some key ingredients still remain conjectural. (See Section 14-5 for the toric Sarkisov program, where we have all the necessary ingredients established in all dimensions.)

As a possible application we expect to have a better understanding of the group of self-birational maps of a Mori fiber space, e.g., the Cremona transformations of the projective space $\phi : \mathbb{P}^n \to \operatorname{Spec} \mathbb{C}$, by analyzing the structures of the links. This turns out to be, however, a formidable task even in dimension 3. In Section 13-3 we present Takahashi's work deriving the classical theorem of Jung on the structure of the automorphism group of the affine space $\mathbb{A}^2$ in dimension 2 as an application of the logarithmic generalization of the Sarkisov program. We hope that this will at least give the reader a taste of what a future application of the Sarkisov program might look like.

K. Matsuki, *Introduction to the Mori Program*

13.1 Sarkisov Program

In Section 1-8 we already discussed the Castelnuovo–Noether theorem, factoring birational maps among Mori fiber spaces in dimension 2, in the form of the Sarkisov program in dimension 2. The main idea of Sarkisov is to untwist a birational map between Mori fiber spaces into certain elementary transformations, called "links." We measure how far a given Mori fiber space is from the reference Mori fiber space via the Sarkisov degree. This main idea and the basic strategy using the Sarkisov degree remain the same in higher dimension. In dimension 2 our description of each link is quite explicit, though its construction in Section 1-8 is ad hoc based upon the classification of extremal rays in dimension 2. In this section we give a systematic treatment of the construction of each link in the framework of the log MMP. Our description of each link, however, becomes less explicit in dimension 3 or higher. Giving a better and more satisfactory description is one of the tasks of future research.

In this section we denote by $\mathfrak{C}$ the category of normal projective varieties with only $\mathbb{Q}$-factorial and terminal singularities.

Theorem 13-1-1 (Sarkisov Program in Dimension 3). *Let*

$$\begin{array}{ccc} X & \underset{\text{birat}}{\overset{\Phi}{\dashrightarrow}} & X' \\ \downarrow\phi & & \downarrow\phi' \\ Y & & Y' \end{array}$$

be a birational map between two Mori fiber spaces in dimension 3*:*

$$\phi : X \to Y,$$
$$\phi' : X' \to Y'.$$

Then there exists an algorithm, called the **Sarkisov program,** *for decomposing* Φ *as a composite of the following four types of* "**links**"*:*

Type (I)*: A link of* Type (I) *consists of a diagram of the form*

$$\begin{array}{ccccc} & & Z & \dashrightarrow & X_1 \\ & \swarrow & & & \\ X & & & & \downarrow \\ \downarrow & & & & \\ Y & & \leftarrow & & Y_1 \end{array}$$

where $Z \to X$ *is a divisorial contraction of an extremal ray with respect to* K_Z *with* $Z \in \mathfrak{C}$*, the birational map* $Z \dashrightarrow X_1$ *is a sequence of log flips,* $\phi_1 : X_1 \to Y_1$ *is a new Mori fiber space with* $X_1 \in \mathfrak{C}$*, and* $Y_1 \to Y$ *is a morphism with connected fibers between normal projective varieties with only* $\mathbb{Q}$*-factorial singularities and* $\rho(Y_1/Y) = 1$.

Type (II): *A link of* Type (II) *consists of a diagram of the form*

$$\begin{tikzcd}
 & Z \arrow[r, dashed] \arrow[dl] & Z' \arrow[dr] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
Y & & & Y_1 \arrow[lll, "\sim"']
\end{tikzcd}$$

where $Z \to X$ is a divisorial contraction of an extremal ray with respect to K_Z with $Z \in \mathfrak{C}$, the birational map $Z \dashrightarrow Z'$ is a sequence of log flips, $Z' \to X_1$ is a divisorial contraction of an extremal ray with respect to $K_{Z'}$ with $Z' \in \mathfrak{C}$, and $\phi_1 : X_1 \to Y_1$ is a new Mori fiber space with $X_1 \in \mathfrak{C}$.

Type (III): *A link of* Type (III) *is the inverse of a link of* Type (I) *and hence consists of a diagram of the form*

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[dd] & Z' \arrow[dr] & \\
 & & X_1 \arrow[d] \\
Y \arrow[rr] & & Y_1 .
\end{tikzcd}$$

Type (IV): *A link of* Type (IV) *consists of a diagram of the form*

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d] & & X_1 \arrow[d] \\
Y \arrow[dr] & & Y_1 \arrow[dl] \\
 & T &
\end{tikzcd}$$

where the birational map $X \dashrightarrow X_1$ is a sequence of log flips, $\phi_1 : X_1 \to Y_1$ is a new Mori fiber space with $X_1 \in \mathfrak{C}$, while $Y \to T$ and $Y_1 \to T$ are morphisms with connected fibers from normal projective varieties Y and Y_1 with only $\mathbb{Q}$-factorial singularities to a normal projective variety T with $\rho(Y/T) = \rho(Y_1/T) = 1$.

We remark that the importance of the theorem lies in the mechanism of how the algorithm works, rather than in the mere fact that a birational map can be decomposed into a composite of links. At present, the description of the types of links is not satisfactory enough to characterize them as those that necessarily appear in some process of untwisting a birational map between Mori fiber spaces via the Sarkisov program. A more detailed study of links is indispensable for and crucial to the future application of the program.

The rest of the section will be spent in showing the general mechanism of the Sarkisov program, leaving the verification of its termination to the next section.

Strategy for Untwisting

The strategy for decomposing Φ into a composite of links, which we call "untwisting" of Φ, is to set up a good invariant of Φ with reference to the fixed Mori fiber space $\phi' : X' \to Y'$ (called the **Sarkisov degree**, the triplets of numbers (μ, λ, e) in lexicographical order, as will be defined below), which should tell us how far ϕ is from being an isomorphism of Mori fiber spaces.

It is the **Noether–Fano–Iskovskikh criterion** that allows us to judge precisely when Φ is an isomorphism of Mori fiber spaces in terms of the Sarkisov degree and the canonical divisor of X.

Starting with a given birational map Φ between two Mori fiber spaces, we ask whether Φ is actually an isomorphism of Mori fiber spaces via the NFI criterion. If the answer is *yes*, then there is nothing more to do and we proceed to *end* of the program. If the answer is *no*, then we "untwist" Φ by an appropriate link of Type (I), (II), (III), or (IV) to obtain a new birational map Φ_1. If Φ_1 is an isomorphism of Mori fiber spaces via the NFI criterion, we proceed to *end.* If not, we repeat the process with Φ_1. Each time we "untwist," the Sarkisov degree should strictly drop, i.e.,

$$(\mu, \lambda, e) = (\mu_0, \lambda_0, e_0) > (\mu_1, \lambda_1, e_1) > (\mu_2, \lambda_2, e_2) > \cdots,$$

where these are the Sarkisov degrees of $\Phi = \Phi_0, \Phi_1, \Phi_2, \ldots$ with respect to the fixed reference Mori fiber space $\phi' : X' \to Y'$:

$$\begin{tikzcd}
X = X_0 \arrow[r, dashed] \arrow[d, "\phi"] \arrow[rrrr, dashed, bend left=60, "\Phi = \Phi_0"] & X_1 \arrow[r, dashed] \arrow[d, "\phi_1"] \arrow[rrr, dashed, bend left=50, "\Phi_1"] & X_2 \arrow[d, "\phi_2"] \arrow[rr, dashed, bend left=40, "\Phi_2"] & \cdots \arrow[r, dashed] & X' \arrow[d, "\phi'"] \\
Y = Y_0 & Y_1 & Y_2 & \cdots & Y'
\end{tikzcd}$$

Finally by observing that this process has to come to an end after finitely many repetitions, the property that we verify in the next section via the analysis of the Sarkisov degree, we reach the Mori fiber space $\phi' : X' \to Y'$ expressing Φ as a composite of links.

Sarkisov degree.

First we choose and fix $\mu' \in \mathbb{N}$ and an ample divisor A' on Y' such that

$$H_{X'} = -\mu' K_{X'} + \phi'^{*} A'$$

is a very ample divisor on X'. (Note that $-K_{X'}$ is relatively ample. We refer the reader to Hartshorne [3], Chapter II, Proposition 7.10, or Iitaka [5], Theorem 7.11.)

We take a nonsingular projective variety V that dominates both X and X' by birational morphisms (which are compatible with Φ) via Hironaka's elimination of points of indeterminacy,

$$X \overset{\sigma}{\leftarrow} V \overset{\sigma'}{\rightarrow} X' \text{ with } \sigma' \circ \sigma^{-1} = \Phi.$$

For a member

$$\mathcal{H}_{X'} \in |H_{X'}|$$

we define the **"homaloidal" transform** $\mathcal{H}_X$ on X of $\mathcal{H}_{X'}$ to be

$$\mathcal{H}_X = \sigma_* \sigma'^* \mathcal{H}_{X'}.$$

We note that the homaloidal transform does not depend on the choice of V. We are ready to define the Sarkisov degee.

(i) μ : **the quasi-effective threshold**

The first of the triplet, the quasi-effective threshold μ, is defined to be a rational number (necessarily positive) such that

$$\mu K_X + \mathcal{H}_X \equiv_Y 0,$$

that is to say,

$$(\mu K_Y + \mathcal{H}_X) \cdot F = 0 \text{ for any curve } F \text{ in a fiber of } \phi : X \to Y.$$

Note that μ is independent of the choice of a member $\mathcal{H}_X$ (cf. Lemma 13-1-5) and that since all the curves contracted by ϕ are numerically proportional, we have to check $(\mu K_X + \mathcal{H}_X) \cdot F = 0$ for only one curve F in a fiber of ϕ.

Note that μ' is the quasi-effective threshold for the special case Φ the identity map of the Mori fiber space $\phi' : X' \to S'$.

In dimension 2 in Section 1-8 we saw that the quasi-effective threshold has a bounded denominator $\mu \in \frac{1}{3!}\mathbb{N}$. In dimension 3 we will also see the fact that the quasi-effective thresholds are discrete in the whole set of rational numbers, though the verification becomes much more subtle, involving some boundedness results of Kawamata [11] for a certain family of 3-folds called $\mathbb{Q}$-Fano 3-folds with Picard number one. Discreteness of the set of quasi-effective thresholds is still conjectural in higher dimensions.

(ii) λ : **the maximal multiplicity**

In order to define the second member of the triplet, the maximal multiplicity λ, we consider the linear system consisting of the homaloidal transforms $\mathcal{H}_X$ for $\mathcal{H}_{X'} \in |H_{X'}|$, which we denote by $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$. (We note that this linear system may be smaller than the complete linear system $|\mathcal{H}_X|$.) In dimension 2, λ was simply the maximum multiplicity of a general member $\mathcal{H}_X$ at the base points of $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$. In dimension 3 or higher, it is the reciprocal $1/\lambda$ that has the more natural and intrinsic description, which we use as the definition.

Definition 13-1-2 (Log Pair (X, c$\mathcal{H}_X$) Being Canonical) (cf. Definition 4-4-2). *Let the situation be as above. We write the ramification formula*

$$K_V = \sigma^* K_X + \Sigma a_k E_k,$$
$$\mathcal{H}_V = \sigma'^* \mathcal{H}_{X'} = \sigma^* \mathcal{H}_X - \Sigma b_k E_k,$$

for the homaloidal transform $\mathcal{H}_X$ *of a general member* $\mathcal{H}_{X'} \in |H_{X'}|$*. For* $c \in \mathbb{Q}_{\geq 0}$ *we say that the log pair* $(X, c\mathcal{H}_X)$ *is* **canonical** *if writing down the log ramification formula*

$$K_V + c\mathcal{H}_V = \sigma'^*(K_X + c\mathcal{H}_X) + \Sigma(a_k - cb_k)E_k,$$

we have

$$a_k - cb_k \geq 0 \quad \forall E_k.$$

Noting that the number $a_k - cb_k$ *coincides with the usual log discrepancy*

$$a_k - cb_k = a(E_k; X, c\mathcal{H}_X),$$

which does not depend on the common resolution V *but only on* E_k*, the log pair* $(X, c\mathcal{H}_X)$ *is canonical if*

$$a(E; X, c\mathcal{H}_X) \geq 0 \quad \forall E \text{ exceptional over } X.$$

Now we are ready to define the invariant λ.

If $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$ does not have any base point, then $\lambda = 0$ by definition. If $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$ has some base points, by taking the common resolution as before and writing the ramification formulae we define

$$\frac{1}{\lambda} = \max\{c \in \mathbb{Q}_{\geq 0}; a_k - cb_k \geq 0 \quad \forall k\} = \min\left\{\frac{a_k}{b_k}\right\}$$

Note that since $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$ has some base points, we have $b_k > 0$ for some E_k. Note also that the number $1/\lambda$ depends only on the linear system $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$ and is independent of the choice of a general member $\mathcal{H}_{X'}$ and hence of $\mathcal{H}_X$, since the numbers b_k stay the same for general members. It has a description free of the particular choice of the common resolution using the notion of log pairs being canonical:

$$\begin{aligned}\frac{1}{\lambda} &= \max\{c \in \mathbb{Q}_{\geq 0}; (X, c\mathcal{H}_X) \text{ is canonical}\}\\ &= \max\{c \in \mathbb{Q}_{\geq 0}; a(E; X, c\mathcal{H}_X) \geq 0 \quad \forall E \text{ exceptional over } X\}.\end{aligned}$$

The reciprocal $1/\lambda$ is called the **canonical threshold** of X with respect to the linear system $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$.

(iii) e: **the number of crepant exceptional divisors**

When $\lambda > 0$, we define

$$\begin{aligned}e &= \#\left\{E_k; a_k - \frac{1}{\lambda}b_k = 0 \text{ or equivalently } \frac{b_k}{a_k} = \lambda\right\}\\ &= \#\left\{E; E \text{ is exceptional over } X \text{ and } a\left(E; X, \frac{1}{\lambda}\mathcal{H}_X\right) = 0\right\}.\end{aligned}$$

(Exercise! Check the second equality, which implies that the number e is independent of the choice of a common resolution V.)

When $\lambda = 0$, i.e., when $\Phi^{-1}_{\text{homaloidal}}|H_{X'}|$ has no base points, all the exceptional divisors E_k have $b_k/a_k = 0 = \lambda$ for any common resolution V, and e is not well-defined. But when $\lambda = 0$, we are always in the case $\lambda \leq \mu$ in the algorithm of the Sarkisov program. The termination of untwistings in this case does not involve the invariant e, and we leave e undefined in this case.

This completes the definition of the triplet (μ, λ, e) associated to Φ with reference to $\phi' : X' \to Y'$ after fixing μ' and A'. We consider these tripletes in lexicographical order. The next criterion tells us when Φ is an isomorphism of Mori fiber spaces, in terms of the Sarkisov degree (μ, λ, e) and the canonical divisor K_X of X.

In the following the notation $\mathcal{H}_{X'}$ always refers to a *general* member of $|H_{X'}|$, and $\mathcal{H}_X$ to its homaloidal transform.

