10. CHERN CLASSES OF Q–SHEAVES

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In this chapter we introduce the notions of Q-varieties, Q-sheaves, Chern classes for Q-sheaves, and we extend some results, such as the condition for semistability and the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$, from smooth varieties to Q-varieties. One of our main aims is to calculate the Chern classes of the Q-sheaves of log differentials. Kawamata's original approach was more analytic, using Chern forms; we take a different, algebraic approach. This also enables us to define Chern classes for Q-sheaves in general, not just Q-vector bundles.

We work over an algebraically closed field of characteristic 0 throughout.

10.1 Definition. [Mumford83, §2.] A *Q-variety* is an irreducible, normal, quasiprojective algebraic variety X with only quotient singularities, together with a finite atlas of charts

$$\begin{tikzcd} X_\alpha \arrow[dd, "p_\alpha"'] \arrow[dr] & \\ & X_\alpha/G_\alpha \arrow[dl, "p'_\alpha"] \\ U_\alpha & \end{tikzcd}$$

where U_α is a Zariski open subset of X, $X = \cup_\alpha U_\alpha$, p'_α is étale, quasifinite, Galois, surjective, and finite in a neighbourhood of any singular point, X_α is smooth and quasiprojective, G_α is a finite group acting faithfully on X_α, freely in codimension one, so that $X_\alpha \to X_\alpha/G_\alpha$ is finite, Galois and étale in codimension 1. We also require the compatibility condition that the natural projections from the normalisation $X_{\alpha\beta}$ of $X_\alpha \times_X X_\beta$ to X_α and X_β should be étale.

X can also be constructed globally as the quotient of a quasiprojective variety $\tilde{X}$ by a finite group. Take a Galois extension of the function field $k(X)$ containing all the function fields $k(X_\alpha)$, and let $\tilde{X}$ be the normalisation of X in this field. Then $G = \mathrm{Gal}(k(\tilde{X})/k(X))$ acts faithfully on $\tilde{X}$, and $X = \tilde{X}/G$.

Let p be the projection morphism $p : \tilde{X} \to X$, and $\tilde{X}_\alpha = p^{-1}(U_\alpha)$. We have the following diagram

$$\begin{tikzcd}
\tilde{X} \arrow[dddd, "p"'] & \supset & \tilde{X}_\alpha \arrow[dd, "q_\alpha"'] \arrow[dr] & \\
& & & \tilde{X}_\alpha/H_\alpha \arrow[dl, "\cong"'] \\
& & X_\alpha \arrow[dd, "p_\alpha"'] \arrow[dr] & \\
& & & X_\alpha/G_\alpha \arrow[dl, "p'_\alpha"] \\
X & \supset & U_\alpha\,, &
\end{tikzcd}$$

where $\tilde{X}_\alpha$ is open in $\tilde{X}$, $H'_\alpha \leq G$ is the stabilizer of $\tilde{X}_\alpha$ as a set, $H_\alpha \lhd H'_\alpha$ with $G_\alpha = H'_\alpha/H_\alpha$.

This global cover $\tilde{X}$ constructed above need not be smooth in general. However, in the case most important for us, that of surfaces, $\tilde{X}$ is always Cohen–Macaulay, because a normal surface is Cohen–Macaulay; we need this fact later.

10.2 Definition. (cf. [Mumford83, §2.]) A *Q-sheaf* on a Q-variety X is a coherent sheaf on X together with a collection of G_α-linearized coherent sheaves $\mathcal{F}_\alpha$ on the X_α, such that $p'^*_\alpha(\mathcal{F}\mid_{U_\alpha}) = \mathcal{F}_\alpha^{G_\alpha}$, with isomorphisms $\mathcal{F}_\alpha \otimes_{\mathcal{O}_{X_\alpha}} \mathcal{O}_{X_{\alpha\beta}} \cong \mathcal{F}_\beta \otimes_{\mathcal{O}_{X_\beta}} \mathcal{O}_{X_{\alpha\beta}}$, satisfying the natural compatibility conditions on triple overlaps. The $\mathcal{F}_\alpha$ pull back to coherent sheaves on the $\tilde{X}_\alpha$ with compatible H'_α actions, so they glue together to a coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ with G action, and $\mathcal{F} = \tilde{\mathcal{F}}^G$.

$\mathcal{F}$ is *Q-locally free* or a *Q-vector bundle* if all the $\mathcal{F}_\alpha$ are locally free. A sequence $\mathcal{E} \to \mathcal{F} \to \mathcal{G}$ of Q-sheaves is said to be Q-exact if the corresponding sequence $\mathcal{E}_\alpha \to \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ is exact on each X_α.

A subvariety Y of a Q-variety X need not be a Q-variety in a way compatible with the Q-variety structure of X; even if Y is smooth and irreducible, its cover $Y_\alpha = p_\alpha^{-1}(Y\mid_{U_\alpha}) \subset X_\alpha$ may be singular or disconnected. Y is still the quotient of Y_α by G_α; the group action, however, need not be free in codimension 1 any more. Considering the reduced scheme structure on the Y_α, the $\mathcal{O}_{(Y_\alpha)_{red}}$ define a Q-sheaf structure on $\mathcal{O}_Y$. We can define a Q-sheaf on Y to be a coherent sheaf $\mathcal{F}$ on Y together with a collection of coherent sheaves $\mathcal{F}_\alpha$ on the Y_α with a G_α action, such that $\mathcal{F}\mid_{Y\cap U_\alpha} = \mathcal{F}_\alpha^{G_\alpha}$ and the $\mathcal{F}_\alpha$ satisfy the same compatibility condition as required for a Q-sheaf on X.

10.3 Examples. (*i*) $\mathcal{F} = \hat{\Omega}^1_X$ is a Q-sheaf, with $\mathcal{F}_\alpha = \Omega^1_{X_\alpha}$.
(*ii*) If X is a surface with quotient singularities and $C \subset X$ is a curve, then $\mathcal{I}_C = \mathcal{O}(-C)$ can be given a Q-vector bundle structure by setting

$\mathcal{F}_\alpha = (\mathcal{I}_{C|_{U_\alpha}} \cdot \mathcal{O}_{X_\alpha})^{\vee\vee}$. There is also a short Q-exact sequence of Q-sheaves $0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$.

(iii) If (X, B) is a log canonical surface, where X has only quotient singularities, then $\hat{\Omega}^1_X(\log B)$ exists as a Q-vector bundle. Let $C_\alpha = p_\alpha^{-1}(B|_{U_\alpha})$. By the classification of Chapter 3, there are three possibilities in the neighbourhood of a point of C_α.

(a) (X_α, C_α) is analytically isomorphic to $(\mathbb{A}^2, x = 0)$, $G_\alpha \cong \mathbb{Z}_n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$, where ζ is a primitive n-th root of unity, $(a, n) = 1$,

(b) $(X_\alpha, C_\alpha) \cong (\mathbb{A}^2, xy = 0)$, $G_\alpha \cong \mathbb{Z}_n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$, or

(c) $(X_\alpha, C_\alpha) \cong (\mathbb{A}^2, xy = 0)$, G_α is the binary dihedral group of order $4n$ acting by $(x, y) \to (\zeta x, \zeta^a y)$ and $(x, y) \to (-y, x)$.

In each case C_α has normal crossings, therefore $\mathcal{F}_\alpha = \Omega^1_X(\log C_\alpha)$ is a locally free sheaf, so $\hat{\Omega}^1_X(\log B)$ is Q-locally free.

Considering the normalization C_α^ν of C_α, we see that the G_α action extends naturally to $\mathcal{O}_{C_\alpha^\nu}$. Therefore we can define the Q-sheaf $\mathcal{O}_{B^\nu}$, the Q-normalisation of B, by the collection of sheaves $\mathcal{O}_{C_\alpha^\nu}$ on the X_α. If $B_1, \dots, B_s$ are the components of B, then $\mathcal{O}_{B^\nu} = \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu}$, and we have a Q-exact sequence

$$0 \to \hat{\Omega}^1_X \to \hat{\Omega}^1_X(\log B) \to \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu} \to 0,$$

whose Q-exactness follows from the exactness of

$$0 \to \Omega^1_{X_\alpha} \to \Omega^1_{X_\alpha}(\log C_\alpha) \to \mathcal{O}_{C_\alpha^\nu} \to 0.$$

For any quasiprojective variety Z we can define the Chow ring $A_*(Z) = \bigoplus_{k=0}^{\dim Z} A_k(Z)$, where A_k is the group of k dimensional cycles on Z modulo rational equivalence, and for Y smooth, we can also define $A^*(Y) = \bigoplus_{k=0}^{\dim Y} A^k(Y)$, where A^k is the group of k codimensional cycles on Y modulo rational equivalence. A morphism $h : Z \to Y$ induces a cap product $A^k(Y) \times A_l(Z) \xrightarrow{\cap} A_{l-k}(Z)$, [Fulton75, §2].

10.4 Definition. For V a possibly singular quasiprojective variety, we define

$$A^*(V) = \mathrm{Im}\Big\{ \varinjlim_{f: V \to Y} A^*(Y) \to \prod_{g: Z \to V} \mathrm{End}(A_*(Z)) \Big\}$$

where Y, Z are quasiprojective, Y is smooth, and the map is induced by the cap product (cf. [Fulton75, §2.] or the definition of $opA^{\cdot}$ in [Mumford83, §1.]).

This definition agrees with the original one for V smooth. Moreover, A^* is a contravariant functor, $A^*(V)$ inherits a natural ring structure, cap products can be defined, and most importantly for our purposes, for any coherent sheaf $\mathcal{F}$ on V with finite locally free resolution, we can define Chern classes $c_k(\mathcal{F}) \in A^k(V)$ [Fulton75, §3.2].

In some of the following we need that $\tilde{X}$ is Cohen–Macaulay, therefore we assume it from now on. As remarked above, this assumption is satisfied for surfaces. The following lemma explains its significance.

10.5 Lemma. [Mumford83, Proposition 2.1.] If X is quasi projective and $\tilde{X}$ is Cohen–Macaulay, then any coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ arising from a Q-sheaf $\mathcal{F}$ on X has a finite locally free resolution.

Proof. Let $n = \dim X$. Let $0 \to \tilde{\mathcal{E}}_n \to \tilde{\mathcal{E}}_{n-1} \to \ldots \to \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_0 \to \tilde{\mathcal{F}}$ be a resolution of $\tilde{\mathcal{F}}$, with $\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_{n-1}$ locally free $\mathcal{O}_{\tilde{X}}$-modules. As X_α is smooth, $\mathcal{F}_\alpha$ has a locally free resolution of length at most n. The morphism $\tilde{X}_\alpha \to X_\alpha$ is flat, since $\tilde{X}_\alpha$ is Cohen–Macaulay and X_α is smooth, therefore the resolution of $\mathcal{F}_\alpha$ pulls back to a locally free resolution of $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$ of length at most n. By Schanuel's lemma, if $\tilde{\mathcal{F}}|_{\tilde{X}_\alpha}$ has a locally free resolution of length at most n, then $\tilde{\mathcal{E}}_0|_{\tilde{X}_\alpha}, \tilde{\mathcal{E}}_1|_{\tilde{X}_\alpha}, \ldots, \tilde{\mathcal{E}}_{n-1}|_{\tilde{X}_\alpha}$ locally free implies that $\tilde{\mathcal{E}}_n|_{\tilde{X}_\alpha}$ is also locally free. Hence $\tilde{\mathcal{E}}_n$ is locally free and so $\tilde{\mathcal{F}}$ has a finite locally free resolution. □

Hence for any coherent sheaf on $\tilde{X}$ we can define Chern classes in $A^*(\tilde{X})$, and using this we can define Chern classes for Q-sheaves on X.

10.6 Definition. The Chern classes $\hat{c}_k$ of the Q-sheaf $\mathcal{F}$ on X are given by $\hat{c}_k(\mathcal{F}) = \dfrac{1}{|G|} c_k(\tilde{\mathcal{F}}) \in A^k(\tilde{X}) \otimes \mathbb{Q}$.

By [Mumford83, Theorem 3.1] there exist canonical isomorphisms $\gamma : A_{n-k}(X) \otimes \mathbb{Q} \to A^k(\tilde{X})^G \otimes \mathbb{Q}$ for $0 \le k \le n$, where $n = \dim X$. Identifying the Chow groups via γ, $A_*(X) \otimes \mathbb{Q}$ obtains a ring structure and we can define Chern classes in it. There exists a degree map $\deg : A^n(\tilde{X})^G \otimes \mathbb{Q} \to \mathbb{Q}$; to get the correct intersection numbers on X we have to take into account that $p : \tilde{X} \to X$ has degree $|G|$, so we define $\deg : A_0(X) \otimes \mathbb{Q} \to \mathbb{Q}$ by $\deg Z = \deg \gamma(Z)/|G|$ for $Z \in A_0(X) \otimes \mathbb{Q}$. We can define the total Chern class by $\hat{c}(\mathcal{E}) = \sum_{k=0}^n \hat{c}_k(\mathcal{E})$. As a Q-exact sequence of Q-sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ on X pulls back to a short exact sequence of sheaves $0 \to \tilde{\mathcal{E}} \to \tilde{\mathcal{F}} \to \tilde{\mathcal{G}} \to 0$ on $\tilde{X}$, we have $\hat{c}(\mathcal{F}) = \hat{c}(\mathcal{E})\hat{c}(\mathcal{G})$.

For a Q-sheaf on a subvariety of X we can not in general define Chern classes in this way. We need this only in one case, for Q-sheaves on a curve B on a surface X with only quotient singularities such that (X, B) is log canonical; then the cover $C_\alpha \subset X_\alpha$ is a curve with at most simple nodes as singularities and we can define $\hat{c}_1$ for a Q-sheaf on B.

By considering the sheaves in codimension 1 only, we see that $\hat{c}_1(\hat{\Omega}^1_X) = K_X$, and if (X, B) is log canonical, then $\hat{c}_1(\hat{\Omega}^1_X(\log B)) = K_X + B$. Calculating $\hat{c}_2(\hat{\Omega}^1_X)$ and $\hat{c}_2(\hat{\Omega}^1_X(\log B))$ is one of the main aims of this chapter. For this, we need the notion of the orbifold Euler number.

10.7 Definition. Let X be a quasiprojective variety with only isolated quotient singularities and let Y be an open or closed subset of X. The orbifold Euler number of Y is defined as

$$e_{orb}(Y) = e_{top}(Y) - \sum_{P \in Y \cap \operatorname{Sing} X} \left(1 - \frac{1}{r(P)}\right),$$

where e_{top} is the usual topological Euler number and $r(P)$ is the order of the local fundamental group. It should be noted that if Y is closed then $e_{orb}(Y)$ depends not only on Y but also on the embedding $Y \subset X$. In our case, this does not lead to any confusion.

