

Some Presentations of Planar Algebras with Applications to Invariant Theory

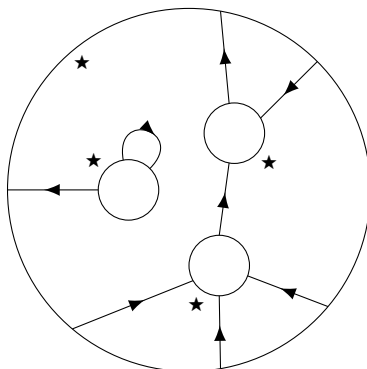
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1 Introduction

2 Background

An **oriented planar tangle (OPT)** is a template for combining elements of an **oriented planar algebra (OPA)**. Input discs and an output disc are connected by oriented strands specifying the type of the inputs and outputs, and a boundary interval is marked (\star) on each disc to align inputs rotationally. These tangles are defined up to planar isotopy of the strands.



The space of OPTs form an operad, where composition is performed by inserting some tangle into an input disc of another tangle with each \star aligned. This composition is defined when the number of strands and their orientations match on the input and output discs.

(picture)

An OPA is a collection of vector spaces on which the operad acts. We group elements of an OPA by type into vector spaces B_σ called **box spaces**, indexed by elements $\sigma \in \{+, -\}^n$. The index carries the information of the number of strands n , and the orientations of each strand at the disc boundary, ordered counterclockwise from the \star (+ for incoming and $-$ for outgoing). We consider oriented Temperley-Lieb as a first example.

2.1 Oriented Temperley-Lieb, $OTL(\delta)$

Consider the planar algebra of all oriented tangles with no input discs, and where the value of the oriented circle is δ . The box space $B_{(+,-,+,-)}$ is generated as a vector space by the following 2 tangles.

(picture of tangles)

2.2 Oriented Symmetric Planar Algebras (OSPA)

In a symmetric planar algebra we want to allow strand crossings. We make the following alteration to the definition of planar isotopy.

Definition 1. A *symmetric isotopy* is a planar isotopy allowing the introduction and removal of transverse strand crossings, Reidemeister moves, and naturality (we can pull function nodes across strands).

- picture example of symmetric isotopy
- example $OTL(\delta)$ with symmetric structure is $OSTL(\delta)$

2.3 Oriented Symmetric Planar Algebras Generated by a Set of Vertices

Definition 2. Let S be our generating set of vertices, where each vertex has some identifying symbol, and tuples of incoming and outgoing oriented strands called the degrees. The OSPA generated by S is the collection of locally oriented graphs on the vertex set S respecting the degrees of each vertex (i.e. the degrees of each vertex are fixed and part of the data of the vertex). We call a path along any graph a (strand) component.

- generic example
- free OSPA on dots and brackets
- dual of vertex
- symmetric and antisymmetric self dual
- rotational symmetry/eigenvectors

2.4 Disoriented Temperley Lieb, $DTL(\delta)$

2.5 Pivotal Symmetric Tensor Categories and OSPAs

Given a pivotal tensor category C and some object V in C , we can define an OPA $P(C, V)$. We think of the oriented strands of the planar algebra as being copies of V or V^* for positively or negatively oriented strands respectively. A disjoint union of n oriented strands is then a tensor product of copies of V and V^* with n total factors. Elements of the planar algebra are morphisms in C between tensor products involving

V and V^* , and the index of the box spaces indicate the domain and target of our maps (e.g. $(+, -, -, +) \mapsto V \otimes V^* \otimes V^* \otimes V$).

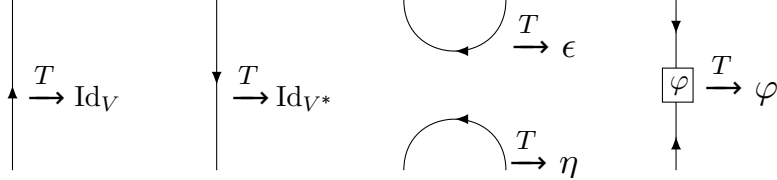
(pictures)

Additional structure of C can be reflected in the diagrams of the planar algebra. If C is symmetric we get an OSPA; a strand crossing is then interpreted as a component map of the natural isomorphism defining the symmetric structure. We will be in the case of a symmetric pivotal tensor category, specifically a category of representations of a group.