Proposition 13-1-3 (Noether–Fano–Iskovskikh Criterion). *Suppose*

$$\lambda \leq \mu \quad \textit{and} \quad K_X + \frac{1}{\mu}\mathcal{H} \textit{ is nef on } X.$$

Then Φ is an isomorphism of Mori fiber spaces, i.e., there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\Phi}{\underset{\longrightarrow}{\sim}} & X' \\ \downarrow\phi & & \downarrow\phi' \\ Y & \overset{\sim}{\longrightarrow} & Y' \end{array}$$

PROOF.

Step 1. Show that $\mu = \mu'$.

First we claim that

$$\mu = \mu'.$$

We take a common resolution as before:

$$\begin{array}{ccccc} & & V & & \\ & \sigma \swarrow & & \searrow \sigma' & \\ X & & \overset{\Phi}{\dashrightarrow} & & X' \\ \downarrow\phi & & & & \downarrow\phi' \\ Y & & & & Y'. \end{array}$$

We write the log ramification formulas for the log pairs $(X, \frac{1}{\mu'}\mathcal{H}_X)$ and $(X', \frac{1}{\mu'}\mathcal{H}_{X'})$:

$$\begin{aligned} K_V + \frac{1}{\mu'}\mathcal{H}_V &= \sigma^*\left(K_X + \frac{1}{\mu'}\mathcal{H}_X\right) + \Sigma r_{k,\mu'} E_k \\ &= \sigma'^*\left(K_{X'} + \frac{1}{\mu'}\mathcal{H}_{X'}\right) + \Sigma r'_{j,\mu'} E'_j, \end{aligned}$$

where the E_k (respectively E'_j) are the exceptional divisors for σ (respectively σ'). Note that since $\mathcal{H}_V = \sigma'^* \mathcal{H}_{X'}$ and since X' has terminal singularities,

$$r'_{j,\mu'} = a'_j > 0 \quad \forall j,$$

where the usual ramification formula for X' reads

$$K_V = \sigma'^* K_{X'} + \Sigma a'_{j'} E'_{j'}.$$

Now if we take a general curve F in the general fiber of ϕ avoiding all $\sigma(E_k)$ and not contained in any of $\sigma(E'_j)$, then F can be considered to lie also on V, and we compute

$$\begin{aligned}\left(K_X + \frac{1}{\mu'}\mathcal{H}_X\right) \cdot F &= \left\{\sigma^* \left(K_X + \frac{1}{\mu'}\mathcal{H}_X\right) + \Sigma r_{k,\mu'} E_k\right\} \cdot F \\ &= \left\{\sigma'^* \left(K_{X'} + \frac{1}{\mu'}\mathcal{H}_{X'}\right) + \Sigma r'_{j,\mu'} E'_j\right\} \cdot F \\ &= \left(\frac{1}{\mu'}\phi'^* A' + \Sigma r'_{j,\mu'} E'_j\right) \cdot F \geq 0.\end{aligned}$$

(Note that we can take the general curve F to be the intersection of a general fiber $\phi^{-1}(p)$ and $\dim \phi^{-1}(p)$ general members of a very ample divisor on X.)

This implies

$$\mu \geq \mu'.$$

Note that this inequality always holds without any extra assumption.

In order to see the inequality in the opposite direction, we write down a similar ramification formula, replacing $1/\mu'$ by $1/\mu$:

$$\begin{aligned}K_V + \frac{1}{\mu}\mathcal{H}_V &= \sigma^* \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \Sigma r_{k,\mu} E_k \\ &= \sigma'^* \left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \Sigma r'_{j,\mu} E'_j.\end{aligned}$$

This time, since $\lambda \leq \mu$ by assumption, we have

$$r_{k,\mu} \geq 0 \quad \forall k.$$

Now if we take a general curve F' in the general fiber of ϕ' avoiding all $\sigma'(E'_j)$ and not contained in any of $\sigma'(E_k)$, then F' can be considered to lie also on V, and we compute

$$\begin{aligned}\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) \cdot F' &= \left\{\sigma'^* \left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \Sigma r'_{j,\mu} E'_j\right\} \cdot F' \\ &= \left\{\sigma^* \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \Sigma r_{k,\mu} E_k\right\} \cdot F' \geq 0,\end{aligned}$$

since $K_X + \frac{1}{\mu}\mathcal{H}_X$ is nef. This implies

$$\mu \leq \mu'.$$

Therefore, we finally conclude that

$$\mu = \mu'.$$

This completes Step 1.

We write down the ramification formulas

$$\begin{aligned} K_V &= \sigma^* K_X + R, \\ K_V &= \sigma'^* K_{X'} + R', \\ K_V + \frac{1}{\mu}\mathcal{H}_V &= \sigma^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \Sigma r_k E_k, \\ K_V + \frac{1}{\mu}\mathcal{H}_V &= \sigma'^*\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \Sigma r'_j E'_j. \end{aligned}$$

Note that

$$R \geq \Sigma r_k E_k \geq 0, \text{ since } \lambda \leq \mu,$$

and that

$$R' = \Sigma r'_j E'_j, \text{ since } \mathcal{H}_V = \sigma'^* \mathcal{H}_{X'}.$$

Step 2. Show that in the obvious equality

$$\sigma^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \Sigma r_k E_k = \sigma'^*\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \Sigma r'_j E'_j$$

we have

$$\sigma^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) = \sigma'^*\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right),$$

which is equivalent to

$$\Sigma r_k E_k = \Sigma r'_j E'_j.$$

We introduce the notation

$$(\cup E_k) \cup (\cup E'_j) = \cup F_l$$

and write

$$\begin{aligned} \Sigma r_k E_k &= \Sigma_{l \in \mathcal{S}_\sigma} r_l F_l + \Sigma_{l \in \mathcal{S}_{\sigma,\sigma'}} r_l F_l, \\ \Sigma r'_j E'_j &= \Sigma_{l \in \mathcal{S}_{\sigma,\sigma'}} r'_l F_l + \Sigma_{l \in \mathcal{S}_{\sigma'}} r'_l F_l, \end{aligned}$$

where

$$\text{the divisor } F_l \text{ is } \begin{cases} \sigma\text{-exceptional but } \textit{not } \sigma'\text{-exceptional}, \\ \sigma\text{-exceptional and } \sigma'\text{-exceptional}, \\ \textit{not } \sigma\text{-exceptional but } \sigma'\text{-exceptional}, \end{cases} \text{ iff } \begin{cases} l \in \mathcal{S}_\sigma, \\ l \in \mathcal{S}_{\sigma,\sigma'}, \\ l \in \mathcal{S}_{\sigma'}. \end{cases}$$

Then by equalities

$$\Sigma_{l \in \mathcal{S}_\sigma} r_l F_l + \Sigma_{l \in \mathcal{S}_{\sigma,\sigma'}} (r_l - r'_l) F_l$$

$$\equiv -\sigma^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \sigma'^*\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \Sigma_{l\in\mathcal{S}_{\sigma'}} r'_l F_l,$$

$$\Sigma_{l\in\mathcal{S}_{\sigma,\sigma'}}(r'_l - r_l)F_l + \Sigma_{l\in\mathcal{S}_{\sigma'}} r'_l F_l$$

$$\equiv -\sigma'^*\left(K_{X'} + \frac{1}{\mu}\mathcal{H}_{X'}\right) + \sigma^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) + \Sigma_{l\in\mathcal{S}_\sigma} r_l F_l,$$

and by the following negativity lemma applied to σ over X and to σ' over X', we conclude that

$$\begin{aligned} r_l &\leq 0 \text{ if } l \in \mathcal{S}_\sigma, \\ r_l - r'_l &\leq 0 \text{ if } l \in \mathcal{S}_{\sigma,\sigma'}, \\ r'_l - r_l &\leq 0 \text{ if } l \in \mathcal{S}_{\sigma,\sigma'}, \\ r'_l &\leq 0 \text{ if } l \in \mathcal{S}_{\sigma'}, \end{aligned}$$

and hence

$$\begin{aligned} r_l &= 0 \text{ if } l \in \mathcal{S}_\sigma, \\ r_l &= r'_l \text{ if } l \in \mathcal{S}_{\sigma,\sigma'}, \\ r'_l &= 0 \text{ if } l \in \mathcal{S}_{\sigma'}. \end{aligned}$$

Therefore, we have

$$\Sigma r_k E_k = \Sigma_{l\in\mathcal{S}_{\sigma,\sigma'}} r_l F_l = \Sigma_{l\in\mathcal{S}_{\sigma,\sigma'}} r'_l F_l = \Sigma r'_j E'_j.$$

Lemma 13-1-4 (Negativity Lemma). *Let $f : V \to T$ be a birational morphism from a nonsingular projective variety V to another projective variety T. Suppose that a divisor $\Sigma\alpha_i E_i$ consisting of the divisors exceptional for f is numerically of the form*

$$\Sigma\alpha_i E_i \equiv N + G,$$

where N is a $\mathbb{Q}$-divisor that is f-nef (i.e., $N \cdot C \geq 0$ for any curve $C \subset V$ with $f(C) = \text{pt.}$) and where G is an effective $\mathbb{Q}$-divisor none of whose components are exceptional for f. Then

$$\alpha_i \leq 0 \quad \forall i.$$

Proof. Take very ample divisors A and B on T and V, respectively. We take any exceptional divisor E_i with $d_i = \dim f(E_i)$. Then by restricting the equality to the surface $S = \cap_{i=1}^{d_i} f^*A \cap \cap_{j=1}^{\dim V - d_i - 2} B_i$, the nonpositivity of α_i follows from the negativity lemma in dimension 2 (cf. Lemma 1-8-10). □

Step 3. Show that Φ induces an isomorphism of the Mori fiber spaces.

First we claim that Φ induces a morphism $\tau : Y \to Y'$ such that $\tau \circ \phi = \phi' \circ \Phi$. By Lemma 1-8-1 we have only to show that

$$(\phi' \circ \sigma')((\phi \circ \sigma)^{-1}(p)) \text{ is a point} \quad \forall p \in Y.$$

Since $(\phi \circ \sigma)_* \mathcal{O}_V = \mathcal{O}_Y$ (Exercise! Why?), $(\phi \circ \sigma)^{-1}(p)$ is connected by Zariski's Main Theorem (cf. Theorem 1-2-17) and hence so is $(\phi' \circ \sigma')((\phi \circ \sigma)^{-1}(p))$. Thus we have only to show that

$$\dim(\phi' \circ \sigma')((\phi \circ \sigma)^{-1}(p)) = 0.$$

Suppose $\dim(\phi' \circ \sigma')((\phi \circ \sigma)^{-1}(p)) > 0$ for some $p \in Y$. Then there exists a curve $C \subset (\phi \circ \sigma)^{-1}(p)$ such that $(\phi' \circ \sigma')(C)$ is not a point. (Exercise! Find such a curve C.) Then by the equality

$$\sigma^* \left(K_X + \frac{1}{\mu} \mathcal{H}_X \right) = \sigma'^* \left(K_{X'} + \frac{1}{\mu} \mathcal{H}_{X'} \right)$$

we compute

$$\begin{aligned} 0 &= \sigma^* \left(K_X + \frac{1}{\mu} \mathcal{H}_X \right) \cdot C \\ &= \sigma'^* \left(K_{X'} + \frac{1}{\mu} \mathcal{H}_{X'} \right) \cdot C \\ &= \frac{1}{\mu} (\phi' \circ \sigma')^* A' \cdot C \\ &= \frac{1}{\mu} A' \cdot (\phi' \circ \sigma')_*(C) > 0, \end{aligned}$$

a contradiction!

Therefore, we have such a morphism $\tau : Y \to Y'$.

Observe that the inequality and equalities

$$R \geq \Sigma r_k E_k = \Sigma r'_j E'_j = R'$$

by Step 2 imply that all the exceptional divisors for σ' are also exceptional for σ. In dimension 2 we could immediately conclude from this that Φ^{-1} is a morphism. In dimension 3 or higher, though the final conclusion is the same, we have to struggle technically a little more.

Lemma 13-1-5 (Homaloidal Transform Preserves Numerical Equivalence). *Let*

$$X \overset{\sigma}{\leftarrow} V \overset{\sigma'}{\rightarrow} X'$$

be as above. Let D'_1, D'_2 be two $\mathbb{Q}$-divisors on X' that are numerically equivalent, i.e.,

$$D'_1 \equiv D'_2 \text{ on } X'.$$

Then their homaloidal transforms are also numerically equivalent on X, i.e.,

$$D_1 = \sigma_* \sigma'^* D'_1 \equiv \sigma_* \sigma'^* D'_2 = D_2 \text{ on } X'.$$

Proof. We can write

$$\sigma'^* D'_1 = \sigma^* D_1 + \Sigma \alpha_k E_k,$$

$$\sigma'^* D_2' = \sigma^* D_2 + \Sigma \beta_k E_k,$$

where the E_k are the exceptional divisors for σ.

Since obviously

$$\sigma'^* D_1' \equiv \sigma'^* D_2',$$

we have

$$\Sigma(\alpha_k - \beta_k)E_k \equiv \sigma^*(D_2 - D_1) \equiv 0 \text{ over } X,$$
$$\Sigma(\beta_k - \alpha_k)E_k \equiv \sigma^*(D_1 - D_2) \equiv 0 \text{ over } X.$$

Thus by the negativity lemma applied to $\sigma : V \to X$ we conclude that

$$\Sigma(\alpha_k - \beta_k)E_k = 0$$

and hence

$$\sigma^* D_1 \equiv \sigma^* D_2.$$

Therefore, we conclude that

$$D_1 \equiv D_2.$$

□

Now we go back to the discussion of Step 3.

We claim that Φ^{-1} is an isomorphism in codimension one.

If not, then there would be a divisor $E_k \subset R$ such that E_k is exceptional for σ but not for σ'. We observe that $\sigma'(E_k)$ is a divisor that is ϕ'-ample, i.e., for any curve F' in a fiber of ϕ' we have

$$\sigma'(E_k) \cdot F > 0.$$

In fact, if not, then $\phi'(\sigma'(E_k))$ is a divisor on Y' such that $\sigma'(E_k) = \phi'^{-1}(\phi'(\sigma'(E_k)))$, since $\rho(X'/Y') = 1$. (Exercise! Why?) Since E_k is σ-exceptional, the set $(\tau \circ \phi)^{-1}(\phi'(\sigma'(E_k)))$ contains a divisor G whose strict transform on V is different from E_k. On the other hand, since $\sigma'(\sigma^{-1}(G)) \subset \phi'^{-1}(\phi'(\sigma'(E_k))) = \sigma'(E_k)$, the strict transform of G on V is σ'-exceptional but not σ-exeptional. By Step 2, there is no such divisor on V, a contradiction.

Therefore, $\sigma'(E_k)$ is ϕ'-ample, and hence there exist $e, a \in \mathbb{N}$ such that

$$D_1' = e\sigma'(E_k) + a\phi'^* A' \text{ is very ample on } X'.$$

Then for a curve C in the general fiber of ϕ we have

$$D_1 \cdot C = (\sigma_* \sigma'^* D_1') \cdot C = a\phi^* \tau^* A' \cdot C = 0.$$

On the other hand, since $e\sigma'(E_k) + a\phi'^* A'$ is very ample, for a general member $D_2' \in |e\sigma'(E_k) + a\phi'^* A'|$ we have

$$D_2 \cdot C = (\sigma_* \sigma'^* D_2') \cdot C > 0,$$

contradicting $D_1 \equiv D_2$, a consequence of Lemma 13-1-5.