10.8 Theorem. *Let X be a normal projective surface with only quotient singularities, B a reduced Weil divisor on X such that (X, B) is log canonical. Then*

$$\hat{c}_2(\hat{\Omega}^1_X(\log B)) = e_{orb}(X \setminus B).$$

Proof. First we consider the case $B = \emptyset$ to prove that $\hat{c}_2(\hat{\Omega}^1_X) = e_{orb}(X)$.

Fix a projective embedding of X. A generic pencil of hyperplane sections has reduced elements only, and its base locus is reduced and disjoint from $\operatorname{Sing} X$ and B. Blowing up this base locus we obtain a morphism $f : \tilde{X} \to \mathbb{P}^1$ with reduced fibres. Since both sides of the required equality increase by 1 under blowing up a smooth point, we may assume that in fact we have a morphism from X, $f : X \to \mathbb{P}^1$ with reduced fibres. Let g be the genus of the general fiber.

There exists a Q-exact sequence

$$0 \to f^*\Omega^1_{\mathbb{P}^1} \to \hat{\Omega}^1_X \to \hat{\omega}_{X/\mathbb{P}^1} \to \mathcal{O}_Z \to 0, \tag{10.8.1}$$

where Z is a 0-dimensional scheme supported on $\operatorname{Sing} X$ together with the nonsingular points where $df(x) = 0$.

Let $P \in Z$, $P \in U_\alpha$. Assume that $f(P) = 0$. $f_\alpha = f \circ p_\alpha$ is a G_α-invariant function on X_α. P has $\deg p_\alpha / r(P)$ inverse images in X_α. For $0 < |t| \ll 1$, $f_\alpha^{-1}(t)$ has the homotopy type of a wedge of μ_P circles in the neighbourhood of each point $Q \in p_\alpha^{-1}(P)$, hence its Euler number is $1 - \mu_P$. Therefore if we fix a small neighbourhood of P, the intersection of $f^{-1}(t)$ with this neighbourhood has orbifold Euler number $\dfrac{1-\mu_P}{r(P)}$ for $0 < |t| \ll 1$. Thus

$$e_{top}(X) = 2(2-2g) + \sum_{P\in Z}\left(\frac{\mu_P - 1}{r(P)} + 1\right)$$

and

$$e_{orb}(X) = 2(2-2g) + \sum_{P\in Z}\frac{\mu_P}{r(P)}. \tag{10.8.2}$$

μ_P can also be calculated as length$(\mathcal{O}_{X_\alpha,Q}/(\partial f_\alpha/\partial x, \partial f_\alpha/\partial y))$ by Milnor's Theorem [Milnor68, §7]. Define a 0-dimensional subscheme Z_α of X_α with ideal $(\partial f_\alpha/\partial x, \partial f_\alpha/\partial y))$ at each $Q \in p_\alpha^{-1}(P)$. The $\mathcal{O}_{Z_\alpha}$ define the Q-sheaf structure of $\mathcal{O}_Z$.

Z_α is a local complete intersection as X_α is smooth, so we can define $\tilde{Z}$ by $\tilde{Z}\,|_{\tilde{X}_\alpha} = q_\alpha^*(Z_\alpha)$, where q_α^* is the scheme theoretic inverse image. $\tilde{Z}$ is also a local complete intersection. We have the following lemma.

10.9 Lemma. *If $\tilde{Z}$ is a zero dimensional local complete intersection subscheme of $\tilde{X}$, then $c_2(\mathcal{O}_{\tilde{Z}}) = -\deg \tilde{Z}$.*

Proof. Both sides are clearly additive over subschemes with disjoint supports.

If $\tilde{Z}$ is a (reduced) smooth point P, then there exist smooth hyperplane sections H_1, H_2 such that $P \in H_1 \cap H_2$, every point of intersection of H_1 and H_2 is smoothin X and H_1, H_2 meet transversally there. Let $Y = H_1 \cap H_2$. From the exact sequence

$$0 \to \mathcal{O}(-H_1 - H_2) \to \mathcal{O}(-H_1) \oplus \mathcal{O}(-H_2) \to \mathcal{I}_Y \to 0$$

we can calculate $c_2(\mathcal{I}_Y) = H_1 \cdot H_2$, hence $c_2(\mathcal{O}_Y) = -H_1 \cdot H_2$. c_2 is invariant in an algebraic family, all points of Y are algebraically equivalent on H_1, therefore $c_2(\mathcal{O}_P) = -1$.

In the general case, since $\tilde{Z}$ is a local complete intersection, there exists a sufficiently ample divisor H such that $\mathcal{O}_{\tilde{Z}}(H)$ is generated by global sections and there exist $H_1, H_2 \in |H|$ whose local equations generate the ideal of $\tilde{Z}$ in $\mathcal{O}_{\tilde{X},P}$ for each point $P \in \operatorname{Supp}\tilde{Z}$, and all their other intersections are transversal and lie at smooth points of $\tilde{X}$. Let Y be the scheme theoretic intersection of H_1 and H_2; then from the exact sequence we have $c_2(\mathcal{O}_Y) = -H_1 \cdot H_2$ as before, and each point of $\operatorname{Supp} Y \setminus \operatorname{Supp}\tilde{Z}$ contributes -1. $c_2(\mathcal{O}_{\tilde{Z}}) = -\deg\tilde{Z}$ in the general case. □

In our case

$$\deg \tilde{Z} = \sum_{P \in Z} \frac{\mu_P \deg q_\alpha \deg p_\alpha}{r(P)} = |G| \sum_{P \in Z} \frac{\mu_P}{r(P)},$$

hence

$$\hat{c}_2(\mathcal{O}_Z) = \frac{1}{|G|} c_2(\mathcal{O}_{\tilde{Z}}) = -\frac{1}{|G|} \deg \tilde{Z} = -\sum_{P \in Z} \frac{\mu_P}{r(P)}. \tag{10.8.3}$$

From (10.8.1), (10.8.2) and (10.8.3) we obtain

$$\begin{aligned} \hat{c}_2(\hat{\Omega}^1_X) &= \hat{c}_1(f^*\Omega^1_{\mathbb{P}^1})\hat{c}_1(\hat{\Omega}_{X/\mathbb{P}^1}) - \hat{c}_2(\mathcal{O}_Z) \\ &= 2(2-2g) + \sum_{P \in Z} \frac{\mu_P}{r(P)} = e_{orb}(X). \end{aligned} \tag{10.8.4}$$

Let $B_1, B_2, \dots, B_s$ be the components of B. There exist Q-exact sequences

$$0 \to \mathcal{O}(-B_i) \to \mathcal{O}_X \to \mathcal{O}_{B_i} \to 0, \tag{10.8.5}$$

$$0 \to \mathcal{O}_{B_i} \to \mathcal{O}_{B_i^\nu} \to \mathcal{O}_{W_i} \to 0 \tag{10.8.6}$$

and

$$0 \to \hat{\Omega}^1_X \to \hat{\Omega}^1_X(\log B) \to \bigoplus_{i=1}^{s} \mathcal{O}_{B_i^\nu} \to 0, \tag{10.8.7}$$

where W_i is a 0-dimensional subscheme of X supported at those points of B_i which are either nodes of B_i or singular points of X of type (c) in Examples 10.3. (iii) on B_i. The Q-sheaf structure of $\mathcal{O}_{W_i}$ is given by $\mathcal{O}_{p_\alpha^{-1}(W_i \cap U_\alpha)}$ on X_α, where p_α^{-1} denotes the set theoretic inverse image.

From (10.8.5) we see that $\hat{c}_1(\mathcal{O}_{B_i}) = B_i$ and $\hat{c}_2(\mathcal{O}_{B_i}) = B_i^2$, while from (10.8.6) and (10.9), $\hat{c}_2(\mathcal{O}_{B_i^\nu}) = \hat{c}_2(\mathcal{O}_{B_i}) + \hat{c}_2(\mathcal{O}_{W_i}) = B_i^2 - \sum_{P \in W_i} \frac{1}{r(P)}$. Thus from (10.8.7) we obtain

$$\hat{c}_2(\hat{\Omega}^1_X(\log B)) = \hat{c}_2(\hat{\Omega}^1_X) + K_X \cdot B + \sum_{1 \le i \le j \le s} B_i \cdot B_j - \sum_{i=1}^{s} \sum_{P \in W_i} \frac{1}{r(P)}. \tag{10.8.8}$$

We have Q-exact sequences

$$0 \to \hat{\mathcal{N}}^\vee_{B_i/X} \to \hat{\Omega}^1_X|_{B_i} \to \hat{\Omega}^1_{B_i} \to 0,$$

where $\hat{\mathcal{N}}^\vee_{B_i/X}$ is the conormal Q-sheaf, obtained by taking the G_α-invariants of $\mathcal{N}^\vee_{C_\alpha/X_\alpha}$, where $C_\alpha = p_\alpha^{-1}(B_i\,|_{U_\alpha})$. Now $\hat{c}_1(\hat{\Omega}^1_X\,|_{B_i}) = K_X \cdot B_i$, while $\hat{c}_1(\hat{\mathcal{N}}^\vee_{B_i/X}) = -B_i^2 + \sum_{P\in W_i} \frac{1}{r(P)}$, since each simple node of C_α contributes $+1$ on X_α. Hence

$$\hat{c}_1(\hat{\Omega}^1_{B_i}) = K_X \cdot B_i + B_i^2 - \sum_{P\in W_i} \frac{1}{r(P)}. \tag{10.8.9}$$

10.10 Lemma. $\hat{c}_1(\hat{\Omega}^1_{B_i}) = -e_{orb}(B_i)$.

Proof. By an argument similar to the above, we can find a morphism $f : B_i \to \mathbb{P}^1$ such that f has only ordinary ramification points and these are all smooth points of X and not nodes of B_i. Let d be the degree of this map, a the number of ramification points, b the number of nodes of B_i. Then $e_{top}(B_i) = 2d - a - b$, and hence $e_{orb}(B_i) = 2d - a - b - \sum_{P\in B_i\cap \operatorname{Sing} X} \left(1 - \frac{1}{r(P)}\right)$.

We determine $\hat{c}_1(\hat{\Omega}^1_{B_i})$ from the Q-exact sequence

$$0 \to f^*\Omega^1_{\mathbb{P}^1} \to \hat{\Omega}^1_{B_i} \to \hat{\Omega}^1_{B_i/\mathbb{P}^1} \to 0;$$

the argument is similar to Hurwitz's formula.

Note first that $\hat{c}_1(f^*\Omega^1_{\mathbb{P}^1}) = -2d$. Each ramification point of f and each node of B_i which is a smooth point of X contributes 1 to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$. If a node $P \in B_i$ is a singular point of X, then it is type (c) in Examples 10.3. (iii). Let $P \in U_\alpha$, $f_\alpha = f \circ p_\alpha$. On X_α, $C_\alpha = p_\alpha^{-1}(B_i\,|_{U_\alpha})$ has a simple node at each $Q \in p_\alpha^{-1}(P)$, and Q is a ramification point of index $r(P)$ of f_α on each branch, therefore the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ at P is $\frac{2(r(P)-1)+1}{r(P)} = \left(2 - \frac{1}{r(P)}\right)$. If $P \in B_i$ is a singular point of X which is not a node of B_i then it is of type (a) or (c) in Examples 10.3. (iii). Let $P \in U_\alpha$. If P is of type (a), then on X_α each $Q \in p_\alpha^{-1}(P)$ is a ramification point of f_α of index $r(P)$, so the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ is $1 - \frac{1}{r(P)}$. If P is of type (c), then $r(P) = 4l$, C_α has a node at each $Q \in p_\alpha^{-1}(P)$, and Q is a ramification point of index $2l$ on both branches of C_α, so the contribution to $\hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})$ at P is $\frac{2(2l-1)+1}{4l} = \left(1 - \frac{1}{r(P)}\right)$ in this case too.

Hence

$$\begin{aligned}\hat{c}_1(\hat{\Omega}^1_{B_i}) &= \hat{c}_1(f^*\Omega_{\mathbb{P}^1}) + \hat{c}_1(\hat{\Omega}^1_{B_i/\mathbb{P}^1})\\ &= -2d + a + b + \sum_{P\in B_i\cap \operatorname{Sing} X} \left(1 - \frac{1}{r(P)}\right) = -e_{orb}(B_i). \qquad \square\end{aligned}$$

For $i \neq j$, $B_i \cdot B_j = \sum_{P \in B_i \cap B_j} \frac{1}{r(P)} = \sum_{P \in B_i \cap B_j} e_{orb}(P)$. Hence by (10.8.9) and (10.10),

$$e_{orb}(B) = \sum_{i=1}^{s} e_{orb}(B_i) - \sum_{1 \leq i \lneq j \leq s} \sum_{P \in B_i \cap B_j} e_{orb}(P)$$

$$= -K_X \cdot B - \sum_{1 \leq i \lneq j \leq s} B_i \cdot B_j + \sum_{i=1}^{s} \sum_{P \in W_i} \frac{1}{r(P)}.$$

Combining this with (10.8.4), we obtain

$$\hat{c}_2(\hat{\Omega}^1_X(\log B)) = e_{orb}(X) - e_{orb}(B) = e_{orb}(X \setminus B). \quad \square$$

Next we generalize some results from sheaves on smooth varieties to Q-sheaves on Q-varieties.

10.11 Lemma. *Let $\mathcal{F}$ be a Q-vector bundle of rank r on a normal projective surface X with only quotient singularities. If*

$$(r-1)(\hat{c}_1(\mathcal{F}))^2 - 2r\hat{c}_2(\mathcal{F}) > 0,$$

then $\mathcal{F}$ is H-unstable for any numerically nontrivial nef divisor H on X, i.e., there exists a saturated Q-subsheaf $\mathcal{E}$ of $\mathcal{F}$ such that

$$\frac{\hat{c}_1(\mathcal{E}) \cdot H}{\operatorname{rk} \mathcal{E}} > \frac{\hat{c}_1(\mathcal{F}) \cdot H}{r}. \tag{10.11.1}$$

Proof. Let $\pi : X^* \to \tilde{X}$ be the minimal resolution of the singularities of $\tilde{X}$. Let $\tilde{H} = p^*H$, $H^* = \pi^*\tilde{H}$, $\mathcal{F}^* = \pi^*\tilde{\mathcal{F}}$. Then

$$(r-1)(c_1(\mathcal{F}^*))^2 - 2rc_2(\mathcal{F}^*) = (r-1)(c_1(\tilde{\mathcal{F}}))^2 - 2rc_2(\tilde{\mathcal{F}}) > 0,$$

so by [Bogomolov79, Theorem 3], there exists a saturated subsheaf $\mathcal{E}^*$ of $\mathcal{F}^*$ such that

$$\left(\frac{c_1(\mathcal{E}^*)}{\operatorname{rk}\mathcal{E}} - \frac{c_1(\mathcal{F}^*)}{r}\right)^2 > 0 \qquad \text{and} \qquad \left(\frac{c_1(\mathcal{E}^*)}{\operatorname{rk}\mathcal{E}} - \frac{c_1(\mathcal{F}^*)}{r}\right) \cdot D > 0$$

for any ample divisor D on X^*. Therefore also

$$\left(\frac{c_1(\mathcal{E}^*)}{\operatorname{rk}\mathcal{E}} - \frac{c_1(\mathcal{F}^*)}{r}\right) \cdot H^* > 0,$$

since H^* is nef and not numerically trivial.

