BOUNDEDNESS QUESTIONS FOR CALABI-YAU'S

1. GENTLE INTRO TO THE MMP

2. BOUNDEDNESS

3. - 4. BOUNDEDNESS for CY's (elliptic)

   + log CY pairs

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY IS TO PRODUCE A COMPLETE CLASSIFICATION OF PROJECTIVE VARIETIES.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

ONE OF THE MAIN GOALS OF ALGEBRAIC GEOMETRY IS TO PRODUCE A COMPLETE CLASSIFICATION OF PROJECTIVE VARIETIES. TO THIS END, WE CAN PARTITION VARIETIES INTO EQUIVALENCE CLASSES USING 2 POSSIBLE EQUIVALENCE RELATIONS :

① ISOMORPHISM :

$$X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$$

② BIRATIONAL ISOMORPHISM :

$$X \underset{g^{-1}}{\overset{g}{\dashrightarrow\!\!\!\!\dashleftarrow}} Y$$

g, g^{-1} are rational maps.

$\mathbb{C}(X) =$ field of rat'l functions on X

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY $/ \mathbb{C}$

CANONICAL BUNDLE: $\det(\Omega^1_X)$

CANONICAL DIVISOR: Any Weil divisor K_X s.t.

$\mathcal{O}_X(K_X) \cong$ canonical bundle.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X SMOOTH PROJECTIVE VARIETY / $\mathbb{C}$

CONJECTURE STARTING FROM X, THERE EXISTS AN ALGORITHMIC WAY OF CONSTRUCTING A DIAGRAM OF THE FOLLOWING FORM

$$
\begin{tikzcd}
X \arrow[r, dashed, "\text{special birat'l}", "\text{transformations}"'] & X' \arrow[d, "f"] \\
 & Z
\end{tikzcd}
$$

special morphism

SUCH THAT f SATISFIES 1 OF THE FOLLOWING:

1. f is BIRATIONAL
   K_Z is ample — Z is a can. model.
2. f is a FIBRATION & $K_{X'} = f^* H$ ample on Z

   the fibers will be K-trivial var's
3. f is a FIBR. & $-K_{X'}|_F$ is ample on F

   the fibers will be Fano

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

HENCE IN THE BIRATIONAL CLASSIFICATION OF ALGEBRAIC VARIETIES WE HAVE 3 IMPORTANT CLASSES OF VARIETIES THAT ACT AS BUILDING BLOCKS

1. CANONICAL MODELS (K is ample)
2. K-TRIVIAL VAR'S ($K \equiv 0$)
3. FANO VAR'S ($-K$ is ample)

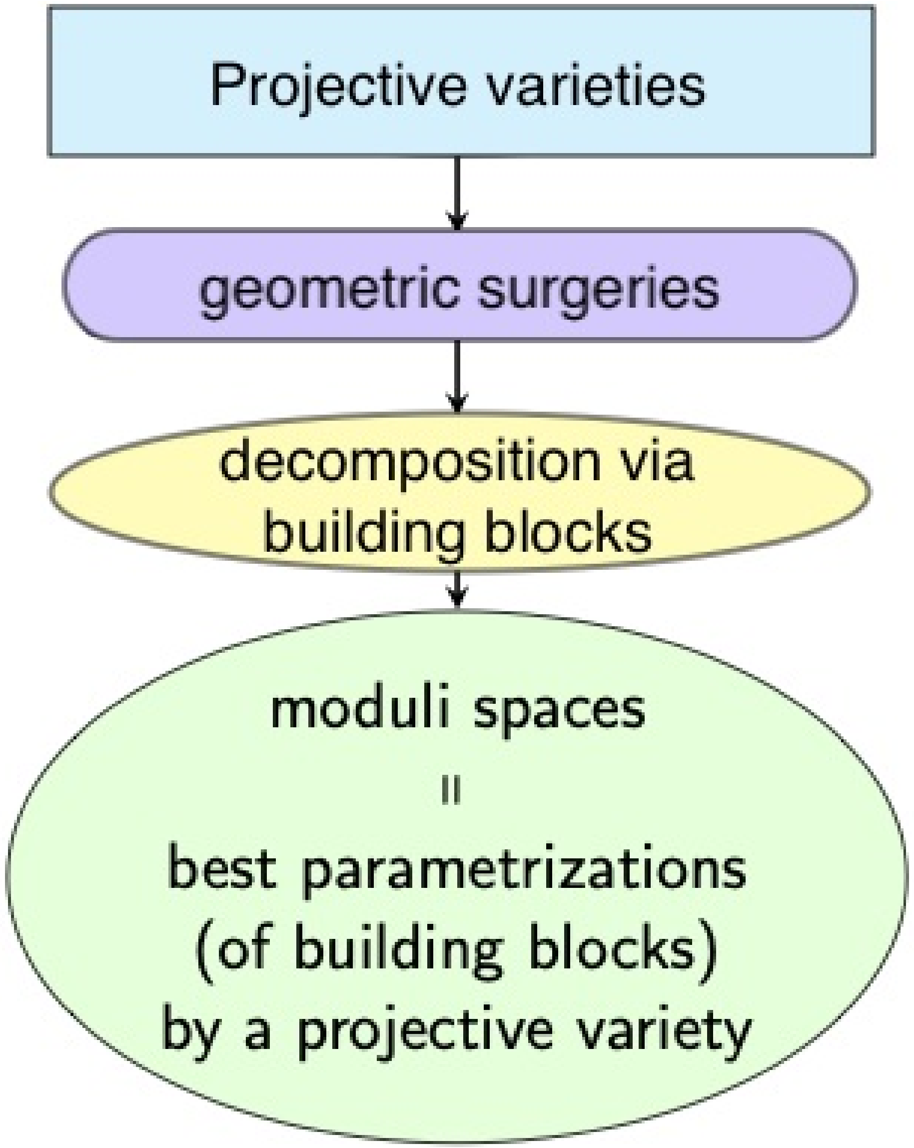
MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

