Problems in the classification of algebraic varieties

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Geometries

Geometry [Klein, Grothendieck]

- A class $\mathfrak{C}$ of **objects** to classify, and
- **Maps/Transformations** between objects of $\mathfrak{C}$.
- Invariant quantities [to be identified]

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Example: Triangles

$\mathfrak{C} = \{$ triangles $\subset \mathbb{R}^2\}$.

1. rigid motions $\longrightarrow$ lengths of sides
2. rigid motions + homotheties $\longrightarrow$ internal angles

In both cases: triangles are classified by (at most) 3 parameters!!

Algebraic geometry

Study of algebraic varieties + polynomial functions & maps.

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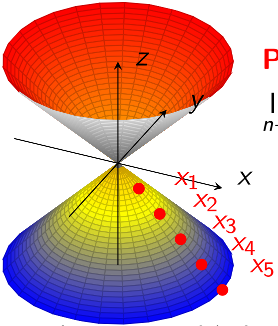
Affine algebraic varieties

$$\left\{ (x_1, \dots, x_n) \in \mathbb{C}^n \;\middle|\; \forall 1 \le j \le r \;:\; f_j(x_1, \dots, x_n) = 0 \right\}$$

where $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$ polynomials with coefficients in $\mathbb{C}$.

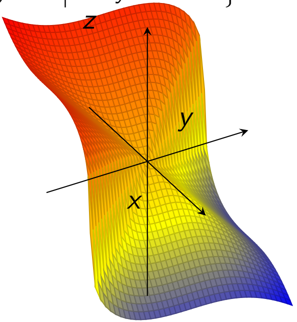


Problem:

$$\lim_{n\to\infty} x_n \notin Q$$

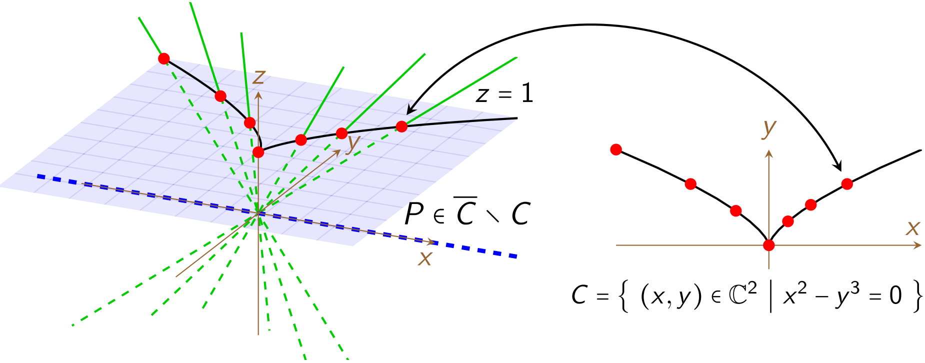
$Q = \left\{ (x, y, z) \in \mathbb{C}^3 \;\middle|\; x^2 + y^2 - z^2 = 0 \right\}$

Fermat surface:
$\left\{ (x, y, z) \in \mathbb{C}^3 \;\middle|\; x^3 + y^3 + z^3 = 0 \right\}$



Compact solutions: projective varieties

Closures of affine varieties in $\mathbb{P}^n$(=space of lines in $\mathbb{C}^{n+1}$ through $\underline{0}$).



$C = \{ \, (x, y) \in \mathbb{C}^2 \mid x^2 - y^3 = 0 \, \}$

$\overline{C} = \{ \, [x, y, z] \in \mathbb{P}^2 \mid x^2 z - y^3 = 0 \, \}$

A naive classification scheme

Definitive goal

Arrive at a classification of projective varieties.

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Projective varieties
(modulo suitable equivalence rel'n)

↓

? algorithmic operations:
get "best" representative ?

