Some Presentations of Planar Algebras with Applications to Invariant Theory

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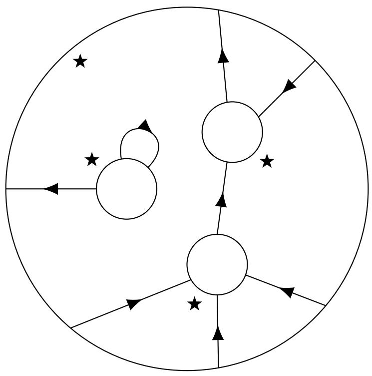
1 Introduction

2 Background

An **oriented planar tangle (OPT)** is a template for combining elements of an **oriented planar algebra (OPA)**. Input discs and an output disc are connected by oriented strands specifying the type of the inputs and outputs, and a boundary interval is marked ($\star$) on each disc to align inputs rotationally. These tangles are defined up to planar isotopy of the strands.



The space of OPTs form an operad, where composition is performed by inserting some tangle into an input disc of another tangle with each $\star$ aligned. This composition is defined when the number of strands and their orientations match on the input and output discs.

(picture)

An OPA is a collection of vector spaces on which the operad acts. We group elements of an OPA by type into vector spaces B_σ called **box spaces**, indexed by elements $\sigma \in \{+,-\}^n$. The index carries the information of the number of strands n, and the orientations of each strand at the disc boundary, ordered counterclockwise from the $\star$ (+ for incoming and $-$ for outgoing). We consider oriented Temperley-Lieb as a first example.

2.1 Oriented Temperley-Lieb, OTL(δ)

Consider the planar algebra of all oriented tangles with no input discs, and where the value of the oriented circle is δ. The box space $B_{(+,-,+,-)}$ is generated as a vector space by the following 2 tangles.

(picture of tangles)

2.2 Oriented Symmetric Planar Algebras (OSPA)

In a symmetric planar algebra we want to allow strand crossings. We make the following alteration to the definition of planar isotopy.

Definition 1. *A **symmetric isotopy** is a planar isotopy allowing the introduction and removal of transverse strand crossings, Reidemeister moves, and naturality (we can pull function nodes across strands).*

- picture example of symmetric isotopy
- example OTL(δ) with symmetric structure is OSTL(δ)

2.3 Oriented Symmetric Planar Algebras Generated by a Set of Vertices

Definition 2. *Let S be our generating set of vertices, where each vertex has the data*

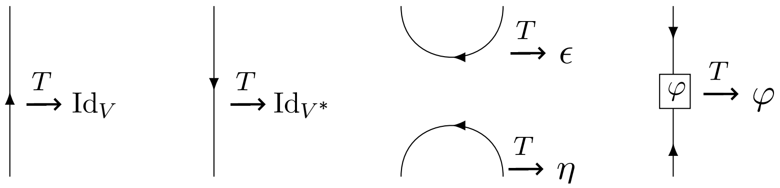
- generic example
- free OSPA on dots and brackets
- dual of vertex
- symmetric and antisymmetric self dual
- rotational symmetry/eigenvectors

2.4 Disoriented Temperley Lieb, DTL(δ)

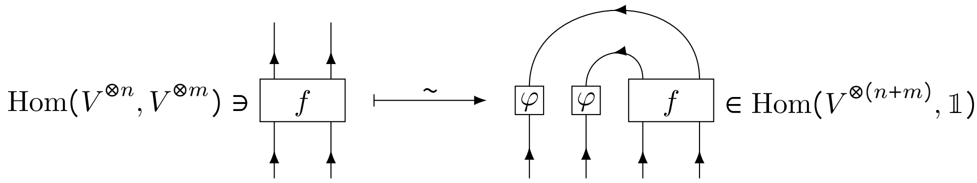
2.5 Rep(G,V,k) as an OSPA

Fix a group G, a field k, and some object $V \in \mathrm{Rep}_k(G)$, the category of k-linear representations of G. We study $\mathrm{Alg}(G,V,k)$, the full subcategory of $\mathrm{Rep}_k(G)$ whose objects are finite tensor products of V and V^*. Our goal will be to give a diagrammatic presentation of $\mathrm{Alg}(G,V,k)$, i.e. to give a map of planar algebras $T : \mathrm{Diag}(G,V,k) \to \mathrm{Alg}(G,V,k)$ for some diagrammatic planar algebra. In all examples considered it will be true that $V \simeq V^*$, so we can restrict to studying the spaces $\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$. Further, since $\mathrm{Hom}(A, B \otimes C) \simeq \mathrm{Hom}(A \otimes B^*, C)$ for A, B, C finite dimensional representations of a group, we can restrict to studying the spaces $B_n := \mathrm{Hom}(V^{\otimes n}, \mathbb{1})$.