THE 2^{nd} PART OF THE CLASSIFICATION AIMS THEN TO CLASSIFY THOSE VARIETIES THAT BELONG TO ONE OF THE 3 BUILDING BLOCKS

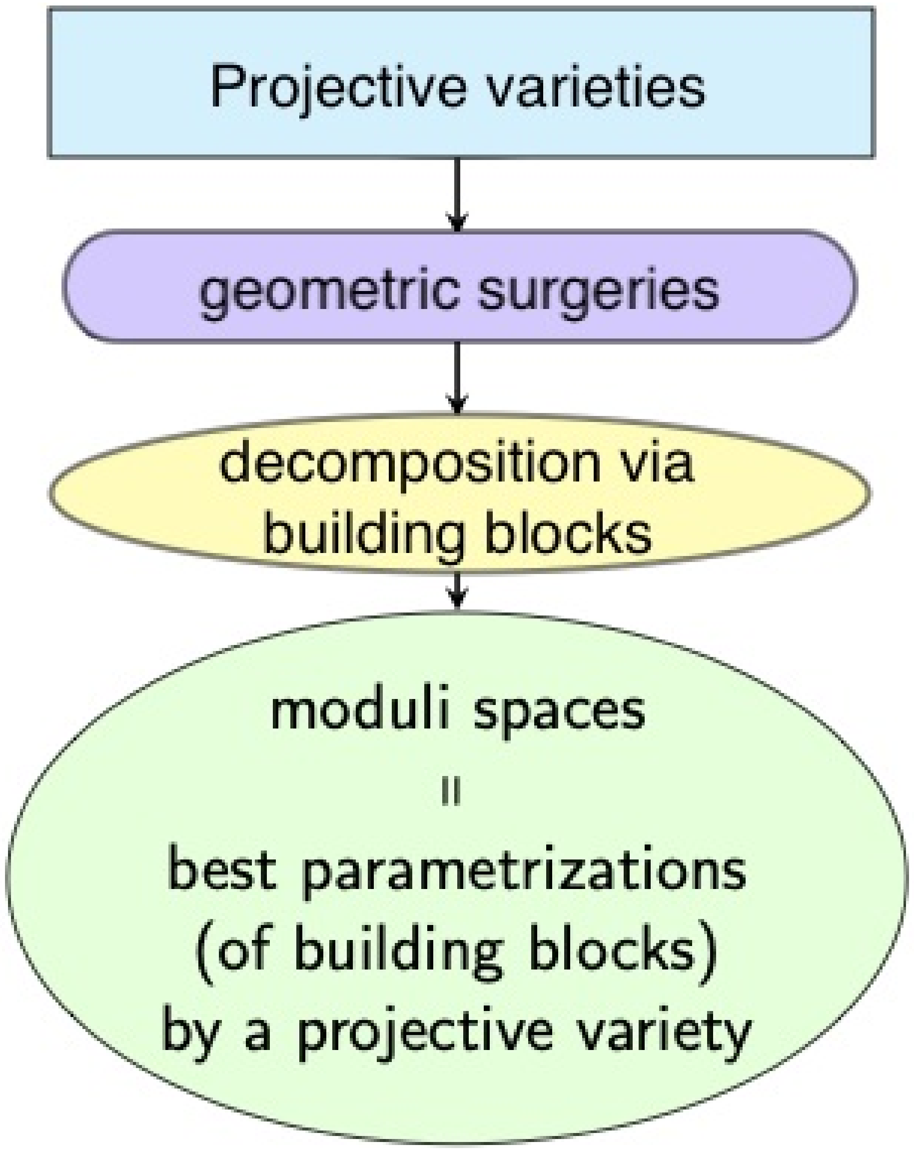
1. CANONICAL MODELS (K IS AMPLE)
2. K-TRIVIAL VAR'S ($K \equiv 0$)
3. FANO VAR'S ($-K$ IS AMPLE)

MODULI SPACES:

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION



MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION



IN ORDER TO COMPLETE SUCH CLASSIFICATION SCHEME, WE NEED ALSO TO WORK WITH SINGULAR VARIETIES.

