Hyperbolicity and log pairs

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Algebraic Geometry Seminar
University of Warwick
February 24 2016

Overview

Hyperbolicity

We will work over the field of complex numbers.

Definition (Kobayashi, Brody)

Let X be a complex manifold. We say that X is hyperbolic if any holomorphic map $f\colon \mathbb{C} \to X$ is constant.

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Hyperbolicity, II

The original idea of Kobayashi was to construct a pseudo-metric on the tangent bundle of a complex manifold X.
For $u \in T_xX$

$$d_K(u) = \frac{1}{\lambda}, \lambda := \sup_{\phi}\{|\mu|,\ \phi_*(\frac{\partial}{\partial t}) = \mu u\}$$

where ϕ ranges in the set of maps $\phi\colon \Delta \to X,\ \phi(0) = x$.
Hence, a manifold X is hyperbolic iff the pseudo-metric d_K is an actual metric on the tangent bundle. This is a consequence of Brody's reparametrization lemma.

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Examples

Hyperbolicity is a very strong property for a complex manifold.
In fact, on a complex manifold X, hyperbolicity forbids the presence of rational curves and complex tori, for example.

- $\dim X = 1 \begin{cases} g(X) = 0, \ X = \mathbb{P}^1 \supset \mathbb{C} \\ g(X) = 1, \ X = \mathbb{C}/\Lambda \\ g(X) > 2, \ \widetilde{X} = \Delta \subset \mathbb{C} \end{cases}$

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Examples, II

- $\dim X = 2$: We have a pretty good understanding of the distribution of rational and elliptic curves on surfaces. In particular, for general type surfaces, we actually have a nice criterion to prove their finiteness, namely,

$$c_1^2(X) > c_2(X) \qquad \text{[Bogomolov, M}^c\text{Quillan]}$$

- $\dim X > 2$: Not very much is known. Lang's Conjecture predicts that Ω^1_X should be "positive". But already in dimension 3 we do not know whether or not Calabi-Yau manifolds (i.e. $K_X \sim 0$) are hyperbolic.

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Three classical conjectures

In the category of complex smooth projective varieties, hyperbolicity is expected to have very strong consequences.

Conjecture (Green-Griffiths)

Let X be a projective manifold of general type. Then, there exists a subvariety $Y \subset X$ s.t. for any holomorphic map $f \colon \mathbb{C} \to X$, $f(\mathbb{C}) \subset Y$.

A projective manifold X is of general type, if

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Conjecture (Lang)

Let X be a projective manifold. Then, X is hyperbolic if and only if all subvarieties of X are general type.

Conjecture (Kobayashi)

Let X be a projective manifold. If X is hyperbolic then K_X is ample.

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Hyperbolicity and positivity

Moral of the conjectures

Hyperbolicity should correspond to (or at least imply) positivity properties of X $(\Omega^1_X,\ K_X)$.

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The Cone Theorem

Cone Theorem (Mori, Kawamata, Kollár, Shokurov, Ambro, Fujino)

Let X be a smooth projective variety.

Let us consider the cone $\overline{\mathrm{NE}}(X) \subset H_2(X, \mathbb{R})$ which is the closure of the cone generated by classes of holomorphic curves $C \subset X$.

There exist (countably many) K_X-negative rational curves R_i on X such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{>0}[R_i].$$

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One can immediately recover the following corollary.

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Let X be a smooth projective variety. Assume that X contains no rational curves.
Then K_X is nef, i.e. $K_X \cdot C \geq 0$ for any irreducible curve $C \subset X$.

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A quest for weaker conditions

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What can we way in the case of quasi-projective varieties?

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What can we way in the case of quasi-projective varieties?

Log smooth pairs and quasi projective varieties

A pair (X, D) is said to be log smooth if X is a smooth proper variety and $D = \sum_i D_i$ is a snc divisor on X.

Strategy

We will use log smooth pairs to give some answers to the previous questions.

If U is a smooth quasi-projective variety, by Hironaka's desingularization theorem, we can always compactify U to a smooth projective variety X and the pair $(X, X \setminus U)$ is log smooth.