Then $\pi_*\mathcal{E}^* \hookrightarrow \pi_*\mathcal{F}^* = \mathcal{F}$ and $c_1(\pi_*\mathcal{E}^*)\cdot\tilde{H} = c_1(\tilde{\mathcal{E}})\cdot\tilde{H}$, since $\tilde{H}$ is ample, so some multiple of it can be moved away from the singular locus of $\tilde{X}$, and the sheaves $\pi_*\mathcal{E}^*, \tilde{\mathcal{E}}$ agree on the smooth locus of $\tilde{X}$. Having obtained the instability of $\tilde{\mathcal{F}}$ on $\tilde{X}$, we can now choose a G-invariant destabilizing subsheaf, namely the first step $\mathcal{E}_1$ in the Harder–Narasimhan filtration of $\tilde{\mathcal{F}}$ for $\tilde{H}$ [Miyaoka87b, Theorem 2.1], which is unique, therefore G-invariant. Taking G-invariants, we obtain the required destabilizing Q-subsheaf $\mathcal{E} = \tilde{\mathcal{E}}_1^G$ of $\mathcal{F}$, then (10.11.1) follows from

$$\frac{c_1(\mathcal{E}_1)\cdot\tilde{H}}{\operatorname{rk}\mathcal{E}_1} > \frac{c_1(\tilde{\mathcal{F}})\cdot\tilde{H}}{r}. \quad \square$$

10.12 Proposition. *Let $\mathcal{E}$ be a Q-locally free sheaf on a normal projective surface X with only quotient singularities such that $\hat{c}_1(\mathcal{E})$ is nef and $\mathcal{E}$ is generically semipositive, i.e., for any nef divisor D on X and for any torsion free quotient Q-sheaf $\mathcal{F}$, $\hat{c}_1(\mathcal{F})\cdot D \geq 0$. Then $\hat{c}_2(\mathcal{E}) \geq 0$.*

Proof. Let H be an ample divisor on X, t a positive rational number, then $H_t = \hat{c}_1(\mathcal{E}) + tH$ is an ample $\mathbb{Q}$-divisor. Let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$ be the Harder–Narasimhan filtration for $\mathcal{E}$ with respect to H_t, which is obtained by taking the G-invariants of the Harder–Narasimhan filtration for $\tilde{\mathcal{E}}$ with respect to p^*H_t. Let $\mathcal{G}_i = (\mathcal{E}_i/\mathcal{E}_{i-1})^{\vee\vee}$, $r_i = \operatorname{rk}\mathcal{G}_i$. $\mathcal{E}_i/\mathcal{E}_{i-1} \subset \mathcal{G}_i$ with skyscraper cokernel, therefore $\hat{c}_2(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq \hat{c}_2(\mathcal{G}_i)$ by (10.9), while $\hat{c}_1(\mathcal{E}_i/\mathcal{E}_{i-1}) = \hat{c}_1(\mathcal{G}_i)$, since they agree in codimension 1. $\hat{c}(\mathcal{E}) = \prod_{i=1}^{s}\hat{c}(\mathcal{E}_i/\mathcal{E}_{i-1})$, where $\hat{c}$ is the total Chern class, therefore

$$\begin{aligned}\hat{c}_2(\mathcal{E}) &\geq \prod_{1\leq i<j\leq s}\hat{c}_1(\mathcal{G}_i)\hat{c}_1(\mathcal{G}_j) + \sum_{i=1}^{s}\hat{c}_2(\mathcal{G}_i)\\ &= \frac{1}{2}(\hat{c}(\mathcal{E}))^2 + \sum_{i=1}^{s}\hat{c}_2(\mathcal{G}_i) - \frac{1}{2}\sum_{i=1}^{s}(\hat{c}_1(\mathcal{G}_i))^2\\ &\geq \frac{1}{2}(\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s}\frac{1}{2r_i}(\hat{c}_1(\mathcal{G}_i))^2,\end{aligned}$$

where in the last step we used the semistability of the $\mathcal{G}_i$ and Lemma 10.11.

Let $\alpha_i = \dfrac{\hat{c}_1(\mathcal{G}_i)\cdot H_t}{r_i H_t^2}$; then $\alpha_1 > \alpha_2 > \ldots \alpha_s \geq 0$ by definition of the Harder–Narasimhan filtration and the generic semipositivity of $\mathcal{E}$. By the

Hodge Index Theorem, $(\hat{c}_1(\mathcal{G}_i))^2 \leq r_i^2 \alpha_i^2 H_t^2$. Hence

$$
\begin{aligned}
\hat{c}_2(\mathcal{E}) &\geq \frac{1}{2}(\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s} \frac{1}{2r_i}(\hat{c}_1(\mathcal{G}_i))^2 \\
&\geq \frac{1}{2}\left((\hat{c}_1(\mathcal{E}))^2 - \sum_{i=1}^{s} r_i \alpha_i^2 H_t^2\right) \\
&\geq \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + \left(1 - \sum_{i=1}^{s} r_i \alpha_i^2\right) H_t^2\right) \\
&\geq \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + \left(1 - \alpha_1 \sum_{i=1}^{s} r_i \alpha_i\right) H_t^2\right) \\
&= \frac{1}{2}\left(((\hat{c}_1(\mathcal{E}))^2 - H_t^2) + (1 - \alpha_1) H_t^2\right).
\end{aligned}
$$

Now $\alpha_1 = \dfrac{\hat{c}_1(\mathcal{G}_1) \cdot H_t}{r_1 H_t^2} \leq \dfrac{\hat{c}_1(\mathcal{E}) \cdot H_t}{r_1 H_t^2} < 1$, whereas $(\hat{c}_1(\mathcal{E}))^2 - H_t^2 \to 0$ as $t \to 0$, so that $\hat{c}_2(\mathcal{E}) \geq 0$. □

10.13 Theorem. *Let X be a normal projective threefold, B a reduced Weil divisor on X, such that (X, B) is log canonical, $(X, \emptyset)$ is log terminal, $K_X + B$ is nef and X is not uniruled. Let S be a general hyperplane section of X; then $\hat{c}_2(\hat{\Omega}_X^1(\log B)\,|_S) \geq 0$.*

Proof. X has quotient singularities in codimension 2, so $\hat{\Omega}_X^1(\log B)$ can be defined as a Q-vector bundle except at finitely many points. S has only quotient singularities, $(S, B\,|_S)$ is log canonical, so $\hat{\Omega}_X^1(\log B)\,|_S$ is a Q-vector bundle. $\hat{\Omega}_X^1\,|_S$ is generically semipositive by (9.0.1), therefore so is $\hat{\Omega}_X^1(\log B)\,|_S$. $\hat{c}_1(\hat{\Omega}_X^1(\log B)\,|_S) = (K_X + B)\,|_S$ is nef by assumption, therefore we can apply (10.12) to deduce the result. □

We prove a generalization of the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$. This inequality was proved for smooth surfaces of general type in [Miyaoka77, Theorem 4] and for smooth surfaces with c_1 negative in [Yau77, Theorem 4.]. It was generalised to $c_1^2(\Omega_X^1(\log B)) \leq 3c_2(\Omega_X^1(\log B))$ in [Sakai80, Theorem 7.6] for the case when X is a smooth surface and $B \subset X$ is a semistable curve, which implies that $K_X + B$ is nef and (X, B) is log canonical. [Miyaoka84, Theorem 1.1] deals with the log case on surfaces with quotient singularities when the curve B does not pass through the singular points of the surface. A version of this inequality for log canonical surfaces with fractional boundary divisor with $K_X + B$ ample is proved in [KNS89, Theorem 12]. We give a new method of proof for the case when X has only quotient singularities, (X, B) is log canonical, $K_X + B$ is nef. Our result is more general than [Miyaoka84] in that we also allow the curve B to pass through the singular points.

10.14 Theorem. *Let X be a normal projective surface with only quotient singularities, $B \subset X$ a curve such that (X, B) is log canonical and $K_X + B$ is nef. Then*

$$\hat{c}_1^2(\hat{\Omega}_X^1(\log B)) \leq 3\hat{c}_2(\hat{\Omega}_X^1(\log B)).$$

Proof. We prove this theorem by reducing it to the smooth case. Let $\mathcal{F} = \hat{\Omega}_X^1(\log B)$. Let $\pi : X^* \to \tilde{X}$ be an embedded resolution of $(\tilde{X}, p^{-1}(B)_{red})$, let $B^* = ((\pi \circ p)^{-1}(B))_{red}$, $\mathcal{F}^* = \pi^* \tilde{\mathcal{F}}$. Since $c_i(\mathcal{F}^*) = \pi^* c_i(\tilde{\mathcal{F}})$, it is sufficient to prove that $c_1^2(\mathcal{F}^*) \leq 3c_2(\mathcal{F}^*)$.

$\mathcal{F}$ is locally free of rank 2, therefore so is $\mathcal{F}^*$. $\mathcal{F}|_{\tilde{X}_\alpha} = q_\alpha^* \Omega_{X_\alpha}^1(\log C_\alpha)$, where $C_\alpha = p_\alpha^{-1}(B|_{U_\alpha})$, hence $\mathcal{F}^*|_{\pi^{-1}\tilde{X}_\alpha} = \pi^* q_\alpha^* \Omega_{X_\alpha}^1(\log C_\alpha) \subset \Omega_{X^*}^1(\log B^*)|_{\pi^{-1}X_\alpha}$, therefore $\mathcal{F}^* \subset \Omega_{X^*}^1(\log B^*)$. If $B = \emptyset$, $\mathcal{F}^* \subset \Omega_{X^*}^1$.

If $\omega \in H^0(X^*, \Omega_{X^*}^1(\log B^*))$, then ω is d-closed by [Deligne71]. (See also [Griffiths–Schmid73, 6.5] for a simpler proof.) Thus we can prove that if $\mathcal{L} \hookrightarrow \Omega_{X^*}^1(\log B^*)$ is an invertible sheaf, then $h^0(X, \mathcal{L}^{\otimes n}) \leq cn$ for some constant c [Sakai80, Lemma 7.5]. Using this, and the fact that $c_1(\mathcal{F}) = \pi^* p^*(K_X + B)$ is nef, we can follow Miyaoka's original proof for the non-log case [Miyaoka77, Theorem 4] to obtain $c_1^2(\mathcal{F}^*) \leq 3c_2(\mathcal{F}^*)$. □

10.15 Corollary. [Miyaoka84, Proposition 2.1.1] Let $\hat{X}$ be a minimal surface of nonnegative Kodaira dimension. Then the number of disjoint smooth rational curves on $\hat{X}$ is at most $\dfrac{2}{9}(3c_2(\hat{X}) - c_1^2(\hat{X}))$.

Proof. $K_{\hat{X}}$ is nef as $\hat{X}$ is minimal, so $C^2 \leq -2$ for any smooth rational curve on $\hat{X}$ by the adjunction formula. Let X be the surface obtained by contracting some disjoint smooth rational curves to singular points. Contracting a smooth rational curve with selfintersection $-n$ increases $\hat{c}_1^2$ by $\dfrac{(n-2)^2}{n}$, decreases $\hat{c}_2$ by $2 - \dfrac{1}{n}$, so $3\hat{c}_2 - \hat{c}_1^2$ decreases by at least $\dfrac{9}{2}$. K_X is still nef, so by the previous theorem $3\hat{c}_2(X) - \hat{c}_1^2(X) \geq 0$, which gives the bound on the number of contracted curves. □

11. LOG ABUNDANCE FOR SURFACES

LUNG-YING FONG and JAMES MCKERNAN

11.1 INTRODUCTION

Chapters 11–14 present Kawamata and Miyaoka's proof of the abundance theorem for threefolds.

11.1.1 Abundance Theorem. *A three dimensional minimal model X has a free pluricanonical system, that is, there exists a positive integer m such that $|mK_X|$ has no base points.*

(1.22–29) contains a general introduction to Abundance, and to the contents of Chapters 11–14. The division of labour indicated by the authors listed for each chapter is somewhat arbitrary; every author has made a significant contribution to each chapter. We would like to thank Kawamata for answering questions regarding his original version of [Kawamata91b]. We would also like to thank Shepherd-Barron, and Corti among others for helpful discussions and comments.

The purpose of this chapter is to gather together and prove some facts concerning log abundance for surfaces. These facts will be needed in Chapters 12–14 to prove the abundance conjecture for threefolds. We collect together some standard definitions and notation.

11.1.2 Notation

$\kappa(X, D)$ denotes the Iitaka dimension of the pair (X, D). By definition $\kappa(X, D) = -\infty$ iff $h^0(\mathcal{O}_X(nD)) = 0$ for every $n > 0$, and $\kappa(X, D) = k > -\infty$ iff

$$0 < \limsup \frac{h^0(\mathcal{O}_X(nD))}{n^k} < \infty.$$

One can see that $\kappa(X, D) \in \{-\infty, 0, 1, \ldots, \dim X\}$.

$\kappa(X) = \kappa(X, K_X)$ is the Kodaira dimension of X.

In case the divisors are nef, we can define the numerical counterparts (cf. (1.28)):

$\nu(X, D) = \max\{\, n \in \mathbb{N} \cup 0 \mid (D^n) \text{ not numerically } 0 \,\}$.

$\nu(X) = \nu(X, K_X)$.

The log abundance theorem for a normal surface X asserts the following:

11.1.3 Theorem. *Let (X, Δ) be a normal surface with boundary Δ (see (2.2.4) for a definition). If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, nef and log canonical then $|m(K_X + \Delta)|$ is basepoint free for some m (and in particular $\nu(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$).*

We need (11.1.3) in the cases $\nu(K_X + \Delta) = 0$ and 1, and content ourselves with proving these cases only. Readers interested in seeing the other case should consult [Fujita84]. The proof presented here is different from that in [Fujita84] at various points, and is adapted from Miyaoka's proof of the abundance theorem in the threefold case, as will be evident to the readers. Following Miyaoka's idea, we extend (11.1.3) to the semi log canonical case in Chapter 12.

