Some Presentations of Planar Algebras with Applications to Invariant Theory

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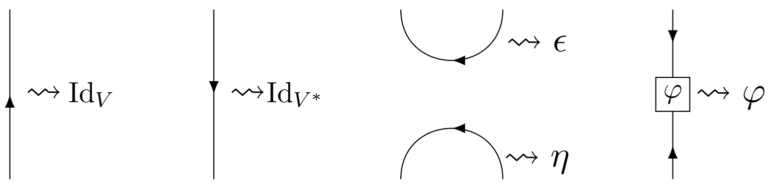
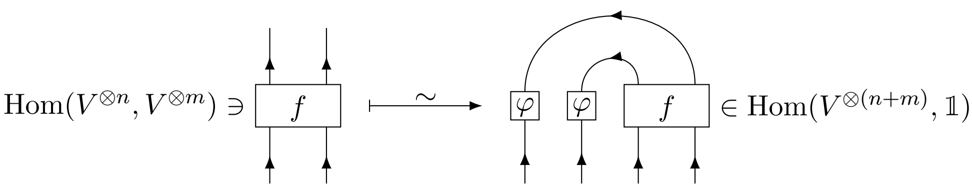
1 Introduction

Fix a group G, a field k, and some object $V \in \mathrm{Rep}_k(G)$, the category of k-linear representations of G. We study $\mathrm{Alg}_{(G,V,k)}$, the full subcategory of $\mathrm{Rep}_k(G)$ whose objects are finite tensor products of V and V^*. We denote by $M_{G,V,k}$ the morphisms of $\mathrm{Alg}_{(G,V,k)}$, and by $P_{G,V,k}$ the planar algebra structure on $M_{G,V,k}$. Our goal will be to give a diagrammatic presentation of $P_{G,V,k}$. In all examples considered it will be true that $V \simeq V^*$, so we can restrict to studying the spaces $\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$. Further, since $\mathrm{Hom}(A, B \otimes C) \simeq \mathrm{Hom}(A \otimes B^*, C)$ for A, B, C finite dimensional representations of a group, we can restrict to studying the spaces $B_n := \mathrm{Hom}(V^{\otimes n}, \mathbb{1})$.

Diagrammatically, we have oriented strands connecting function blocks. We will assign an upwards oriented strand to the identity map on V and a downward oriented strand to the identity map on V^*, keeping track of the type of data flowing along a strand and consequently the type of inputs and outputs to function blocks. We have evaluation and coevaluation maps $V^* \otimes V \xrightarrow{\epsilon} k$, $k \xrightarrow{\eta} V \otimes V^*$ drawn by arcs connecting appropriately oriented strands. We fix an isomorphism $\varphi : V \xrightarrow{\sim} V^*$ and draw this between the upward and downward oriented strand.



Using φ along with evaluation we can turn outgoing strands into incoming ones, illustrating the isomorphism $\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \mathrm{Hom}(V^{\otimes(n+m)}, \mathbb{1})$, and allowing us to focus on pictures where all strands are 'attached to the ground'.



Our goal is to determine a minimal generating set of pictures and relations so that the resulting diagrammatic planar algebra, call it $\text{Diag}_{(G,V,k)}$, will be isomorphic to $P_{G,V,k}$. In each example we use the following procedure:

1. Define generating pictures and relations for $\text{Diag}_{(G,V,k)}$, and give a map of planar algebras $\text{Diag}_{(G,V,k)} \xrightarrow{T} P_{G,V,k}$.
2. For each n find a subset $D_n \subset \text{Diag}_{(G,V,k)}$ such that $T(D_n)$ is a basis for B_n. Exhibiting such D_n shows that T is surjective, since as argued above $M_{G,V,k}$ is determined by the spaces B_n.
3. Show that an arbitrary picture in $\text{Diag}_{(G,V,k)}$ can be rewritten linearly in terms of elements of the D_n using the presented relations. This shows that T is injective, so that $\text{Diag}_{(G,V,k)} \simeq P_{G,V,k}$ as planar algebras.