↓

decomposition using
building blocks

↓

construct moduli spaces:
best parametrizations (of building blocks) by a projective variety

Hope: parametrized by finitely/countably many parameters

Example of a moduli space:

$\mathbb{P}^n$ = moduli space of lines in $\mathbb{C}^{n+1}$ through the origin

A little history

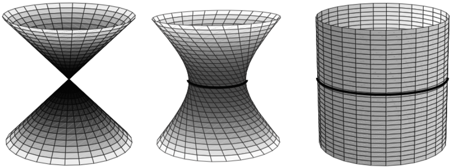
1. the Italian School, ca. 1890-1940:
   Castelnuovo, Enriques, Fano, Severi, and many others;
2. the early Japanese School, ca. 1950-1970:
   Kodaira (Fields Medal 1954), Fujita, Ueno, and many others;
3. MMP's reinassance, ca. 1980-1990:
   Mori (Fields Medal 1990), Kollàr, Shokurov, Kawamata, Miyaoka, and many others;
4. MMP's coming of age, ca. 2000-present:
   Hacon–M^{c}Kernan (Breakthrough Prize 2018), Birkar (Fields Medal 2018), Cascini, Xu, and many others.

Equivalence of varieties

Equivalence relations on varieties

1. X, Y are **isomorphic** if there are morphisms $f : X \to Y$ and $g : Y \to X$ that are inverse of each other.

2. X and Y are **birationally equivalent** if $\exists$ subvarieties $Z \subsetneq X$, $W \subsetneq Y$ s.t. $X \backslash Z$ is isomorphic to $Y \backslash W$.



Fundamental questions/goals

Definitive goal

Classify projective varieties up to birational equivalence.

1. Can we find "best representatives" in each equivalence class?
2. Can we reduce every algebraic variety to a finite number of building blocks?

$$\text{building blocks} : \text{alg. varieties} = \text{simple groups} : \text{finite groups}$$

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Example: Conics in $\mathbb{C}^2$

Conics: solutions to degree 2 polynomials

$$f(x,y) = a_0x^2 + a_1y^2 + a_2xy + a_3x + a_4y + a_5, \;\; a_i \in \mathbb{C}.$$

Finitely many parameters: $(a_0 : a_1 : \cdots : a_5) \in \mathbb{P}^5_{\mathbb{C}}$.

Plane Curves

Question

How do we distinguish the structure of different algebraic varieties?

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Example: plane curves

Consider $X = \{f = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$, for

$$f(X_0, X_1, X_2) = \sum a_{ijk} X_0^i X_1^j X_2^k, \quad i + j + k = d.$$

X is a projective curve (1 dimensional object).
If X is smooth $\implies$ X compact **Riemann surface**.

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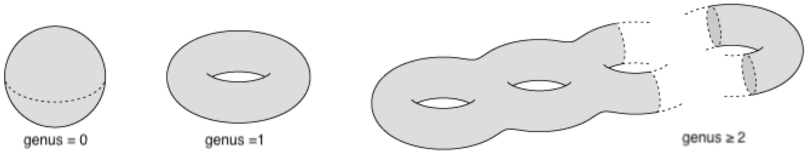
Question

How does the structure of X vary with d?

Projective curves



deg(f)	g(X)	Variety	Universal cover	Curvature
1,2	0	$\mathbb{P}^1$	S^2	> 0
3	1	Elliptic curves	$\mathbb{R}^2$	$= 0$
> 3	≥ 2	Hyperbolic curves	$\lvert x \rvert \leq 1 \subset \mathbb{R}^2$	< 0

The protagonist: positivity

Let X be smooth/mildly singular projective variety.

Fundamental Question # 1

How to find a “best birational model” in the birational equivalence class of X?

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Fundamental principle of birational geometry

The (birational) geometry of X is governed by the curvature of T_X.
Main character: the **canonical divisor**

$K_X = \det(\Omega^1_X) = Ric(g) = -c_1(X)$ (topology/diff geometry meets AG)

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MMP's Goal: maximize the positivity of K_X!
Nefness: we would like that $K_X \cdot C \geq 0$, $\forall C \subset X$ compact curve.
If X is contains lots of rational curves, K_X cannot possibly be nef, and neither can any of its birational models.

The main protagonist: *positivity*

Question

How to find a "best birational model" for X?