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Iitaka's philosophy

There should be a "correspondence" between

$$\left\{\begin{array}{l}\text{Statements involving}\\ \text{a projective variety}\\ \text{and positivity properties}\\ \text{of } \Omega^1_X,\ K_X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Statement involving}\\ \text{a q.-proj. var. } U = X/D\\ \text{and positivity properties of}\\ \Omega^1_X(\log D),\ K_X + D\end{array}\right\}.$$

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The Cone Theorem, log-version

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Let (X, D) be a log smooth projective pair.

There exist (countably many) $(K_X + D)$-negative rational curves R_i on X such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+D\geq 0} + \sum_i \mathbb{R}_{>0}[R_i].$$

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Log smooth pairs and quasi projective varieties, II

Given a log smooth pair (X, D), we can induce a natural stratification of the variety X.

The closed strata will be X itself, the components of D, D_i, $i = 1, \ldots, k$, and then the irreducible components of the intersections of the D_i.

The open strata will be the closed strata minus the intersections with all other strata not containing them.That is, $X \setminus D$, $D_i \setminus (\cup_{j \neq i} D_j)$ and more in general

$$\cap_{i \in I} D_i \setminus (\cup_{j \in \{1,\ldots,k\} \setminus I} D_j) \text{ for } I \subset \{1, \ldots, k\}$$

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A first theorem

Theorem (Lu-Zhang, -)

Let (X, D) be a projective log smooth pair.
If $K_X + D$ is not nef, there exists a non-constant morphism $f\colon \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the strata of (X, D).
That is to say,

$$f(\mathbb{A}^1) \subset (\cap_{i\in I} D_i \setminus (\cup_{j\in\{1,\dots,k\}\setminus I} D_j)) \text{ for } I \subset \{1, \dots, k\} \text{ and } f(\mathbb{A}^1) \neq pt.$$

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Definition (L-Z)

A pair (X, D) is said to be Mori hyperbolic if there exists no non-constant morphism $f \colon \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the strata of (X, D).

Corollary

Let (X, D) be a projective log smooth Mori hyperbolic pair.
Then $K_X + D$ is nef.

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Cone Theorem and hyperbolicity

Theorem (-)

Let (X, D) be a projective log smooth pair. There exist countalby many $(K_X + D)$-negative curves C_i s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0}[C_i].$$

Moreover, one of the two following conditions holds:

- $C_i \cap (X \setminus D)$ *contains the image of a non-constant map* $f\colon \mathbb{A}^1 \to X$;
- *there exists an open stratum* W *of* D *such that* $C_i \cap W$ *contains the image of a non-constant map* $f\colon \mathbb{A}^1 \to W$.

Some known examples

Mori hyperbolic pairs are not so rare in algebraic geometry:

- Hyperbolic varieties;
- Toric varieties;
- Ramified coverings of surfaces along ample curves of genus > 2 [Zaidenberg, Rousseau, Liu];
- Non-singular compactifications of neat quotients of the n-dimensional hyperbolic space $\mathcal{H}_n$ [Ash-Mumford-Rapoport-Tai, Siu-Yau, Mok].

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Ingredients for a proof

- Adjunction: we can restrict to the strata of (X, D) and work inductively. Hence we can assume that $K_X + D$ is nef when restricted to the components of D.
- Cone Theorem: If $K_X + D$ is not nef along a stratum W, by the Cone Thm there exists a K_X-negative rational curve R. We will use R to obtain a morphism $f : \mathbb{A}^1 \to X$ intersecting D in at most one point. We can actually assume even more, i.e., that there exists a morphism

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Ingredients for a proof, II

- KV-vanishing: Fibers for π are either 0- or 1-dimensional. When they are 1-dimensional, Kawamata-Viehweg vanishing implies that the generic one is a smoothly embedded copy of $\mathbb{P}^1$.
- For a rational curve R in the (positive-dimensional) fibers of π, it is enough to show now that the intersection with D is supported in just one point.

Connectedness Theorem (Shokurov, Kollár et al.)