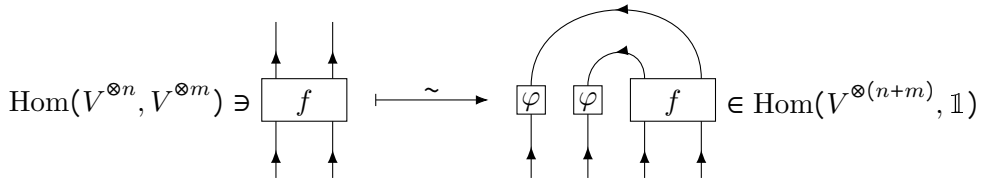
2.6 $\text{Rep}(G, V, k)$ as an OSPA

Fix a group G , a field k , and some object $V \in \text{Rep}_k(G)$, the category of k -linear representations of G . We study $\text{Alg}(G, V, k)$, the full subcategory of $\text{Rep}_k(G)$ whose objects are finite tensor products of V and V^* . Our goal will be to give a diagrammatic presentation of $\text{Alg}(G, V, k)$, i.e. to give a map of planar algebras $T : \text{Diag}(G, V, k) \rightarrow \text{Alg}(G, V, k)$ for some diagrammatic planar algebra. In all examples considered it will be true that $V \simeq V^*$, so we can restrict to studying the spaces $\text{Hom}(V^{\otimes n}, V^{\otimes m})$. Further, since $\text{Hom}(A, B \otimes C) \simeq \text{Hom}(A \otimes B^*, C)$ for A, B, C finite dimensional representations of a group, we can restrict to studying the spaces $B_n := \text{Hom}(V^{\otimes n}, \mathbb{1})$.

To construct $\text{Diag}(G, V, k)$ we start with an OSPA generated by some vertex set, and take a quotient to include relations. In defining T we will map an upwards oriented strand to the identity map on V and a downward oriented strand to the identity map on V^* . We have evaluation and coevaluation maps $V^* \otimes V \xrightarrow{\epsilon} k, k \xrightarrow{\eta} V \otimes V^*$ which will be the image of arcs connecting appropriately oriented strands. We fix an isomorphism $\varphi : V \xrightarrow{\sim} V^*$ and make this the image under T of the vertex φ between an upward and downward oriented strand. (clean these pictures to $T(X)=Y$ form)



Using φ along with evaluation we can turn outgoing strands into incoming ones, illustrating the isomorphism $\text{Hom}(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \text{Hom}(V^{\otimes(n+m)}, \mathbb{1})$, and allowing us to focus on pictures where all strands are ‘attached to the ground’. (define ground prior to this)



2.7 Outline of Technique for Finding a Diagrammatic Presentation of $\text{Rep}(G, V, k)$

Our goal is to determine a minimal generating set of pictures and relations so that the resulting diagrammatic planar algebra, call it $\text{Diag}(G, V, k)$, will be isomorphic to $P_{G, V, k}$. In each example we use the following procedure:

1. Define generating pictures and relations for $\text{Diag}(G, V, k)$, and give a map of planar algebras $\text{Diag}(G, V, k) \xrightarrow{T} P_{G, V, k}$.
2. For each n find a subset $D_n \subset \text{Diag}(G, V, k)$ such that $T(D_n)$ is a basis for B_n . Exhibiting such D_n shows that T is surjective, since as argued above $M_{G, V, k}$ is determined by the spaces B_n .
3. Show that an arbitrary picture in $\text{Diag}(G, V, k)$ can be rewritten linearly in terms of elements of the D_n using the presented relations. This shows that T is injective, so that $\text{Diag}(G, V, k) \simeq P_{G, V, k}$ as planar algebras.

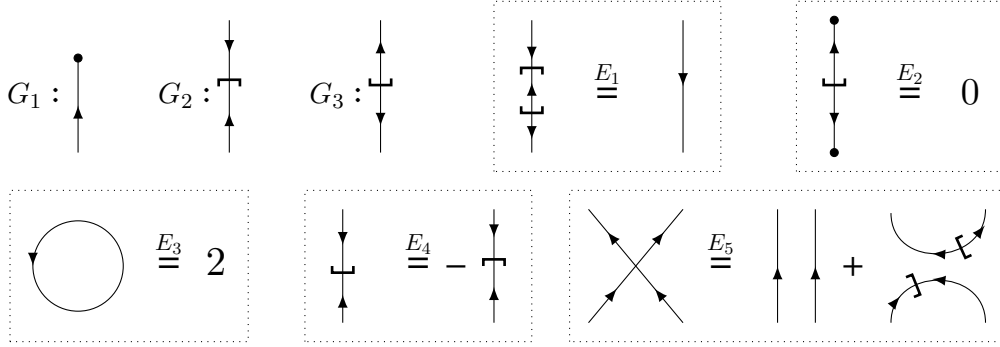
If we fix coordinates on V , when describing the spaces B_n we are also describing a subring of vector invariants for G , i.e. each f in B_n is also in $(V^{\oplus n})^G = \{f \in k[x_1, \dots, x_n] : \forall g \in G, f(\bar{x}) = f(g \cdot \bar{x})\}$. This follows from observing that the defining property for an element of $(V^{\oplus n})^G$ is the same as the defining property for a map of G representations from $V^{\otimes n}$ to the trivial representation. Giving a presentation of $\text{Alg}(G, V, k)$ is then related to giving a first and second fundamental theorem of invariant theory for $(V^{\oplus n})^G$, and in each example we discuss this relationship.