Therefore, Φ^{-1} is an isomorphism in codimension one and hence so is Φ. Applying Lemma 13-1-5 to the homaloidal transforms $\sigma_* \sigma'$ and $\sigma'_* \sigma^*$, which are nothing but the strict transforms by Φ^{-1} and Φ, we conclude that they induce the isomorphism

$$N^1(X) \underset{\sigma_*\sigma'^*}{\overset{\sigma'_*\sigma^*}{\overset{\sim}{\longleftrightarrow}}} N^1(X').$$

Therefore, the inclusions

$$N^1(Y') \overset{\tau^*}{\hookrightarrow} N^1(Y) \hookrightarrow N^1(X) \cong N^1(X')$$

and

$$1 = \rho(X'/Y') = \dim N^1(X') - \dim N^1(Y')$$

imply

$$N^1(Y') = N^1(Y).$$

This implies that $\tau : Y \to Y'$ is a finite morphism and hence an isomorphism, since $\mathbb{C}(Y')$ is algebraically closed in $\mathbb{C}(X') = \mathbb{C}(X)$, i.e.

$$\tau : Y \overset{\rightarrow}{\sim} Y'.$$

Now if we take a (relatively) very ample divisor G' on X' over Y', then its strict transfrom G on X is also relatively ample over Y. Thus we finally conclude that

$$X = Proj_Y \oplus_{m\geq 0} \phi_*\mathcal{O}_X(mG) \underset{\Phi}{\overset{\sim}{\rightarrow}} Proj_{Y'} \oplus_{m\geq 0} \phi'_*\mathcal{O}_{X'}(mG') = X'.$$

This completes the proof of Proposition 13-1-3. □

Flowchart for the Sarkisov Program

In the following we present a flowchart to untwist a birational map via the Sarkisov program,

$$\begin{array}{ccc} X & \overset{\Phi}{\underset{\text{birat}}{\dashrightarrow}} & X \\ \downarrow\phi & & \downarrow\phi' \\ Y & & Y' \end{array}$$

between two Mori fiber spaces. (See Flowchart 13-1-9 for an illustration of the entire algorithm.)

We

Start.

The first question to ask is:

$$\lambda \leq \mu\ ?$$

According to whether the answer to this question is *yes* or *no*, we proceed separately into the case $\lambda \leq \mu$ or into the case $\lambda > \mu$.

$\boxed{\text{Case: } \lambda \leq \mu}$

If $\lambda \leq \mu$, then the next question to ask is:

$$K_X + \frac{1}{\mu}\mathcal{H}_X \text{ nef?}$$

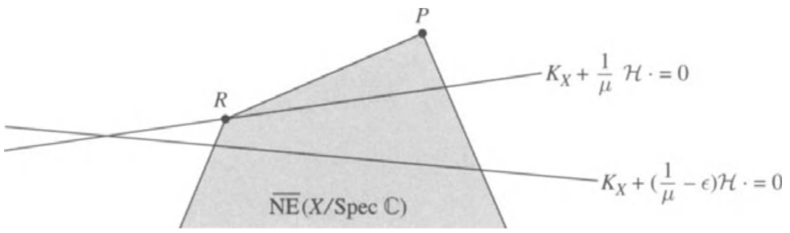
If the answer to this question is *yes*, then $K_X + \frac{1}{\mu}\mathcal{H}_X$ is nef and $\lambda \leq \mu$ by the case assumption. Thus the Noether–Fano–Iskovskikh criterion applies to the situation to conclude that Φ is an isomorphism of Mori fiber spaces. This leads to an

End.

If $K_X + \frac{1}{\mu}\mathcal{H}_X$ is not nef, then we construct as follows a normal projective variety T dominated by a morphism with connected fibers $Y \to T$ s.t. $K_X + \frac{1}{\mu}\mathcal{H}_X$ is not relatively nef over T and $\rho(X/T) = 2$, so that we run $K + \frac{1}{\mu}\mathcal{H}$-MMP over T to have an untwisting link.

In fact, we construct such a morphism $Y \to T$ as follows. We pick a $K_X + \frac{1}{\mu}\mathcal{H}_X$-negative extremal ray P of $\overline{\text{NE}}(X/\text{Spec } \mathbb{C})$ so that the span $F := P + R$ is a 2-dimensional extremal face, where R is the K_X-negative but $K_X + \frac{1}{\mu}\mathcal{H}_X$-trivial extremal ray giving the Mori fiber space $\phi : X \to Y$. (Exercise! Show why we can pick such an extremal ray P.)

Figure 13-1-6.

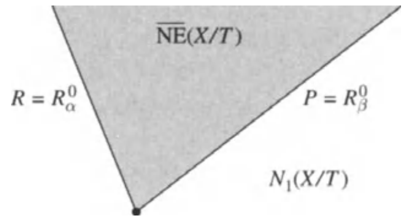


The face F is $K_X + (\frac{1}{\mu} - \epsilon)\mathcal{H}_X$-negative for $0 < \epsilon \ll 1$; thus we have the contraction morphism $\text{cont}_F : X \to T$ to obtain T (cf. Theorem 8-1-3 and Exercise 8-1-4). Since $F \supset R$, the contraction cont_F factors through $\text{cont}_R = \phi : X \to Y$, and by construction T satisfies all the required conditions.

2-Ray Game

Now we run $K + \frac{1}{\mu}\mathcal{H}$-MMP over T. It is a special kind of MMP, which we call the **2-ray game**. Since $\rho(X/T) = 2$, the cone $\overline{\text{NE}}(X/T)$ has two extremal rays (edges), one $R = R^0_\alpha$ for the original Mori fiber space $\phi = \text{cont}_R : X \to Y(\to T)$ and the other $P = R^0_\beta$ on $X = X^0$. Note that we identify $\overline{\text{NE}}(X/T)$ with the face $P + R = F$ through the inclusion $N_1(X/T) \hookrightarrow N_1(X/\text{Spec } \mathbb{C})$ (cf. Example-Exercise 3-5-1).

Figure 13-1-7.



The ray R^0_α is $K + \frac{1}{\mu}\mathcal{H}$-trivial, so in order to run $K + \frac{1}{\mu}\mathcal{H}$-MMP we just have to look at the other ray R^0_β.

If the contraction is either of divisorial type or fibering type, then the $K + \frac{1}{\mu}\mathcal{H}$-MMP necessarily comes to an end. If it is of (log) flipping type, then we continue the 2-ray game with the flip $X^1 = (X^0)^+$. The ray $R^1_\alpha = (R^0_\alpha)^+$ is necessarily $K + \frac{1}{\mu}\mathcal{H}$-positive, so in order to run $K + \frac{1}{\mu}\mathcal{H}$-MMP we just have to look at the other ray R^1_β. This process must come to an end, since there is no infinite sequence of log flips, and the MMP terminates.

We reach at the end either a log minimal model or a log Mori fiber space (with respect to $K + \frac{1}{\mu}\mathcal{H}$ and over T).

So we ask the next question:

Do we reach a log Mori fiber space?

First we deal with the case where the answer is *yes*, i.e., where we reach a log Mori fiber space $X_1 \to Y_1$.

Then the next question to ask just in order to separate the types of links is:

Is the last birational contraction divisorial?

If the answer is *yes*, the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K + \frac{1}{\mu}\mathcal{H}$-negative divisorial contraction $Z' \to X_1$. Since $\rho(X_1/T) = 1$ and by the case assumption, $\phi_1 : X_1 \to Y_1 = T$ is a log Mori fiber space with respect to $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$, i.e., ϕ_1 is a $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fiber space:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d] & Z' \arrow[dr] & \\
Y \arrow[dr] & & X_1 \arrow[d] \\
& T \arrow[r, "\sim"] & Y_1
\end{tikzcd}$$

If the answer is *no*, then the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fibering contraction $\phi_1 : Z' = X_1 \to Y_1$. Since $\rho(X_1/T) = \rho(X/T) = 2$, we have

$\rho(Y_1/T) = 1$:

$$\begin{array}{ccccc} X & & \dashrightarrow & & Z' = X_1 \\ \downarrow & & & & \downarrow \\ Y & & & & Y_1 \\ & \searrow & & \swarrow & \\ & & T & & \end{array}$$

We claim in both cases that X_1 has only $\mathbb{Q}$-factorial and terminal singularities. First, $\mathbb{Q}$-factoriality of X_1 is automatic from the construction via the (log) MMP (cf. Section 8-2). In order to see that X_1 has only terminal singularities, let I be the locus of indeterminacy of the birational map $X_1 \dashrightarrow X$. If E is a discrete valuation whose center on X_1 is not contained in I (and has codimension ≥ 2), then

$$a(E; X_1, \emptyset) = a(E; X, \emptyset) > 0.$$

If the center of E on X_1 is contained in I, then

$$a(E; X_1, \emptyset) \geq a(E; X_1, \frac{1}{\mu}\mathcal{H}_{X_1}) > a(E; X, \frac{1}{\mu}\mathcal{H}_X) \geq 0.$$

We note that the second inequality follows from the logarithmic version of Proposition 8-2-1 (ii) and the observation of Shokurov in Chapter 9 that the discrepancy does not decrease under the process of the (log) MMP and it strictly increases if the center of the exceptional divisor is on the locus that is modified by the process of the (log) MMP. We also note that the third inequality follows from the case assumption $\lambda \leq \mu$.

Thus we have the claim.

Therefore, we have a link of Type (III) in the former case and a link of Type (IV) in the latter.

Moreover, since $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is negative over Y_1, we conclude in both cases that

$$\mu_1 < \mu.$$

Therefore, after untwisting Φ by a link of Type (III) or Type (IV), we go back to *Start* with a strictly decreased quasi-effective threshold.

Second, we deal with the case where the answer to the question; Do we reach a log Mori fiber space? is *no*, i.e., where we reach a log minimal model over T.

Then just in the previous case of reaching a log Mori fiber space the next question to ask is:

Is the last birational contraction divisorial?

If the answer is *yes*, the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z'$ followed by a $K + \frac{1}{\mu}\mathcal{H}$-negative divisorial contraction $Z' \to X_1$. We will see below that $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is trivial over $T = Y_1$. Since $\rho(X_1/Y_1) = 1$,

$\phi_1 : X_1 \to Y_1$ is a K_{X_1}-negative fiber space:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & Z' \arrow[dr] & \\ Y \arrow[dr] & & X_1 \arrow[d] \\ & T \arrow[r, "\sim"] & Y_1 \end{tikzcd}$$

If the answer is *no*, then the $K + \frac{1}{\mu}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips $X \dashrightarrow Z' = X_1$. In this case we will see that there is an extremal ray of $\overline{\mathrm{NE}}(X_1/T)$ that is $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-trivial and K_{X_1}-negative and that is of fibering type. Let $\phi_1 : X_1 \to Y_1$ be the contraction of the extremal ray and $Y_1 \to T$ the induced morphism with $\rho(Y_1/T) = 1$:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d] & & Z' = X_1 \arrow[d] \\ Y \arrow[dr] & & Y_1 \arrow[dl] \\ & T & \end{tikzcd}$$

Just as before it follows that in both cases X_1 has only $\mathbb{Q}$-factorial and terminal singularities.

Now we verify the claim that in both cases there is a curve F_1 on X_1 mapping to a point on T such that

$$\left(K_{X_1} + \frac{1}{\mu}\mathcal{H}\right) \cdot F_1 = 0 \quad \text{and} \quad K_{X_1} \cdot F_1 < 0.$$

In fact, if we take a general curve F_1 in the general fiber of the morphism $X_1 \to T$ away from the locus of indeterminacy of the birational map $X_1 \dashrightarrow X$ (i.e., in the first case the union of the image of the exceptional divisor of the divisorial contraction and all the flipped curves and in the second case all the flipped curves) and not contained in the base locus of $\mathcal{H}_{X_1}$, then F_1 can be considered to lie also on X, and thus

$$0 \geq \left(K_X + \frac{1}{\mu}\mathcal{H}\right) \cdot F_1 = \left(K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}\right) \cdot F_1 \geq 0,$$
$$0 > \left(K_X + \left(\frac{1}{\mu} - \epsilon\right)\mathcal{H}_X\right) \cdot F_1 = \left(K_{X_1} + \left(\frac{1}{\mu} - \epsilon\right)\mathcal{H}_{X_1}\right) \cdot F_1 \geq K_{X_1} \cdot F_1.$$

When the last birational contraction is divisorial, since $\rho(X_1/Y_1) = 1$, the claim implies that $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is trivial over Y_1.

When the last birational contraction is flipping and we reach a log minimal model, $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is nef over T and cannot be trivial over T. (If it were trivial, then so would be $K_X + \frac{1}{\mu}\mathcal{H}_X$ over T, a contradiction!) Thus the claim implies the existence of an extremal ray containing $[F_1]$ of $\overline{\mathrm{NE}}(X_1/T)$ which is $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$-

trivial and K_{X_1}-negative. The extremal ray is also of fibering type, since F_1 is a general curve in the general fiber of the morphism $X_1 \to T$.

Therefore, we have a link of Type (III) in the former case and a link of Type (IV) in the latter. We observe that in either case of untwisting by a link of Type (III) or (IV) reaching a log minimal model, the morphism $Y \to T$ is birational. In fact, suppose that the morphism $Y \to T$ is not birational, i.e., $\dim Y > \dim T$. Then F_1 being a general curve as before, $\phi(F_1)$ becomes a curve (not a point) under the assumption. But then

$$0 > \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) \cdot F_1 = \left(K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}\right) \cdot F_1 = 0,$$

a contradiction! (The first strict inequality follows from the fact that $K_X + \frac{1}{\mu}\mathcal{H}_X$ is nonpositive on $\overline{\mathrm{NE}}(X/T)$ and that it is zero only on the extremal ray $R \not\ni [F_1]$.) Note that this observation implies that the case $\lambda \leq \mu$ reaching a log minimal model after running $K + \frac{1}{\mu}\mathcal{H}$-MMP is impossible in dimension 2. (cf. Section 1-8.)

The morphism $Y \to T$ being birational implies that both links are "square." (See the end of this section for the definition of a "square" link.) In fact, say that the morphism $Y \to T$ is an isomorphism over a dense open subset $Y \supset U \subset T$. In the case of a link of Type (III), the $K + \frac{1}{\mu}\mathcal{H}$-MMP does not affect the locus over U (Exercise! Why?) Therefore, $\phi : X \to Y$ and $\phi_1 : X_1 \to Y_1 = T$ are identical over U. In the case of a link of Type (IV), the $K + \frac{1}{\mu}\mathcal{H}$-MMP does not affect the locus over U and the last contraction of fibering type contracts all the curves in fibers over U. Therefore, again $\phi : X \to Y$ and $\phi_1 : X_1 \to Y_1$ are identical over U.