Let $\pi\colon X \to Y$ a morphism. Assume that $-(K_X + D)$ is ample along every fiber of π.
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Log-hyperbolicity and ampleness

Theorem (Kleiman, Nakai, Moishezon)

Let X be a proper variety and D a Cartier divisor on X.
D is ample iff $D^{\dim V} \cdot V > 0$, $\forall V \subset X$ irreducible subvariety.

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Let (X, D) be a log-smooth pair.
Assume that the pair is Mori hyperbolic (X, D).
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Log canonical pairs

Let (X, Δ) be a pair of a normal variety X and en effective $\mathbb{R}$-divisor $\Delta = \sum d_i \Delta_i,\ d_i \in (0, 1]$ s.t. $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Given a log-resolution $\psi\colon Z \to X$, we can write

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Let (X, Δ) be a pair of a normal variety X and en effective $\mathbb{R}$-divisor $\Delta = \sum d_i \Delta_i,\ d_i \in (0, 1]$ s.t. $K_X + \Delta$ is $\mathbb{R}$-Cartier.
Given a log-resolution $\psi \colon Z \to X$, we can write

$$K_Z + \widetilde{\Delta} = \psi^*(K_X + \Delta) + \sum a_i E_i$$

Definition

The pair (X, Δ) is a log canonical pair if $a_i \geq -1,\ \forall i$.

Non-klt locus

In the previous equation, the components E_i of coefficient -1 are very important.

Their image via ψ on X is called the non-klt locus of the pair (X, Δ), $\mathrm{Nklt}(X, \Delta)$.

$\mathrm{Nklt}(X, \Delta)$ comes equipped with a stratification analogous to the one defined in the log smooth case and in fact induced by the one on $(Z, \lfloor \widetilde{\Delta} - \sum a_i E_i \rfloor)$.

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The main theorems

Theorem (-)

Let (X, Δ) be a projective log canonical pair.
If $K_X + \Delta$ is not nef, there exists a non-constant morphism $f\colon \mathbb{A}^1 \to X$ and $f(\mathbb{A}^1)$ is contained in one of the strata of the non-klt locus of (X, Δ).

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The main theorems, II

Theorem (-)

Let (X, Δ) be a projective log canonical pair. Then there exist countably many $(K_X + \Delta)$-negative rational curves C_i such that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0}[C_i].$$

Moreover, one of the two following conditions holds:

- $C_i \cap (X \setminus \mathrm{Nklt}(\Delta))$ *contains the image of a non-constant map* $f\colon \mathbb{A}^1 \to X$;
- *there exists an open stratum W of $\mathrm{Nklt}(\Delta)$ such that $C_i \cap W$ contains the image of a non-constant map $f\colon \mathbb{A}^1 \to W$.*

Differences with the log smooth case

- Adjunction: log canonical pairs are much more general and singular than log smooth ones.

  Hence, one needs a better theory of adjunction in order to apply the previous strategy of proof.

Theorem (-)

Let (X, Δ) be log canonical pair. Let W be a lc center and let W^ν be the normalization.
On W^ν there exists a divisor Δ_{W^ν} such that (W^ν, Δ_{W^ν}) is a log pair,

$$K_{W^\nu} + \Delta_{W^\nu} \sim (K_X + \Delta)|_{W^\nu}$$

and $\mathrm{Nklt}(W^n u, \Delta_{W^\nu}) = \mathrm{Nklt}(X, \Delta)|_{W^\nu}$

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The ample case

Theorem (-)

Let (X, Δ) be a dlt pair.
Assume that there is no non-constant morphism $f\colon \mathbb{A}^1 \to X$ such that $f(\mathbb{A}^1)$ is contained in one of the open strata of (X, Δ).
The following are equivalent:

- *$K_X + \Delta$ is ample*
- *$(K_X + \Delta)^{\dim W} \cdot W > 0,\ \forall W \subset X$ stratum of (X, D)*
- *$K_X + \Delta$ is big and its restriction along $\lfloor \Delta \rfloor$ is ample.*

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Thanks for your attention!!