3 A 2-dimensional \mathbb{Z} -representation over \mathbb{C}

Let $V_n = (\mathbb{C}^n, \phi_n)$ where $\phi_n : \mathbb{Z} \rightarrow GL(\mathbb{C}^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\text{Alg}(\mathbb{Z}, V, \mathbb{C})$. Taking the standard basis of \mathbb{C}^2 , $v_0 = (1, 0)$ and $v_1 = (0, 1)$, we use the isomorphism $\varphi : V \rightarrow V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \text{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

3.1 Presentation of $\text{Diag}(\mathbb{Z}, V, \mathbb{C})$ and map into $\text{Alg}(\mathbb{Z}, V, \mathbb{C})$

Definition 3. We define $\text{Diag}(\mathbb{Z}, V, \mathbb{C})$ to be the OSPA quotient G/E for the generating set of vertices $G = \{G_1, G_2, G_3\}$ and relations $E = \{E_1, \dots, E_5\}$ below.



Proposition 1. *There is a map of planar algebras $T : \text{Diag}(\mathbb{Z}, V, \mathbb{C}) \rightarrow \text{Alg}(\mathbb{Z}, V, \mathbb{C})$ determined by the values $T(G_1) = v_1^*$, $T(G_2) = \varphi$ and $T(G_3) = \varphi^{-1}$.*

Proof. We need to show each of E_1 through E_5 hold in the image of T so that this map is well defined.

$$E_1: \varphi \cdot \varphi^{-1} = \text{Id}_{V^*}$$

$$E_2: 1 \mapsto v_1^* \mapsto v_0 \mapsto 0$$

$$E_3: \epsilon \cdot \tau \cdot \eta(1) = \epsilon \cdot \tau(v_1 \otimes v_1^* + v_0 \otimes v_0^*) = \epsilon(v_1^* \otimes v_1 + v_0^* \otimes v_0) = 2$$

E_4 : This follows from taking the dual (rotation) of G_2 and a coordinate computation:

$$\begin{array}{c} * \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \downarrow \\ \text{---} \end{array} = \begin{array}{c} \downarrow \\ \text{---} \\ \uparrow \end{array}$$

$$\begin{aligned} & (\text{Id}_{V^*} \otimes \epsilon)(\text{Id}_{V^*} \otimes \varphi \otimes \text{Id}_V)(\tau \cdot \eta \otimes \text{Id}_V)(v_0) & = \\ & (\text{Id}_{V^*} \otimes \epsilon)(\text{Id}_{V^*} \otimes \varphi \otimes \text{Id}_V)(v_0^* \otimes v_0 \otimes v_0 + v_1^* \otimes v_1 \otimes v_0) & = \\ & (\text{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_0 - v_1^* \otimes v_0^* \otimes v_0) & = -v_1^* \end{aligned}$$

$$\begin{aligned} & (\text{Id}_{V^*} \otimes \epsilon)(\text{Id}_{V^*} \otimes \varphi \otimes \text{Id}_V)(\tau \cdot \eta \otimes \text{Id}_V)(v_1) & = \\ & (\text{Id}_{V^*} \otimes \epsilon)(\text{Id}_{V^*} \otimes \varphi \otimes \text{Id}_V)(v_0^* \otimes v_0 \otimes v_1 + v_1^* \otimes v_1 \otimes v_1) & = \\ & (\text{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_1 - v_1^* \otimes v_0^* \otimes v_1) & = v_0^* \end{aligned}$$

E_5 : The second term on the RHS takes the values $v_0 \otimes v_0 \mapsto 0, v_1 \otimes v_1 \mapsto 0, v_0 \otimes v_1 \mapsto v_1 \otimes v_0 - v_0 \otimes v_1, v_1 \otimes v_0 \mapsto v_0 \otimes v_1 - v_1 \otimes v_0$, and adding the identity map gives us the LHS.