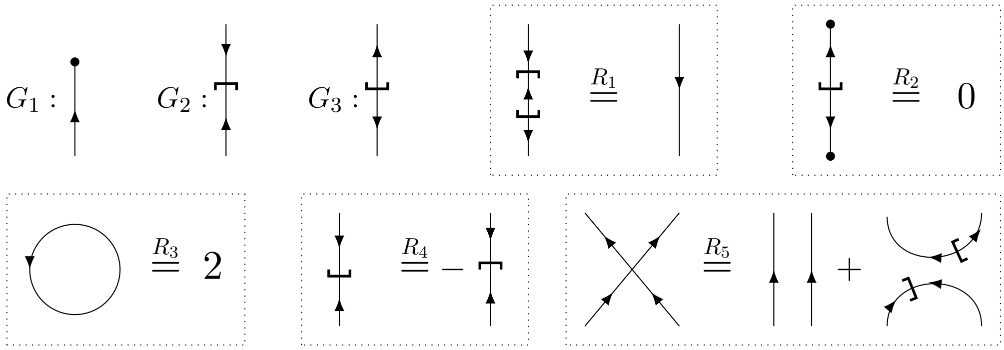
If we fix coordinates on V, when describing the spaces B_n we are also describing a subring of vector invariants for G, i.e. each f in B_n is also in $(V^{\oplus n})^G = \{f \in k[x_1, \dots, x_n] : \forall g \in G, f(\bar{x}) = f(g \cdot \bar{x})\}$. This follows from observing that the defining property for an element of $(V^{\oplus n})^G$ is the same as the defining property for a map of G representations from $V^{\otimes n}$ to the trivial representation. Giving a presentation of $\text{Alg}_{(G,V,k)}$ is then related to giving a first and second fundamental theorem of invariant theory for $(V^{\oplus n})^G$, and in each example we discuss this relationship.

2 A 2-dimensional $\mathbb{Z}$-representation over $\mathbb{C}$

Let $V_n = (\mathbb{C}^n, \phi_n)$ where $\phi_n : \mathbb{Z} \to GL(\mathbb{C}^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\text{Alg}_{(\mathbb{Z},V,\mathbb{C})}$. Taking the standard basis of $\mathbb{C}^2$, $v_0 = (1,0)$ and $v_1 = (0,1)$, we use the isomorphism $\varphi : V \xrightarrow{\sim} V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \text{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

2.1 Presentation of $\text{Diag}_{(\mathbb{Z},V,\mathbb{C})}$ and map into $\text{Alg}_{(\mathbb{Z},V,\mathbb{C})}$

Definition 1. *We present $\text{Diag}_{(\mathbb{Z},V,\mathbb{C})}$ by the generators and relations below:*



Proposition 1. *There is a map of planar algebras $T : Diag_{(\mathbb{Z},V,\mathbb{C})} \to Alg_{(\mathbb{Z},V,\mathbb{C})}$ determined by the values $T(G_1) = v_1^*, T(G_2) = \varphi$ and $T(G_3) = \varphi^{-1}$.*

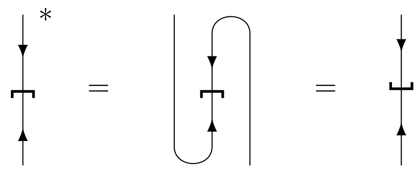
Proof. We need to show each of R_1 through R_5 hold in the image of T so that this map is well defined.