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

X NORMAL PROJECTIVE VARIETY $/ \mathbb{C}$

CANONICAL BUNDLE : $i_* \det(\Omega^1_{X^{sm}})$ rank 1 - sheaf

$i : X^{sm} \hookrightarrow X$

CANONICAL DIVISOR : Zar. closure of any

$K_{X^{sm}}$

K_X will be a Weil divisor
(not. nec. Cartier)

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION ① A PAIR IS THE DATUM OF :

(i) X a normal variety

(ii) B an effective Weil divisor with coefficients $\in (0,1]$ BOUNDARY

② A LOG PAIR IS A PAIR (X,B) SUCH THAT

$K_X + B$ is $\mathbb{R}$-CARTIER
(can be written as a $\mathbb{R}$-sum of Cartier divisors)

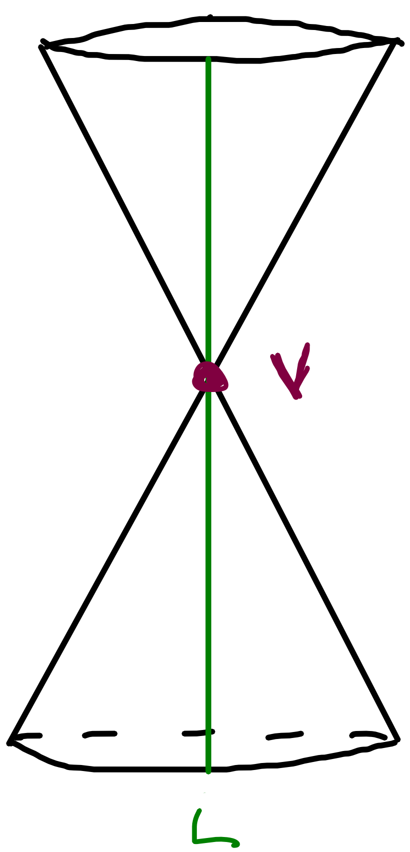
MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

CONSIDER FOR EXAMPLE



$C = V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^3$, $H \subseteq \mathbb{P}^3$ hyperpl.

C is SING. at v

$2L \in |H|_C|$ $[2L = V(x_0, x_0x_1 - x_2^2)$

L IS NOT CARTIER AT v

ADJ. FORM: X SMOOTH

$D \subseteq X$ SMOOTH codim 1

$(K_X + D)|_D = K_D$

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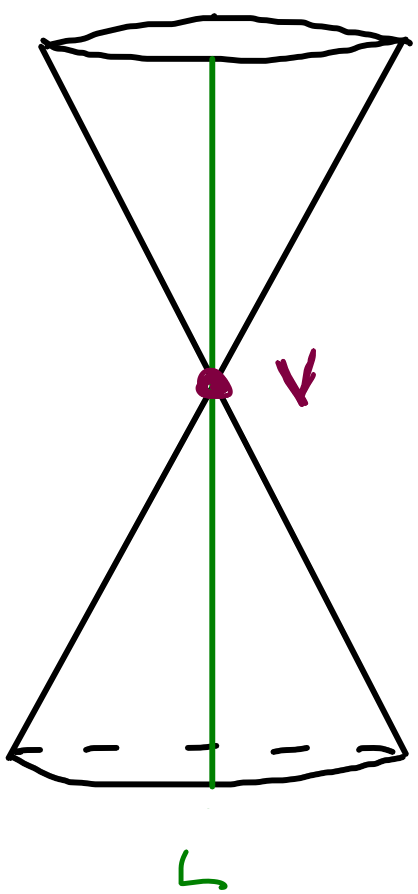

$C = V(x_0 x_1 - x_2^2) \subseteq \mathbb{P}^3$

$H \subseteq \mathbb{P}^3$ hyperpl.

$2L \sim H|_C$

ADJ. FORM: X SMOOTH, $D \subseteq X$ SMOOTH codim 1

$$(K_X + D)|_D = K_D$$

$$\left(K_{C\setminus\{v\}} + L|_{C\setminus\{v\}}\right)\Big|_{L|_{C\setminus\{v\}}} = K_{L|_{C\setminus\{v\}}}$$

$$\underset{\left(-2H|_C + \frac{1}{2}H|_C\right)\cdot L}{\overset{\parallel}{(K_C + L)\cdot L}} = -\frac{3}{2} \Longrightarrow (K_C + L)|_L = K_L + \frac{1}{2} v$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WHY IS IT USEFUL TO CONSIDER LOG PAIRS?