□

3.2 Exhibiting bases D_n for each B_n

The indecomposable representations that appear in \otimes -powers of V are exhausted by the sequence V_n . When $i \geq 2$ we have the rule

$$V \otimes V_i \simeq V_{i+1} \oplus V_{i-1} \quad (1)$$

so that the fusion graph Γ for V is



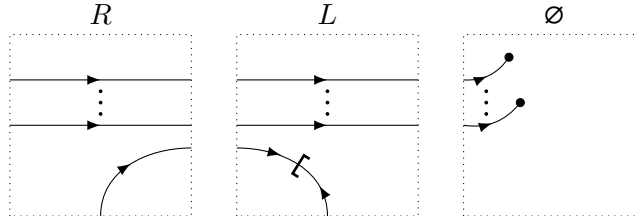
Definition 4. Let P_n be the set of paths of length n on Γ based at V_1 . Label edges directed from V_i to V_{i+1} with R , and edges directed from V_i to V_{i-1} by L . Then we can describe P_n as the set of words w of length n in the alphabet $\{R, L\}$ where no initial segment of w has more L s than R s.

Proposition 2. $\#(P_n) = \dim(B_n)$

Proof. We have $\dim(B_n) = \dim \text{Hom}(V^{\otimes n}, \mathbb{1}) = \dim \text{Hom}(\sum \alpha_i V_i, \mathbb{1}) = \sum \alpha_i \dim \text{Hom}(V_i, \mathbb{1})$. Since $\dim \text{Hom}(V_j, \mathbb{1}) = 1$ for any indecomposable V_j , we get $\dim(B_n) = \sum \alpha_i$, which is the number of summands of $V^{\otimes n}$. We can see summands of $V^{\otimes n}$ are in bijection with P_n by induction on n . Assume we have a direct sum decomposition of $V^{\otimes n}$ and a bijection between summands of $V^{\otimes n}$ and P_n . By definition of Γ the summands of $V^{\otimes(n+1)}$ will be the indecomposables that are adjacent to the summands of $V^{\otimes n}$, so append the adjacency edge to the path from the bijection at level n . □

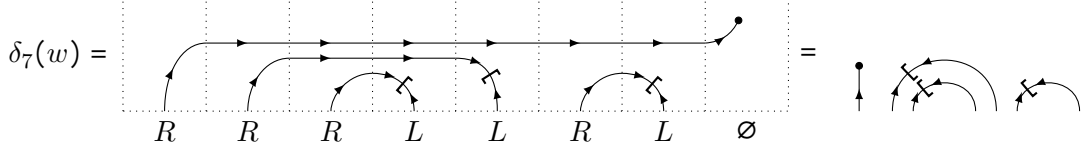
We will then construct a set of maps in bijection with P_n , and show these maps are independent, so that this set forms a basis for B_n . We first construct a map from the set of paths to the diagrammatic category.

Definition 5. We define a map $\delta_n : P_n \rightarrow \text{Diag}(\mathbb{Z}, V, \mathbb{C})$. Identify concatenation in a path word w with composition of diagrams, and identify each letter of the alphabet with a portion of a picture as below, where \emptyset signifies the end of the word.



The number of horizontal strands in each picture varies, and is equal to the excess of R s to L s in the segment prior to the current letter of w . To define $\delta_n(w)$ replace each letter of w with its identified picture, and then glue each portion end to end, including the picture for \emptyset at the end (glue dots onto any remaining strands at the end).

For example, take $w = RRRLRL$:



The images of δ_n form our sequence D_n . Composing δ_n with T we get $T_n : P_n \rightarrow B_n$. We want to show the values of T_n are linearly independent, and will use the following lemma.

Lemma 1. *Let $(S, <)$ be a finite ordered set, and V a vector space. If there are maps $f : S \rightarrow V$ and $g : S \rightarrow V^*$ such that for all $x, y \in S$:*

1. $g(x)(f(x)) \neq 0$
2. $x < y \implies g(x)(f(y)) = 0$

Then both the values of f and the values of g are linearly independent.