To construct $\mathrm{Diag}(G,V,k)$ we start with an OSPA generated by some vertex set, and take a quotient to include relations. In defining T we will map an upwards oriented strand to the identity map on V and a downward oriented strand to the identity map on V^*. We have evaluation and coevaluation maps $V^* \otimes V \xrightarrow{\epsilon} k$, $k \xrightarrow{\eta} V \otimes V^*$ which will be the image of arcs connecting appropriately oriented strands. We fix an isomorphism $\varphi : V \xrightarrow{\sim} V^*$ and make this the image under T of the vertex φ between an upward and downward oriented strand. (clean these pictures to T(X)=Y form)



Using φ along with evaluation we can turn outgoing strands into incoming ones, illustrating the isomorphism $\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \mathrm{Hom}(V^{\otimes(n+m)}, \mathbb{1})$, and allowing us to focus on pictures where all strands are 'attached to the ground'. (define ground prior to this)



Our goal is to determine a minimal generating set of pictures and relations so that the resulting diagrammatic planar algebra, call it $\mathrm{Diag}(G, V, k)$, will be isomorphic to $P_{G,V,k}$. In each example we use the following procedure:

1. Define generating pictures and relations for $\mathrm{Diag}(G, V, k)$, and give a map of planar algebras $\mathrm{Diag}(G, V, k) \xrightarrow{T} P_{G,V,k}$.

2. For each n find a subset $D_n \subset \mathrm{Diag}(G, V, k)$ such that $T(D_n)$ is a basis for B_n. Exhibiting such D_n shows that T is surjective, since as argued above $M_{G,V,k}$ is determined by the spaces B_n.

3. Show that an arbitrary picture in $\mathrm{Diag}(G, V, k)$ can be rewritten linearly in terms of elements of the D_n using the presented relations. This shows that T is injective, so that $\mathrm{Diag}(G, V, k) \simeq P_{G,V,k}$ as planar algebras.

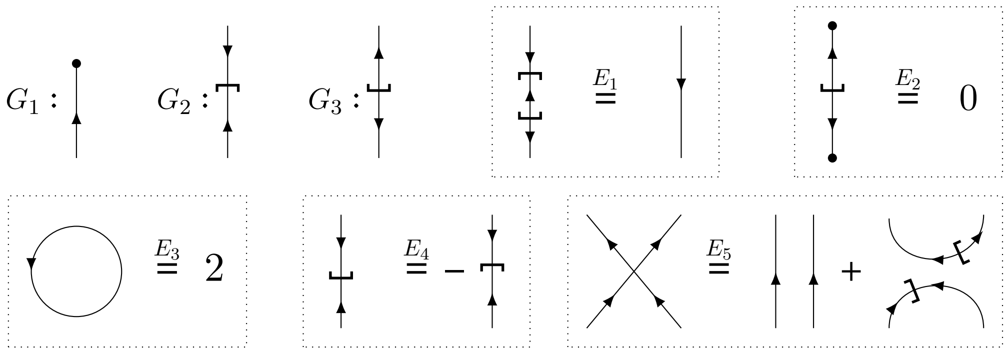
If we fix coordinates on V, when describing the spaces B_n we are also describing a subring of vector invariants for G, i.e. each f in B_n is also in $(V^{\oplus n})^G = \{f \in k[x_1, \ldots, x_n] : \forall g \in G, f(\bar{x}) = f(g \cdot \bar{x})\}$. This follows from observing that the defining property for an element of $(V^{\oplus n})^G$ is the same as the defining property for a map of G representations from $V^{\otimes n}$ to the trivial representation. Giving a presentation of $\mathrm{Alg}(G, V, k)$ is then related to giving a first and second fundamental theorem of invariant theory for $(V^{\oplus n})^G$, and in each example we discuss this relationship.