CONSIDER A FIBRATION IN K-TRIVIAL VARIETIES

$$f : Y \longrightarrow X$$

MINIMAL MODEL PROGRAM: A GENTLE INTRODUCTION

WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.

TO DO THAT, WE NEED TO MEASURE THEIR SINGULARITIES.

WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

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DEFINITION LET (X,B) BE A PAIR. A BIRAT'L MORPHISM

$$\pi : X' \longrightarrow X$$

IS A LOG RESOLUTION WHEN :

① π IS A PROPER RESOLUTION OF X

② $\tilde{B} + \mathrm{Exc}(\pi)$ is SIMPLE NORMAL CROSSING (S N C) $\rightarrow$ $\forall x' \in X$ locally analytically

$$(X', \tilde{B} + \mathrm{Exc}(\pi)) \cong \left(\mathbb{C}^n, \sum_{i=1}^{k} \{z_i = 0\}\right)$$

strict transform of B on X'

$\tilde{B}$ = strict transform of B on X'; "supp" is written above $\tilde{B}+\mathrm{Exc}(\pi)$.

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IS A LOG RESOLUTION WHEN:

① X' IS SMOOTH;

② $(X', \widetilde{B} + \mathrm{Exc}(\pi))$ IS SNC.

THEOREM [HIRONAKA] LOG RESOLUTION EXIST IN CHAR 0.

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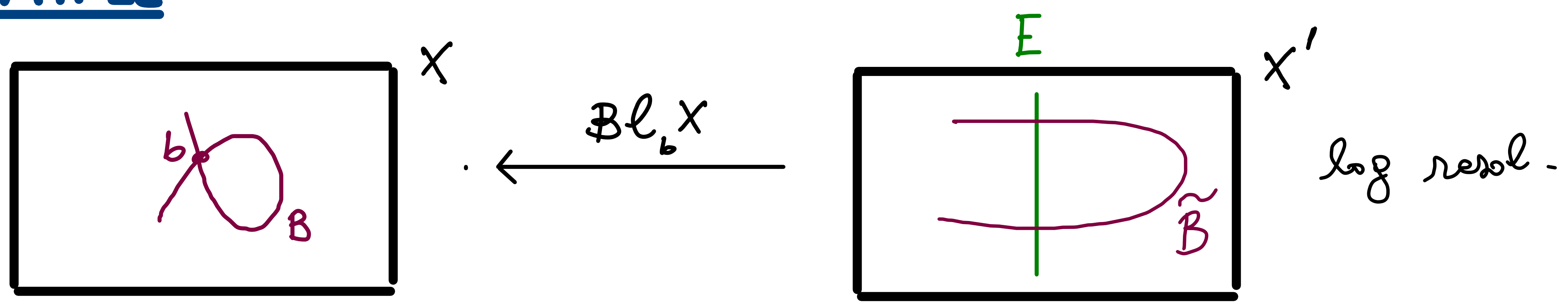

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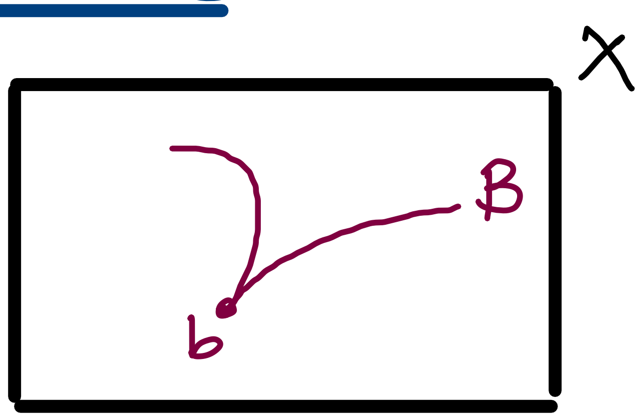

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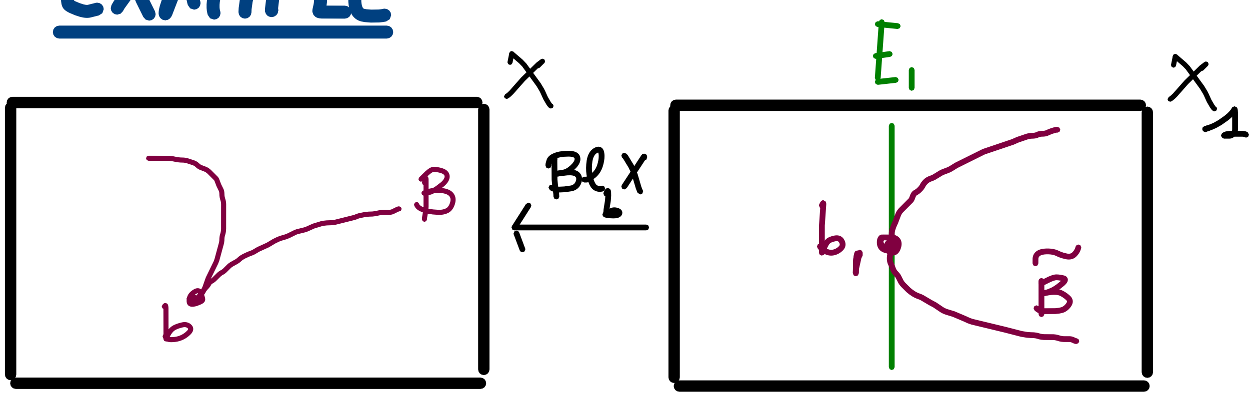