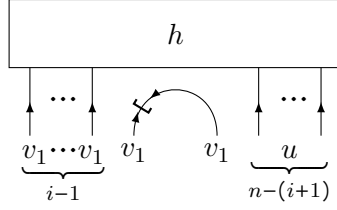
Proof. For simplicity replace S with the ordered set $(1, 2, \dots, n)$. Consider the $n \times n$ matrix A defined by $A_{i,j} = g(i)(f(j))$. Condition (1) implies entries on the main diagonal of A are non-zero. Condition (2) implies A is upper triangular, so together (1) and (2) imply A is invertible. Any linear dependence among the rows of A implies a linear dependence among the values of g , and a dependence among columns of A implies a dependence among the values of f , so we have our result. \square

Proposition 3. *The values of T_n are linearly independent.*

Proof. As in Lemma 1, let S be P_n with lexicographic order ($R > L$), and let g be T_n . To define f assign to each $x \in P_n$ a vector $f(x) \in V^{\otimes n}$ by identifying R with v_1 , L with v_0 , and concatenation with \otimes (e.g. $f(RRLRLRL) = v_1 \otimes v_1 \otimes v_0 \otimes v_1 \otimes v_0 \otimes v_0 \otimes v_1 \otimes v_0$). By construction of the pairing we have $g(x)(f(x)) = 1$, so condition 1 of Lemma 1 holds. To show condition 2 holds, note that if $x < y$ then

$$\begin{aligned} f(y) &= v_1^{\otimes i} \otimes v_1 \otimes u, & u &\in V^{\otimes(n-(i+1))} \\ f(x) &= v_1^{\otimes i} \otimes v_0 \otimes w, & w &\in V^{\otimes(n-(i+1))} \end{aligned}$$

so that $g(x)(f(y))$ will have the form



which vanishes since looking at the value of the map on positions i and $i+1$, $(\epsilon)(-\varphi \otimes \text{Id}_V)(v_1 \otimes v_1) = \epsilon(v_0^* \otimes v_1) = 0$. \square

We then have a basis for each B_n described as images of a set of diagrams $D_n \subset \text{Diag}(\mathbb{Z}, V, \mathbb{C})$ under the map T .

3.3 Showing D_n spans $\text{Diag}_n(\mathbb{Z}, V, \mathbb{C})$.

Let $\text{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ be the box space of n positively oriented strands. We need to give a description of the pictures in D_n , and show that if we take an arbitrary picture in $\text{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ we can reduce it to the span of D_n .

Definition 6. We consider $U \in \text{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ to be a member of D_n if:

1. There are no crossings in U .
2. For every dot, there is a path from that dot to the sky which does not intersect U .
3. Any strand component in U has at most 1 vertex.
4. Any bracket is directed towards the leftmost endpoint of a strand component.


Proposition 4. D_n spans $\text{Diag}_n(\mathbb{Z}, V, \mathbb{C})$.

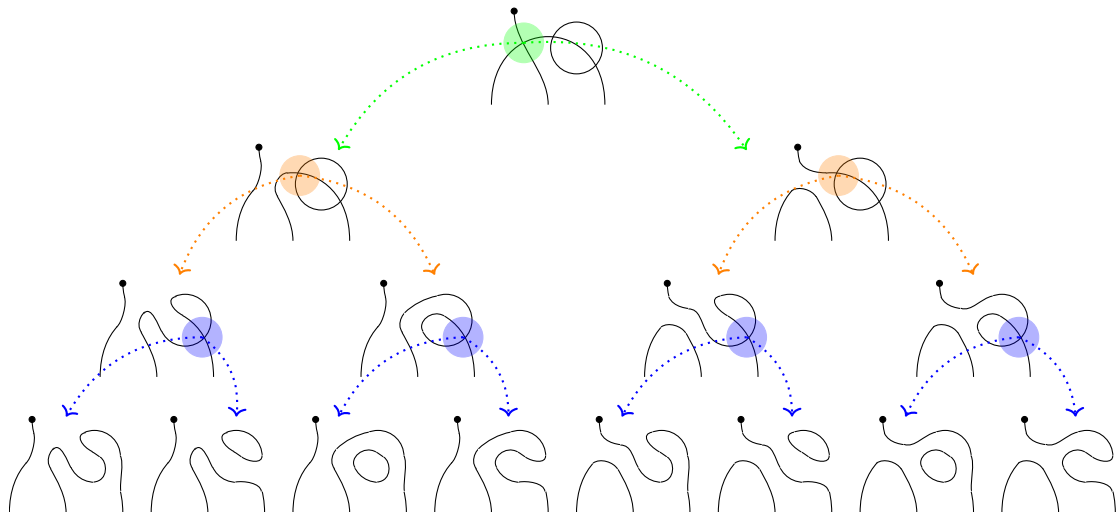
Proof. We perform the following algorithm to reduce an arbitrary picture to D_n :

1. Pull all dots to the sky via symmetric isotopy.
2. Use relation E_5 (crossing resolution) to remove all crossings.
3. Use E_1 (bracket cancellation) and E_4 (bracket reversal) to reduce the number of brackets on any path to at most 1, and to direct any bracket towards the leftmost endpoint (if it has one) of its strand component.
4. Use E_2 (dot pair removal) to remove any floating double dots.
5. Use E_3 (circle removal) to remove any floating circles.