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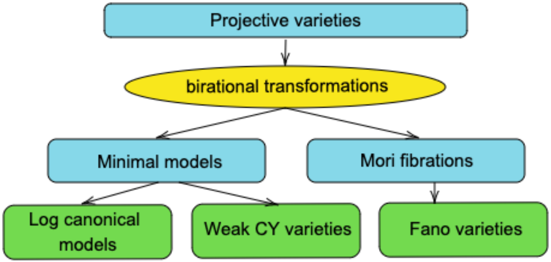
Conjecture (Existence of minimal models)

$\exists$ *an algorithm that for any smooth projective variety constructs:*

- *either a birational model where K_X is nef (**minimal model**),*
- *or a birational model fibred with fibers that have < 0 canonical divisor (**Mori fibration**).*

The Minimal Model Program



Curvature of KE metric	< 0	= 0	> 0
Rational points	Few [Lang conjecture: {rat'l points} $\subseteq Z \subsetneq X$]	?	Many [Manin conj: |{rat'l pts of height $< B$}| ~ $cB(\log B)^{b_2-1}$]
Fundamental group	Anything	Virtually abelian	Finite

Moduli spaces

Projective varieties come in families.

Moduli spaces

For a class $\mathfrak{D}$ of projective varieties, a moduli space $\mathfrak{M}_{\mathfrak{D}}$ is an algebraic variety such that

$$\{\text{Points of } \mathfrak{M}_{\mathfrak{D}}\} \xleftrightarrow{1:1} \{\text{Isomorphism classes of varieties in } \mathfrak{D}\}$$

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How to construct a compact moduli space: 3-step-recipe

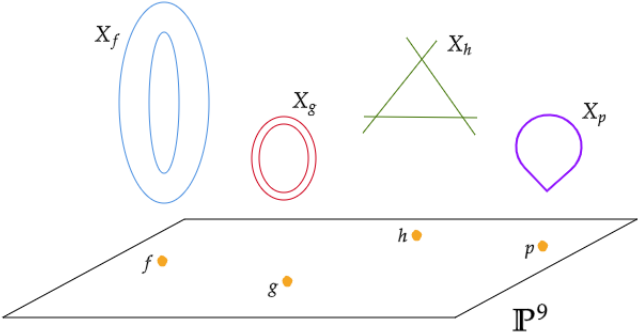
1. Need to check that we are not trying to parametrize too many varieties! Key word: Boundedness
2. Need to choose what kind of degenerations will be admitted for varieties in $\mathfrak{D}$. Key word: Functor
3. Need to choose a way to construct the moduli space.
   Key word: Quotient
   Many available techniques: GIT, VGIT, KSBA, BB, ...

Families of cubics

Cubics: $X_f = (f = 0) \subset \mathbb{P}^2$

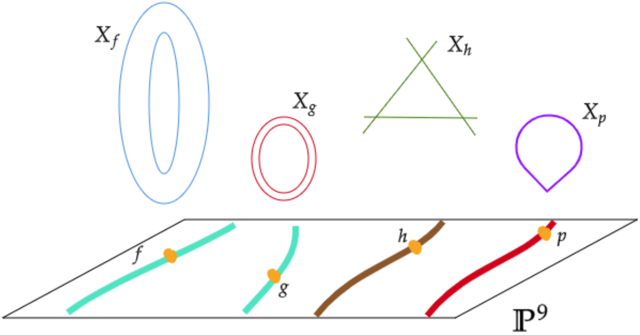
$$f(X_0, X_1, X_2) = a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_0X_1^2 + a_4X_0X_2^2 + a_5X_1X_0^2 + \\ + a_6X_1X_2^2 + a_7X_2X_0^2 + a_8X_2X_1^2 + a_9X_0X_1X_2.$$

Parameters for cubics: $(a_0 : a_1 : \cdots : a_9) \in \mathbb{P}^9_{\mathbb{C}}$.



Cubics: $X_f = (f = 0) \subset \mathbb{P}^2$

Parameters for f: $(a_0 : a_1 : \cdots : a_9) \in \mathbb{P}^9_{\mathbb{C}}$. Too many points still!



Moduli for curves.

Elliptic curves: j-invariant $\longrightarrow$ $\mathbb{P}^1_j$ = moduli space of ell. curves

$\overline{\mathfrak{M}}_g$ = compact moduli space of curves of genus g [Deligne-Mumford]

Boundedness

Let $\mathfrak{D}$ be a set of projective varieties of fixed dimension d.