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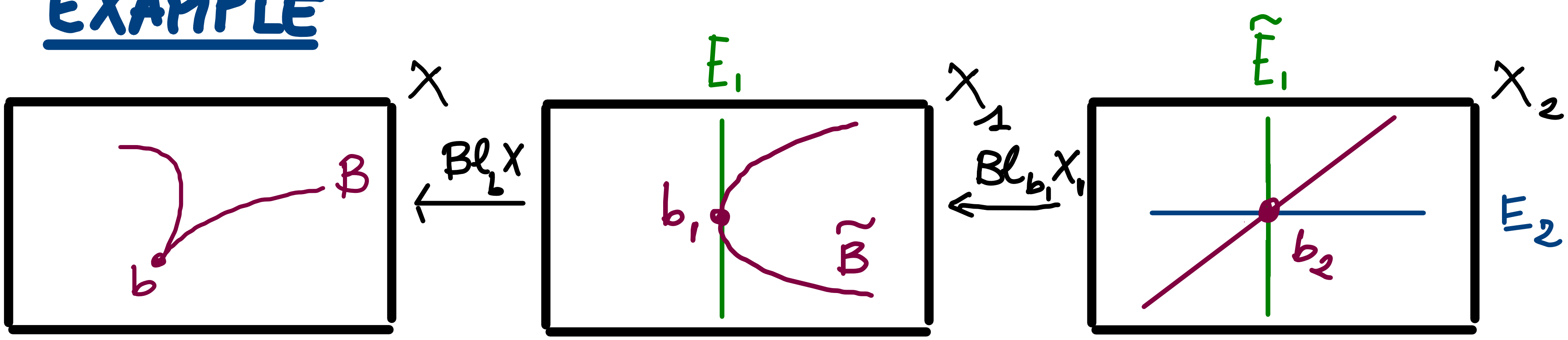

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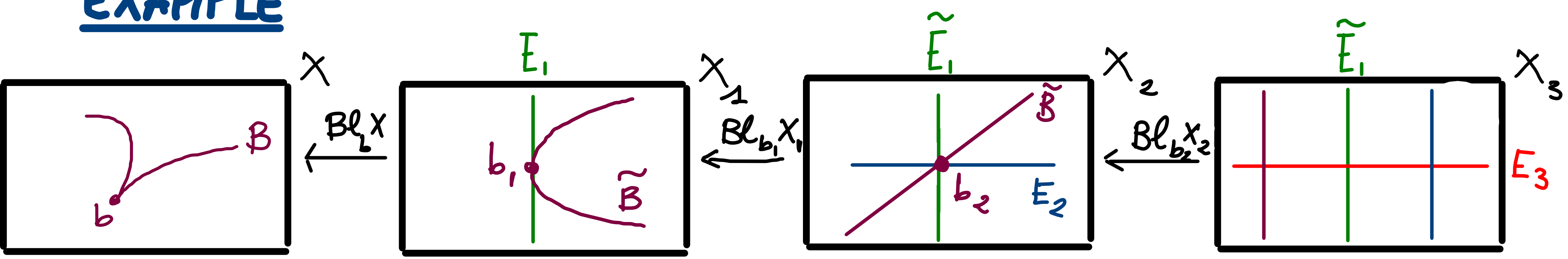