After steps 1 and 2 have been performed, there will be no crossings and all dots will touch the sky. Steps 3 through 5 will not change these properties as they cannot introduce any new dots or crossings. In step 3 we use bracket cancellation and reversal as necessary to remove the number of brackets to 1 in the odd case or 0 in the even case on any

component. If a component has two dots on it, it will be oriented outward at both ends and so must have an odd number of brackets and hence only one bracket. Similarly if a component is a closed loop, then in order for it to be consistently oriented there must be an even number of brackets, hence zero brackets. Bracket reduction is done only at the cost of a sign so that it effects no previous steps. At this point we must be left with a diagram that meets items 1 through 3 of Definition 4, and possibly has some oriented circles and dot pairs which don't meet the boundary. Steps 4 and 5 remove these circles at the cost of a constant, and our diagram has been reduced to D_n . □

We illustrate the algorithm and definition with the example . The leaves of the tree below give all terms which appear in the reduction of our initial diagram.



The result of this subsection concludes the construction of a bijection $\text{Diag}(\mathbb{Z}, V, \mathbb{C}) \rightarrow \text{Alg}(\mathbb{Z}, V, \mathbb{C})$.

3.4 Language for B_n and computation in Alg

To aid in computation we use the following language for the basis B_n .

The map $v_0 \mapsto 0, v_1 \mapsto 1$, projection onto the second coordinate, and use the symbol \bullet to represent this map.

we use a pair of matched parenthesis $\langle \rangle$ to denote the cap with left-facing bracket (determinant) map. We write standard basis vectors in $V_2^{\otimes n}$ by concatenating the indices of the vectors v_0 and v_1 and using bold (e.g $v_0 \otimes v_1 \otimes v_1 \otimes v_0 \otimes v_1 = \mathbf{01101}$). Some example values on the standard basis of $V^{\otimes 2}$ are:

	$\bullet\bullet$	$\langle \rangle$
00	0	0
01	0	-1
10	0	1
11	1	0

A table of values for B_3 on $V^{\otimes 3}$ follows.

	$\bullet\bullet\bullet$	$\bullet\langle \rangle$	$\langle \bullet \rangle$	$\langle \rangle\bullet$
000	0	0	0	0
001	0	0	0	0
010	0	0	0	0
100	0	0	0	0
110	0	1	1	0
101	0	-1	0	1
011	0	0	-1	-1
111	1	0	0	0

We can see that $\langle \bullet \rangle = \bullet\langle \rangle + \langle \rangle\bullet$, and the maps $\bullet\bullet\bullet, \bullet\langle \rangle, \langle \bullet \rangle$ form a basis of B_3 , as expected.

Definition 7. Consider the alphabet $A = \{\bullet, \langle, \rangle\}$ and denote an empty word by ϵ . We define a language X via BNF as

$$w ::= \epsilon \mid \bullet \mid \langle w \rangle \mid ww$$

and define a language L as the quotient of X by the relation $\langle \bullet \rangle = \bullet\langle \rangle + \langle \rangle\bullet$.

L contains all the information needed to write expressions and perform computations in Alg, and a simplified expression in L is the same as a diagram reduced in terms of B_n .

4 A 2-dimensional \mathbb{Z}_p -representation over \mathbb{F}_p

Let $V_n = (\mathbb{F}_p^n, \phi_n)$ where $\phi_n : \mathbb{Z}_p \rightarrow GL(\mathbb{F}_p^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\text{Alg}(\mathbb{Z}_p, V, \mathbb{F}_p)$. Taking the standard basis of \mathbb{F}_p^2 , $v_0 = (1, 0)$ and $v_1 = (0, 1)$, we use the isomorphism $\varphi : V \xrightarrow{\sim} V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \text{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

4.1 Presentation of $\text{Diag}(\mathbb{Z}_p, V, \mathbb{F}_p)$ and map into $\text{Alg}(\mathbb{Z}_p, V, \mathbb{F}_p)$

We will include all the generators and the relations from (Ex 1) for all p in our presentation. We need one new generator and one new relation which are dependent on p .