Moreover, since $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is trivial over Y_1, we conclude in both cases that

$$\mu_1 = \mu.$$

Here we don't quite have control over the Sarkisov degree. (It is the subtle point we have to face when we try to verify "termination of Sarkisov program" in the next section.) But since we start from a canonical pair $K_X + \frac{1}{\mu}\mathcal{H}_X$ running $K + \frac{1}{\mu}\mathcal{H}$-MMP over T to reach X_1, we conclude that $K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}$ is also canonical. (Note that in general a canonical pair (X, B) may not stay canonical when we contract a component of the boundary B through $K + B$-MMP. But in our case, the boundary $\mathcal{H}$ is the strict transform of a general member of one unique base point free system, and thus we may assume that the contracted divisor, if any, is not contained in $\mathcal{H}$. Therefore, all the pairs in the process stay canonical.) Thus we conclude that

$$\lambda_1 \leq \mu = \mu_1.$$

Therefore, after untwisting Φ by a link of Type (III) or Type (IV), we stay in the case $\lambda \leq \mu$.

$\boxed{\text{Case: } \lambda > \mu}$

In this case we take what we call a **maximal divisorial blowup** $p : Z \to X$ with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$, i.e., p is a projective birational morphism from Z with only $\mathbb{Q}$-factorial and terminal singularities such that

(i) $\rho(Z/X) = 1$,
(ii) the exceptional locus of p is a prime divisor E,
(iii) p is $K + \frac{1}{\lambda}\mathcal{H}$-crepant, i.e.,

$$K_Z + \frac{1}{\lambda}\mathcal{H}_Z = p^*\left(K_X + \frac{1}{\lambda}\mathcal{H}_X\right).$$

Proposition 13-1-8 (Existence of a Maximal Divisorial Blowup). *A maximal divisorial blowup $p : Z \to X$ with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$ exists.*

Before giving a proof of the proposition, we make a remark. Note that the exceptional divisor E of p is necessarily one of the $K + \frac{1}{\lambda}\mathcal{H}$-crepant divisors $\{E_1, E_2, \ldots, E_e\}$ counted for the number e in the triplet (μ, λ, e) defining the Sarkisov degree. As long as we require Z to have only terminal singularities, we can't specify which E_i is to appear as the exceptional divisor for the maximal divisorial blowup p. However, if we allow Z to have canonical singularities, for each E_i we can construct a maximal divisorial blowup $p_i : Z_i \to X$ with the exceptional divisor being E_i. Moreover, once we fix E_i, such a maximal divisorial blowup $p_i : Z_i \to X$ is unique. (Exercise! Verify the assertions in this remark.)

Proof. Take a resolution $V \to X$ from a nonsingular projective variety V s.t.

(a) the exceptional locus is a divisor with only normal crossings,
(b) V dominates X', so that the strict transform $\mathcal{H}_V$ coincides with the total transform of $\mathcal{H}_{X'}$ and a general member $\mathcal{H}_V$ is smooth and crosses normally with the exceptional locus.

We run the $K + \frac{1}{\lambda}\mathcal{H}$-MMP over X to get a log minimal model $f : (Z', \frac{1}{\lambda}\mathcal{H}_{Z'}) \to (X, \frac{1}{\lambda}\mathcal{H}_X)$. As before, it is easy to see that Z' has only $\mathbb{Q}$-factorial and terminal singularities. Since both Z' and X are $\mathbb{Q}$-factorial, the exceptional locus of f is purely one-codimensional (cf. Shafarevich [1], Theorem 2 in Section 4 of Chapter II and the remark in the proof of Theorem 4-1-3). An easy application of the negativity lemma shows that the exceptional locus is actually $\cup_{i=1}^{e} E_i$ and that f is $K + \frac{1}{\lambda}\mathcal{H}$-crepant, i.e., $K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'} = f^*(K_X + \frac{1}{\lambda}\mathcal{H}_X)$.

Now we run the K-MMP starting from Z' over X ending necessarily with a divisorial contraction $p : Z \to X$. It is immediate that $p : Z \to X$ is a maximal divisorial blowup with respect to $K_X + \frac{1}{\lambda}\mathcal{H}_X$. (If we want to specify the exceptional divisor to be E_i allowing Z to have canonical singularities, then we run $K + \frac{1}{\lambda}\mathcal{H} + \epsilon\Sigma_{j \neq i} E_j$-MMP starting from Z' over X for sufficiently small $\epsilon > 0$ instead, ending necessarily with a minimal model $p_i : Z_i \to X$, which is the maximal divisorial blowup with the exceptional divisor E_i and Z_i having only canonical singularities.) □

We go back to the discussion of the flowchart in the case $\lambda > \mu$.

After taking a maximal divisorial blowup $p : Z \to X$, we run $K + \frac{1}{\lambda}\mathcal{H}$-MMP on Z over Y.

A priori we reach either a log minimal model or a log Mori fiber space (with respect to $K + \frac{1}{\lambda}\mathcal{H}$ over Y). So we ask the following question:

Do we reach a log Mori fiber space?

First we show that it is impossible to have the answer *no* to the above question, i.e., it is impossible to reach a log minimal model.

Suppose we did.

Then according to whether the last birational contraction is divisorial or not we should have two different diagrams reaching a log minimal model $X_1 \to Y_1 = Y$:

$$\begin{tikzcd}
& Z \arrow[r, dashed] \arrow[dl, "p"'] & Z' \arrow[dr, "q"] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
Y & & & Y_1 \arrow[lll, "\sim"']
\end{tikzcd}$$

$$\begin{tikzcd}
& Z \arrow[r, dashed] \arrow[dl] & X_1 \arrow[dd] \\
X \arrow[d] & & \\
Y & & Y_1 \arrow[ll, "\sim"']
\end{tikzcd}$$

We take a general curve F on X in the general fiber of ϕ, away from $p(E)$ (and thus can be considered to lie on Z) and away from all the flipping curves (and thus can be considered to lie on Z').

In the first case, we have

$$\begin{aligned}
0 &\leq \left(K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}\right) \cdot q_* F \\
&= \left\{\left(K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'}\right) - aE_q\right\} \cdot F \ (a > 0) \\
&\leq \left(K_X + \frac{1}{\lambda}\mathcal{H}_X\right) \cdot F \\
&< \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) \cdot F = 0,
\end{aligned}$$

a contradiction!

In the second case, we have

$$0 \leq \left(K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}\right) \cdot F$$

$$
\begin{aligned}
&= \left(K_Z + \frac{1}{\lambda}\mathcal{H}_Z\right) \cdot F \\
&< \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) \cdot F = 0,
\end{aligned}
$$

again a contradiction! (Note that in each of the above cases the last strict inequality follows from the case assumption $\lambda > \mu$.)

Second, we deal with the case where the answer to the question, Do we reach a log Mori fiber space? is *yes*, i.e., where we reach a log Mori fiber space $X_1 \to Y_1$. Then the next question to ask just in order to separate the types of links is:

Is the last birational contraction divisorial?

If the answer is *yes*, then the $K + \frac{1}{\lambda}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\lambda}\mathcal{H}$-flips $Z \dashrightarrow Z'$ followed by a $K + \frac{1}{\lambda}\mathcal{H}$-negative contraction $Z' \to X_1$. Since $\rho(X_1/Y) = 1$, the morphism $\phi_1 : X_1 \to Y_1 = Y$ is a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fiber space:

$$
\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & Z' \arrow[dr] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
Y & & \arrow[ll, "\sim"'] & Y_1
\end{tikzcd}
$$

We note that the exceptional divisors E and E_q are distinct, since otherwise X and X_1 are isomorphic in codimension one, which would imply that X and X_1 are indeed isomorphic over $Y = Y_1$. (Exercise! Why?). But while the divisor E_q is *not* $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-crepant, the divisor E is $K_X + \frac{1}{\lambda}\mathcal{H}_X$-crepant; absurd!

If the answer is *no*, then the $K + \frac{1}{\lambda}\mathcal{H}$-MMP consists of a sequence of $K + \frac{1}{\lambda}\mathcal{H}$-flips $Z \dashrightarrow Z'$ followed by a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-negative and thus K_{X_1}-negative fibering contraction $Z' = X_1 \to Y_1$. Since $\rho(X_1/Y) = \rho(Z/Y) = 2$, we have $\rho(Y_1/T) = 1$:

$$
\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & X_1 \arrow[dd] \\
X \arrow[d] & & \\
Y & & Y_1 \arrow[ll]
\end{tikzcd}
$$

In both cases X_1 has only $\mathbb{Q}$-factorial and terminal singularities, and thus we have a link of Type (II) or a link of Type (I), respectively.

Now we study how the Sarkisov degree (μ, λ, e) changes after untwisting by a link of Type (II) or Type (I).

We claim that

$$\mu_1 \leq \mu$$

with equality holding only if either $\dim Y_1 > \dim Y$ or $\dim Y_1 = \dim Y$ and ψ_1 is square. Here ψ_1 is defined to be square if in the diagram

$$\begin{array}{ccc} X & \overset{\psi_1}{\underset{\text{birat}}{\dashrightarrow}} & X' \\ \downarrow\phi & & \downarrow\phi_1 \\ Y & \overset{\pi}{\dashleftarrow} & Y' \end{array}$$

there exists a birational map π that makes the diagram commute and $\psi_\eta : X_\eta \dashrightarrow (X_1)_\eta$ an isomorphism, where η is the generic point of Y.

First, by definition of λ and the assumption of this case $\lambda > \mu$, it follows that

$$p^*\left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) = K_Z + \frac{1}{\mu}\mathcal{H}_Z + bE$$

for some $b > 0, b \in \mathbb{Q}$.

We take a general curve F_1 in the general fiber of $\phi_1 : X_1 \to Y_1$ away from the locus of indeterminacy of the birational map $X_1 \dashrightarrow Z$ (i.e., in the case of a link of Type (II) the union of $q(E_q)$ and all the flipped curves and in the case of a link of Type (I) the union of all the flipped curves) and not contained in the strict transform of E. Then F_1 can be considered to lie on Z and

$$\begin{aligned} 0 &= \left(K_X + \frac{1}{\mu}\mathcal{H}_X\right) \cdot p_* F_1 \\ &= \left(K_Z + \frac{1}{\mu}\mathcal{H}_Z + bE\right) \cdot F_1 \\ &\geq \left(K_Z + \frac{1}{\mu}\mathcal{H}_Z\right) \cdot F_1 \\ &= \left(K_{X_1} + \frac{1}{\mu}\mathcal{H}_{X_1}\right) \cdot F_1, \end{aligned}$$

which implies

$$\mu_1 \leq \mu.$$

Moreover, if $\mu_1 = \mu$ and $\dim Y = \dim Y_1$ (which implies that $\pi : Y_1 \to Y$ is a birational morphism, since both field extensions $k(X)/k(Y)$ and $k(X_1) = k(X)/k(Y_1)$ are algebraically closed), then we necessarily have $E \cdot F_1 = 0$, which is equivalent to saying that ϕ_1 (the strict transform of E) is not equal to Y_1. Thus $\phi(p(E)) \neq Y$. It also follows easily that the process of $K + \frac{1}{\lambda}\mathcal{H}$-MMP only modifies the locus over $\phi(p(E))$. Therefore, ψ_1 is square.

We also claim that

$$\lambda_1 \leq \lambda$$

and

$$\text{if } \lambda_1 = \lambda \text{ then } e_1 < e.$$

First, $(X_1, \frac{1}{\lambda}\mathcal{H}_{X_1})$ is canonical, since it is obtained from a canonical pair $(Z, \frac{1}{\lambda}\mathcal{H}_Z)$ through $K + \frac{1}{\lambda}\mathcal{H}$-MMP. Thus $\lambda_1 \leq \lambda$. (See also the note at the end of the discussion of the case $\lambda \leq \mu$.)

Moreover, if $\lambda_1 = \lambda$, then in the case of untwisting by a link of Type (II) the ramification formula

$$K_{Z'} + \frac{1}{\lambda}\mathcal{H}_{Z'} = q^* \left(K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1} \right) + aE_q \qquad (a > 0)$$

implies that E_q is not a $K_{X_1} + \frac{1}{\lambda}\mathcal{H}_{X_1}$-crepant divisor (and E is a divisor on X_1 and thus not exceptional) and thus

$$e_1 \leq e - 1 < e.$$

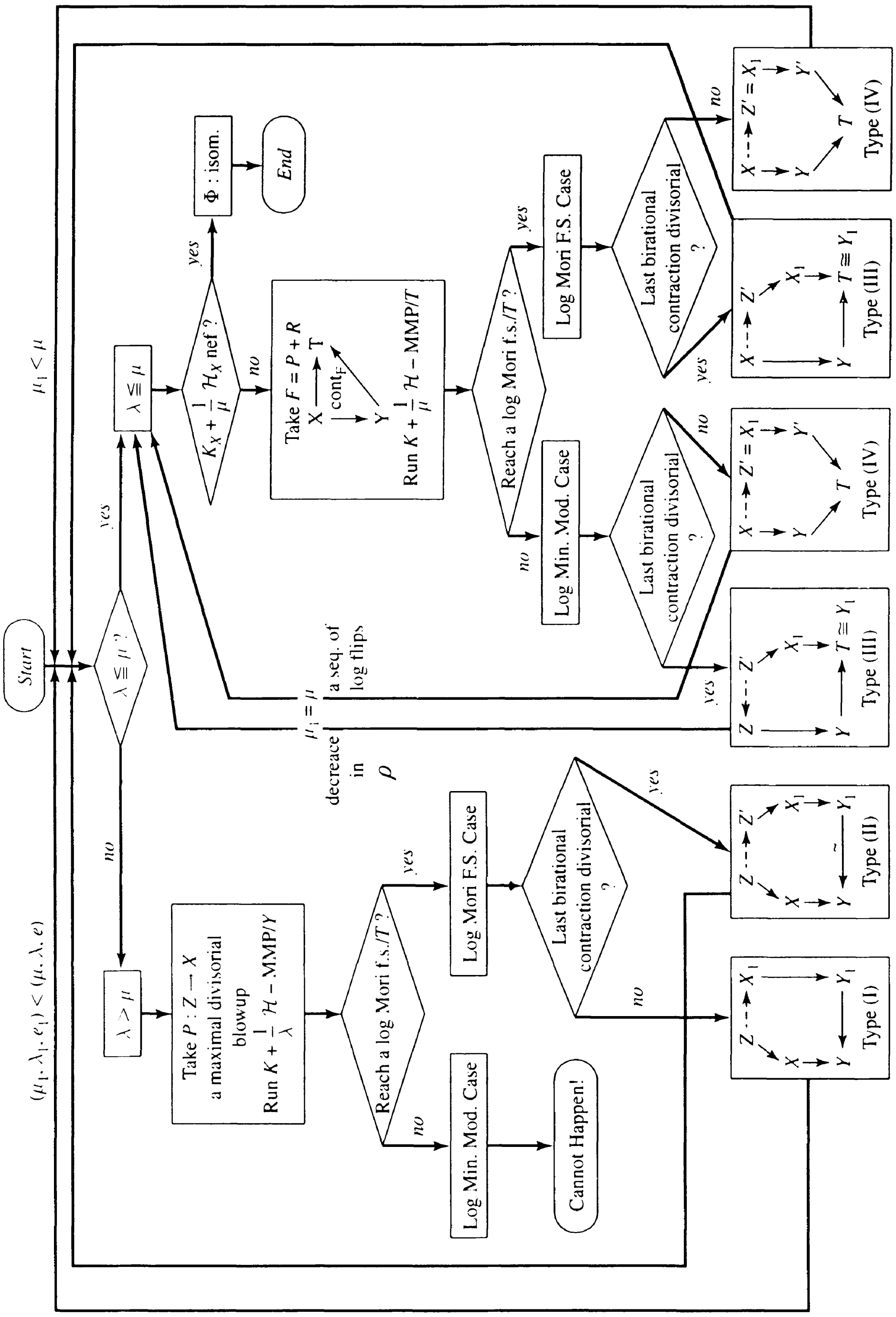
In the case of untwisting by a link of Type (I) E is a divisor on X_1 (and thus not exceptional), and hence we have the same conclusion (cf. Lemma 9-1-3 and its logarithmic version).