3 A 2-dimensional $\mathbb{Z}$-representation over $\mathbb{C}$

Let $V_n = (\mathbb{C}^n, \phi_n)$ where $\phi_n : \mathbb{Z} \to GL(\mathbb{C}^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\mathrm{Alg}(\mathbb{Z}, V, \mathbb{C})$. Taking the standard basis of $\mathbb{C}^2$, $v_0 = (1,0)$ and $v_1 = (0,1)$, we use the isomorphism $\varphi : V \xrightarrow{\sim} V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \mathrm{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

3.1 Presentation of $\mathsf{Diag}(\mathbb{Z}, V, \mathbb{C})$ and map into $\mathsf{Alg}(\mathbb{Z}, V, \mathbb{C})$

Definition 3. *We define Diag$(\mathbb{Z}, V, \mathbb{C})$ to be the OSPA quotient G/E for the generating set of vertices $G = \{G_1, G_2, G_3\}$ and relations $E = \{E_1, \dots, E_5\}$ below.*



Proposition 1. *There is a map of planar algebras T : Diag$(\mathbb{Z}, V, \mathbb{C}) \to$ Alg$(\mathbb{Z}, V, \mathbb{C})$ determined by the values $T(G_1) = v_1^*, T(G_2) = \varphi$ and $T(G_3) = \varphi^{-1}$.*

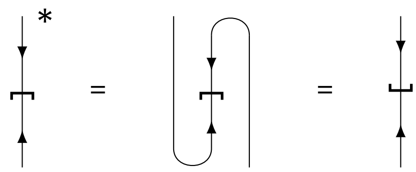
Proof. We need to show each of E_1 through E_5 hold in the image of T so that this map is well defined.

E_1: $\varphi \cdot \varphi^{-1} = \mathrm{Id}_{V^*}$

E_2: $1 \mapsto v_1^* \mapsto v_0 \mapsto 0$

E_3: $\epsilon \cdot \tau \cdot \eta(1) = \epsilon \cdot \tau(v_1 \otimes v_1^* + v_0 \otimes v_0^*) = \epsilon(v_1^* \otimes v_1 + v_0^* \otimes v_0) = 2$

E_4: This follows from taking the dual (rotation) of G_2 and a coordinate computation:



$$
\begin{aligned}
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(\tau \cdot \eta \otimes \mathrm{Id}_V)(v_0) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(v_0^* \otimes v_0 \otimes v_0 + v_1^* \otimes v_1 \otimes v_0) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_0 - v_1^* \otimes v_0^* \otimes v_0) &&= -v_1^*
\end{aligned}
$$

$$
\begin{aligned}
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(\tau \cdot \eta \otimes \mathrm{Id}_V)(v_1) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(v_0^* \otimes v_0 \otimes v_1 + v_1^* \otimes v_1 \otimes v_1) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_1 - v_1^* \otimes v_0^* \otimes v_1) &&= v_0^*
\end{aligned}
$$

E_5: The second term on the RHS takes the values $v_0 \otimes v_0 \mapsto 0, v_1 \otimes v_1 \mapsto 0, v_0 \otimes v_1 \mapsto v_1 \otimes v_0 - v_0 \otimes v_1, v_1 \otimes v_0 \mapsto v_0 \otimes v_1 - v_1 \otimes v_0$, and adding the identity map gives us the LHS.

□

3.2 Exhibiting bases D_n for each B_n

The indecomposable representations that appear in ⊗-powers of V are exhausted by the sequence V_n. When $i \geq 2$ we have the rule

$$V \otimes V_i \simeq V_{i+1} \oplus V_{i-1} \tag{1}$$

so that the fusion graph Γ for V is

$$V_1 \leftrightarrow V_2 \leftrightarrow V_3 \leftrightarrow V_4 \leftrightarrow V_5 \leftrightarrow V_6 \leftrightarrow V_7 \leftrightarrow \cdots$$

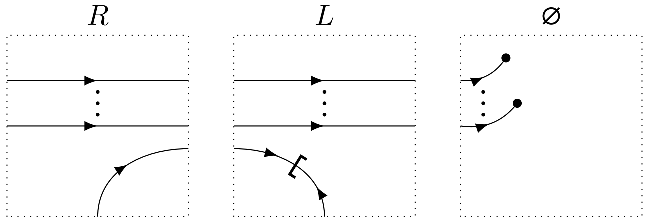
Definition 4. *Let P_n be the set of paths of length n on Γ based at V_1. Label edges directed from V_i to V_{i+1} with R, and edges directed from V_i to V_{i-1} by L. Then we can describe P_n as the set of words w of length n in the alphabet $\{R, L\}$ where no initial segment of w has more Ls than Rs.*