Definition 8. We present $\text{Diag}(\mathbb{Z}_p, V, \mathbb{F}_p)$ by the same generators and relations of Definition 3, along with the new generator G_4 and relation E_5

$$\begin{array}{c}
 G_4 : \quad \begin{array}{c} \bullet \\ \curvearrowright \quad \curvearrowright \quad \dots \quad \curvearrowright \quad \curvearrowright \\ \underbrace{\hspace{10em}}_{2p-1} \end{array} \\
 \\
 \begin{array}{c} \curvearrowright \quad \dots \quad \curvearrowright \\ \vdots \\ \curvearrowright \quad \curvearrowright \end{array} = \begin{array}{c} \bullet \\ \uparrow \end{array} \begin{array}{c} \bullet \\ \curvearrowright \quad \curvearrowright \quad \dots \quad \curvearrowright \quad \curvearrowright \end{array} + \begin{array}{c} \bullet \\ \curvearrowright \quad \curvearrowright \quad \dots \quad \curvearrowright \quad \curvearrowright \end{array} \begin{array}{c} \bullet \\ \uparrow \end{array} \quad \boxed{DTL(2)}
 \end{array}$$

Before discussing the map into Alg we need the following proposition.

Proposition 5. Consider V with \mathbb{F}_q^+ ($q = p^n$) action given by $x \cdot v_0 = v_0, x \cdot v_1 = xv_0 + v_1$. Then $V^{\otimes n}$ has basis $B = \{0, 1\}^{\times n} \xrightarrow{\sim} \{w_1 \otimes \dots \otimes w_n \mid w_i \in \{v_0, v_1\}\}$. Let $l(v) = \sum_v v_i$ and define a linear map φ_n on B for any $n \in \mathbb{N}, q = p^i$:

$$\varphi_n(v) = \begin{cases} 1 & q-1 \mid l(v), l(v) \neq 0, n \\ 0 & \text{else} \end{cases}$$

Then $\varphi_n : V^{\otimes n} \rightarrow V_1 = \mathbb{F}_p$ is a map of \mathbb{F}_q^+ representations with the action described above, and the trivial action on V_1 .

Proof. We need to show $x \cdot \varphi_n(v) = \varphi_n(x \cdot v) \ \forall x \in \mathbb{F}_q, \forall v \in B$. We compute the action

of $x \in \mathbb{F}_q$ on some $v \in B$:

$$x \cdot v = v + \sum_{i=1}^{l(v)-1} x^i \sum_{l(w)=l(v)-i} w + x^{l(v)}(0, 0, \dots, 0)$$

Applying φ_n to the above expression gives

$$\varphi_n(x \cdot v) = \varphi_n(v) + \sum_{i=1}^{l(v)-1} x^i \sum_{l(w)=l(v)-i} \varphi_n(w) = \varphi_n(v) + \sum_{0 < j(q-1) < l(v)} \binom{l(v)}{j(q-1)} x^{l(v)-j(q-1)}$$

Since this sum is in $\overline{\mathbb{F}}_p$ with $x \in \mathbb{F}_q$, we have $x^{q-1} = 1$ and can simplify the above expression:

$$\varphi_n(x \cdot v) = \varphi_n(v) + x^{l(v)} \sum_{0 < j(q-1) < l(v)} \binom{l(v)}{j(q-1)}$$

We then need to check that $x \cdot \varphi_n(v) = \varphi_n(v) + x^{l(v)} \sum_{0 < j(q-1) < l(v)} \binom{l(v)}{j(q-1)}$, and since \mathbb{F}_q^+ acts trivially on L_1 , $x \cdot \varphi_n(v) = \varphi_n(v)$ so we just need each $x \in \mathbb{F}_q$ to be a root of $x^{l(v)} \sum_{0 < j(q-1) < l(v)} \binom{l(v)}{j(q-1)}$. We see that 0 is a root, so assume $x \in \mathbb{F}_q^\times$ and cancelling $x^{l(v)}$ we must show $S = \sum_{0 < j(q-1) < l(v)} \binom{l(v)}{j(q-1)} \equiv_p 0$.

The generating function for $\binom{j}{k}$ is $(1+t)^j$. We would like to exclude the constant term and t^j , and then take the sum of coefficients of each $t^{j(q-1)}$. This can be done by fixing a primitive root of unity ξ_{q-1} of order $q-1$ and replacing t by $\xi_{q-1}^m t$ in $(1+t)^j - (1+t^j)$, then summing over m from 0 to $q-2$ and evaluating at $t = 1$. This leaves

$$S = \frac{1}{q-1} \sum_{m=0}^{q-2} [(1 + \xi_{q-1}^m)^{l(v)} - (1 + \xi_{q-1}^{m \cdot l(v)})] = - \sum_{m=0}^{q-2} [(1 + \xi_{q-1}^m)^{l(v)} - (1 + \xi_{q-1}^{m \cdot l(v)})]$$

