Eigenvectors and eigenvalues, Part II

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Eigenvalues and eigenvectors

Let V be a finite dimensional $\mathbb{F}$-vector space and $T\colon V \to V$ a linear transformation.

Our goal: answering 2 important questions

1. Do eigenvalues and eigenvectors exist for every choice of T?
2. How do we compute eigenvalues and eigenvectors of T?

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2. How do we compute eigenvalues and eigenvectors of T?

Definition 2.1

We say that $\lambda \in \mathbb{F}$ is an eigenvalue of T if there exists $w \in V$, $w \neq \underline{0}$ such that

$$Tw = \lambda w.$$

If λ is an eigenvalue of T, then any $w \in V$, $w \neq \underline{0}$ such that $Tw = \lambda w$ is called an eigenvector of T corresponding to the eigenvalue λ.

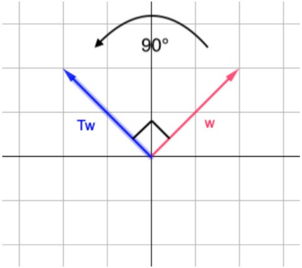
Existence of eigenvalues and eigenvectors

Question 2.1

Do eigenvalues/eigenvectors exist for any $T\colon V \to V$? **No!!**

Consider $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation of 90° around $\underline{0} \in \mathbb{R}^2$

$$T(y_1, y_2) := \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$



For any $w \in \mathbb{R}^2$, $w \neq \underline{0}$,
Tw and w **are never parallel** and
w **is not an eigenvector** of T.

T does not admit any eigenvector in $\mathbb{R}^2$ nor any eigenvalue in $\mathbb{R}$!

Computing eigenvalues

Fix $\lambda \in \mathbb{F}$. Then λ is an eigenvalue of T if and only if, by definition,

$$\exists w \in V, w \neq \underline{0} \text{ such that } Tw = \lambda w$$

Let's work on the equation $Tw = \lambda w$:

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Let's work on the equation $Tw = \lambda w$:

$$\begin{aligned}
& Tw = \lambda w \\
\iff \quad & Tw - \lambda w = 0 \\
\iff \quad & Tw - \lambda I_V w = 0 \\
\iff \quad & (T - \lambda I_V) w = 0 \\
\iff \quad & w \in \ker(T - \lambda I_V).
\end{aligned}$$

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$[w = I_V w]$

$[(A+B)w = Aw + Bw]$

DEF'N of Ker

Conclusion: λ is an eigenvalue of T if and only if

$$\exists w \in V, w \neq \underline{0} \text{ such that } w \in \ker(T - \lambda I_V)$$

Computing eigenvalues, II

We can summarize the previous discussion with the following statement.

Theorem 2.1

$\lambda \in \mathbb{F}$ is an eingenvalue of $T\colon V \to V$ if and only $\det(T - \lambda I_V) = 0$. *Eigenvectors corresponding to λ are all $w \in V, w \neq \underline{0}$ such that*

$$w \in \ker(T - \lambda I_v).$$

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Characteristic equation. We can define a function $p_T: \mathbb{F} \to \mathbb{F}$ as

$$p_T(x) := \det(T - xI_V).$$

Part 3: we will show that $p_T(x)$ is a polynomial in x of degree $\dim V$.

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Theorem 2.1 (2nd version)

$\lambda \in \mathbb{F}$ is an eingenvalue of $T: V \to V$ if and only if λ is a solution of the characteristic equation

$$p_T(x) = 0.$$

Example

Take $A = \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$ and define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ as

$$T(y_1, y_2) := \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4y_1 + 2y_2 \\ 3y_2 \end{pmatrix}.$$

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The characteristic function of T is

$$p_T(x) = \det(T - xI_{\mathbb{R}^2}) = \det(A - xI_2) = \begin{vmatrix} 4-x & 2 \\ 0 & 3-x \end{vmatrix}$$

and the characteristic equation is

$$p_T(x) = (4-x)(3-x) - 2 \cdot 0 = 0$$

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The roots of the degree 2 polynomial $p_T(x) = (4-x)(3-x)$ are 4 and 3: these are the eigenvalues of T.

To compute the eigenvectors for the eigenvalue 4 of T, we need to compute

$$\ker(T - 4I_{\mathbb{R}^2}) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 4-4 & 2 \\ 0 & 3-4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

To compute the eigenvectors for the eigenvalue 4 of T, we need to compute

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On the other hand, $\operatorname{rank}\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} = 1$ and $\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
Hence, by the Rank-Nullity Thm,

$$\begin{aligned} \dim\ker(T - 4I_{\mathbb{R}^2}) &= \dim \mathbb{R}^2 - \operatorname{rank}(T - 4I_{\mathbb{R}^2}) = 2 - 1 = 1, \text{ and} \\ \ker(T - 4I_{\mathbb{R}^2}) &= \{(\mu, 0) \in \mathbb{R}^2 \mid \mu \in \mathbb{R}\} \end{aligned}$$

In the same way for the eigenvalue 3, we compute

$$\ker(T - 3I_{\mathbb{R}^2}) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 4-3 & 2 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

by resolving the associated homogenous linear system

$$\begin{cases} 1 \cdot y_1 + 2y_2 = 0 \\ \cancel{0 \cdot y_1 + 0 \cdot y_2 = 0} \end{cases} \longrightarrow \quad y_1 = -2y_2$$

Hence, $\ker(T - 3I_{\mathbb{R}^2}) = \{(-2\mu, \mu) \in \mathbb{R}^2 \mid \mu \in \mathbb{R}\}$.

Existence of eigenvalues and eigenvectors, II

Questions

Do eigenvalues always exist for $T\colon V \to V$? **No!!**

Let us consider $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of 90° around the origin

$$T(y_1, y_2) := \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

The characteristic function of T is

$$p_T(x) = \begin{vmatrix} -x & -1 \\ 1 & -x \end{vmatrix} = x^2 + 1$$

and the polynomial $x^2 + 1$ **has no roots in** $\mathbb{R}$. Hence, again:
T does not have eigenvalues in $\mathbb{R}$.