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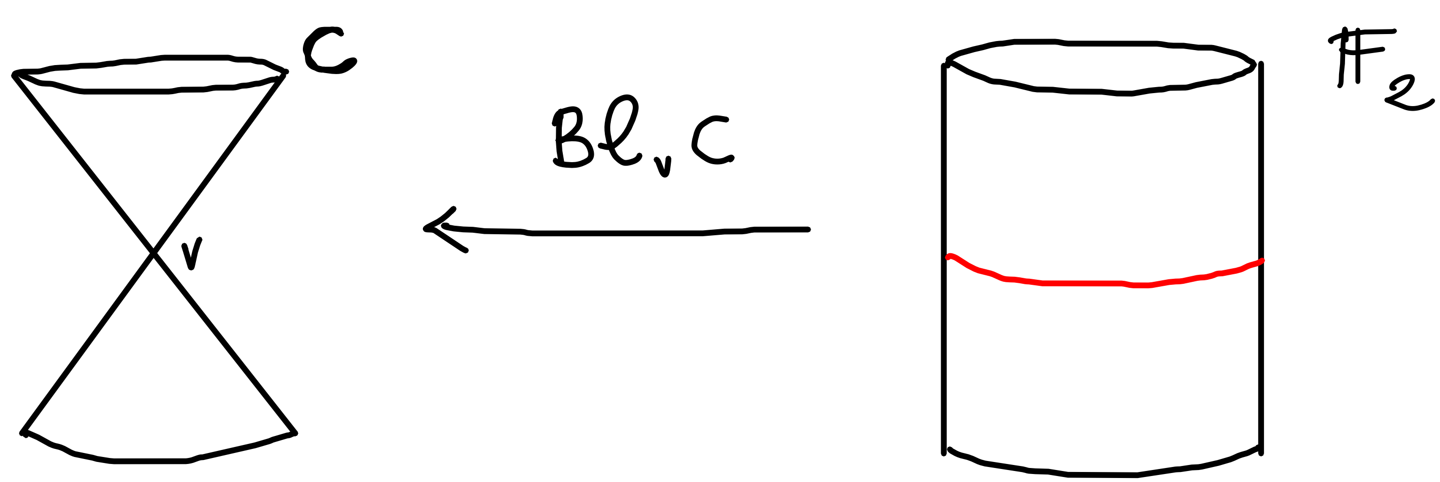

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

WE SAW THAT WE WILL NEED TO CONSIDER SINGULAR OBJECTS.
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WE WILL DO SO USING (BIRATIONAL) RESOLUTIONS.

GIVEN (X, B) A LOG PAIR AND A LOG RESOL.

$$\pi : X' \longrightarrow X$$

WE CAN ALWAYS WRITE IN A UNIQUE WAY

$$K_{X'} + \tilde{B}' = \pi^*(K_X + B) + \sum_{\substack{\text{exc} \\ \text{div.}}} a_i E_i$$

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WE CAN ALWAYS WRITE IN A UNIQUE WAY

$$K_{X'} + B' = \pi^*(K_X + B)$$

WHERE $B' = \pi^{-1}_* B + \sum_{\substack{\text{exc.}\\ \text{div.}}} a_i E_i$

DEFINITION LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$a(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'$$

log discrepancy

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

GIVEN (X,B) A LOG PAIR AND A LOG RESOL.

$$\pi : X' \longrightarrow X$$

WE CAN ALWAYS WRITE IN A UNIQUE WAY

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DEFINITION ① LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$a(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'.$$

[THIS IS CALLED THE LOG DISCREPANCY OF D wrt (X,B)]

② THE TOTAL LOG DISCREPANCY OF (X,B) is

$$a(X,B) := \text{INF } a(D; X, B)$$

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION ① LET $D \subseteq X'$ BE A PRIME DIVISOR, THEN

$$a(D; X, B) := 1 - \text{coeff. of } D \text{ in } B'.$$

[THIS IS CALLED THE LOG DISCREPANCY OF D wrt (X, B)]

② THE TOTAL LOG DISCREPANCY OF (X, B) is

$$a(X, B) := \inf a(D; X, B)$$

SPECIAL DICHOTOMY : $a(X, B) = -\infty$

OR

$a(X, B) \geq 0$ MMP :)

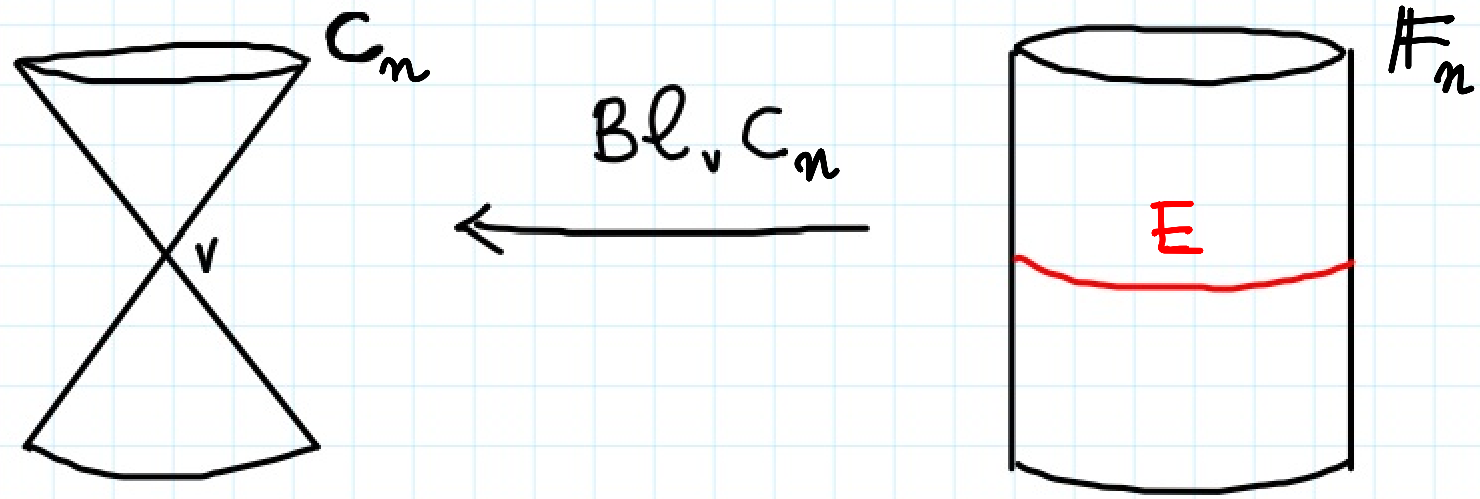
MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