R_1: $\varphi \cdot \varphi^{-1} = \mathrm{Id}_{V^*}$

R_2: $1 \mapsto v_1^* \mapsto v_0 \mapsto 0$

R_3: $\epsilon \cdot \tau \cdot \eta(1) = \epsilon \cdot \tau(v_1 \otimes v_1^* + v_0 \otimes v_0^*) = \epsilon(v_1^* \otimes v_1 + v_0^* \otimes v_0) = 2$

R_4: This follows from taking the dual (rotation) of G_2 and a coordinate computation:



$$\begin{aligned}
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(\tau \cdot \eta \otimes \mathrm{Id}_V)(v_0) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(v_0^* \otimes v_0 \otimes v_0 + v_1^* \otimes v_1 \otimes v_0) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_0 - v_1^* \otimes v_0^* \otimes v_0) &&= -v_1^*
\end{aligned}$$

$$\begin{aligned}
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(\tau \cdot \eta \otimes \mathrm{Id}_V)(v_1) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(\mathrm{Id}_{V^*} \otimes \varphi \otimes \mathrm{Id}_V)(v_0^* \otimes v_0 \otimes v_1 + v_1^* \otimes v_1 \otimes v_1) &&= \\
&(\mathrm{Id}_{V^*} \otimes \epsilon)(v_0^* \otimes v_1^* \otimes v_1 - v_1^* \otimes v_0^* \otimes v_1) &&= v_0^*
\end{aligned}$$

R_5: The second term on the RHS takes the values $v_0 \otimes v_0 \mapsto 0, v_1 \otimes v_1 \mapsto 0, v_0 \otimes v_1 \mapsto v_1 \otimes v_0 - v_0 \otimes v_1, v_1 \otimes v_0 \mapsto v_0 \otimes v_1 - v_1 \otimes v_0$, and adding the identity map gives us the LHS.

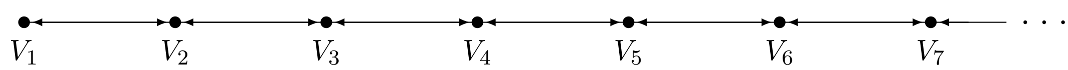
□

2.2 Exhibiting bases D_n for each B_n

The indecomposable representations that appear in $\otimes$-powers of V are exhausted by the sequence V_n. When $i \geq 2$ we have the rule

$$V \otimes V_i \simeq V_{i+1} \oplus V_{i-1} \tag{1}$$

so that the fusion graph Γ for V is



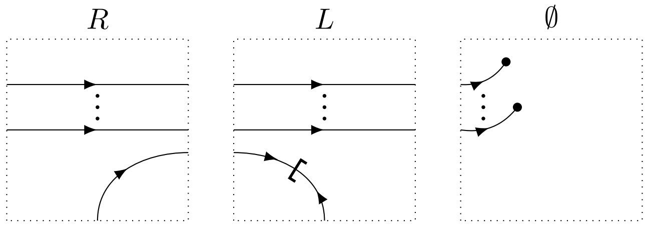
Definition 2. *Let P_n be the set of paths of length n on Γ based at V_1. Label edges directed from V_i to V_{i+1} with R, and edges directed from V_i to V_{i-1} by L. Then we can describe P_n as the set of words w of length n in the alphabet $\{R, L\}$ where no initial segment of w has more Ls than Rs.*

Proposition 2. $\#(P_n) = dim(B_n)$

Proof. We have $\dim(B_n) = \dim\mathrm{Hom}(V^{\otimes n}, \mathbb{1}) = \dim\mathrm{Hom}\big(\sum \alpha_i V_i, \mathbb{1}\big) = \sum \alpha_i \dim\mathrm{Hom}(V_i, \mathbb{1})$. Since $\dim\mathrm{Hom}(V_j, \mathbb{1}) = 1$ for any indecomposable V_j, we get $\dim(B_n) = \sum \alpha_i$, which is the number of summands of $V^{\otimes n}$. We can see summands of $V^{\otimes n}$ are in bijection with P_n by induction on n. Assume we have a direct sum decomposition of $V^{\otimes n}$ and a bijection between summands of $V^{\otimes n}$ and P_n. By definition of Γ the summands of $V^{\otimes(n+1)}$ will be the indecomposables that are adjacent to the summands of $V^{\otimes n}$, so append the adjacency edge to the path from the bijection at level n. □