Therefore, after untwisting by a link of Type (II) or Type (I), we go back to the *Start* with strictly decreased Sarkisov degree.

We summarize and visualize the algorithm in Flowchart 13-1-9.

Flowchart 13-1-9.

Sarkisov Program in Dimension 3



Remark 13-1-10 (A Logarithmic Viewpoint to Interpret What the Sarkisov Program Is Doing to Untwist a Birational Map). Roughly speaking, the Sarkisov program tries to bring the log pair $(X, \frac{1}{\mu}\mathcal{H}_X) \to Y$ closer and closer to the reference log pair $(X', \frac{1}{\mu'}\mathcal{H}_{X'}) \to Y'$. Note that the latter is the morphism of a log minimal model $(X', \frac{1}{\mu'}\mathcal{H}_{X'})$ to its unique log canonical model Y' with relative Picard number one. Note that we transformed a Mori fiber space to a log minimal model by considering the extra boundary divisors. When $\lambda > \mu$, the singularities of the log pair $(X, \frac{1}{\mu}\mathcal{H}_X)$ are bad in the sense that the pair is not canonical, so what we try to do in this case is to improve its singularities by extracting the exceptional divisor with the worst discrepancy (a maximal divisorial blowup). When $\lambda \leq \mu$ and the sigularities are good, we try to apply $K + \frac{1}{\mu}\mathcal{H}$-MMP to bring the pair closer to a log minimal model until we finally reach a stage where the Noether–Fano–Iskovskikh criterion tells us that the two log minimal models (the one we obtain and the original reference) are indeed isomorphic over their (common) log canonical model.

13.2 Termination of the Sarkisov Program

The purpose of this section is to discuss **termination of the flowchart for the Sarkisov program**, i.e., the problem of showing that there is no infinite loop in the flowchart and thus after a finite number of untwistings the algorithm gives a factorization of any given birational map between two Mori fiber spaces. Once we have the (log) MMP in dimension n, together with (termination of log flips), the key points of showing termination of the flowchart of the Sarkisov Program in dimension n are:

(i) the discreteness of the quasi-effective thresholds μ, which follows from the boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n$, and
(ii) Corti's [1] ingenious argument to reduce the problem to a property of log canonical thresholds called S_n (local) (cf. Alexeev [1][2], Kollár et al. [1]) when the quasi-effective threshold of the Sarkisov degree stabilizes.

In dimension 3, where we have all the necessary ingredients, termination of the flowchart is a theorem by Corti [1]. We restrict ourselves to dimension 3 in the following presentation, but we carry out the argument in such a way that it works almost verbatim in arbitrary dimension once all the necessary but still conjectural ingredients are established (cf. toric Sarkisov program in Section 14-5).

Claim 13-2-1. *There is no infinite number of untwistings (successive or unsuccessive) by the links under the case $\lambda \leq \mu$.*

Proof. First we assert that there are not infinitely many links (successive or unsuccessive) under the case $\lambda \leq \mu$ and obtained by reaching a log Mori fiber space running $K + \frac{1}{\mu}\mathcal{H}$-MMP over T. Suppose there are infinitely many such links (successive or unsuccessive):

$$\begin{array}{ccc} X_i & \overset{\psi_i}{\underset{\text{birat}}{\dashrightarrow}} & X_{i+1} \\ \downarrow \phi_i & & \downarrow \phi_{i+1} \\ Y_i & & Y_{i+1}. \end{array}$$

Note that in the case $\lambda \leq \mu$ we have $\dim Y_i \geq 1$ (unless Φ_i becomes an isomorphism of Mori fiber spaces via the NFI criterion). When $\dim Y_i = 2$, the general fiber l of ϕ_i is a rational curve, and hence we have

$$K_{X_i} \cdot l = -2,$$
$$(\mu K_{X_i} + \mathcal{H}_{X_i}) \cdot l = 0,$$

which implies

$$\mu \in \frac{1}{2}\mathbb{N}.$$

When $\dim Y_i = 1$, we can take a rational curve l in the general fiber, which is a Del Pezzo surface s.t. (cf. Theorem 1-4-8 classification theorem of extremal rays in dimension 2)

$$K_{X_i} \cdot l = -1, -2, \text{ or } -3,$$
$$(\mu K_{X_i} + \mathcal{H}_{X_i}) \cdot l = 0,$$

which implies

$$\mu \in \frac{1}{3!}\mathbb{N}.$$

Since after any link under the case $\lambda \leq \mu$ and obtained by reaching a log Mori fiber space running $K + \frac{1}{\mu}\mathcal{H}$-MMP over T the quasi-effective threshold strictly decreases and since it does not increase after any link in any other case, an infinite sequence of links of the assumed type would lead to a strictly decreasing sequence in $\frac{1}{3!}\mathbb{N}$

$$\mu = \mu_0 > \mu_1 > \mu_2 > \cdots > 0,$$

a contradiction!

In general, we have only to use boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n - 1$ to derive the discreteness of μ and thus a contradiction to establish the first assertion. (Exercise! Show that the above-mentioned boundedness would imply boundedness of the indices and hence that via Theorem 10-3-1 it would imply the discreteness of the quasi-effective thresholds.)

Note that after untwisting by a link under the case $\lambda \leq \mu$ and obtained by reaching a log minimal model running $K + \frac{1}{\mu}\mathcal{H}$-MMP over T we will always go back to the case $\lambda \leq \mu$, though we lose track of the Sarkisov degree (μ, λ, e), not being able to control λ. (See the case analysis in the flowchart of Section 13-1.) We argue in the following way. The first assertion and the above note imply that in order for us to have an infinite number of untwisting by the links under the case $\lambda \leq \mu$ (successive or unsuccessive) we would have to have an infinite and successive sequence of untwisting by the links under the case $\lambda \leq \mu$ and obtained by reaching log minimal models running $K + \frac{1}{\mu}\mathcal{H}$-MMP over T:

$$\begin{array}{ccccccccc} X = X_0 \dashrightarrow & X_1 \dashrightarrow & X_2 \dashrightarrow & \cdots & \dashrightarrow X_i & \overset{\psi_i}{\dashrightarrow} & X_{i+1} \cdots \\ \downarrow\phi & \downarrow\phi_1 & \downarrow\phi_2 & \cdots & \downarrow\phi_i & & \downarrow\phi_{i+1} \cdots \\ Y = Y_0 & Y_1 & Y_2 & \cdots & Y_i & & Y_{i+1} \cdots \end{array}$$

If the link ψ_i is of Type (III), then the Picard number drops by one:

$$\rho(X_{i+1}) = \rho(X_i) - 1.$$

If the link ψ_i is of Type (IV), then the Picard number stays the same:

$$\rho(X_{i+1}) = \rho(X_i).$$

Thus there are not infinitely many untwistings of Type (III). Thus we may assume that it is an infinite sequence consisting purely of links of Type (IV). But then the sequence

$$X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots$$

would be an infinite sequence of $K + \frac{1}{\mu}\mathcal{H}$-flips, which contradicts termination of log flips (cf. Shokurov [2], Kawamata [15], Kollár et al. [1]). (Note that each transformation $X_i \dashrightarrow X_{i+1}$ is a sequence of log flips over T_i but also a sequence of log flips over Spec $\mathbb{C}$.)

Therefore, there is no infinite number of untwisting (successive or unsuccessive) by the links under the case $\lambda \leq \mu$.

This completes the proof of Claim 13-2-1. □

Claim 13-2-2. *There is no infinite (successive) sequence of untwisting by the links under the case $\lambda > \mu$ with stationary quasi-effective threshold.*

PROOF. This is the heart of the ingenious argument by Corti [1]. Suppose there is such an infinite sequence

$$\begin{array}{ccccccccccccc} X = X_0 \dashrightarrow & & X_1 \dashrightarrow & & X_2 \dashrightarrow & & \cdots & & \dashrightarrow X_i & \overset{\psi_i}{\dashrightarrow} & X_{i+1} \cdots \\ \downarrow\phi & & \downarrow\phi_1 & & \downarrow\phi_2 & & \cdots & & \downarrow\phi_i & & \downarrow\phi_{i+1} \cdots \\ Y = Y_0 & \leftarrow & Y_1 & \leftarrow & Y_2 & \leftarrow & \cdots & \leftarrow & Y_i & \leftarrow & Y_{i+1} \cdots \end{array}$$

Since $\mu_i = \mu_{i+1}$ for each i by the assumption, we have (see the case analysis in the flowchart of Section 13-1) either $\dim Y_{i+1} > \dim Y_i$ or $\dim Y_{i+1} = \dim Y_i$ and ψ_i is square. The first cannot happen infinitely many times, and thus we may assume that we are in the second case for all i. Note that $\dim Y_i \geq 1$, since if $\dim Y_i = \dim Y_{i+1} = 0$, then ψ_i being square would imply that ψ_i is an isomorphism of Mori fiber spaces, which is absurd!

We also know that since $\{\lambda_i\}$ is a nonincreasing sequence and since if $\lambda_i = \lambda_{i+1}$, then $e_{i+1} < e_i$, the value of λ_i cannot be stationary. Therefore, we have a sequence

$$\left\{\frac{1}{\lambda_i}\right\}, \quad \frac{1}{\lambda_i} < \frac{1}{\mu_i} = \frac{1}{\mu_0},$$

which accumulates from below to (but never equals) the value α with the property

$$\alpha \leq \frac{1}{\mu_0}.$$

Step 1. We claim that the log pair $(X_i, \alpha\mathcal{H}_{X_i})$ and $(Z_i, \alpha\mathcal{H}_{Z_i})$ ($p_i : Z_i \to X_i$ is a maximal divisorial blowup with respect to $K + \frac{1}{\lambda_i}\mathcal{H}$ extracting the exceptional divisor E_i) have only log canonical singularities for i sufficiently large (and thus we may assume that this holds for all i).

Let α_i be the **log canonical threshold** of the pair X_i with respect to $\mathcal{H}_i$, i.e.,

$$\alpha_i = \max\{c \in \mathbb{Q}_{>0}; (X_i, c\mathcal{H}_{X_i}) \text{ is log canonical}\},$$

where the log pair $(X_i, c\mathcal{H}_{X_i})$ being **log canonical** means by definition (cf. Definition 4-4-2)

$$a(E; X_i, c\mathcal{H}_{X_i}) \geq -1 \quad \forall E \text{ exceptional over } X_i.$$

If $\alpha > \alpha_i$ ($> \frac{1}{\lambda_i}$) for infinitely many i's (i.e., if the above claim fails to hold), then there is a strictly increasing subsequence $\{\alpha_l\}$ of log canonical thresholds accumulating to α. This contradicts S_3(local) proved by Alexeev [1][2]. The same argument applies to $(Z_i, \alpha\mathcal{H}_{Z_i})$.

Theorem 13-2-3 (A Special Case of S_3 (local) by Alexeev [1][2]). *The log canonical thresholds of 3-folds with terminal singularities with respect to linear systems without base components satisfy the ascending chain condition.*

PROOF. We refer the reader to Kollár et al. [1] and the original papers of Alexeev [1][2]. □

Every link $X_i \dashrightarrow X_{i+1}$ is an outcome of $K + \frac{1}{\lambda_i}\mathcal{H}$-MMP over Y_i (after taking a maximal divisorial blowup $p_i = p_i^0 : Z_i \to X_i$) consisting of a finite number of $K + \frac{1}{\lambda_i}\mathcal{H}$-flips

$$Z_i = Z_i^0 \overset{t^0}{\dashrightarrow} Z_i^1 \overset{t^1}{\dashrightarrow} \cdots \overset{t^{m-2}}{\dashrightarrow} Z_i^{m-1} \overset{t^{m-1}}{\dashrightarrow} Z_i^m,$$

possibly followed by a divisorial contraction $q_i^m : Z_i^m \to X_i^{m+1} = X_{i+1}$ (otherwise, $Z_i^m = X_i^{m+1}$).

Step 2. We claim that every step

$$\begin{tikzcd} Z_i^k \arrow[rr, dashed] \arrow[dr, "q_i^k"'] & & Z_i^{k+1} \arrow[dl, "p_i^{k+1}"] \\ & X_i^{k+1} & \end{tikzcd}$$

is a step of $K + \alpha\mathcal{H}$-MMP.

We prove this by induction on k.

First note that since $\alpha > 1/\lambda_i$, we have

$$K_{Z_i} + \alpha\mathcal{H}_{Z_i} = p_i^{0*}(K_{X_i} + \alpha\mathcal{H}_{X_i}) - aE_i \ (a > 0).$$

Therefore, we have

$$(K_{Z_i^0}, +\alpha\mathcal{H}_{Z_i^0}) \cdot P_i^0 > 0,$$

where P_i^0 is the extremal ray corresponding to the morphism p_i^0.

Suppose we have

$$(K_{Z_i^k}, +\alpha\mathcal{H}_{Z_i^k}) \cdot P_i^k > 0,$$

where P_i^k is the extremal ray corresponding to the morphism p_i^k.

Note that $K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}$ is never relatively nef over Y_i. We see this fact as follows: First, $\alpha \leq 1/\mu_i = 1/\mu_0$ by the assumption of Claim 13-2-2. Suppose $\alpha = 1/\mu_i$. Then

$$K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k} \equiv_{Y_i} -a \text{ (the strict transform of } E_i) \quad (a > 0)$$

is never relatively nef over Y_i. Suppose $\alpha < 1/\mu_i$. Then by taking a general curve F_i in the general fiber of $\phi_i : X_i \to Y_i$ away from the locus of indeterminacy of the birational map $X_i \dashrightarrow Z_i^k$ (thus the curve can be considered to lie on Z_i^k) we have

$$\begin{aligned}(K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}) \cdot F_i &= (K_{X_i} + \alpha\mathcal{H}_{X_i}) \cdot F_i \\ &< (K_{X_i} + \frac{1}{\mu_i}\mathcal{H}_{X_i}) \cdot F_i = 0.\end{aligned}$$

Therefore, $K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}$ is never relatively nef over Y_i.

Since one of the two extremal rays, namely P_i^k, of $\overline{\mathrm{NE}}(Z_i^k/Y_i)$ has positive intersection with $K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}$, this implies that the other extremal ray, Q_i^k, has negative intersection with $K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}$:

$$(K_{Z_i^k} + \alpha\mathcal{H}_{Z_i^k}) \cdot Q_i^k < 0.$$

This proves the claim.