DEFINITION LET (X, B) BE A LOG PAIR.

THEN (X, B) IS

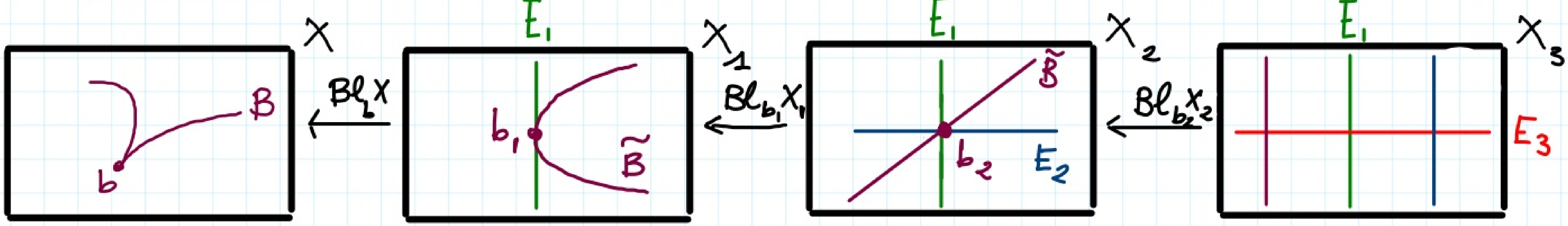
LOG CANONICAL		≥ 0
KLT (KAWAMATA LOG TERM.)	IF $a(X, B)$	> 0
CANONICAL		≥ 1
TERMINAL		> 1

EXAMPLE



C_n = CONE OVER RAT'L NORMAL CURVE OF $\deg n$

EXAMPLE



X
B
b
$Bl_b X$
E_1
X_1
b_1
$\tilde{B}$
$Bl_{b_1} X_1$
$\tilde{E}_1$
X_2
$\tilde{B}$
b_2
E_2
$Bl_{b_2} X_2$
$\tilde{E}_1$
X_3
E_3

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

~~X SMOOTH PROJECTIVE VARIETY / $\mathbb{C}$~~ (X, B) LOG CANONICAL PAIR

CONJECTURE STARTING FROM (X,B), THERE EXISTS AN ALGORITHMIC WAY OF CONSTRUCTING A DIAGRAM OF THE FOLLOWING FORM

$$\begin{array}{ccc} X & \overset{\text{composition of special}}{\underset{\text{birat'l maps}}{\dashrightarrow}} & (X',B') \\ & & \downarrow f \\ & & Z \end{array}$$

SUCH THAT f SATISFIES 1 OF THE FOLLOWING:

1. f BIRAT'L & $\cancel{K_Z}$ is AMPLE
2. f FIBRATION & $K_{X'}+B' = f^* H$ ample
3. f FIBRATION & $-(K_{X'}+B')$ AMPLE on fibers

MINIMAL MODEL PROGRAM : A GENTLE INTRODUCTION

IN ORDER TO ACHIEVE SUCH RESULT, WE WILL TRY TO CONTROL & TUNE THE POSITIVITY/NEGATIVITY OF LOG DIVISORS.

(X, B) LOG PAIR $\qquad K_X + B$ is $\mathbb{R}$-Cartier

$\forall\, C \subseteq X$ proper curve

$$(K_X + B) \cdot C$$

$K_X + B \cdot C \geq 0$
$\forall\, C \subseteq X$ proper curve

GOAL: either make $K_X + B$ nef after many birational transformations

OR

die trying (but in finite time)

THE CONE THEOREM

THEOREM [Mori, Kollár, Shokurov, ...]

LET (X,B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \overline{NE}(X)_{K_X+B<0}$$

the closure the cone of effective curves

THE CONE THEOREM

THEOREM

LET (X, B) BE A LOG CANONICAL PAIR.