Now, let g be a generator of \mathbb{F}_q^\times , and let $\phi_{q-1}(u)$ be the cyclotomic polynomial, $\phi_{q-1}(u) = \prod_{d|q-1} (u^d - 1)^{\mu(\frac{q-1}{d})}$. We consider our sum as an element of the ring $\mathbb{Z}[\xi_{q-1}] \simeq \mathbb{Z}[u]/\phi_{q-1}(u)$, and since $\phi_{q-1}(g) = 0$ we have a ring homomorphism $\gamma : \mathbb{Z}[\xi_{q-1}] \rightarrow \mathbb{F}_q^\times$ given by $\xi_{q-1} \mapsto g$. Rewriting S via γ we have

$$\gamma(S) = - \sum_{m=0}^{q-2} [(1 + g^m)^{l(v)} - (1 + g^{m \cdot l(v)})]$$

If $q-1$ divides $l(v)$ we know $(1 + g^m)^{l(v)} = 1$ for all but one value of m , where $g^m = -1 \implies (1 + g^m)^{l(v)} = 0$, and $(1 + g^{m \cdot l(v)}) = 2$. In this case we get $\gamma(S) = -(-(q-2) - 2) = q = 0$. If $q-1$ does not divide $l(v)$ we have $\sum_{m=0}^{q-2} (1 + g^{m \cdot l(v)}) = q-1 = -1$, and $1 + g^m$ will range over $\mathbb{F}_q - \{1\}$. This lets us write

$$\gamma(S) = -1 - \sum_{m=0}^{q-2} (1 + g^m)^{l(v)} = -1 - \sum_{y \in \mathbb{F}_q - \{1\}} y^{l(v)} = \sum_{y \in \mathbb{F}_q} y^{l(v)} = \sum_{z \in \mathbb{F}_q} z = 0$$

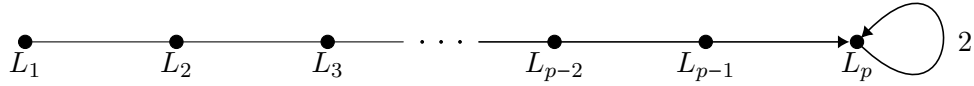
□

Proposition 6. *There is a map of planar algebras $T : \text{Diag}(\mathbb{Z}, V, \mathbb{C}) \rightarrow \text{Alg}(\mathbb{Z}, V, \mathbb{C})$ determined by the values $T(G_1) = v_1^*$, $T(G_2) = \varphi$ and $T(G_3) = \varphi^{-1}$.*

Proof. The computations for relations E_1 through E_4 all hold in any field, so we need to define the image $T(G_4)$ and check that E_5 is satisfied. □

4.2 Exhibiting bases D_n for each B_n

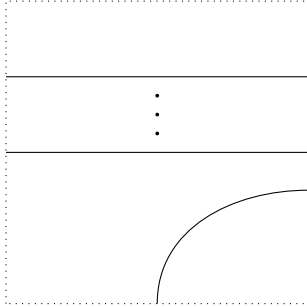
The indecomposable representations that appear in \otimes powers of V are exhausted by the sequence $(V_n)_{n \in [1..p]}$. We have $V_2 \otimes V_n \simeq V_{n-1} \oplus V_{n+1}$ when $1 < n < p$, and $V_2 \otimes V_p \simeq V_p \oplus V_p$, so every indecomposable appears as a direct summand of some $V_2^{\otimes n}$ and all indecomposables are self dual. We get the following fusion graph.



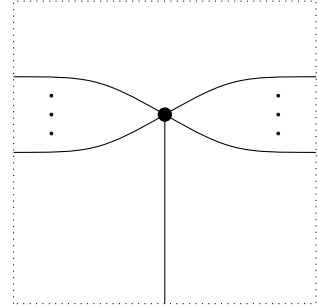
4.3 Generators and Bases for $B_{n,p}$

We would like to present bases for the $B_{n,p}$ and will construct a diagrammatic language to do this. We want to define a function T from walks on the fusion graph G_p to diagrams in some planar algebra. We will define T on each letter R, L, A, B , and then extend by concatenation and by a natural composition in the planar algebra. First make the identifications:

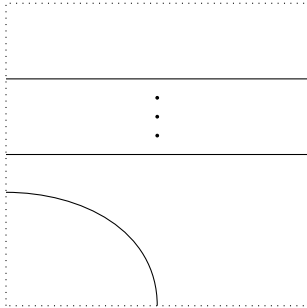
$$R : l \rightarrow (l+1)$$



$$A : (p-1) \rightarrow (p-1)$$



$$L : (l+1) \rightarrow l$$



$$B : (p-1) \rightarrow (p-1)$$