A consequence of this claim is that (cf. Shokurov's observation, i.e., Lemma 9-1-3 and its logarithmic version in Chapter 9)

$$a(\nu; X_0, \alpha\mathcal{H}_{X_0}) \leq a(\nu; X_i, \alpha\mathcal{H}_{X_i})$$

for any discrete valuation ν of the function field $k(X)$, and strict inequality holds iff ψ_l is not an isomorphism at the center of ν on X_l for some $l < i$.

Step 3. We claim that the log pair $(X_i, \alpha\mathcal{H}_{X_i})$ has purely log terminal singularities for i sufficiently large (and thus we may assume that this holds for all i).

Assume to the contrary that there exist infinitely many indices i such that $(X_i, \alpha\mathcal{H}_{X_i})$ is not purely log terminal. Since $(X_i, \alpha\mathcal{H}_{X_i})$ is log canonical by Step 1, the assumption is equivalent to saying that for each of these i there exists a valuation ν_i of $k(X)$ with

$$a(\nu_i; X_i, \alpha\mathcal{H}_{X_i}) = -1,$$

which implies by the consequence that

$$a(\nu_i; X_0, \alpha\mathcal{H}_{X_0}) = -1$$

and that at the center $z(\nu_i, X_0)$ of ν_i on X_0, the birational map $\psi_{i-1} \circ \cdots \circ \psi_1 \circ \psi_0 : X_0 \dashrightarrow X_i$ is an isomorphism. Thus the local (w.r.t. the Zariski topology) canonical thresholds are the same:

$$c(z(\nu_i, X_i); X_i, \mathcal{H}_{X_i}) = c(z(\nu_i, X_0); X_0, \mathcal{H}_{X_0}).$$

On the other hand, by definition $1/\lambda_i$ is the global canonical threshold and hence is less than or equal to the local canonical threshold

$$\frac{1}{\lambda_i} \leq c(z(\nu_i, X_i), X_i, \mathcal{H}_{X_i}).$$

Since $K_{X_i} + \alpha\mathcal{H}_{X_i}$ is not canonical at the center $z(\nu_i, X_i)$, we have

$$c(z(\nu_i, X_i); X_i, \mathcal{H}_i) < \alpha.$$

Therefore,

$$\frac{1}{\lambda_i} \leq c(z(\nu_i, X_1); X_1, \mathcal{H}_{X_1}) < \alpha.$$

But $\{\frac{1}{\lambda_i}\}$ is a nondecreasing and nonstationary sequence converging to α, and it is easy to see (Exercise! Why?) that the set $\{c(x; X_0, \mathcal{H}_{X_0}); x \in X_0 = X\}$ is finite, a contradiction!

This proves the claim of Step 3.

Step 4. Conclusion of the argument.

We remark that the valuations of $k(X)$ corresponding to the divisors E_i extracted by the maximal divisorial blowups are all distinct. In fact, suppose that E_i and E_j coincide and hence that Z_i and Z_j are isomorphic in a neighborhood of the generic points of E_i and E_j. This would imply

$$a(E_i; X_i, \alpha\mathcal{H}_{X_i}) = a(E_j; X_j, \alpha\mathcal{H}_{X_j}).$$

On the other hand, from Step 2 we have

$$a(E_i; X_i, \alpha\mathcal{H}_{X_i}) < a(E_j; X_j, \alpha\mathcal{H}_{X_j}),$$

a contradiction!

Finally, we conclude the proof of Claim 13-2-2 as follows: From Step 3 we may assume that $(X_0, \alpha\mathcal{H}_{X_0})$ has only purely log terminal singularities. But on the other hand, for infinitely many E_i with distinct corresponding discrete valuations,

$$a(E_i; X_0, \alpha\mathcal{H}_{X_0}) \leq a(E_i; X_i, \alpha\mathcal{H}_{X_i}) < 0,$$

contradictiong the following lemma. □

Lemma 13-2-4. *Let (X, D) be a log pair with only purely log terminal singularities. Then there are only finitely many discrete valuations ν with negative log discrepancy $a(\nu, X, D) < 0$.*

PROOF. The proof is easy and is left to the reader as an exercise. □

In order to prove Claim 13-2-2 in dimension n, we need only S_n(local), which is still a missing ingredient when $n > 3$.

Remark 13-2-5. If we knew that the set of canonical thresholds satisfied the ascending chain condition, the proof of Claim 13-2-2 would be immediate by looking at the sequence $\{\frac{1}{\lambda_i}\}$. However, the behavior of the set of canonical thresholds seems to be much harder to grasp than that of the set of log canonical thresholds (cf. Remark 14-5-5).

Claim 13-2-6. *There is no infinite (successive) sequence of untwisting by the links under the case $\lambda > \mu$ with nonstationary quasi-effective thresholds.*

Suppose there is such an infinite sequence with nonstationary quasi-effective thresholds

$$\begin{array}{ccccccccccc}
X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \dashrightarrow & \cdots & \dashrightarrow & X_i & \overset{\psi_i}{\dashrightarrow} & X_{i+1} \cdots \\
\downarrow\phi & & \downarrow\phi_1 & & \downarrow\phi_2 & & \cdots & & \downarrow\phi_i & & \downarrow\phi_{i+1} \cdots \\
Y = Y_0 & \leftarrow & Y_1 & \leftarrow & Y_2 & \leftarrow & \cdots & \leftarrow & Y_i & \leftarrow & Y_{i+1} \cdots
\end{array}$$

Case: For some i_0, we have $\dim Y_{i_0} \geq 1$.

In this case, for all $i \geq i_0$ we have $\dim Y_i \geq 1$ and hence

$$\mu_i \in \frac{1}{3!}\mathbb{N}$$

as before, and $\{\mu_i\}$ is a nonstationary and nonincreasing infinite sequence $\mu_0 \geq \mu_i > 0$, a contradiction!

In order to prove Claim 13-2-6 in dimension n and in the above case assumption, we need only the boundedness of $\mathbb{Q}$-Fano d-folds for $d \leq n - 1$.

Finally, we consider

Case: For all i, we have $\dim Y_i = 0$.

In this case, the X_i's are all $\mathbb{Q}$-Fano varieties of dimension n with $\rho(X_i) = 1$. In dimension 3, we quote the following result of Kawamata [11].

Theorem 13-2-7 (Boundedness of $\mathbb{Q}$-Fano 3-Folds with Picard Number 1). *The family of $\mathbb{Q}$-Fano 3-folds (normal projective 3-folds with only $\mathbb{Q}$-factorial and terminal singularities having ample anticanonical divisors) with Picard number equal to one is bounded.*

Therefore, there exists $r \in \mathbb{N}$ such that rK_{X_i} is Cartier $\forall i$. (Exercise! Why?) Then Theorem 10-3-1 on the lengths of the extremal rational curves says that there exists a rational curve L_i on X_i s.t. $0 < -K_{X_i} \cdot L_i \leq 2 \dim X_i$, which implies

$$\mu_i \in \frac{1}{(r \cdot 2 \dim X)!q}\mathbb{N}.$$

Again $\{\mu_i\}$ is a nonstationary and nonincreasing infinite sequence with $\mu_0 \geq \mu_i > 0$, a contradiction!

We remark that this last step is the only place where we use boundedness of $\mathbb{Q}$-Fano n-folds (with $\rho = 1$). □

Claims 13-2-1, 13-2-2, and 13-2-6 show that there is no infinite loop in the flowchart of the Sarkisov program.

This completes the discussion of termination of the flowchart.

13.3 Applications

In this section we discuss some applications of the Sarkisov program. While it is expected that the Sarkisov program should shed new light on the birational structure of Mori fiber spaces in higher dimensions (e.g., in dimension 3), little is known or has been done at the moment (except for a marvelous recent work of Corti–Puklikov–Reid [1] on the structure of some special $\mathbb{Q}$-Fano 3-folds), and much is left for future research. Here we will present **Takahashi [3]'s work deriving the classical theorem of Jung** on the structure of the group of automorphisms $\mathrm{Aut}(\mathbb{A}^2)$ of the affine 2-space $\mathbb{A}^2$ **via the logarithmic Sarkisov program in dimension 2**.

First we will go over the logarithmic version of the Sarkisov program (called the log Sarkisov program for short) in dimension 2 very quickly, as the reader should have no difficulty modifying the argument in the previous sections into the language of the logarithmic category. Second, we will show how Jung's theorem can be derived by decomposing given birational maps into the links via the log Sarkisov program.

Theorem 13-3-1 (Log Sarkisov Program in Dimension 2 with Log Terminal Singularities) (cf. Bruno–Matsuki [1], Takahashi [1][2]). *Let*

$$\begin{array}{ccc} (X, B_X) & \stackrel{\Phi}{\dashrightarrow} & (X', B_{X'}) \\ \downarrow\phi & & \downarrow\phi' \\ Y & & Y' \end{array}$$

be a birational map between two log Mori fiber spaces in dimension 2 *with only log terminal singularities, inducing a log MMP relation between them (cf. Definition* 2-2-9 *or Definition* 11-3-1*). Let* (W, B_W) *be a log pair consisting of a nonsingular projective surface* W *and a boundary divisor* B_W *with only normal crossings, which gives the log MMP relation. That is to say, the birational map* Φ *can be* *decomposed as $\Phi = \tau' \circ \tau^{-1}$ where τ (respectively τ') is a process of $K + B$-MMP starting from (W, B_W) finishing with the end result (X, B_X): (respectively $(X', B_{X'})$):*

$$\begin{tikzcd}[column sep=small]
 & (W, B_W) \arrow[dl, "K+B-\mathrm{MMP}"', "\tau"] \arrow[dr, "\tau'", "K+B-\mathrm{MMP}"'] & \\
(X, B_X) \arrow[d, "\phi"] \arrow[rr, dashed, "\Phi"] & & (X', B_{X'}) \arrow[d, "\phi'"] \\
Y & & Y'.
\end{tikzcd}$$

Take the log birational category $\mathcal{D}_{(W,B_W)}$ of our choice to be the one consisting of all the log pairs obtained from (W, B_W) via $K + B$-MMP.

The log birational category $\mathfrak{D}_{(W,B_W)}$ satisfies the following properties:

(i) *$\mathfrak{D}_{(W,B_W)}$ contains the original two log Mori fiber spaces we consider.*

(ii) *For any finite family $\{(X_l, B_{X_l})\}_{l \in L} \subset \mathfrak{D}_{(W,B_W)}$ there exists an object $(V, B_V) \in \mathfrak{D}_{(W,B_W)}$ such that (V, B_V) dominates each (X_l, B_{X_l}) by a projective birational morphism $(V, B_V) \to (X_l, B_{X_l})$ that is a process of $K + B$-MMP over X_l (and hence over* Spec $\mathbb{C}$*).*

(iii) *Any $K + B$-MMP over* Spec$\mathbb{C}$ *starting from an object in $\mathfrak{D}_{(W,B_W)}$ stays inside of the category $\mathfrak{D}_{(W,B_W)}$, and so does any $K + B + c\mathcal{H}$-MMP over* Spec$\mathbb{C}$ *starting from an object in $\mathcal{D}_{(W,B_W)}$, where $\mathcal{H}$ is a linear system without base components (i.e., no divisor is contained in the base locus of the linear system) and c is a positive rational number.*

As a consequence, there is an algorithm to decompose Φ into a composite of the four types of links (as described in the Sarkisov program) in the log birational category $\mathfrak{D}_{(W,B_W)}$, where the Noether-Fano-Iskovskikh criterion and termination also hold.

PROOF. It is straightforward to check properties (i) and (ii) for the log birational category $\mathfrak{D}_{(W,B_W)}$. (In checking property (ii) note that any process of $K + B + c\mathcal{H}$-MMP must also be a process of $K + B$-MMP, since a linear system without base components is nef, a feature unique to dimension 2.)

Thus the general mechanism of the log Sarkisov program works in this log birational category. (Exercise! Check that in order for the general mechanism of the (log) Sarkisov program in Sections 13-1 and 13-2 to work we need only these 3 properties (i) (ii) (iii).)

The definition of the log Sarkisov degree is more subtle (than the general mechanism) and is explained below.

The quasi-effective threshold is defined just as before.

The maximal multiplicity λ of the linear system of homaloidal transforms is defined as

$$\frac{1}{\lambda} = \max\{c \in \mathbb{Q}_{>0}; (K_X + B_X) + c\mathcal{H}_X \text{ is } \mathfrak{D}_{(W,B_W)}\text{-canonical}\},$$

where $(K_X + B_X) + c\mathcal{H}_X$ being $\mathfrak{D}_{(W,B_W)}$-canonical means by definition

$$a(E; X, B_X + c\mathcal{H}_X) \geq a(E; W, B_W) \quad \forall E \text{ exceptional over } X.$$

Now, in order to define the number e of the $\mathfrak{D}_{(W,B_W)}$-crepant exceptional divisors we consider only those exceptional divisors that appear as divisors on W, i.e., only those exceptional divisors that have codimension-one centers on W. (If we consider all the exceptional divisors, then the number could be infinite and not well-defined.)

The Noether–Fano–Iskovskikh criterion holds without any change.

Finally, we check termination of the program. Claim 13-2-1 follows easily, since the general fibers of the fibering morphisms ϕ_i with $\dim Y_i = 1$ are all $\mathbb{P}^1$ and we have the discreteness of the quasi-effective thresholds.

The proof for Claim 13-2-2 goes without change for Steps 1 and 2 replacing K with $K + B$. We disregard Step 3. (Note that the assertion of Step 3 was used to guarantee the finiteness of the divisors extracted by the maximal divisorial blowups, which follow easily in our case.) In Step 4 we conclude the argument as follows. First note that the discrete valuations of the function field corresponding to the exceptional divisors E_i of the maximal divisorial blowups are all distinct as before. Moreover, all the E_i are divisors on W. On the other hand,

$$a(E_i; X_0, B_{X_0} + \alpha\mathcal{H}_{X_0}) \leq a(E_i; X_i, B_{X_i} + \alpha\mathcal{H}_{X_i}) < 0.$$

But there are only finitely many divisors on W with negative discrepancies with respect to the boundary divisor $B_{X_0} + \alpha\mathcal{H}_{X_0}$, a contradiction!

In order to show Claim 13-2-5 for the case with $\dim Y_{i_o} = 1$ for some i_o (and thus for all $i \geq i_o$), we show the discreteness of the quasi-effective thresholds, again noting that the general fibers of ϕ_i are all $\mathbb{P}^1$. For the case $\dim Y_i = 0 \quad \forall i$, we note that the X_i are all log Del Pezzo surfaces (normal projective surfaces with quotient singularities having ample anticanonical divisors) that are dominated by one fixed nonsingular projective surface W. Therefore, it is easy to see that the X_k belong to a bounded family, from which fact the discreteness of the quasi-effective thresholds follows just as before.

This completes the discussion for termination of the log Sarkisov program in the log birational category $\mathfrak{D}_{(W,B_W)}$.

This completes the proof of Theorem 13-3-1. □

Now we discuss the application by Takahashi [3] of the log Sarkisov program in dimension 2 to the following classical theorem of Jung:

Theorem 13-3-2 (Structure of Aut($\mathbb{A}^2$)). *Any automorphism of* $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x, y]$ *is a composition of linear (affine) transformations and de Jonquiéres transformations, where the latter consist of the transformations of type*

$$\begin{aligned} X &= x, \\ Y &= y + f(x), \qquad \textit{where } f(x) \textit{ is a polynomial in } x. \end{aligned}$$

The rest of the section will be spent on the proof of this theorem in the framework of our log Sarkisov program in dimension 2 with log terminal singularities, following Takahashi.