The idea of the proof is as follows: we first show that the linear system $|m(K_X+\Delta)|$ contains a divisor D (11.2.1). Then we replace D with $B = D_{\mathrm{red}}$, and apply the log minimal model program to $(X, \Delta + B)$, so that $K_X + \Delta + B$ becomes nef (11.3.2). Then we use a further series of log extremal contractions to make each connected component of B irreducible (11.3.4). Next we make a cyclic cover of a neighborhood of a connected component of B, to improve how it sits inside X (11.3.6). Finally using some simple cohomological arguments, one can show that this component moves to any infinitesimal order (11.3.7).

11.2 Existence of an effective member

We start with the following lemma.

11.2.1 Lemma. *Let (X, Δ) be a smooth surface with boundary Δ. If $K_X+\Delta$ is nef then $\kappa(X, K_X + \Delta) \geq 0$. In other words, there exists a member $D \in |m(K_X + \Delta)|$ for some $m > 0$.*

11.2.2 Remark An analog of this result for threefolds is proved in Chapter 9.

Proof. (cf. [Fujita84, §2]) If $\kappa(X, K_X) \geq 0$ then the conclusion is clear. Thus we may assume that X is ruled. There are two cases to consider, X is rational or irrational.

First consider the case when X is rational. Let $G = K_X+\Delta$. G is nef by assumption. Since X is rational, $h^1(\mathcal{O}_X) = 0$. Therefore if G is numerically trivial, then $mG \sim 0$ for some m. Otherwise $h^2(mG) = h^0(-(m-1)G - \Delta) = 0$ for $m \geq 2$ and sufficiently divisible. Now $\chi(\mathcal{O}_X) = 1$, and so Riemann–Roch reads

$$h^0(X, mG) = h^1(X, mG) + \frac{1}{2}mG \cdot (mG - K_X) + 1.$$

Note that $mG - K_X = (m-1)G + \Delta$ and Δ is a sum of effective divisors. Since G is nef, we have $G \cdot (mG - K_X) \geq 0$ and therefore $h^0(X, mG) > 0$. This proves the lemma for rational surfaces.

Next consider the case when X is irrational. We write $\Delta = \Delta_1 + \Delta_2$, where Δ_1 and Δ_2 are boundaries, in such a way that Δ_1 has no vertical components, and furthermore $(K_X + \Delta_1) \cdot F = 0$. (11.2.1) follows if we show that $|m(K_X + \Delta_1)| \neq \emptyset$. Thus we may as well assume that $(K_X + \Delta) \cdot F = 0$ to start with, i.e., we prove the stronger statement:

11.2.3 lemma. *Let (X, Δ) be an irrational ruled surface with boundary Δ. Ssuppose that Δ has no vertical components and $(K_X + \Delta) \cdot F = 0$. Then $\kappa(X, K_X + \Delta) \geq 0$.*

The proof is by induction on the Picard number $\rho(X)$. Consider the case when X is a $\mathbb{P}^1$-bundle.

$\rho(X) = 2$, and the cone $\overline{\mathrm{NE}}(X)$ has two edges. One is the class generated by F, a fibre of the ruling $\pi : X \to C$ with C of genus $g > 0$. Suppose that the other edge is generated by H. Since $F^2 = 0$ and $H^2 \leq 0$ (see [CKM88, 4.4]), we must have $H \cdot F > 0$. We normalize H by taking $H \cdot F = 1$.

Let $\Delta = \sum k_i \Delta_i$, where the Δ_i are the prime components of Δ. We have $\Delta_i \equiv a_i H + F_i$, where $a_i \in \mathbb{Z}$ and $F_i = \pi^*(D_i)$ for some divisor D_i on C. Let $b_i = \deg(D_i)$. Since Δ_i is not a vertical component, $a_i > 0$. We also know that $K_X \equiv -2H + F_0$, with $F_0 = \pi^*(D_0)$, $\deg(D_0) = H^2 + 2g - 2$. Hence

$$K_X + \Delta \equiv \left(-2 + \sum k_i a_i\right) H + \sum F_i. \tag{11.2.3.1}$$

By assumption $(K_X + \Delta) \cdot F = 0$, and so $\sum k_i a_i = 2$. Now $\sum F_i = \pi^*(\sum D_i)$ and $\deg(\sum D_i) = H^2 + 2g - 2 + \sum k_i b_i$.

Look at $H \cdot \Delta_i$. If $H \cdot \Delta_i \geq 0$, then $b_i \geq -a_i H^2 \geq 0$. Otherwise $H \cdot \Delta_i < 0$, but since H is an edge of $\overline{\mathrm{NE}}(X)$, this implies that $\Delta_i^2 < 0$. Hence Δ_i is a section of the ruling of X with negative selfintersection. Moreover according to [CKM88, 4.5], the class of Δ_i is an edge, and so Δ_i is proportional to H. By the normalization $H \cdot F = 1$, H is the class generated by Δ_i, and we can replace numerical equivalence in (11.2.3.1) by linear equivalence. In particular, $a_i = 1$ and $D_i = 0$.

Now we have the following two cases:

Case (i). If $H \cdot \Delta_i \geq 0$ for all i, then $\sum k_i b_i \geq -(\sum a_i k_i) H^2 = -2H^2$, and so $H^2 + 2g - 2 + \sum k_i b_i \geq -H^2 + 2g - 2$.

Case (ii). The other possibility is that $H \cdot \Delta_1 < 0$, in which case H is generated by Δ_1 and $H \cdot \Delta_i \geq 0$ for all $i \neq 1$. Then $\sum k_i b_i \geq -2H^2 + k_1 H^2$, and so $H^2 + 2g - 2 + \sum k_i b_i \geq -(1 - k_1)H^2 + 2g - 2$. Note that since Δ is a boundary, $k_1 \leq 1$.

When $g > 1$, in either case, deg $(\sum D_i) > 0$, which implies (11.2.3).

We are left with the case $g = 1$ and deg $(\sum D_i) = 0$. This implies, in case (ii), that $\Delta_1 = H$ is an elliptic curve, $H^2 < 0$, $k_1 = 1$ and $\Delta_i \cdot H = 0$ for $i \geq 2$. Thus H is disjoint from Δ_i, for $i \geq 2$ and so $(K_X + \Delta)|_H = K_H \sim 0$. Therefore $K_X + \Delta \sim 0$.

In case (i), this implies $H^2 = 0$ and $b_i = 0$ for all i. Then H is the class of a section with selfintersection 0 and we denote the section by H. We also replace the numerical equivalence by linear equivalence. Then $K_X \sim -2H$, and $\Delta_i \cdot \Delta_j = 0$ for any i and j. Applying adjunction to Δ_i, we see that each Δ_i is a smooth elliptic curve. Now π restricts to an étale map from Δ_i to C of degree a_i.

As the Δ_i are disjoint, we can find an étale cover $p : \tilde{C} \to C$, so that on the fibre product $\tilde{\pi} : \tilde{X} \to \tilde{C}$, the pull back by $\tilde{p}$ of Δ is a disjoint union of $n = \sum a_i$ sections of $\tilde{\pi}$. Since $\tilde{p}$ is étale, $\tilde{p}^*(K_X) = K_{\tilde{X}}$. Now if $n \geq 3$, then $\tilde{X}$ is actually $\tilde{C} \times \mathbb{P}^1$, and $\tilde{p}^*(K_X + \Delta)$ is trivial. If $n < 3$, as $\sum k_i a_i = 2$, n must be 2, and on $\tilde{X}$, $\tilde{p}^*(\Delta) = \tilde{\Delta}_1 + \tilde{\Delta}_2$. It is then clear that both $K_{\tilde{X}}$ and $\mathcal{O}_{\tilde{X}}(-\tilde{\Delta}_1 - \tilde{\Delta}_2)$ are the relative dualizing sheaf for $\tilde{\pi}$. Thus $\tilde{p}^*(K_X + \Delta)$ is still trivial. But $\tilde{p}^*(K_X + \Delta) = \tilde{\pi}^* p^*(\sum D_i)$. Therefore $p^*(\sum D_i) \sim 0$, and $r(\sum D_i) = p_* p^*(\sum D_i) \sim 0$, where r is the degree of p, that is $\sum D_i$ is a torsion class on C. This finishes the proof of (11.2.3) when X is a $\mathbb{P}^1$-bundle.

Now suppose that π has a singular fibre and E is a component of the singular fibre. If E is not a -1-curve, then $E \cdot K_X \geq 0$. Since Δ contains no vertical component, $(K_X + \Delta) \cdot E \geq 0$. By assumption, $(K_X + \Delta) \cdot F = 0$, hence $(K_X + \Delta) \cdot E \leq 0$ for some exceptional curve E of the singular fibre. We may blow down E to get $p : X \to X'$. Set $\Delta' = p_* \Delta$. We have $K_X + \Delta = p^*(K_{X'} + \Delta') + \alpha E$, with $\alpha = -E \cdot (K_X + \Delta) \geq 0$. Clearly $K_{X'} + \Delta'$ satisfies the inductive assumption, hence $\kappa(X', K_{X'} + \Delta') \geq 0$. It follows at once that $\kappa(X, K_X + \Delta) \geq 0$. □

The log abundance theorem for the case $\nu(X, K_X + \Delta) = 0$ is a direct consequence of (11.2.1).

11.2.4 Lemma. *Let X be a proper surface and assume that (X, Δ) is log canonical. If $K_X + \Delta$ is nef then $\kappa(X, K_X + \Delta) \geq 0$.*

Proof. We want to find a member in $|m(K_X + \Delta)|$. For this let $\phi : X' \to X$ be the minimal resolution, and write $K_{X'} + \Delta_{X'} = \phi^*(K_X + \Delta)$. Since $K_X + \Delta$ is log canonical and ϕ is minimal, $\Delta_{X'}$ is a boundary. As $K_{X'} + \Delta_{X'}$ is nef, (11.2.1) implies that $|m(K_{X'} + \Delta_{X'})| \neq \emptyset$. But $\mathrm{H}^0(m(K_{X'} + \Delta_{X'})) = \mathrm{H}^0(m(K_X + \Delta))$, and so we can find $D \in |m(K_X + \Delta)|$. □

11.3 The case $\nu(K_X + \Delta) = 1$

This section is devoted to a proof of the following result.

11.3.1 Theorem. *Let (X, Δ) be a normal surface with boundary Δ. If $K_X + \Delta$ is nef, $\mathbb{Q}$-Cartier, log canonical and $\nu(X, K_X + \Delta) = 1$, then $|m(K_X + \Delta)|$ is free for some m.*

We first observe that to prove (11.3.1), it is enough to show that $\kappa(X, K_X + \Delta) = 1$. In fact suppose $M + B \in |m(K_X + \Delta)|$, where M moves in a pencil, and B is the fixed part. Now $M \cdot B \geq 0$, as $|M|$ has no one dimensional base locus, and since $(M + B)^2 = 0$, this implies $M(M + B) = M^2 = 0$. Thus $|M|$ is free, and so it defines a map of S to a smooth curve C. As $M \cdot B = 0$ and the numerical class of M is equivalent to a multiple of a fibre, the divisor B is linearly equivalent to the pullback of a divisor from C. But then some multiple of B is base point free.

Here is the first step of (11.3.1).

11.3.2 Lemma. *There exists a surface $\hat{X}$ birational to X, and divisors $\hat{\Delta}$, $\hat{B}$ and $\hat{D}$ such that:*

(1) *$(\hat{X}, \hat{\Delta} + \hat{B})$ is $\mathbb{Q}$-factorial and log canonical and $\hat{D} \in |m(K_{\hat{X}} + \hat{\Delta} + \hat{B})|$. Moreover $\hat{B} = \hat{D}_{\mathrm{red}}$.*
(2) *$K_{\hat{X}} + \hat{\Delta} + \hat{B}$ is nef.*
(3) *$\nu(X, K_X + \Delta) = \nu(X, K_{\hat{X}} + \hat{\Delta} + \hat{B})$ and $\kappa(X, K_X + \Delta) = \kappa(\hat{X}, K_{\hat{X}} + \hat{\Delta} + \hat{B})$.*

Proof. By (11.2.4), we may find $D \in |m(K_X + \Delta)|$. Pick a minimal good resolution $\mu : X_0 \longrightarrow X$ of the pair $(X, D + \Delta)$, and write $K_{X_0} + \tilde{\Delta} = \mu^*(K_X + \Delta)$. As (X, Δ) is log canonical, $\tilde{\Delta}$ is effective. Set $B_0 = (\mu^* D)_{\mathrm{red}}$ and replace $\tilde{\Delta}$ with Δ_0, where we only include those components of $\tilde{\Delta}$ which are not components of B_0. With this choice of Δ_0, $\Delta_0 + B_0$ is a boundary, and there is a divisor $D_0 \in |m(K_{X_0} + \Delta_0 + B_0)|$.

We now apply the log minimal model program to $(X_0, \Delta_0 + B_0)$. We inductively construct a sequence X_i, Δ_i, B_i and D_i satisfying (1). If $K_{X_i} + \Delta_i + B_i$ is not nef, then there is a divisorial contraction ϕ_i associated to some log extremal ray of $K_{X_i} + \Delta_i + B_i$ (clearly ϕ_i is not of fibre type), and we put $B_{i+1} = \phi_{i*}(B_i)$, $\Delta_{i+1} = \phi_{i*}(\Delta_i)$, and $D_{i+1} = \phi_{i*}(D_i)$. (By [KMM87, 5-1-6] $(X_{i+1}, \Delta_{i+1} + B_{i+1})$ is $\mathbb{Q}$-factorial and log terminal.)

Since at each step the Picard number drops by one, this process must terminate at some i, and we set $\hat{X} = X_i$, $\hat{B} = B_i$, $\hat{\Delta} = \Delta_i$ and $\hat{D} = D_i$.

Conditions (1) and (2) are automatic from the construction. (3) follows from the (11.3.3) applied to the pullbacks of the divisors $m(K_X + \Delta)$ and D_i to X_0 (cf. (13.2.4)). $\square$

Note that in fact the pair $(\hat{X}, \hat{\Delta} + \hat{B})$ is log terminal; we do not need this.

11.3.3 Lemma. *Let X be a proper variety of dimension n and G_1, G_2 two effective nef divisors with the same support. Then $\nu(X, G_1) = \nu(X, G_2)$ and $\kappa(X, G_1) = \kappa(X, G_2)$.*

Proof. Let $\nu(X, G_i) = \nu_i$ and $\kappa(X, G_i) = \kappa_i$. Choose a_1 so that $a_1G_1 - G_2$ is effective.