Proposition 2. $\#(P_n) = dim(B_n)$

Proof. We have $\dim(B_n) = \dim\mathrm{Hom}(V^{\otimes n}, \mathbb{1}) = \dim\mathrm{Hom}\big(\sum \alpha_i V_i, \mathbb{1}\big) = \sum \alpha_i \dim\mathrm{Hom}(V_i, \mathbb{1})$. Since $\dim\mathrm{Hom}(V_j, \mathbb{1}) = 1$ for any indecomposable V_j, we get $\dim(B_n) = \sum \alpha_i$, which is the number of summands of $V^{\otimes n}$. We can see summands of $V^{\otimes n}$ are in bijection with P_n by induction on n. Assume we have a direct sum decomposition of $V^{\otimes n}$ and a bijection between summands of $V^{\otimes n}$ and P_n. By definition of Γ the summands of $V^{\otimes(n+1)}$ will be the indecomposables that are adjacent to the summands of $V^{\otimes n}$, so append the adjacency edge to the path from the bijection at level n. □

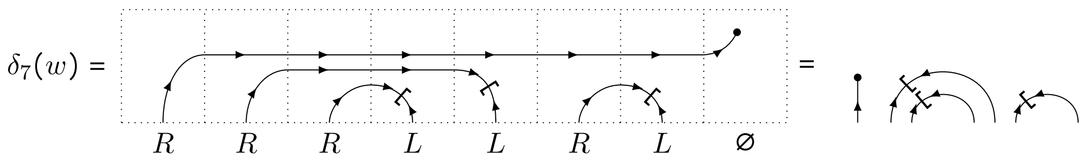
We will then construct a set of maps in bijection with P_n, and show these maps are independent, so that this set forms a basis for B_n. We first construct a map from the set of paths to the diagrammatic category.

Definition 5. *We define a map $\delta_n : P_n \to Diag(\mathbb{Z}, V, \mathbb{C})$. Identify concatenation in a path word w with composition of diagrams, and identify each letter of the alphabet with a portion of a picture as below, where $\varnothing$ signifies the end of the word.*



The number of horiztonal strands in each picture varies, and is equal to the excess of Rs to Ls in the segment prior to the current letter of w. To define $\delta_n(w)$ replace each letter of w with its identified picture, and then glue each portion end to end, including the picture for $\varnothing$ at the end (glue dots onto any remaining strands at the end).

For example, take $w = RRRLLRL$:



The images of δ_n form our sequence D_n. Composing δ_n with T we get $T_n : P_n \to B_n$. We want to show the values of T_n are linearly independent, and will use the following lemma.

Lemma 1. *Let $(S, <)$ be a finite ordered set, and V a vector space. If there are maps $f : S \to V$ and $g : S \to V^*$ such that for all $x, y \in S$:*

1. $g(x)(f(x)) \neq 0$
2. $x < y \implies g(x)(f(y)) = 0$

Then both the values of f and the values of g are linearly independent.

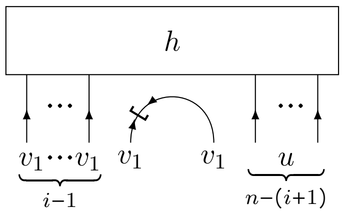
Proof. For simplicity replace S with the ordered set $(1, 2, \ldots, n)$. Consider the $n \times n$ matrix A defined by $A_{i,j} = g(i)(f(j))$. Condition (1) implies entries on the main diagonal of A are non-zero. Condition (2) implies A is upper triangular, so together (1) and (2) imply A is invertible. Any linear dependence among the rows of A implies a linear dependence among the values of g, and a dependence among columns of A implies a dependence among the values of f, so we have our result. ☐