Take an automorphism

$$\sigma \in \mathrm{Aut}(\mathbb{A}^2).$$

Then taking a compactification $\mathbb{P}^2$ of $\mathbb{A}^2$ by adding the hyperplane H at infinity we realize that σ induces a self-birational map Φ of the log Mori fiber space

$$(\mathbb{P}^2, H) \to \mathrm{Spec}\ \mathbb{C}.$$

Therefore, by setting

$$\begin{array}{rl} (X, B_X) = (\mathbb{P}^2, H), \\ Y = \mathrm{Spec}\ \mathbb{C}, \end{array} \quad \text{and} \quad \begin{array}{rl} (X', B_{X'}) = (\mathbb{P}^2, H), \\ Y' = \mathrm{Spec}\ \mathbb{C}, \end{array}$$

we obtain the diagram

$$\begin{array}{ccc} (X, B_X) & \stackrel{\Phi}{\dashrightarrow} & (X', B_{X'}) \\ \downarrow\phi & & \downarrow\phi' \\ Y & & Y'. \end{array}$$

We take a common resolution

$$(X, B_X) \stackrel{\pi}{\leftarrow} (W, B_W) \stackrel{\pi'}{\rightarrow} (X', B_{X'})$$

so that it induces the isomorphisms

$$\mathbb{A}^2 \cong X - B_X \cong W - B_W \cong X' - B_{X'} \cong \mathbb{A}^2$$

and so that

$$\pi^{-1}(B_X)_{\mathrm{red}} = B_W = \pi'^{-1}(B_{X'})_{\mathrm{red}}$$

is a divisor with only normal crossings. It is straightforward to check that both π and π' are processes of $K + B$-MMP starting from (W, B_W). Thus Φ induces a log MMP relation, and we take the log birational category $\mathcal{D}_{(W, B_W)}$ as in Theorem 13-3-1.

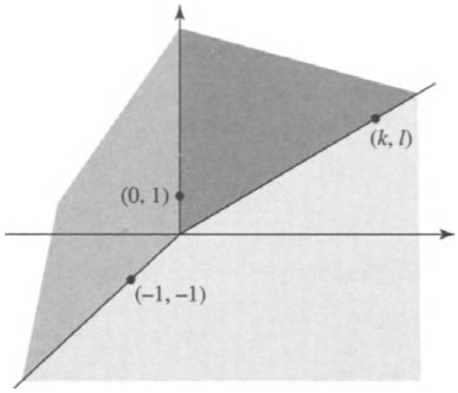
Our strategy now is to use the untwisting of the birational map Φ to deduce the decomposition of the automorphism σ.

(Takahashi's Main Idea)
We describe each link in terms of the toric geometry by choosing some appropriate coordinate system. The coordinate system chosen depends on each link. The difference between the chosen coordinate systems for the adjacent links is described by a linear (affine) transformation or by a de Jonquiéres transformation.

We use the notation $T(k, l)$ for the toric projective surface defined by the complete fan (in $N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathbb{Z}^2 \otimes \mathbb{R}$) whose 1-dimensional cones are defined by $(k, l), (0, 1)$, and $(-1, -1)$ (where $k > l \geq 0$ and $\gcd(k, l) = 1$ unless

$(k, l) = (1, 0)$), and the notation $B(k, l)$ for the (closed) divisor corresponding to the 1-dimensional cone spanned by (k, l).

Figure 13-3-3.



Note that

$$\begin{aligned} T(k, l) - B(k, l) &= \text{the toric variety associated to the fan} \\ &\qquad \text{spanned by the vectors } (0, 1) \text{ and } (-1, -1) \\ &\cong \mathbb{A}^2. \end{aligned}$$

We fix the last isomorphism and choose the coordinates x and y for $\mathbb{A}^2$ such that $x = (-1, 1)$ and $y = (-1, 0)$ in the dual space $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. According to this coordinate system we have $B(0, 1) = \{x = 0\}$ and $B(-1, -1) = \{y = 0\}$.

Theorem 13-3-4 (after Takahashi [3]). *Let*

$$\Phi = \psi_{N-1} \circ \cdots \circ \psi_1 \circ \psi_0$$

be the untwisting of the birational map

$$\Phi : (X, B_X) = (X_0, B_{X_0}) \dashrightarrow (X_N, B_{X_N}) = (X', B_{X'})$$

by the links

$$\psi_i : (X_i, B_{X_i}) \dashrightarrow (X_{i+1}, B_{X_{i+1}}) \textit{ for } i = 0, 1, \ldots, N-1$$

in the log birational category $\mathfrak{D}_{(W, B_W)}$ *via the process of the log Sarkisov program. Then there exist isomorphisms of log pairs*

$$\alpha_i : (T(k_i, l_i), B(k_i, l_i)) \xrightarrow{\sim} (X_i, B_{X_i}) \textit{ for } i = 0, 1, \ldots, N-1$$

such that

$$\alpha_{i+1}^{-1} \circ \psi_i \circ \alpha_i : (T(k_i, l_i), B(k_i, l_i)) \dashrightarrow (T(k_{i+1}, l_{i+1}), B(k_{i+1}, l_{i+1})) \quad \textit{for } i = 0, 1, \ldots, N-1$$

induces an automorphism

$$\alpha_{i+1}^{-1} \circ \psi_i \circ \alpha_i : \mathbb{A}^2 \cong Y(k_i, l_i) - B(k_i, l_i) \to Y(k_{i+1}, l_{i+1}) - B(k_{i+1}, l_{i+1}) \cong \mathbb{A}^2,$$

which is described in terms of the precribed coordinate system as

(a) $(x, y) \mapsto (x, y + ax)$ *for some* $a \in \mathbb{C}$ *or* $(x, y) \mapsto (y, x)$ *if* $(k_i, l_i) = (1, 0)$,
(b) *identity* *if* $k_i > l_i + 1$,
(c) $(x, y) \mapsto (x, y + ax^{k_i})$ *for some* $a \in \mathbb{C}$ *if* $k_i = l_i + 1 > 1$.

Proof. We prove the theorem by induction on i.

Suppose we have constructed

$$\alpha_i : (T(k_i, l_i), B(k_i, l_i)) \xrightarrow{\sim} (X_i, B_{X_i})$$

for $i = 0, 1, \ldots, n$ with the property claimed. We have only to construct

$$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}}),$$

which also satisfies the property.

Since $\rho(X_n) = \rho(T(k_n, l_n)) = 1$, the link ψ_n must be either of Type (I) or Type (II) via some maximal divisorial blowup

$$p_n : (Z_n, B_{Z_n}) \to (X_n, B_n)$$

with the exceptional divisor E_n extracted by p_n. Furthermore, since by induction

$$X_0 - B_{X_0} \overset{\psi_0}{\cong} X_1 - B_{X_1} \overset{\psi_1}{\cong} \ldots \overset{\psi_{n-1}}{\cong} \quad X_{n-1} - B_{X_{n-1}} \overset{\psi_{n-1}}{\cong} X_n - B_{X_n} \cong X_N - B_{X_N},$$

we conclude that the image of E_n, which is in the base locus of the homaloidal transforms of a very ample divisor on X_N, must be on the boundary, i.e.,

$$p_n(E_n) \in B_{X_n}.$$

Let

$$V_{0,n} \to X_n$$

be the minimal resolution and

$$V_{m,n} \to V_{m-1,n} \to \cdots \to V_{0,n}$$

the succession of blowups at the centers of E_n until E_n shows up as a divisor on $V_{m,n}$.

Observe that the log pair (Z_n, B_{Z_n}), where $B_{Z_n} = p_{n*}^{-1}(B_{X_n}) + E_n$, has only log terminal singularities. We recall one fact, which follows from the classification of log terminal singularities in dimension 2 (cf. Exercise-Theorem 4-6-30).

Fact 13-3-5. *Let* $p \in (S, D)$ *be a germ of a log terminal singularity in dimension* 2, *where the boundary divisor* D *consists of two irreducible components* D_1 *and* D_2 *with coefficient* 1 *intersecting at* p. *Then* p *is necessarily a smooth point of the surface* S, *while* D_1 *and* D_2 *intersect transversally at* p.

Since $p_{n*}^{-1}(B_{X_n}) \cong B_{X_n} \cong \mathbb{P}^1$, we conclude by the fact above that $p_{n*}^{-1}(B_n)$ intersects $E_n = p_n{}^{-1}(p_n(E_n))$ at one point only and the intersection point is a smooth point of Z_n. Therefore, noting that the minimal resolution of Z_n would factor through $V_{m,n}$, we conclude that the center of each blowup $V_{j+1,n} \to V_{j,n}$ must be the point of intersection of the strict transform of B_{X_n} and the exceptional locus of $V_{j,n} \to X_n$.

Actually, this completely determines the way we obtain (Z_n, B_{Z_n}) from (X_n, B_{X_n}) as given by the following recipe:

(i) Get the minimal resolution

$$V_{0,n} \to X_n.$$

(ii) Keep blowing up the intersection point of the strict transform of B_{X_n} and the exceptional locus of $V_{j,n} \to X_n$ until E_n appears as a divisor.

(iii) Contract all the exceptional divisors other than E_n to get Z_n, with B_{Z_n} the union of the strict transform of B_{X_n} and E_n.

Now we analyze and construct α_{n+1} case by case looking at the toric description of

$$(X_n, B_n) \overset{\alpha_n^{-1}}{\underset{\sim}{\to}} (T(k_n, l_n), B(k_n, l_n))$$

according to the recipe.

$\boxed{\text{Case (a) } (k_n, l_n) = (1, 0).}$

In this case, $X_n \cong T(k_n, l_n)$ is nonsingular and $V_{0,n} = X_n$ is the minimal resolution.

Subcase (a.1) $p_n(E_n) = B(1, 0) \cap B(0, 1)$.

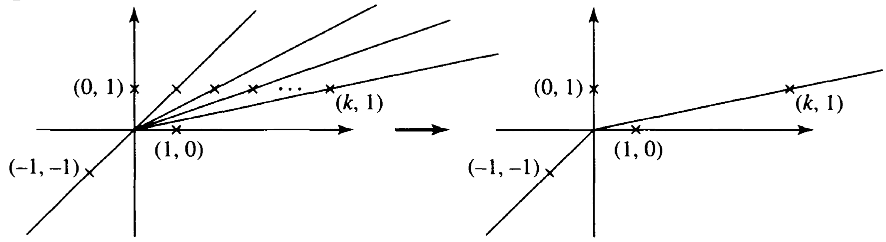
In this subcase, according to the recipe (ii) we successively blowup the points

$$\begin{aligned} P_0 &= B(0, 1) \cap B(1, 0), \\ P_1 &= B(1, 1) \cap B(1, 0), \\ &\dots \\ P_{k-1} &= B(k-1, 1) \cap B(1, 0), \end{aligned}$$

until E_n shows up as the divisor $B(k, 1)$ and then blow down all the exceptional divisors other than $B(k, 1)$ to obtain Z_n.

In terms of the cones in $N_{\mathbb{R}}$, the procedure can be illustrated as in Figure 13-3-6.

Figure 13-3-6.



Finally,we blow down $B(1,0)$ to obtain X_{n+1} by the contraction morphism $q_n : Z_n \to X_{n+1}$. (We show in Claim 13-3-7 that it is impossible to have $(k, 1) = (1, 1)$, in which case there would be no birational contraction blowing down all the exceptional divisors other than $B(k, 1)$.)

This way we obtain the link

$$\begin{aligned}\psi_n : (X_n, B_{X_n}) &\overset{\alpha_{n-1}}{\underset{\to}{\sim}} (T(1,0), B(1,0)) \\ &\overset{p_n}{\leftarrow} (Z_n, B_{Z_n}) \\ &\overset{q_n}{\to} (T(k_{n+1} = k, l_{n+1} = 1), B(k_{n+1}, l_{n+1})) \overset{\alpha_{n+1}}{\underset{\to}{\sim}} (X_{n+1}, B_{X_{n+1}})\end{aligned}$$

and

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \text{identity},$$

mapping $(x, y) \mapsto (x, y + 0 \cdot x)$.

Claim 13-3-7. *Suppose* $(k_n, l_n) = (1, 0)$ *and* $p_n(E_n) = B(0, 1) \cap B(1, 0)$ *as above. Then the last ray* $(k, 1)$ *to add to obtain* E_n *as a divisor cannot be* $(1, 1)$. *That is to say,*

$$E_n \neq B(1, 1).$$

PROOF. If $(k, 1) = (1, 1)$, then the log Mori fiber space $(X_{n+1}, B_{X_{n+1}}) \to Y_{n+1}$ is nothing but $(\mathbb{F}_1, \sigma_1 + F_1) \to \mathbb{P}^1$ from the Hirzebruch surface $\mathbb{F}_1$, where $\sigma_1 = B(1, 1)$ is the unique (-1)-curve and $F_1 = B(1, 0)$ is a fiber of the ruling.

Since the log Mori fiber space $\phi_n : (X_{n+1}, B_{X_{n+1}}) \to Y_{n+1}$ does not coincide with the reference log Mori fiber space $\phi_N : (X_N, B_{X_N}) \to Y_N = \phi' : (X', B_{X'}) \to Y'$, we have to have another link ψ_{n+1}.

Now, the link ψ_{n+1} cannot be of Type (III). In fact, a link of Type (III) would contract σ_1, which brings us back to (X_n, B_{X_n}), a contradiction.

Therefore, it must be of Type (II) via the maximal divisorial blowup

$$p_{n+1} : (Z_{n+1}, B_{Z_{n+1}}) \to (X_{n+1}, B_{X_{n+1}})$$

with the exceptional divisor E_{n+1}.

(Note also that the link ψ_{n+1} must be in the case where we reach a log Mori fiber space running $K + \frac{1}{\mu}\mathcal{H}$-MMP when $\lambda \leq \mu$ and running $K + \frac{1}{\lambda}\mathcal{H}$-MMP when $\lambda > \mu$. See Flowchart 13-1-9.)

Since

$$X_{n+1} - B_{X_{n+1}} \subset X' - B_{X'},$$

we have

$$p_{n+1}(E_{n+1}) \in B_{X_{n+1}} = \sigma_1 + F_1.$$

On the other hand,

$$p_{n+1}(E_{n+1}) \neq \sigma_1 \cap F_1,$$

since $\sigma_1 \cap F_1$ is a point that is log terminal but not purely log terminal of the logarithmic pair $(X_{n+1}, B_{X_{n+1}})$. (Recall that (W, B_W) dominates both $(Z_{n+1}, B_{Z_{n+1}})$ and $(X_{n+1}, B_{X_{n+1}})$ by a process of the log MMP.) We can also see this fact by observing via direct computation that if $p_{n+1}(E_{n+1}) = \sigma_1 \cap F_1$, then $K_{Z_{n+1}} + B_{Z_{n+1}}$ would not be p_{n+1}-negative, contradicting the property of the maximal divisorial blowup. We also see that

$$p_{n+1}(E_{n+1}) \not\in F_1 - \sigma_1,$$

since otherwise $K_{Z_{n+1}} + B_{Z_{n+1}}$ would not be q_{n+1}-negative via direct computation, where

$$q_{n+1} : (Z_{n+1}, B_{Z_{n+1}}) \to (X_{n+2}, B_{n+2})$$

is the contraction of the strict transform of F_1.