(1) Let H be any ample divisor. Then

$$\begin{aligned}\left((a_1G_1)^{\nu_2} \cdot H^{n-\nu_2}\right) &\geq \left((a_1G_1)^{\nu_2-1} \cdot G_2 \cdot H^{n-\nu_2}\right)\\ &\vdots\\ &\geq \left(G_2^{\nu_2} \cdot H^{n-\nu_2}\right) > 0,\end{aligned}$$

and therefore $\nu_1 \geq \nu_2$.

(2) $\mathrm{H}^0(mG_2) \hookrightarrow \mathrm{H}^0(ma_1G_1)$, therefore $\kappa_1 \geq \kappa_2$.

Now reverse the roles of G_1 and G_2. $\square$

Now Riemann–Roch for $n\hat{D}$ reads:

$$\begin{aligned}\chi(n\hat{D}) &= \frac{n\hat{D} \cdot (n\hat{D} - K_{\hat{X}})}{2} + \chi(\mathcal{O}_X)\\ &= \frac{n(n-1/m)}{2}(\hat{D}^2) + \frac{n}{2}\hat{D} \cdot (\hat{\Delta} + \hat{B}) + \chi(\mathcal{O}_X).\end{aligned}$$

We know already that $\hat{D}^2 = 0$ and so from now on we can assume $\hat{D} \cdot \hat{\Delta} = 0$, since otherwise (11.3.1) follows immediately (because $h^2(n\hat{D}) = h^2(K_{\hat{X}} - n\hat{D}) = 0$ for large n, as G is not numerically trivial, and we only need to show $\kappa(X, K_X + \Delta) = 1$). Since we have chosen $\hat{B}$ so that $\hat{\Delta}$ and $\hat{B}$ have no components in common, this implies that $\hat{\Delta}$ and $\hat{B}$ do not intersect.

Choose an integer m so that $\hat{L} = \mathcal{O}_{\hat{X}}(m(K_{\hat{X}} + \hat{\Delta} + \hat{B})) \in \mathrm{Pic}\,(\hat{X})$ and $|\hat{L}|$ is non-empty. Note that $\hat{L}$ is nef.

11.3.4 Lemma. *There are X', Δ', B' and D' satisfying (11.3.2.1–3) and in addition*

(4) *Every connected component of B' is irreducible.*

Proof. Pick an irreducible component S of $\hat{B}$. Suppose S meets another component $\hat{S}$ of $\hat{B}$. Now $\nu(X, \hat{L}) = 1$, so that $\hat{L}^2 = 0$. But $\hat{L}^2 = \hat{L} \cdot (\hat{D} - \hat{S}) + \hat{L} \cdot \hat{S}$, and both terms are non-negative as $\hat{L}$ is nef. It follows that $\hat{L} \cdot \hat{S} = 0$ and moreover that $(K_{\hat{X}} + \hat{\Delta} + \hat{B} - S) \cdot \hat{S} < 0$. But then there is a log extremal ray of $(K_{\hat{X}} + \hat{\Delta} + \hat{B} - S)$ associated to $\hat{S}$, and so a log extremal contraction, which must be divisorial. Such a contraction decreases the Picard number of $\hat{X}$, and so eventually we may isolate every component of $\hat{B}$. □

Pick any prime component S of B', and let U be an open subset of X' which retracts to S [BPV84, page 27]. Let L' be the line bundle $\mathcal{O}_U(m(K_U+S)) = \mathcal{O}_U(m(K_U+\Delta'+S))$.

11.3.5 Lemma. *$L'|_S$ is a torsion element of* Pic (S) *(i.e some multiple of $L'|_S$ is isomorphic to $\mathcal{O}_S$).*

Proof. If we apply adjunction to S in U, we get

$$(K_U+S)|_S = K_S + P$$

where $P = \mathrm{Diff}$ is effective. If $P=0$, then S is elliptic or nodal rational and so $K_S+P=0$. If $P \neq 0$ then S is a smooth $\mathbb{P}^1$. □

Now we make a cover of U to improve S and how it sits inside U (compare [Miyaoka88b], where this argument first appears).

11.3.6 Lemma. *Let U be a normal analytic space, and S a compact subspace. If the inclusion $i: S \to U$ induces isomorphisms*

$$i^*: H^j(U,\mathbb{Z}) \cong H^j(S,\mathbb{Z}) \quad \text{for } j=1,2,$$

then

(1) *the kernel of the restriction map*

$$\mathrm{Pic}\,(U) \longrightarrow \mathrm{Pic}\,(S)$$

is a $\mathbb{C}$-vector space. In particular it is divisible, and torsion free.

Moreover if G is a $\mathbb{Q}$-Cartier integral divisor on U such that $G|_S$ is torsion, then

(2) *there is a finite Galois cover $\pi: \tilde{U} \longrightarrow U$, étale in codimension one, such that π^*G is a Cartier divisor, which restricts to a divisor linearly equivalent to zero on π^*S.*

Proof. Compare the cohomology exact sequences of the exponential sequences on S and U:

$$\begin{tikzcd}
H^1(S,\mathbb{Z}) \arrow[r, "\alpha_S"] & H^1(S,\mathcal{O}_S) \arrow[r] & H^1(S,\mathcal{O}_S^*) \arrow[r] & H^2(S,\mathbb{Z}) \\
H^1(U,\mathbb{Z}) \arrow[r, "\alpha_U"] \arrow[u, "\beta_1"] & H^1(U,\mathcal{O}_U) \arrow[r] \arrow[u, "\beta_2"] & H^1(U,\mathcal{O}_U^*) \arrow[r] \arrow[u, "\beta_3"] & H^2(U,\mathbb{Z}). \arrow[u, "\beta_4"]
\end{tikzcd}$$

Now $\operatorname{Pic}(U) = H^1(U, \mathcal{O}_U^*)$, $\operatorname{Pic}(S) = H^1(S, \mathcal{O}_S^*)$, and β_3 is just the restriction map. By assumption, β_1 and β_4 are isomorphisms, and so the kernel of β_3 is isomorphic to the kernel of β_2, which in turn is a subvector space of the $\mathbb{C}$-vector space $H^1(U, \mathcal{O}_U)$. Hence (1) holds.

For (2), let r be the smallest integer such that rG is Cartier and $rG|_S \sim 0$. The class of $rG|_S$ in $H^2(S, \mathbb{Z})$ is zero, and as β_4 is an isomorphism, the class of rG is zero in $H^2(U, \mathbb{Z})$. As $H^1(U, \mathcal{O}_U)$ is divisible, there is a line bundle M on U such that:

$$\mathcal{O}_U(rG) \otimes M^r = \mathcal{O}_U \quad \square$$

We are going to apply (11.3.6.2) to ensure that both the pullback of K_X and the class of S are multiples of the same Cartier divisor $\tilde{G}$, which will itself restrict to a divisor linearly equivalent to zero on the pullback of S.

As S is irreducible, there is a divisor $D \in |m(K_U + S)|$ such that $D = eS$ for some positive integer e. But then $dS \sim mK_U$, where $d = e - m$. Note that either d and m are nonnegative or d is negative, but $-d < m$. Let c be the highest common factor of m and d. We may find integers m', d', b_1 and b_2 such that:

$$m = m'c, \qquad d = d'c, \qquad c = b_1 m + b_2 d.$$

Let G be the Weil divisor $b_1 S + b_2 K_U$. We have

$$c(S - m'G) = (b_1 m + b_2 d)S - m(b_1 S + b_2 K_U) \sim 0 \sim c(K_U - d'G),$$

and so

$$c(K_U + S - (m' + d')G) \sim 0.$$

Thus the three divisors

$$(S - m'G)|_S, \qquad (K_U - d'G)|_S \quad \text{and} \quad G|_S$$

are all torsion (the third by (11.3.5)). Now we apply (11.3.6.2) three times to these divisors. Thus there is a finite Galois cover $\pi : \tilde{U} \longrightarrow U$, étale in codimension one, such that, if we put $\tilde{S} = \pi^* S$ and $\tilde{G} = \pi^* G$,

$$\tilde{G}|_{\tilde{S}} \sim 0 \qquad \tilde{S} \sim m'\tilde{G}, \qquad K_{\tilde{U}} \sim d'\tilde{G},$$

and so

$$\omega_{\tilde{S}} = \mathcal{O}_{\tilde{U}}(\tilde{S})|_{\tilde{S}} = \mathcal{O}_{\tilde{S}}.$$

The next lemma shows that $\tilde{S}$ moves in $\tilde{U}$ infinitesimally (cf. [Miyaoka88b, 4.2]). First some notation; let V be an analytic space, and S a Cartier divisor on V. Denote by S_n the analytic subspace of V defined by the sheaf of ideals $\mathcal{O}_U(-nS)$ and set $A_n = \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon)^n$.

11.3.7 Lemma. *Let V be a Cohen–Macaulay complex space, and S a divisor on V. Assume that K_V and S are both multiples d' and m' of the same Cartier divisor G, and that the following three conditions hold*

(1) $d' + (n-1)m' \neq 0$ *for any* $n \geq 2$,
(2) $\omega_S \simeq \mathcal{O}_S$,
(3) *the restriction* $H^p(S_n, \mathcal{O}_{S_n}) \longrightarrow H^p(S, \mathcal{O}_S)$ *is surjective for every* p

Then S moves infinitesimally in V, to any order.

Proof. We prove the following statements by induction on n.

(i)$_n$ There are proper flat morphisms $\xi_i : S_i \longrightarrow A_i$ $(i \leq n)$ such that the following diagram is commutative

$$\begin{array}{ccc} S_{i-1} & \longrightarrow & S_i \\ {\scriptstyle \xi_{i-1}}\big\downarrow & & {\scriptstyle \xi_i}\big\downarrow \\ A_{i-1} & \longrightarrow & A_i \end{array}$$

(ii)$_n$ The sheaves $R^p\xi_{n*}\mathcal{O}_{S_n}$ are locally free.
(iii)$_n$ $\omega_{S_n} \simeq \mathcal{O}_{S_n}$.

Note that if (i)$_n$ holds for every n, then S moves infinitesimally to any order, by definition.

For $n = 1$, we take ξ_1 to be the structure map. Then (ii)$_1$ is automatic, and (iii)$_1$ is just (2).

Otherwise suppose that all three statements are true for all integers less than n. As $K_V + (n-1)S$ is Cartier and S_{n-1} is Cohen Macaulay (it is a Cartier divisor in a Cohen Macaulay scheme), we may apply adjunction to S_{n-1}:

$$\begin{aligned} \omega_{S_{n-1}} &= \omega_V((n-1)S) \otimes \mathcal{O}_{S_{n-1}} \\ &= \mathcal{O}_{S_{n-1}}((d' + (n-1)m')G). \end{aligned}$$

On the other hand (iii)$_{n-1}$ implies that $\omega_{S_{n-1}}$ is linearly equivalent to zero. We may apply (11.3.6) (1) to S and S_{n-1} to deduce that G is linearly equivalent to zero on S_{n-1}. In particular $\mathcal{O}_{S_{n-1}} \otimes \mathcal{O}_V(-S) \simeq \mathcal{O}_{S_{n-1}}$.

Consider the exact sequence of sheaves on V,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{S_n} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

where $\mathcal{K}$ is defined by exactness. It is clear that the support of $\mathcal{K}$ is S_{n-1}. In fact

$$\mathcal{O}_S = \mathcal{O}_V/\mathcal{O}_V(-S) \qquad \mathcal{O}_{S_n} = \mathcal{O}_V/\mathcal{O}_V(-(n+1)S)$$

and so $\mathcal{K} \cong \mathcal{O}_{S_{n-1}} \otimes \mathcal{O}_V(-S)$ as a sheaf of $\mathcal{O}_V$-modules. Now we have shown that this is the trivial line bundle on S_{n-1}.

Let e be the image of the global section 1 of the sheaf $\mathcal{K}$ in the vector space $H^0(S_n, \mathcal{O}_{S_n})$. Define a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[\epsilon]/(\epsilon)^n$ to $H^0(S_n, \mathcal{O}_{S_n})$, by sending ϵ to e. This gives $H^0(S_n, \mathcal{O}_{S_n})$ a flat $\mathbb{C}[\epsilon]/(\epsilon)^n$-module structure, which since A_n is affine, is equivalent to a proper flat morphism $\xi_n : S_n \longrightarrow A_n$. It is not hard, from the definition of ξ_n, to check that the diagram

$$\begin{array}{ccc} S_{n-1} & \longrightarrow & S_n \\ {\scriptstyle \xi_{n-1}} \downarrow & & {\scriptstyle \xi_n} \downarrow \\ A_{n-1} & \longrightarrow & A_n \end{array}$$

commutes. This proves $(\mathrm{i})_n$.

Condition (3) now implies $(\mathrm{ii})_n$ (see for example [Hartshorne77, III 12.11]). It follows by duality, that $R^p\xi_{n*}\omega_{S_n}$ are also locally free, for every p. (Unfortunately this seems to require relative duality theory, see e.g. [Hartshorne66].) As ω_S is isomorphic to the trivial line bundle, $\xi_{n*}\omega_{S_n}$ has a global non vanishing section, which we may pullback to ω_{S_n}. Thus ω_{S_n} is trivial also, which is $(\mathrm{iii})_n$. $\square$

11.3.8 Example. There is an interesting example which indicates the necessity for the somewhat strange assumptions of (11.3.7). Take X to be a $\mathbb{P}^1$-bundle over an elliptic curve, given by the unique rank two vector bundle of degree zero which does not split. X has a unique section S of selfintersection zero, which does not move. However it does move to first order. Of course there is no divisor G such that both the class of the curve and its dualizing sheaf are multiples of G.

11.3.9. Now we check that the conditions of (11.3.7) apply to $\tilde{S}$ in $\tilde{U}$. In fact (1) follows as m' is always positive, and if d' is negative, $-d' < m'$, (2) has already been verified, and so we are left with (3). But as $\tilde{S}$ is a curve, certainly

$$H^1(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}),$$

is surjective, as the obstruction is the second cohomology of the kernel of the natural map $\mathcal{O}_{\tilde{S}_n} \longrightarrow \mathcal{O}_{\tilde{S}}$, which always vanishes. This leaves

$$H^0(\tilde{S}_n, \mathcal{O}_{\tilde{S}_n}) \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) \simeq \mathbb{C},$$

which is again certainly surjective.