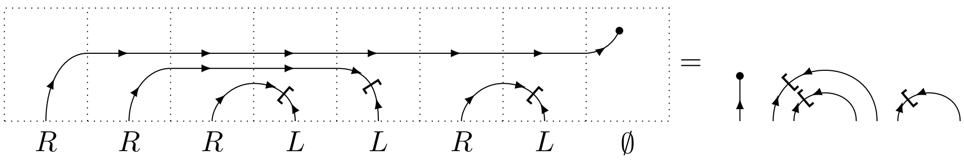
We will then construct a set of maps in bijection with P_n, and show these maps are independent, so that this set forms a basis for B_n. We first construct a map from the set of paths to the diagrammatic category.

Definition 3. *We define a map $\delta_n : P_n \to Diag_{(\mathbb{Z},V,\mathbb{C})}$. Identify concatenation in a path word w with composition of diagrams, and identify each letter of the alphabet with a portion of a picture as below, where $\emptyset$ signifies the end of the word.*



The number of horiztonal strands in each picture varies, and is equal to the excess of Rs to Ls in the segment prior to the current letter of w. To define $\delta_n(w)$ replace each letter of w with its identified picture, and then glue each portion end to end, including the picture for $\emptyset$ at the end (glue dots onto any remaining strands at the end).

For example, take $w = RRRLLRL$:

$\delta_7(w) =$ 

The images of δ_n form our sequence D_n. Composing δ_n with T we get $T_n : P_n \to B_n$. We want to show the values of T_n are linearly independent, and will use the following lemma.

Lemma 1. *Let $(S, <)$ be a finite ordered set, and V a vector space. If there are maps $f : S \to V$ and $g : S \to V^*$ such that for all $x, y \in S$:*

1. $g(x)(f(x)) \neq 0$
2. $x < y \implies g(x)(f(y)) = 0$

Then both the values of f and the values of g are linearly independent.

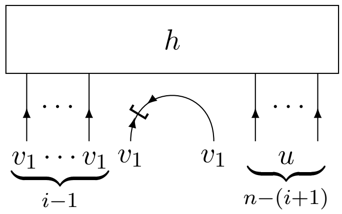
Proof. For simplicity replace S with the ordered set $(1, 2, \ldots, n)$. Consider the $n \times n$ matrix A defined by $A_{i,j} = g(i)(f(j))$. Condition (1) implies entries on the main diagonal of A are non-zero. Condition (2) implies A is upper triangular, so together (1) and (2) imply A is invertible. Any linear dependence among the rows of A implies a linear dependence among the values of g, and a dependence among columns of A implies a dependence among the values of f, so we have our result. $\square$

Proposition 3. *The values of T_n are linearly independent.*

Proof. As in Lemma 1, let S be P_n with lexicographic order ($R > L$), and let g be T_n. To define f assign to each $x \in P_n$ a vector $f(x) \in V^{\otimes n}$ by identifying R with v_1, L with v_0, and concatenation with $\otimes$ (e.g. $f(RRLRLLRL) = v_1 \otimes v_1 \otimes v_0 \otimes v_1 \otimes v_0 \otimes v_0 \otimes v_1 \otimes v_0$). By construction of the pairing we have $g(x)(f(x)) = 1$, so condition 1 of Lemma 1 holds. To show condition 2 holds, note that if $x < y$ then

$$f(y) = v_1^{\otimes i} \otimes v_1 \otimes u, \qquad u \in V^{\otimes (n-(i+1))}$$
$$f(x) = v_1^{\otimes i} \otimes v_0 \otimes w, \qquad w \in V^{\otimes (n-(i+1))}$$

so that $g(x)(f(y))$ will have the form



which vanishes since looking at the value of the map on positions i and $i+1$, $(\epsilon)(-\varphi \otimes \mathrm{Id}_V)(v_1 \otimes v_1) = \epsilon(v_0^* \otimes v_1) = 0$. ☐

We then have a basis for each B_n described as images of a set of diagrams $D_n \subset \mathrm{Diag}_{(\mathbb{Z},V,\mathbb{C})}$ under the map T.