WE HAVE THE FOLLOWING DECOMPOSITION

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \overline{NE}(X)_{K_X+B<0} =$$

$$= \overline{NE}(X)_{K_X+B\geq 0} + \sum_{i\in I} \mathbb{R}_+ [C_i]$$

extremal rays

locally discrete away from $(K_X+B)^\perp$

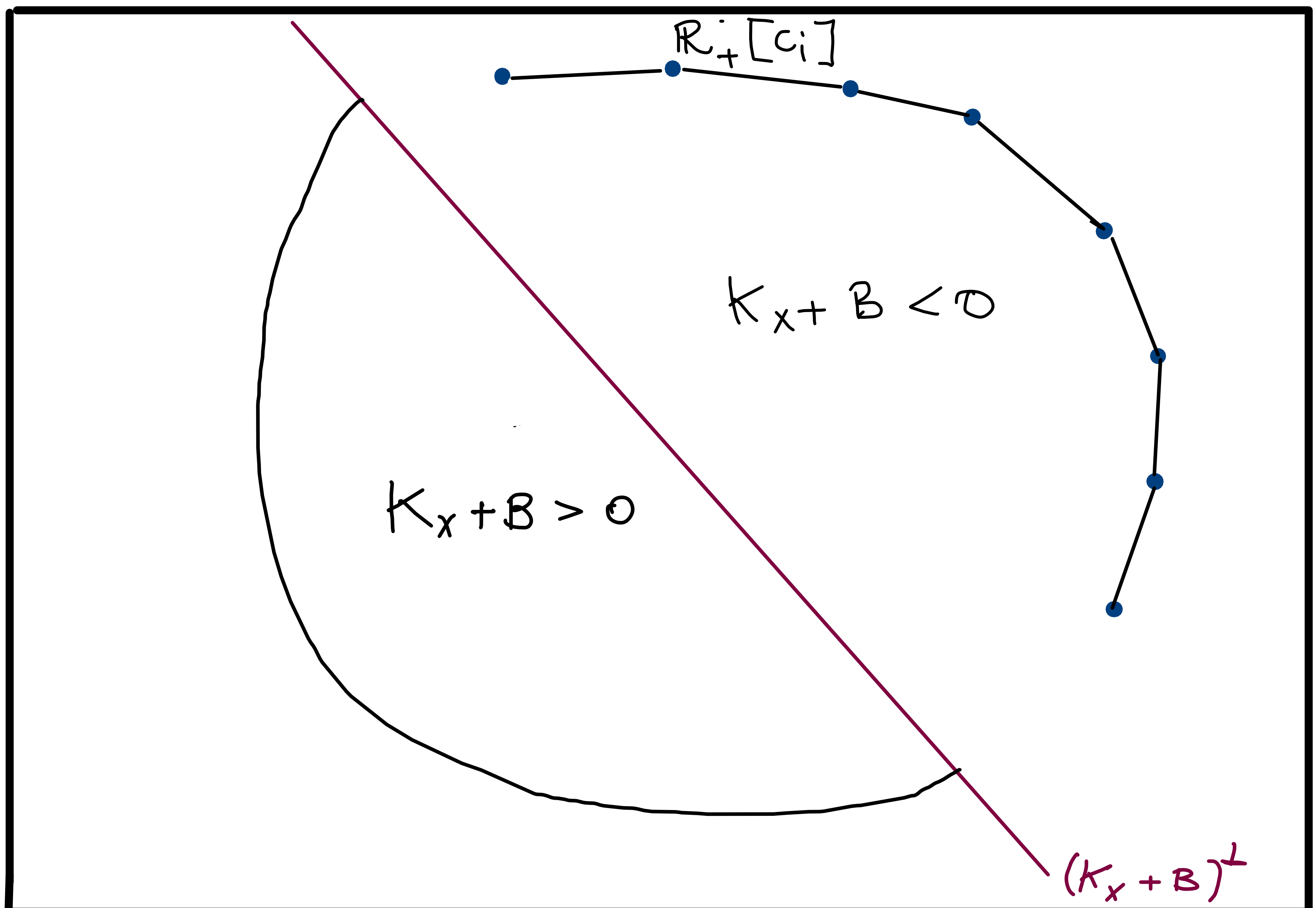
countable set

rat'l curves

$$-2\dim X \leq (K_X+B)\cdot C_i < 0$$

HORIZONTAL SLICE OF $\overline{NE}(X)$

extremal rays $\stackrel{def}{=}$ $v_1 + v_2 \in R_i^- \Rightarrow v_1, v_2 \in R_i^-$



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$$= \overline{NE}(X)_{K_X+B\geq 0} + \sum_{i\in I} \overbrace{\mathbb{R}_+[C_i]}^{R_i}$$

R_i extremal ray

$i\in I$ ← at most countable set

$[C_i]$ ← rat'l curves

$-2\dim X \leq (K_X+B)\cdot C_i < 0$

MOREOVER, $\forall i\in I$, $\exists\ \mathrm{cont}_{R_i} : X \longrightarrow Z_i$ SUCH THAT cont_{R_i} HAS CONNECTED FIBERS & IF $\mathrm{cont}_{R_i}(C) = pt \Rightarrow [C]\in R_i$.