Now we are in a position to finish the proof of (11.3.1). Let G be the Galois group of the cover $\tilde{U} \longrightarrow U$ of degree r. The Cartier divisor $(rS)_n$ pulls back, under π, to the Cartier divisor $\tilde{S}_{nr}$. Thus $\tilde{S}_{nr}$ descends to $(rS)_n$, and moreover G acts naturally on $H^0(\tilde{S}_{nr}, \mathcal{O}_{\tilde{S}_{nr}})$. But this may be identified, via ξ_{nr}, with $\mathbb{C}[\epsilon]/(\epsilon)^{nr}$. It follows that $(rS)_n$ maps to $A_{ns} = \operatorname{Spec}(\mathbb{C}[\epsilon]/(\epsilon)^{nr})^G$, for some s dividing r. Since the Hilbert scheme is of finite type we are done.

12. SEMI LOG CANONICAL SURFACES

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12.1 INTRODUCTION

In this chapter we collect together some results concerning semi log canonical surfaces (see (12.2) for the definitions and basic properties). The first of these is log abundance for semi log canonical surfaces in the cases $\nu = 0$ or $\nu = 1$.

12.1.1 Theorem. *Let S be a reduced projective surface and let Δ be a $\mathbb{Q}$-Weil divisor on S. Assume that $K_S + \Delta$ is $\mathbb{Q}$-Cartier, nef and semi log canonical and $\nu(S, K_S + \Delta) = 0$ or 1.*

Then the linear system $|m(K_S + \Delta)|$ is base point free for suitable $m > 0$ (and in particular $\nu(S, K_S + \Delta) = \kappa(S, K_S + \Delta)$).

The idea is to show that we can descend sections to S from the normalization of S (here we use (11.1.3)). In both cases the arguments are a little delicate; we have to analyze carefully the patching data.

The second result is a version of (1.13) (which is proved in (12.5)).

12.1.2 Theorem. *Let S be a reduced projective surface with semi log canonical singularities. Then the natural map induced by $\mathbb{C}_S \subset \mathcal{O}_S$*

$$i_p : H^p(S, \mathbb{C}_S) \longrightarrow H^p(S, \mathcal{O}_S) \qquad \text{is surjective for every } p.$$

When S is smooth (12.1.2) is a standard result. Therefore we just need to analyze how the cohomology of S differs from the cohomology of a resolution. We split this analysis into two steps; in one step we consider how to resolve the bad singularities at isolated points of S, and in the other step we remove the one dimensional singular locus via a finite map. However we introduce a new twist; rather than first normalizing S for the second step, which loses too much information about the singularities of S, we make S as nice as possible by altering S at a finite set of points, and then normalize.

12.2 Basic results

We collect together here some of the properties of semi log canonical surface singularities.

Let X be a scheme with at worst double normal crossings in codimension one. The next set of definitions introduces the appropriate notion of log canonical (these definitions were given in [KSB88] for surfaces).

12.2.1 Definition.

(1) An n-dimensional singularity $(x \in X)$ is called a *double normal crossing point*, resp. a *pinch point* if it is analytically (or formally) isomorphic to

$$\big(0 \in (x_0x_1 = 0)\big) \subset (0 \in \mathbb{C}^{n+1}) \text{ resp. } \big(0 \in (x_0^2 = x_1x_2^2)\big) \subset (0 \in \mathbb{C}^{n+1}).$$

(2) An n-fold X is *semismooth* if every closed point $(x \in X)$ is either smooth or double normal crossing point or pinch point. The singular locus of X is then a smooth $(n-1)$-fold D_X. The normalization $\nu : X^\nu \to X$ is smooth and $D_\nu = \nu^{-1}(D_X) \to D_X$ is a double cover ramified along the pinch locus.

(3) A morphism $f : Y \to X$ is called a *semiresolution* if f is proper, Y is semismooth, no component of D_Y is f-exceptional, and there is a codimension two closed subset $S \subset X$ such that $f|f^{-1}(X \setminus S) : f^{-1}(X \setminus S) \to X \setminus S$ is an isomorphism.

(4) Let X be a reduced scheme, $\Delta \subset X$ a $\mathbb{Q}$-Weil divisor (cf. (16.2)). Let $f : Y \to X$ be a semiresolution with exceptional divisors E and exceptional set $Ex(f) \subset Y$.

f is a *good* semiresolution (resp. a *good divisorial* semiresolution) of $\Delta \subset X$ if the union $E \cup D_Y \cup f_*^{-1}(\Delta)$ (resp. $Ex(f) \cup D_Y \cup f_*^{-1}(\Delta)$) is a divisor with global normal crossings on Y.

(5) Let S be a reduced surface. A semiresolution $f : T \to S$ is *minimal* if ω_T is f-nef. (In the nonnormal case, minimal resolutions are not unique.)

(6) Let X be a reduced S_2 scheme, $\Delta \subset X$ a boundary (i.e., a $\mathbb{Q}$-Weil divisor $\Delta = \sum d_i\Delta_i$ with $0 \le d_i \le 1$). We say that $K_X + \Delta$ is *semi log terminal* (resp. *divisorial semi log terminal*, resp. *semi log canonical*) (frequently abbreviated as **slt** resp. **dslt** resp. **slc**) if it is $\mathbb{Q}$-Cartier and there is a good semiresolution (resp. a good divisorial semiresolution, resp. a good semiresolution) $f : Y \to X$ of $\Delta \subset X$ such that:

$$K_Y + f_*^{-1}(\Delta) = f^*(K_X + \Delta) + \sum a_iE_i,$$

where the E_i are the f-exceptional divisors and all $a_i > -1$ (resp. $a_i > -1$ resp. $a_i \geq -1$). We leave it to the reader to formulate the analogous definition of the various flavors of semi log terminal.

(7) Let $f : Y \to X$ be a semiresolution. We say X has semirational singularities, if $f_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$. As in the normal case, this is independent of the semiresolution chosen.

(8) A scheme X (over an algebraically closed field) is called seminormal if the following condition holds:

Every finite and surjective morphism $X' \to X$ which is one-to-one on closed points is an isomorphism.

12.2.2 Notation. Let (X, Δ) be slc. Let $\mu : X^\mu \longrightarrow X$ be the normalization. Let $D \subset X$ (resp. $D_\mu \subset X^\mu$) be the double intersection locus. Thus $\mu | D_\mu : D_\mu \longrightarrow D$ is a double cover. Let $\Theta = \mu^{-1}\Delta + D_\mu$. Thus

$$K_{X^\mu} + \Theta = \mu^*(K_X + \Delta).$$

The irreducible components of X^μ are frequently denoted by X_i and then Θ_i denotes the restriction of Θ to X_i.

12.2.3 Proposition. *[vanStraten87] Let S be a surface which is semismooth in codimension one. Then S has a minimal semiresolution. If $\Delta \subset S$ is a Weil divisor then (S, Δ) has a good semiresolution.*

Proof. Let S be a surface, with normal crossings in codimension one, and choose a good resolution (T_0, D_0) of the pair (S^μ, D_μ). If $\Delta = \emptyset$ then D_μ is reduced and we may assume in addition that $K_{T_0} + D_0$ is nef on T_0/S^μ. The map $D_\mu \longrightarrow D_S$ is two-to-one, and defines an involution τ on D_0. It is easy to see (cf. [Artin70] for the general theory) that one can find an analytic (or algebraic) space T, which is obtained from T_0 by gluing together points of D_0 that are conjugate under the involution τ. Moreover it is not hard to see that T is semismooth; pinch points correspond to fixed points of the involution τ. There is a morphism $f : T \longrightarrow S$ with fibres which are either points or curves. Thus f is projective, hence T is also projective and so f is a semiresolution. □

The following is clear from the definitions (cf. (2.6)):

12.2.4 Proposition. *Notation as above. Then*

$$\operatorname{discrep}(X, \Delta) = \operatorname{discrep}(X^\mu, \Theta). \quad \square$$

It might seem from (12.2.4) that one could define the semi log versions of lt, lc etc. by requiring the corresponding notion to hold for the normalization. However, $K_X + \Delta$ is usually not $\mathbb{Q}$-Cartier even when $\Delta = \emptyset$ and (X^μ, Θ) is log canonical. In dimension two one can give the following necessary (and sufficient) condition.

12.2.5 Proposition. *Let (S, Δ) be a slc surface. Let $D_1 \subset S$ be a double curve such that $\mu^{-1}(D_1) = D_1' \cup D_1''$ has two components. Then (see (16.6) for the definition of* Diff*)*

$$\operatorname{Diff}_{D_1'}(\Theta - D_1') = \operatorname{Diff}_{D_1''}(\Theta - D_1'').$$

Proof. Let S_1, S_2 be analytic neighborhoods of D_1', D_1'' respectively. We abuse notation, and identify S_1 and S_2 with their images under μ. Now we may compute the different at any point of D_1' or D_1'', on the surface S, by first restricting to S_1 or S_2. In either case this is equivalent to restricting $K_S + \Delta$ to the double curve D_1. □

12.2.6 Corollary. *Let (S, Δ) be a germ of a slc surface. Assume that S^μ has two irreducible components S_1^μ, S_2^μ. Then*

$$(S_1^\mu, \Theta_1) \cong (\mathbb{C}^2, \mathbb{C}) \quad \Leftrightarrow \quad (S_2^\mu, \Theta_2) \cong (\mathbb{C}^2, \mathbb{C}).$$

Proof. Note that by (16.6) (S_i, Θ_i) is isomorphic to $(\mathbb{C}^2, \mathbb{C})$ iff Θ_i is irreducible and the different is zero. □

12.2.7 Corollary. *Let (S, B) be a germ of a slt surface. Then S has one or two irreducible components.*

Proof. Assume that S has at least three irreducible components. Then there is a component S_1 which intersects at least two other components along curves. Thus $\Theta_i = \Theta | S_i^\mu$ contains at least two reduced curves. By Chapter 3, this implies that (S_i^μ, Θ_i) is not lt. □

12.2.8 Proposition – Definition. *Let (S, Δ) be a germ of an slc surface. Let $f : T \to S$ be a minimal semiresolution (of S). Let $E_i \subset T$ be the exceptional divisors. Then*

(1)
$$K_T + f_*^{-1}(\Delta) = f^*(K_S + \Delta) + \sum a_i E_i,$$
where $0 \geq a_i \geq -1$. Let $E = \sum_{a_i=-1} E_i$.

(2) $R^1 f_* \mathcal{O}_T(-E) = 0$.

(3) *If (S, Δ) is not semirational then $\Delta = \emptyset$ and S is either simple elliptic, a cusp or a degenerate cusp; where we define S to be*

(i) simple elliptic, *if $E = Ex(f)$ is a smooth elliptic curve, and S is normal,*

(ii) a cusp (resp. degenerate cusp), *if S is normal (resp. not normal, but T has no pinch points, locally about E), if $E = Ex(f)$ is a cycle of $\mathbb{P}^1$ or a nodal $\mathbb{P}^1$.*

Proof. Let $\nu : T^\nu \longrightarrow T$ be the normalization of T. We get a commutative diagram

$$\begin{tikzcd} T^\nu \arrow[r, "g"] \arrow[d, "\nu"'] & S^\mu \arrow[d, "\mu"'] \\ T \arrow[r, "f"] & S. \end{tikzcd}$$

Now $g : T^\nu \longrightarrow S^\mu$ is a resolution of S^μ. Thus

$$K_{T^\nu} + g_*^{-1}(\Theta) = g^*(K_{S^\mu} + \Theta) + \sum a_i F_i,$$

where F_i are the exceptional divisors of g and $0 \geq a_i$ follows from (2.19).

Let $F = \sum_{a_i=-1} F_i$. Then $f^*E = F$, and so to show that $R^1 f_* \mathcal{O}_T(-E) = 0$, it is enough to show that $R^1 g_* \mathcal{O}_{T^\nu}(-F) = 0$, as the morphisms ν and μ are finite. But as

$$-F = K_{T^\nu} + \left(g_*^{-1}(\Theta) + \sum_{a_i > -1} -a_i F_i \right) - g^*(K_{S^\mu} + \Theta),$$

this follows by Kawamata–Viehweg vanishing [KMM87, 1-2-3].

Now if S is not semirational, $h^1(\mathcal{O}_E) \geq 1$, by (2). Applying adjunction to E, we have:

$$K_E = (K_T + E)|E = \sum_{0 \geq a_i > -1} (a_i E_i - f_*^{-1}(\Delta))|E,$$

which is negative unless $E = Ex(f)$ and $\Delta = \emptyset$. Thus $H^1(\mathcal{O}_E) = 0$ unless $E = Ex(f)$ and $\Delta = \emptyset$. In the latter case E has arithmetic genus one, and so it is an elliptic curve, a cycle of $\mathbb{P}^1$ or a nodal $\mathbb{P}^1$. Therefore if S is not normal then D^μ has two components on every component of S^μ and every (S^μ, D_μ) falls to case (9) of Figure 3 in the classification of Chapter 3. Thus S is a degenerate cusp. This proves (3). $\square$

12.2.9 Definition. Let (C, Δ) be a semi log canonical curve and Δ a $\mathbb{Q}$-divisor.

Let $n : \bar{C} = \cup C_i \to C$ be the normalization and define Δ_i by

$$n^*(K_C + \Delta)|C_i = K_{C_i} + \Delta_i.$$

Assume that $m(K_{C_i} + \Delta_i)$ is an integral divisor. For every $P \in \llcorner \Delta_i \lrcorner$ let z_P be a local parameter at P. A section $s_i \in \Gamma(C_i, \mathcal{O}(m(K_{C_i} + \Delta_i)))$ is *normalized* if $s_i - (dz_P/z_P)^m$ vanishes at P. This is easily seen to be independent of the choice of z_P.

A section $s \in \Gamma(C, \mathcal{O}(m(K_C + \Delta)))$ is *normalized* if $n^*(s)|C_i$ is normalized for every i.

On the nodal curve $(xy = 0) \subset \mathbb{C}^2$ consider the 1-form $\sigma = dx/x = -dy/y$. Even powers of σ are normalized and there are no normalized sections if m is odd.

All normalized sections form an *affine* subspace in the space of sections. This will be denoted by

$$\Gamma^n(C, \mathcal{O}(m(K_C + \Delta))).$$

12.2.9.1 Complement. If C_i is such that $\llcorner \Delta_i \lrcorner = 0$ then C_i is a smooth connected component of C and the above definition imposes no restrictions on sections of $\mathcal{O}(m(K + \Delta_i))$. For our purposes it will be convenient to make the following convention. Assume that C_i is an elliptic curve such that $\Delta_i = 0$. $\operatorname{Aut}(C)$ acts trivially on $H^0(C, \mathcal{O}_C(12K_C))$. We fix a nonzero section for every elliptic curve and call it (and its powers in $H^0(C, \mathcal{O}_C(12mK_C)))$ normalized.