Thus

$$p_{n+1}(E_{n+1}) \in \sigma_1 - F_1,$$

in which case

$$q_{n+1} : (Z_{n+1}, B_{Z_{n+1}}) \to (X_{n+2}, B_{n+2})$$

is the contraction of the strict transform of the ruling passing through $p_{n+1}(E_{n+1})$. Then $X_{n+2} \cong \mathbb{F}_2$, the Hirzebruch surface with the unique negative section σ_2, and

$$B_{X_{n+2}} = \sigma_2 + F_1 + F_2,$$

where F_1 and F_2 are rulings.

Arguing in the same way, we see that the log Mori fiber space $(X_{n+i}, B_{X_{n+i}}) \to Y_i$ for $i \geq 1$, which is nothing but $(\mathbb{F}_i, \sigma_i + F_1 + F_2 + \cdots + F_i) \to \mathbb{P}^1$ satisfying

$$X_i - B_{X_i} = \mathbb{F}_i - (\sigma_r + F_1 + F_2 + \cdots + F_i) \subset X' - B_{X'},$$

where $\mathbb{F}_i$ is the Hirzebruch surface having the unique section with the minimum self-intersection $\sigma_i^2 = -i$ and where $F_1 + F_2 + \cdots + F_i$ are rulings, would be followed only by a link of Type (II) ending with a log Mori fiber space of the same type. (It cannot be followed by a link of Type (III), since it would contract σ_r, which is not $K_{X_i} + B_{X_i}$-negative. The rest of the reasoning is exactly the same as above.)

This would produce an infinite sequence of links, contradicting termination of the log Sarkisov program.

Therefore, the ray $(k, 1)$ cannot be equal to $(1, 1)$.

This completes the proof of Claim 13-3-7. □

Subcase (a.2) $p_n(E_n) = B(1, 0) \cap B(-1, -1)$.

In this case, we take an automorphism

$$\tau_n^{-1} : (T(k_n, l_n), B(k_n, l_n)) \to (T(k_n, l_n), B(k_n, l_n)),$$

which is induced by the automorphism of the lattice N switching the points $(0, 1)$ and $(-1, -1)$. Then $\alpha_n \circ \tau_n^{-1}$ is the one in Subcase (a.1), and hence we can find

$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}})$ such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ (\alpha_n \circ \tau_n^{-1}) = \text{identity}.$$

Therefore,

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \tau_n,$$

mapping $(x, y) \mapsto (y, x)$.

Subcase (a.3) $p_n(E_n) \in B(1, 0) - \{B(0, 1) \cup B(-1, -1)\}$.

Observe that in this subcase y/x gives the affine coordinate system for $B(1, 0) - \{B(0, 1) \cup B(-1, -1)\} \cong \mathbb{P}^1 - \{0, \infty\} \cong \mathbb{C}^*$. Say $p_n(E_n) = \{\frac{y}{x} = a \in \mathbb{C}^*\}$. We take an automorphism

$$\tau_n^{-1} : (T(k_n, l_n), B(k_n, l_n)) \to (T(k_n, l_n), B(k_n, l_n))$$

that sends $(x, y) \in \mathbb{A}^2 = T(k_n, l_n) - B(k_n, l_n)$ to $(x, y - ax) \in \mathbb{A}^2 = T(k_n, l_n) - B(k_n, l_n)$. Then $\alpha_n \circ \tau_n^{-1}$ is the one in Subcase (a.1), and hence we can find $\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}})$ such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ (\alpha_n \circ \tau_n^{-1}) = \text{identity}.$$

Therefore,

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \tau_n$$

mapping $(x, y) \mapsto (x, y + ax)$.

$\boxed{\text{Case (b) } (k_n, l_n) \text{ with } k_n > l_n + 1.}$

In this case, $T(k_n, l_n)$ has two singular points that are the closed orbits corresponding to the 2-dimensional cones generated by (k_n, l_n) and $(0, 1)$, and by (k_n, l_n) and $(-1, -1)$. They are on the divisor $B(k_n, l_n)$.

Claim 13-3-8. *Suppose* $(X_n, B_n) \overset{\alpha_n^{-1}}{\sim\to} (T(k_n, l_n), B(k_n, l_n))$ *with* $k_n > l_n + 1$. *Then* $p_n(E_n)$ *must be one of the two singular points.*

PROOF. Suppose not. Then after taking the maximal divisorial blowup

$$p_n : (Z_n, B_{Z_n} = p_n^{-1}{}_*(B_{X_n}) + E_n) \to (X_n, B_{X_n}),$$

we have two extremal rays on Z_n with $\rho(Z_n) = 2$, one, which corresponds to the contraction of E_n, and the other, which corresponds to the contraction of $p_{n*}^{-1}(B_{X_n})$ as

$$\{p_{n*}^{-1}(B_{X_n})\}^2 \leq B(k_n, l_n)^2 - 1 = \frac{1}{k_n(k_n - l_n)} - 1 < 0$$

to obtain

$$q_n : (Z_n, B_{Z_n}) \to (X_{n+1}, B_{X_{n+1}}).$$

But then $K_{Z_n} + B_{Z_n}$ is not q_n-negative, since

$$(K_{Z_n}+B_{Z_n})\cdot p_{n*}^{-1}(B_{X_n}) = (K_{T(k_n,l_n)}+B(k_n, l_n))\cdot B(k_n, l_n)+1 = \frac{-2k_n + l_n}{k_n(k_n - l_n)}+1 > 0,$$

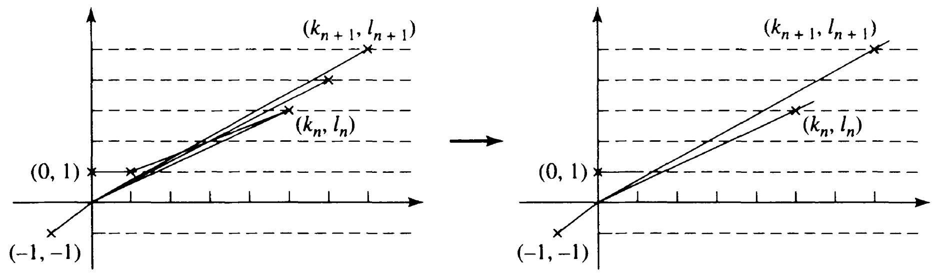
contradicting the construction of links of Type (I) or Type (II) (since we are always in the case where we reach a log Mori fiber space no matter whether $\lambda \leq \mu$ or $\lambda > \mu$). (Exercise! Check the computation of the intersection numbers on the toric surface above.) □

Subcase (b.1) $p_n(E_n)$ is the singularity corresponding to the 2-dimensional cone generated by (k_n, l_n) and $(0, 1)$.

We follow the recipe to construct (Z_n, B_{Z_n}) from (X_n, B_{X_n}) explained right before the case-by-case study.

(i) We take the Newton polygon of the cone generated by (k_n, l_n) and $(0, 1)$ to obtain the minimal resolution.
(ii) Keep taking the subdivision between (k_n, l_n) and the adjacent lattice point until E_n shows up as a divisor corresponding to (k_{n+1}, l_{n+1}). By construction, we have $k_{n+1} > l_{n+1} > 0$ with $l_{n+1}/k_{n+1} > l_n/k_n$.
(iii) Contract all the exceptional divisors other than E_n. This corresponds to obtaining the toric variety associated to the complete fan whose 1-dimensional cones are generated by the vectors $(0, 1)$, $(-1, -1)$, (k_n, l_n), and (k_{n+1}, l_{n+1}), respectively.

Figure 13-3-9.



Finally, by eliminating the 1-dimensional cone generated by (k_n, l_n) we obtain

$$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}})$$

such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \text{identity}.$$

Subcase (b.2) $p_n(E_n)$ is the singularity corresponding to the 2-dimensional cone generated by (k_n, l_n) and $(-1, -1)$.

We follow the recipe to construct (Z_n, B_{Z_n}) from (X_n, B_{X_n}) explained right before the case-by-case study.

(i) We take the Newton polygon of the cone generated by (k_n, l_n) and $(-1, -1)$ to obtain the minimal resolution.
(ii) Keep taking the subdivision between (k_n, l_n) and the adjacent lattice point until E_n shows up as a divisor corresponding to (k_{n+1}, l_{n+1}). By construction, we have $k_{n+1} > l_{n+1} > 0$ with $l_{n+1}/k_{n+1} < l_n/k_n$.
(iii) Contract all the exceptional divisors other than E_n. This corresponds to obtaining the toric variety associated to the complete fan whose 1-dimensional cones are generated by the vectors $(0, 1)$, $(-1, -1)$, (k_n, l_n), and (k_{n+1}, l_{n+1}), respectively.

Finally, by eliminating the 1-dimensional cone generated by (k_n, l_n) we obtain

$$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}})$$

such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \text{identity}.$$

$\boxed{\text{Case (c) } (k_n, l_n) \text{ with } k_n = l_n + 1 > 1.}$

Subcase (c.1) $p_n(E_n) = B(k_n, l_n) \cap B(0, 1)$.

In this subcase, we proceed as in the subcases of the previous case.

We follow the recipe to construct (Z_n, B_{Z_n}) from (X_n, B_{X_n}) explained right before the case-by-case study.

(i) We take the Newton polygon of the cone generated by (k_n, l_n) and $(0, 1)$, i.e., the convex hull whose vertices are (k_n, l_n), $(1, 1)$, $(0, 1)$ in this subcase, to obtain the minimal resolution.
(ii) Keep taking the subdivision between (k_n, l_n) and the adjacent lattice point until E_n shows up as a divisor corresponding to (k_{n+1}, l_{n+1}). By construction, we have $k_{n+1} \geq l_{n+1} > 0$. Moreover, by arguing just as in Claim 13-3-7 we conclude that $(k_n, l_n) \neq (1, 1)$.
(iii) Contract all the exceptional divisors other than E_n. This corresponds to obtaining the toric variety associated to the complete fan whose 1-dimensional cones are generated by the vectors $(0, 1)$, $(-1, -1)$, (k_n, l_n), and (k_{n+1}, l_{n+1}), respectively.

Finally, by eliminating the 1-dimensional cone generated by (k_n, l_n) we obtain

$$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{n+1})$$

such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \text{identity}.$$

Subcase (c.2) $p_n(E_n) \in B(k_n, l_n) \cap B(-1, -1)$.

We follow the recipe to construct (Z_n, B_{Z_n}) from (X_n, B_{X_n}) explained right before the case-by-case study.

(i) In this subcase the cone generated by (k_n, l_n) and $(-1,-1)$ already corresponds to a nonsingular affine surface.
(ii) Keep taking the subdivision between (k_n, l_n) and the adjacent lattice point until E_n shows up as a divisor corresponding to (k_{n+1}, l_{n+1}). By construction, we have $k_{n+1} > l_{n+1} \geq 0$.
(iii) Contract all the exceptional divisors other than E_n. This corresponds to obtaining the toric variety associated to the complete fan whose 1-dimensional cones are generated by the vectors $(0, 1)$, $(-1,-1)$, (k_n, l_n), and (k_{n+1}, l_{n+1}), respectively.

Finally, by eliminating the 1-dimensional cone generated by (k_n, l_n) we obtain

$$\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{n+1})$$

such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \text{identity}.$$

Subcase (c.3) $p_n(E_n) \in B(k_n, l_n) - \{B(0, 1) \cup B(-1,-1)\}$.

Observe that in this subcase y/x^{k_n} gives the affine coordinates for $B(k_n, l_n) - \{B(0, 1) \cup B(-1,-1)\} \cong \mathbb{P}^1 - \{0, \infty\} \cong \mathbb{C}^*$. Say $p_n(E_n) = \{y/x^{k_n} = a \in \mathbb{C}^*\}$. We take an automorphism

$$\tau_n^{-1} : (T(k_n, l_n), B(k_n, l_n)) \to (T(k_n, l_n), B(k_n, l_n))$$

that sends $(x, y) \in \mathbb{A}^2 = T(k_n, l_n) - B(k_n, l_n)$ to $(x, y - ax^{k_n}) \in \mathbb{A}^2 = T(k_n, l_n) - B(k_n, l_n)$. Then $\alpha_n \circ \tau_n^{-1}$ is the one in Subcase (c.1), and hence we can find $\alpha_{n+1} : (T(k_{n+1}, l_{n+1}), B(k_{n+1}, l_{n+1})) \xrightarrow{\sim} (X_{n+1}, B_{X_{n+1}})$ such that

$$\alpha_{n+1}^{-1} \circ \psi_n \circ (\alpha_n \circ \tau_n^{-1}) = \text{identity}.$$

Therefore,

$$\alpha_{n+1}^{-1} \circ \psi_n \circ \alpha_n = \tau_n,$$

mapping $(x, y) \mapsto (x, y + ax^{k_n})$.

This completes the proof of Theorem 13-3-4. □

In order to see Theorem 13-3-2 from Theorem 13-3-4,

$$\beta_0 : (T(1, 0), B(1, 0)) \xrightarrow{\sim} (X_0, B_{X_0}) = (X, B_X),$$
$$\beta_N : (T(1, 0), B(1, 0)) \xrightarrow{\sim} (X_N, B_{X_N}) = (X', B_{X'}),$$

being the isomorphisms such that

$$\sigma = \beta_{N+1}^{-1} \circ \Phi \circ \beta_0,$$

we observe that

$$\sigma = (\beta_N^{-1} \circ \alpha_N) \circ (\alpha_N^{-1} \circ \psi_{N-1} \circ \alpha_{N-1}) \circ \cdots \circ (\alpha_{i+1}^{-1} \circ \psi_i \circ \alpha_i) \circ \cdots \circ (\alpha_1^{-1} \circ \psi_0 \circ \alpha_0) \circ (\alpha_0^{-1} \circ \beta_0),$$

where

$$\alpha_{i+1}^{-1} \circ \psi_i \circ \alpha_i \text{ for } i = 0, 1, \ldots, N-1$$

are de Jonquiéres transformations or linear (affine) transformations by Theorem 13-3-4 and where

$$\beta_N^{-1} \circ \alpha_N \text{ and } \alpha_0^{-1} \circ \beta_0$$

are linear (affine) transformations of $\mathbb{A}^2$, since they are automorphisms of $T(1,0) \cong \mathbb{P}^2$ fixing the plane at infinity $B(1,0)$.

This completes the presentation of Takahashi's work showing Jung's theorem on the structure of $\mathrm{Aut}(\mathbb{A}^2)$ as an application of the log Sarkisov program in dimension 2. □

Remark 13-3-10. For an application of the Sarkisov program in dimension 3 or higher to study the birational properties of the $\mathbb{Q}$-Fano varieties via the structure of their self-birational maps, we refer the reader, e.g., to the recent paper of Corti–Pukhlikov–Reid [1]. See also Example 3-2-13 to taste the flavor of their methods.