Boundedness

$\mathfrak{D}$ is bounded if it can be described by a finite number of families:

$$
\begin{tikzcd}
\exists \text{ a projective family} & \mathcal{X} \arrow[d] & \text{s.t.} & \forall x_i \in \mathfrak{D},\ \exists t_i \in T \\
\text{finitely many comp's} \arrow[r] & T & & \text{such that } X_{t_i} \simeq X_i.
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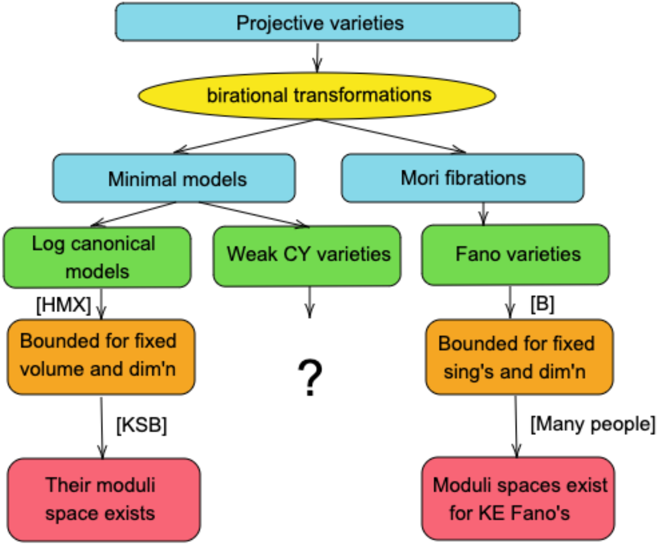
How to show boundedness?

$\mathfrak{D}$ is bounded $\iff$ we can embed all varieties of $\mathfrak{D}$ in some (fixed) $\mathbb{P}^n_{\mathbb{C}}$ with degree $\leq C$.

Very difficult

$\forall X \in \mathfrak{D}$ we need to construct a very ample divisor H_X with $H_X^d \leq C$.

Minimal Model Program, II



Weak Calabi-Yau varieties

Weak Calabi–Yau variety: smooth/mildly singular X with $K_X \equiv 0$.

Theorem (Beauville-Bogomolov)

Up to an étale cover, any smooth weak CY X decomposes as

$$X = \textit{Abelian var.} \times \textit{strict CY var.} \times \textit{Holomoprhic symplectic var.}$$

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Weak CY varieties and boundedness

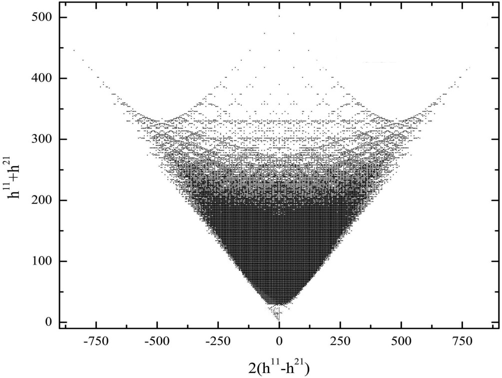
- $\dim X = 1$: Bounded, $\mathbb{P}^1_j$ = j-line.
- $\dim X = 2$: X is K3 or Abelian: finitely many topological types but ∞ many parameters.
- $\dim X \geq 3$: ? [$\Rightarrow$ only consider strict CY varieties]

Strict Calabi–Yau variety: smooth/mildly singular X, simply connected, with $K_X \sim 0$, and

$$H^i(X, \mathcal{O}_X) = \begin{cases} \mathbb{C} & i = 0, \dim X \\ 0 & \text{else} \end{cases}$$

Strict Calabi-Yau varieties

Strict Calabi–Yau varieties: their geometric/topological structure is already very mysterious in dimension 3.



Interest and importance (beyond Algebraic geometry)

- Theoretical Physics: geometric models for strings/QFTs.
- Symplectic geometry: mirror symmetry.

Elliptic Calabi-Yau varieties and boundedness

Theorem (Gross 94)

Elliptic Calabi–Yau threefolds $f : X \to Y$ are bounded up to birational equivalence, assuming that f is non-isotrivial.

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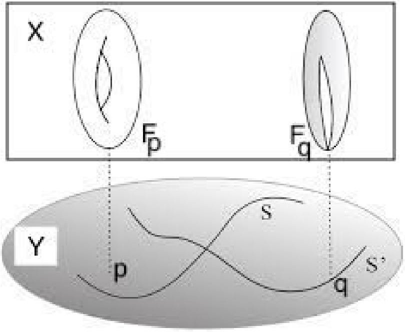

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Elliptic CYs varieties & String Theory

Elliptic fibration $\leftrightarrow$ **gauge action**.
Elliptic CYs are the geometric models of strings.