12.2.10 Definition. Let (X, Δ) be an slc surface. As in (12.2.2) let $n : (X^\mu, \Theta) \to (X, \Delta)$ be the normalization. As section $s \in \Gamma(X, \mathcal{O}(m(K_X + \Delta)))$ is *normalized* if

$$n^* s|_{\llcorner \Theta \lrcorner} \in \Gamma(\llcorner \Theta \lrcorner, \mathcal{O}(m(K_{\llcorner \Theta \lrcorner} + \operatorname{Diff}(\Theta - \llcorner \Theta \lrcorner))))$$

is normalized.

All normalized sections form an affine subspace $\Gamma^n(X, \mathcal{O}(m(K_X + \Delta)))$ in the space of all sections.

12.2.11 Proposition. *Let (C, Δ) be an slc curve and let m be a natural number such that $m\Delta$ is integral. Then*

(12.2.11.1) $\Gamma^n(C, \mathcal{O}(2m(K_C + \Delta))) = \prod_i \Gamma^n(C_i, \mathcal{O}(2m(K_{C_i} + \Delta_i)))$;

(12.2.11.2) If $K_C + \Delta$ is nef then $\Gamma^n(C, \mathcal{O}(12m(K_C + \Delta)))$ generates $\mathcal{O}(12m(K_C + \Delta))$.

Proof. The first part is clear. Using the first part, it is sufficient to prove the second for C irreducible and smooth.

We distinguish two cases:

(12.2.11.3) $\deg(K_C + \Delta) = 0$. Then either $g(C) = 1$ and $\Delta = 0$ or $g(C) = 0$ and $\llcorner \Delta \lrcorner$ is at most two points of C. $\mathcal{O}(12m(K_C + \Delta))$ has one section (up to scalars) and a suitable multiple is normalized if $\llcorner \Delta \lrcorner$ is at most one point. If $\llcorner \Delta \lrcorner = \{0, \infty\}$ then $(dz/z)^{12m}$ is normalized.

(12.2.11.4) $\deg(K_C + \Delta) > 0$. Let P be any point different from $\llcorner \Delta \lrcorner$. Consider the exact sequence

$$0 \to \mathcal{O}(12m(K_C + \Delta) - \llcorner \Delta \lrcorner - P) \to \mathcal{O}(12m(K_C + \Delta)) \to \mathbb{C}(P) + \mathbb{C}(\llcorner \Delta \lrcorner) \to 0.$$

Since

$$\begin{aligned}&\deg(12m(K_C+\Delta)-\llcorner\Delta\lrcorner-P)\\&=\deg(K_C+11m(K_C+\Delta))+\deg((m-1)(K_C+\Delta)+\{\Delta\})-1\\&\geq\deg K_C+11-1=\deg K_C+10,\end{aligned}$$

we conclude that

$$H^0(C,\mathcal{O}(12m(K_C+\Delta)))\to H^0(C,\mathbb{C}(P)+\mathbb{C}(\llcorner\Delta\lrcorner))$$

is surjective. $\square$

12.3 The reduced boundary of lc surfaces

Let (S,Θ) be an lc surface. Our aim is to analyze $\llcorner\Theta\lrcorner$ in the cases when $\nu(S,\Theta)\in\{0,1\}$.

12.3.1 Proposition. *[Shokurov91, 6.9] Let (S,Θ) be a proper lc surface. Assume that $K+\Theta\equiv 0$. Then (S,Θ) satisfies one of the following conditions:*

(1) *$\llcorner\Theta\lrcorner$ is connected and for every $C\in\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is* **not** *klt, (i.e., $\mathrm{Diff}(\Theta-C)$ contains a point with multiplicity 1.)*
(2) *$\llcorner\Theta\lrcorner$ is irreducible and for $C=\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is klt.*
(3) *$\llcorner\Theta\lrcorner$ has two connected components, for every $C\subset\llcorner\Theta\lrcorner$ the pair $(C,\mathrm{Diff}(\Theta-C))$ is klt and there is a morphism onto a curve $g:S\to B$ such that $\llcorner\Theta\lrcorner$ consists of two sections of g. (B is either rational or elliptic.)*

Proof. Let $h:S'\to S$ be an lt modification of S and let $K+\Theta'=h^*(K+\Theta)$. Then (S',Θ') is lt and it is sufficient to prove that the result holds for (S',Θ'). In this case $(C,\mathrm{Diff}(\Theta'-C))$ is not klt iff C intersects another irreducible component of $\llcorner\Theta'\lrcorner$.

We prove a stronger relative version:

12.3.2 Proposition. *Let (S,Θ) be a log terminal surface. Let $f:S\to R$ be a proper morphism with connected fibers. Assume that $K+\Theta$ is numerically f-trivial. Let $r\in R$ be arbitrary. Then one of the following holds:*

(1) *$\llcorner\Theta\lrcorner$ is connected in a neighborhood of $f^{-1}(r)$;*
(2) *$\llcorner\Theta\lrcorner$ has two connected components in a neighborhood of $f^{-1}(r)$, both components are smooth and there is a morphism onto a curve $g:S/R\to B/R$ such that $\llcorner\Theta\lrcorner$ consists of two sections of g.*

Proof. If f is birational then (17.4) implies that we have (1). Thus we may assume that f has positive dimensional fibers and that $\llcorner\Theta\lrcorner \neq \emptyset$.

We apply the $(K+\Theta-\epsilon\llcorner\Theta\lrcorner)$-MMP on S/R for $0<\epsilon\ll 1$. The end result is a proper birational morphism $p: S/R \to Z/R$ such that $K_Z+p(\Theta)$ is lc and $K_Z+p(\Theta)-\epsilon\llcorner p(\Theta)\lrcorner$ is lt. We claim that

$$p(\llcorner\Theta\lrcorner)=\llcorner p(\Theta)\lrcorner.$$

Indeed, since $K+\Theta$ is numerically f-trivial, $K=p^*(K_Z+p(\Theta))-\Theta$. If $z\in p(\llcorner\Theta\lrcorner)-\llcorner p(\Theta)\lrcorner$ then

$$K=p^*(K_Z+p(\Theta))-\Theta=p^*(K_Z+p(\Theta)-\epsilon\llcorner p(\Theta)\lrcorner)-\Theta$$

in a neighborhood of $p^{-1}(z)$, which shows that $K_Z+p(\Theta)-\epsilon\llcorner p(\Theta)\lrcorner$ is not lt at z, a contradiction. In particular $\llcorner p(\Theta)\lrcorner \neq \emptyset$. By (17.4) the fibers of $\llcorner\Theta\lrcorner \to \llcorner p(\Theta)\lrcorner$ are connected, hence $\llcorner p(\Theta)\lrcorner$ is connected iff $\llcorner\Theta\lrcorner$ is connected.

Now we distinguish several cases.

(i) $K_Z+p(\Theta)-\epsilon\llcorner p(\Theta)\lrcorner$ is numerically trivial over R. This can only happen if the fibers of $Z\to R$ are one dimensional and $\llcorner p(\Theta)\lrcorner$ is the union of some fibers, thus $\llcorner p(\Theta)\lrcorner$ is connected near any fiber. Otherwise there is a $(K_Z+p(\Theta)-\epsilon\llcorner p(\Theta)\lrcorner)$-extremal contraction $u: Z/R\to V/R$. Here there are two subcases:

(ii) u contracts Z to a point. Then $\rho(Z)=1$, hence any two curves in Z intersect. Thus $\llcorner p(\Theta)\lrcorner$ is connected.

(iii) u contracts Z to a curve and the generic fiber is $\mathbb{P}^1$. Therefore $\llcorner p(\Theta)\lrcorner$ intersects the generic fiber in at most two points. For any $v\in V$, the fiber $u^{-1}(v)\subset Z$ is an irreducible curve. Thus if $\llcorner p(\Theta)\lrcorner$ is not connected in the neighborhood of a fiber of $Z\to S$ then $\llcorner p(\Theta)\lrcorner$ is the union of two sections of u near that fiber. Thus $\llcorner\Theta\lrcorner$ also has two connected components.

In order to prove (2), consider the morphism $u\circ p: S\to V$. In a neighborhood of $(u\circ p)^{-1}(v)$, $\llcorner\Theta\lrcorner$ consists of two sections and possibly some other curves $C=\cup C_i\subset (u\circ p)^{-1}(v)$ which are p-exceptional. If C is not empty then $(u\circ p)^{-1}(v)-C$ is contractible, and the resulting contraction contradicts (17.4). Thus C is empty and (2) holds. □

As a straightforward corollary we obtain:

12.3.3 Theorem. *Let (S,Δ) be a proper, connected slc surface such that $K+\Delta\equiv 0$. Let (S_i,Θ_i) be the irreducible components of the normalization. Then one of the following conditions is satisfied:*

(1) *$\llcorner\Theta_i\lrcorner$ is connected for every i and for every irreducible curve $C\subset\llcorner\Theta_i\lrcorner$ the different $(C,\mathrm{Diff}(\Theta_i-C))$ is **not** klt.*

(2) *For every i and for every irreducible curve $C \subset \llcorner\Theta_i\lrcorner$ the different $(C, \mathrm{Diff}(\Theta_i - C))$ is klt.* $\square$

The combinatorial description of the intersections of the irreducible components of S is very subtle in case (1). (See [Friedman–Morrison83] for an overview of the special case of semistable degenerations of surfaces.) In the second case the combinatorics is easy but we need further information about the relationship between the two components of $\llcorner\Theta_i\lrcorner$.

12.3.4 Theorem. *Let (S, Θ) be an lc surface. Let $f : S \to B$ be a proper morphism onto a curve, with connected fibers. Assume that $K + \Theta$ is numerically f-trivial and $\llcorner\Theta\lrcorner \supset C_1 \cup C_2$ where the C_i are sections of f. Let $f_i = f|C_i$. Then*

(1) $(f_1)_* \mathrm{Diff}_{C_1}(\Theta - C_1) = (f_2)_* \mathrm{Diff}_{C_2}(\Theta - C_2)$*; let us call this $\mathbb{Q}$-divisor P.*

(2) *For some $m > 0$ we have an isomorphism $\psi : f^*\mathcal{O}_B(mK + mP) \cong \mathcal{O}_S(mK + m\Theta)$.*

(3) *Let ψ_i denote the composite isomorphism*

$$\begin{aligned}\psi_i : \mathcal{O}_B(mK + mP) &\cong f_*(f^*\mathcal{O}_B(mK + mP)) \\ &\overset{\psi}{\cong} f_*\mathcal{O}_S(mK + m\Theta) \\ &\cong f_*(\mathcal{O}_S(mK + m\Theta)|C_i) \\ &\cong (f_i)_*\mathcal{O}_{C_i}(mK + m\,\mathrm{Diff}(\Theta - C_i)).\end{aligned}$$

Then

$$\psi_2 \circ \psi_1^{-1} : (f_1^{-1} \circ f_2)^*\mathcal{O}_{C_1}(mK + m\,\mathrm{Diff}(\Theta - C_1)) \to \mathcal{O}_{C_2}(mK + m\,\mathrm{Diff}(\Theta - C_2))$$

and the natural isomorphism

$$(f_1^{-1} \circ f_2)_* : (f_1^{-1} \circ f_2)^*\mathcal{O}_{C_1}(mK + m\,\mathrm{Diff}(\Theta - C_1)) \to \mathcal{O}_{C_2}(mK + m\,\mathrm{Diff}(\Theta - C_2))$$

differ by the sheaf multiplication (-1).

Proof. Let $h : (S', \Theta') \to (S, \Theta)$ be a proper morphism such that $K + \Theta' \equiv h^*(K + \Theta)$. Then the theorem holds for (S, Θ) iff it holds for (S', Θ'). Thus as in (11.2.4) we may reduce to the case when S is smooth, and then by contracting (-1)-curves in the fibers we may assume that $f : S \to B$ is a $\mathbb{P}^1$-bundle. Thus Θ consists of two sections and some fibers (with coefficients), which clearly implies (1). (2) and (3) are not affected by the vertical components of Θ, thus we may even assume that $\Theta = C_1 \cup C_2$. By further elementary transformations we may also assume that C_1 and C_2 are disjoint. It is now clear that

$$\psi : \mathcal{O}_S(K + C_1 + C_2) \cong f^*\mathcal{O}_B(K).$$

In order to see (3) we may restrict our attention to a local chart on B. Thus S is of the form $\mathbb{P}^1 \times B$. Let $(s : t)$ be coordinates on $\mathbb{P}^1$ and let $C_1 = (s = 0)$ and $C_2 = (t = 0)$. Let z be a parameter on B and let $g(z)dz$ be a 1-form. Under the isomorphism ψ we obtain

$$\psi^*(g(z)dz) = \lambda\frac{ds}{s} \wedge g(z)dz,$$

where λ is an unknown constant. Thus ψ_1 is given by

$$\psi_1(g(z)dz) = \lambda g(f_1^*(z))d(f_1^*(z)).$$

Changing from s to t we obtain

$$\psi^*(g(z)dz) = -\lambda\frac{dt}{t} \wedge g(z)dz,$$

hence

$$\psi_2(g(z)dz) = -\lambda g(f_2^*(z))d(f_2^*(z)).$$

This proves (3). $\square$

12.4 Abundance

In this section we present a proof of (12.1.1).

Let $f : T \to S$ be a minimal semiresolution. By (12.2.8.1) there is a boundary Δ_T on T such that (T, Δ_T) is log canonical and $K + \Delta_T = f^*(K + \Delta)$. Thus abundance for (S, Δ) is equivalent to abundance for (T, Δ_T). In several instances it will be convenient to consider only the case when our surface S is already semismooth.

12.4.1 Claim. *(12.1.1) is true if $\nu = 0$ and we are in case (1) of (12.3.3).*

Proof. We may assume S is semismooth. Choose m such that $m(K + \Theta_i)$ is a linearly trivial Cartier divisor for every i. We claim that $12m(K + \Delta) \sim 0$.

In order to see this we have to choose sections $\sigma_i \in \mathcal{O}_{S_i}(12m(K + \Theta_i))$ such that they patch together along the double curves. By assumption $\llcorner\Theta_i\lrcorner$ is connected and $K + \Theta_i$ is numerically trivial; thus

$$H^0(\llcorner\Theta_i\lrcorner, \mathcal{O}_{\llcorner\Theta_i\lrcorner}(12m(K + \mathrm{Diff}(\Theta_i - \llcorner\Theta_i\lrcorner))))$$

is one dimensional, and it contains a unique normalized section ρ_i. Choose σ_i such that it restricts to ρ_i. If $C \subset \llcorner\Theta_i\lrcorner$ is a proper subcurve then $\rho_i|C$ is the unique normalized section of $\mathcal{O}_C(12m(K + \mathrm{Diff}(\Theta_i - \llcorner\Theta_i\lrcorner))|C)$. Thus the σ_i automatically patch together to a global section $\sigma \in H^0(S, \mathcal{O}(12m(K_S + \Delta)))$. $\square$

12.4.2 Claim. *(12.1.1) is true in the following cases:*

(1) $\nu = 0$ *in case (2) of (12.3.3); and*
(2) $\nu = 1$ *provided* $\nu(S_i, \Theta_i) = 1$ *for every irreducible component* S_i *of* S^ν *and* $S_i \cap S_j$ *has no vertical components for* $i \neq j$.