Proposition 3. *The values of T_n are linearly independent.*

Proof. As in Lemma 1, let S be P_n with lexicographic order ($R > L$), and let g be T_n. To define f assign to each $x \in P_n$ a vector $f(x) \in V^{\otimes n}$ by identifying R with v_1, L with v_0, and concatenation with $\otimes$ (e.g. $f(RRLRLLLRL) = v_1 \otimes v_1 \otimes v_0 \otimes v_1 \otimes v_0 \otimes v_0 \otimes v_1 \otimes v_0$). By construction of the pairing we have $g(x)(f(x)) = 1$, so condition 1 of Lemma 1 holds. To show condition 2 holds, note that if $x < y$ then

$$f(y) = v_1^{\otimes i} \otimes v_1 \otimes u, \qquad u \in V^{\otimes(n-(i+1))}$$
$$f(x) = v_1^{\otimes i} \otimes v_0 \otimes w, \qquad w \in V^{\otimes(n-(i+1))}$$

so that $g(x)(f(y))$ will have the form



which vanishes since looking at the value of the map on positions i and $i+1$, $(\epsilon)(-\varphi \otimes \mathrm{Id}_V)(v_1 \otimes v_1) = \epsilon(v_0^* \otimes v_1) = 0$. □

We then have a basis for each B_n described as images of a set of diagrams $D_n \subset \mathrm{Diag}(\mathbb{Z}, V, \mathbb{C})$ under the map T.

3.3 Showing D_n spans $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$.

Let $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ be the box space of n positively oriented strands. We need to give a description of the pictures in D_n, and show that if we take an arbitrary picture in $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ we can reduce it to the span of D_n.

Definition 6. *We consider $U \in Diag_n(\mathbb{Z}, V, \mathbb{C})$ to be a member of D_n if:*

1. *There are no crossings in U.*
2. *For every dot, there is a path from that dot to the sky which does not intersect U.*
3. *Any path in U has at most 1 vertex.*
4. *Any bracket is directed towards the leftmost endpoint of a path.*

Proposition 4. *D_n spans $Diag_n(\mathbb{Z}, V, \mathbb{C})$.*

Proof. We perform the following algorithm to reduce an arbitrary picture to D_n:

1. Pull all dots to the sky via symmetric isotopy.
2. Using relation E_5 resolve all crossings.

3. Use E_1, E_2, and E_4 to reduce the number of vertices on any path to at most 1 and to direct any bracket towards the left path endpoint.

4. Replace any circles with a constant using E_3.

After steps 1 and 2 have been performed, there will be no crossings and all dots will touch the sky. Steps 3 and 4 will not change these properties as they cannot intoduce any new dots or crossings. After step 3. At this point we must be left with a diagram that meets items 1 through 3 of Definition 4 and possibly has some oriented circles which don't meet the boundary. These circles can have no vertices since any closed path must have an even number of brackets and no dots, and we ensured in step 3 that there is at most 1 vertex along any path. Further we can have no floating intervals since any path with 2 dots has value 0 using relation E_2. Step 4 removes any circles at the cost of a constant, and our diagram has been reduced to D_n. □

We illustrate the algorithm and definition by an example.

(picture of dot/circle intersection removal)

The result of this section concludes the construction of a bijection $\mathrm{Diag}(\mathbb{Z}, V, \mathbb{C}) \to \mathrm{Alg}(\mathbb{Z}, V, \mathbb{C})$.

3.4 The Invariant space $(V^{\oplus n})^{\mathbb{Z}}$ and the Nowicki conjecture

4 A 2-dimensional $\mathbb{Z}_p$-representation over $\mathbb{F}_p$

Let $V_n = ({\mathbb{F}_p}^n, \phi_n)$ where $\phi_n : \mathbb{Z}_p \to GL({\mathbb{F}_p}^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\mathrm{Alg}\mathbb{Z}, V, \mathbb{C}$. Taking the standard basis of $\mathbb{C}^2$, $v_0 = (1, 0)$ and $v_1 = (0, 1)$, we use the isomorphism $\varphi : V \xrightarrow{\sim} V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \mathrm{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

4.1 Presentation of $\mathrm{Diag}(\mathbb{Z}, V, \mathbb{C})$ and map into $\mathrm{Alg}(\mathbb{Z}, V, \mathbb{C})$

We will include all the generators and the relations from (Ex 1) for all p in our presentation. We need one new generator and one new relation which are dependent on p.

Definition 7. *We present $D_{\mathbb{Z}_p, V, \mathbb{F}_p}$ by the same generators and relations of Definition 1, along with the new generator G_4 and relation E_5*

G_4 :