Expectation inspired by MS/ST

If X is a smooth strict CY and $b_2(X) \gg 1$, then

$$X \overset{\text{birational}}{\underset{\text{flops}}{\dashrightarrow}} X' \xrightarrow[\text{fibration}]{\text{elliptic}} Y'.$$

New theorems

New results: [Di Cerbo-S, 19], [Birkar-DC-S, 20], [Filipazzi-Hacon-S, 21]

1. In dimension $n = 3$, elliptic CY varieties are really bounded: there exist finitely many **algebraic** families of elliptic Calabi–Yau threefolds.
2. In fixed dimension $n \geq 4$, there exist finitely many families of elliptic Calabi–Yau varieties when the fibration admits a section, up to birational equivalence.

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Future problems

- Moduli of elliptic CY: We have now good theoretical tools that guarantee the (theoretical) existence of moduli for elliptic CY 3-folds, but the hard part is to actually carry out such construction: Moduli of elliptic CY = Moduli of polarized bases + control of f.
- Kawamata-Morrison Conjecture: this conjecture predicts that there are finitely many (unmarked) images of rational contractions with source a strict CY variety. It has now been proven in the relative case, but many of its consequences still need to be explored.

Foliations

Foliation $\mathcal{F}$ on a variety X

Bundle of algebraic differential equations and their solutions on X.

$$\mathcal{F} \subset T_X : \text{ rank } r \text{ subsheaf} + \text{integrability condition } [\mathcal{F}, \mathcal{F}] \subset \mathcal{F}.$$

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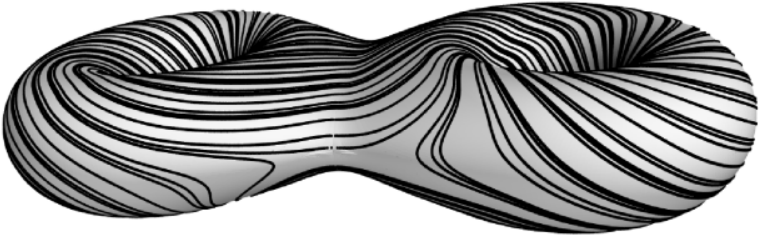

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Hard setup!

We need to manipulate at the same time the underlying variety X and the (analytic) leaves of $\mathcal{F}$.
The latter are not algebraic objects & they may be highly singular.

New results: Cascini, Spicer, Svaldi

The philosophy of the Minimal Model Program works also for foliations on projective varieties of dimension 3.

Foliated surfaces

In dimension 2, we have a pretty good understanding of the birational geometry of foliated surfaces $(X, \mathcal{F})$ [thanks to work of Brunella, Mendes, McQuillan].

MMP for foliated varieties: try to maximize the positivity of

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New idea! [Pereira, Svaldi 16], [Spicer, Svaldi, 21]

Instead of considering the positivity of $K_{\mathcal{F}}$, consider the positivity of perturbations

$$K_{\mathcal{F}} + \epsilon K_X, \quad \text{for } 0 < \epsilon \ll 1$$

Boundedness for foliated surfaces [Spicer, Svaldi 23]

Fix $\epsilon \in (0, \frac{1}{5}), v \in \mathbb{R}_{>0}$. There exists finitely many families of foliated surfaces $(X, \mathcal{F})$ with

$$K_{\mathrm{F}} + \epsilon K_X \text{ ample}, \quad \text{and} \quad \mathrm{vol}(X, K_{\mathcal{F}} + \epsilon K_X) \leq v.$$

Future projects

- Given that boundedness is now known for the canonical models of perturbed divisors of the form $K_{\mathcal{F}}$, it is natural to try and form moduli spaces with respect to these polarizations and study their stability properties when $\epsilon \to 0$ to try and form a moduli space of foliated surfaces of general type (without having to perturb by means of K_X).

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- Foliations on higher dimensional varieties (and rank/corank $\neq 1$) are still rather mysterious from the viewpoint of their birational structure. The existence of moduli spaces for canonical models implies the possibility to construct alterations of foliated varieties with good singularities. Via alterations, one can start to prove the existence of the MMP for higher dimensional (and higher rank) foliated varieties.

Thank you for your attention!