Proof. Let $\mu : S^\mu \to S$ be the normalization and let $D_i \subset S_i$ be the inverse images of the double curves. By assumption D_i has one or two irreducible components. Moreover, except when D_i is irreducible, it makes sense to talk about horizontal and vertical components of Θ_i. If $\nu = 0$ then (12.3.1.3) provides a morphism onto a curve, in the second case the morphism is given by abundance for (S_i, Θ_i).

By suitable indexing of the components S_i $(1 \leq i \leq n)$ of S^μ we may assume the following conditions

$$\llcorner \Theta_i \lrcorner = D_i^- \cup D_i^+ \cup (\text{vertical parts}) \qquad (D_1^- \text{ or } D_n^+ \text{may be empty}); \text{ and}$$
$$D_i^+ \cong \mu(D_i^+) = \mu(D_{i+1}^-) \cong D_{i+1}^- \qquad \text{for } 1 \leq i \leq n-1.$$

We distinguish two cases according to the behaviour of μ on the curves D_1^- and D_n^+.

(chain) $D_1^- \to \mu(D_1^-)$ and $D_n^+ \to \mu(D_n^+)$ are isomorphisms and $\mu(D_1^-) \neq \mu(D_n^+)$. If $D_1^- \to \mu(D_1^-)$ or $D_n^+ \to \mu(D_n^+)$ is two-to-one, let τ_1 (resp. τ_n) denote the corresponding involution of D_1^- (resp. D_n^+). Otherwise let τ_1 and τ_n be the identity.

(cycle) $D_n^+ \cong \mu(D_n^+) = \mu(D_1^-) \cong D_1^-$.

The following obvious proposition describes $H^0(S, \mathcal{O}(mK + m\Delta))$ in terms of S^μ:

12.4.3 Proposition. *Suppose that* m *is sufficiently divisible. Set*

$$\begin{aligned} H(i) &= H^0(S_i, \mathcal{O}(mK + m\Theta_i)) \\ H(i^-) &= H^0(D_i^-, \mathcal{O}(mK + m\,\mathrm{Diff}(\Theta_i - D_i^-)) \\ H(i^+) &= H^0(D_i^+, \mathcal{O}(mK + m\,\mathrm{Diff}(\Theta_i - D_i^+)), \end{aligned} \tag{12.4.3.1}$$

and let

$$\begin{aligned} \psi_i^- &: H(i) \to H(i^-) \\ \psi_i^+ &: H(i) \to H(i^+) \\ \phi_i &: H(i^+) \to H((i+1)^-) \\ \phi_n &: H(n^+) \to H(0^-) \quad \textit{(for cycle only)} \end{aligned} \tag{12.4.3.2}$$

be the natural isomorphisms.

Then the sections of $H^0(S, \mathcal{O}(mK + m\Delta))$ *are exactly those sequences* $\{\eta_i \in H(i)\}$ *which satisfy the following assumptions:*

(chain) $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$, $\phi_1^-(\eta_1)$ *is* τ_1*-invariant and* $\phi_n^+(\eta_n)$ *is* τ_n*-invariant.*

(cycle) $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$ *and* $\psi_1^-(\eta_1) = \phi_n(\psi_n^+(\eta_n))$. □

The choice of η_1 and the compatibility conditions $\psi_{i+1}^-(\eta_{i+1}) = \phi_i(\psi_i^+(\eta_i))$ automatically determine the other η_i uniquely. Let η denote any set $\{\eta_i\}$ which satisfy these compatibility conditions.

We also need the following:

12.4.4 Lemma. *The image G of* $\mathrm{Aut}(D_1^-, \mathrm{Diff}(\Theta_1 - D_1^-))$ *in* $H(1^-)$ *is finite.*

Proof. This is clear unless $D_1^- \cong \mathbb{P}^1$. If this holds then $\mathrm{Diff}(\Theta_1 - D_1^-)$ is klt in case $\nu = 0$ and has degree > 2 in case $\nu = 1$. Thus $\mathrm{Supp}\,\mathrm{Diff}(\Theta_1 - D_1^-)$ consists of ≥ 3 points, hence $\mathrm{Aut}(D_1^-, \mathrm{Diff}(\Theta_1 - D_1^-))$ is itself finite. □

12.4.5 Corollary. *Notation as above. Let* $G = \{g_1, \dots, g_k\}$. *Then*

$$(g_1^*(\eta) \otimes g_2^*(\eta) \otimes \cdots \otimes g_k^*(\eta))^{\otimes 2}$$

descends to a section of

$$\mathcal{O}_S(2kmK + 2km\Delta).$$

Proof. Note first that by (12.3.4) all the pairs $(D_i^-, \mathrm{Diff}(\Theta_i - D_i^-))$ and $(D_i^+, \mathrm{Diff}(\Theta_i - D_i^+))$ are isomorphic, and thus all the corresponding groups are the same. Furthermore, any isomorphism obtained by a combination of the isomorphisms in (12.4.3.2) is, up to a sign, induced by an isomorphism of the underlying pairs. Therefore, the second set of compatibility conditions are satisfied for η up to an element of G and up to a sign.

Therefore, in the cycle case, there is an element $g \in G$ such that

$$\psi_1^-(\eta_1) = \pm g^*(\phi_n(\psi_n^+(\eta_n))),$$

and similarly for chains. By taking the product over all $g_i \in G$ and taking the square we get rid of the ambiguities. □

12.4.6 Claim. *(12.1.1) is true if* $\nu = 1$.

Proof. Let (S, Δ) be slc with $\nu = 1$. As we remarked earlier, it is sufficient to consider the case when S is semismooth, and hence D is smooth. Let $D = D_0 \cup D_1$, where D_0 is the union of those irreducible components D^j such that $\nu(D^j, K_S + \Delta) = 0$ and at least one of the irreducible components S_i containing D^j has $\nu = 1$.

Let $\pi : S' \to S$ be the morphism obtained by normalizing in a neighborhood of D_0. The connected components of (S', Θ') have either $\nu = 0$ or $\nu = 1$ and they satisfy the assumptions of (12.4.2.2). Thus abundance holds for (S', Θ'). We need to analyze the patching of sections along $\pi^{-1}(D_0)$.

12.4.7 Lemma. *Assume that (S, Δ) satisfies the assumptions of (12.4.2.2). Let $p : S \to B$ be the morphism given by a large multiple of $K + \Delta$. Let Δ' be the vertical part of Δ. Then $\llcorner \Delta' \lrcorner$ is the union of fibers of p. In particular for every irreducible $C \subset \llcorner \Delta' \lrcorner$ the restriction $(C, \mathrm{Diff}_C(\Delta - C))$ is either not klt or C is a smooth elliptic curve and $\mathrm{Diff}_C(\Delta - C) = 0$. Furthermore, there are sections*

$$\tau \in H^0(S, \mathcal{O}_S(2mK + 2m\Delta))$$

whose restriction to $\llcorner \Delta' \lrcorner$ is the unique normalized section of

$$\mathcal{O}_{\llcorner \Delta' \lrcorner}(2m(K + \Delta)|\llcorner \Delta' \lrcorner).$$

These sections have no common zeros.

Proof. The first claim follows from (12.3.2) applied to the normalization of S. Let $b_i \in B$ be the points corresponding to $\llcorner \Delta \lrcorner$. For some $m > 1$ we have

$$\mathcal{O}_S(mK + m\Delta) = p^*(\mathcal{O}_B(mK + m\sum[b_i] + mP))$$

for some $\mathbb{Q}$-divisor P. Since $K_B + \sum[b_i] + P$ is ample, for $m \gg 1$, it follows that there are sections of $\mathcal{O}_B(mK + m\sum[b_i] + mP)$ taking any preassigned value at the points b_i. Furthermore these sections will not have any common zeros. $\square$

12.4.7.1 Complement. It is easy to see that (12.4.7) also holds if (S, Δ) is a semi-smooth surface, B is an affine curve and $p : S \to B$ is a proper and flat morphism such that $K + \Delta$ is p-trivial and every double curve of S is horizontal.

Now we can finish the proof of (12.4.6). By (12.4.7) and (12.4.1) we can choose sections of $\mathcal{O}_{S'}(2mK + 2m\Theta')$ which induce the unique normalized section of

$$\mathcal{O}(2mK_{S'} + 2m\Theta'|\pi^{-1}(D_0)).$$

These sections will descend to S and they have no common zeros. $\square$

12.5 Hodge Theory

In this section we prove (12.1.2). The following lemma is useful in comparing the cohomology of S, with that of a partial resolution of S.

12.5.1 Lemma. *Consider the following commutative diagram of Abelian groups*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\alpha"'] & W' \arrow[r] \arrow[d, "\omega"'] & B' \arrow[r, "d'"] \arrow[d, "\beta"'] & C' \arrow[d, "\gamma"'] \\ A \arrow[r] & W \arrow[r] & B \arrow[r, "d"] & C. \end{tikzcd}$$

If the rows are exact, α and β are surjective, and

(12.5.1.1) $$d'(\ker\beta) = \operatorname{im} d' \cap \ker\gamma,$$

then ω is surjective. (The last condition holds for example if there are compatible splittings β' and γ' of the maps β and γ, or if γ is an isomorphism.)

Proof. An easy diagram chase, left to the reader. □

We first prove (12.1.2) assuming that S is semismooth.

12.5.2 Lemma. *If S is semismooth then the natural map*

$$i_p : H^p(S,\mathbb{C}) \longrightarrow H^p(S,\mathcal{O}_S) \qquad \text{is surjective for every } p.$$

Proof. Let $g : S^\mu \longrightarrow S$ be the normalization of S; S^μ is smooth. We compare the cohomology of S and S^μ. There are two relevant exact sequences:

(12.5.3) $$0 \longrightarrow \mathbb{C}_S \longrightarrow g_*\mathbb{C}_{S^\mu} \longrightarrow \mathcal{G} \longrightarrow 0.$$

(12.5.4) $$0 \longrightarrow \mathcal{O}_S \longrightarrow g_*\mathcal{O}_{S^\mu} \longrightarrow \mathcal{F} \longrightarrow 0$$

We identify the sheaves $\mathcal{F}$ and $\mathcal{G}$, which are defined at the moment as cokernels in (12.5.3–4).

D_μ is smooth and maps two-to-one to $D = D_S$. Let τ be the natural involution on D_μ. The involution τ acts naturally on the sheaves $g_*(\mathcal{O}_{D_\mu})$ and $g_*(\mathbb{C}_{D_\mu})$. Under this action, these sheaves decompose into invariant and anti-invariant parts; the sheaves $\mathcal{F}$ and $\mathcal{G}$ are then the anti-invariant parts. Let P be the union of all the pinch points and let $L^2 \cong \mathcal{O}(P)$ the line bundle defining the double cover. It is an easy computation to check that $\mathcal{F} = L^{-1}$.

Now we compare the two long exact sequences of (12.5.3) and (12.5.4):
(12.5.5)

$$\begin{tikzcd}
H^{p-1}(D,\mathcal{G}) \arrow[r] \arrow[d, "k_{p-1}"'] & H^p(S,\mathbb{C}) \arrow[r] \arrow[d, "i_p"'] & H^p(S^\mu,\mathbb{C}) \arrow[r, "c_p"] \arrow[d, "j_p"'] & H^p(D,\mathcal{G}) \arrow[d, "k_p"'] \\
H^{p-1}(D,\mathcal{F}) \arrow[r] & H^p(S,\mathcal{O}_S) \arrow[r] & H^p(S^\mu,\mathcal{O}_{S^\mu}) \arrow[r, "d_p"] & H^p(D,\mathcal{F}).
\end{tikzcd}$$

Here the diagram commutes, and the horizontal sequences are exact. As previously observed, since S^μ is smooth the map j_p is surjective.

Now we have to find compatible splittings of the maps j_p and k_p; these are given by Hodge theory. In fact the cohomology groups $H^p(D_\mu,\mathbb{C})$ decompose into invariant and anti-invariant subspaces under the action of τ and $H^p(D,\mathcal{G})$ is just the anti-invariant part. As such $H^p(D,\mathcal{G})$ inherits a filtration from the natural Hodge filtration on $H^p(D_\mu,\mathbb{C})$. Now consider the commutative square

(12.5.6)
$$\begin{tikzcd}
H^p(S^\mu,\mathbb{C}) \arrow[r, "e_p"] \arrow[d, "j_p"'] & H^p(D_\mu,\mathbb{C}) \arrow[d] \\
H^p(S^\mu,\mathcal{O}_{S^\mu}) \arrow[r, "f_p"] & H^p(D_\mu,\mathcal{O}_{D_\mu}).
\end{tikzcd}$$

Clearly the maps e_p and f_p preserve the Hodge filtrations. But the horizontal maps c_p and d_p of (12.5.5) factor through the horizontal maps e_p and f_p of (12.5.6). Thus there is a natural splitting of the map k_p, compatible with the splitting of j_p. Now apply (12.5.1) to deduce i_p is surjective. □

We are now in a position to prove (12.1.2).

Proof. Let $f : T \longrightarrow S$ be a semiresolution of S. By (12.5.2), the natural maps

$$j_p : H^p(T,\mathbb{C}) \longrightarrow H^p(T,\mathcal{O}_T)$$

are surjective.

We wish to compare the cohomology of T and S. There are two relevant spectral sequences; the Leray–Serre spectral sequences associated to the map f and the sheaves $\mathbb{C}_T$, $\mathcal{O}_T$. The respective E_2 terms of the two spectral sequences are $H^p(S,R^qf_*\mathbb{C}_T)$ and $H^p(S,R^qf_*\mathcal{O}_T)$. Both spectral sequences degenerate at the E_3 level, and converge to $H^*(T,\mathbb{C})$ and $H^*(T,\mathcal{O}_T)$ respectively.

Let F be the exceptional locus of the map f. As F is one dimensional, the only interesting cohomology groups to identify at the E_2 level are

$$H^0(S,R^1f_*\mathbb{C}_T) = H^1(F,\mathbb{C}_F) \quad \text{and} \quad H^0(S,R^1f_*\mathcal{O}_T) = H^1(F,\mathcal{O}_F).$$