2.3 Showing D_n spans $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$.

We need to give a topological description of the pictures in D_n and show that if we take an arbitrary picture in $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$ we can reduce it to the span of D_n.

Definition 4. *Let our ambient space be $X = (0,1) \times [0,1)$ and refer to $(0,1) \times \{0\}$ as the* ***ground****. Consider a union, U, of smooth decorated paths $\alpha_i : [0,1] \to X$. We refer to the connected component of $X - U$ adjacent to $(0,1) \times \{1\}$ as the* ***sky****. We consider U to be a member of D_n if:*

1. *The images of the α_i do not intersect and the α_i are injective (there are no crossings in U).*
2. *For every dot, there is a path from that dot to the sky which does not intersect the image of any α_i.*
3. *For each α_i, either both $\alpha_i(0)$ and $\alpha_i(1)$ are on the ground, or one is on the ground while the other is the position of a dot.*
4. *Any image of an α_i has exactly 1 decoration (a dot or a bracket).*
5. *Any bracket is directed towards the leftmost endpoint of a path image.*
6. *There is a single orientation marking for each intersection of U with the ground, and it is directed away from the ground.*

Proposition 4. *D_n spans $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$.*

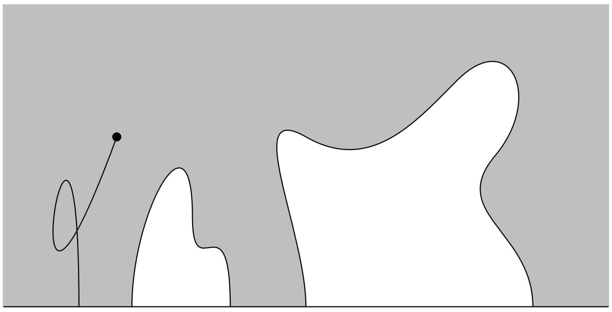
Proof. We perform the following algorithm to reduce an arbitrary picture to D_n:

1. Pull all dots to the sky via smooth homotopy, possibly introducing crossings.
2. Using relation R_5 resolve all crossings.
3. Replace any portions of the picture which do not intersect the ground with a constant using R_2 and R_3.

4. Use R_4 and R_1 to reduce the number of brackets per strand to 1 and to direct this bracket towards the left endpoint.

After steps 1 and 2 have been performed, there will be no crossings and all dots will touch the sky, so that items 1 and 2 of Definition 4 are satisfied. After step 3 item 3 of Definition 4 is satisfied, and items 1 and 2 are not effected. After step 4, items 4 and 5 of the definition will be satisfied, and no crossings or dots can be created or moved, so all other previous items of the definition are preserved. Item 6 is satisfied as a result of starting with a diagram in $\mathrm{Diag}_n(\mathbb{Z}, V, \mathbb{C})$. and the reduction of brackets in step 4. $\square$

We illustrate the algorithm and definition by an example.



The result of this section concludes the construction of a bijection $\mathrm{Diag}(\mathbb{Z}, V, \mathbb{C}) \to \mathrm{Alg}(\mathbb{Z}, V, \mathbb{C})$.

2.4 The Invariant space $(V^{\oplus n})^{\mathbb{Z}}$ and the Nowicki conjecture

3 A 2-dimensional $\mathbb{Z}_p$-representation over $\mathbb{F}_p$

Let $V_n = ({\mathbb{F}_p}^n, \phi_n)$ where $\phi_n : \mathbb{Z}_p \to GL({\mathbb{F}_p}^n)$ is defined by $\phi_n(1) = J_n$, and J_n is the Jordan block of dimension n with eigenvalue 1. In this section we set $V = V_2$ and study $\mathrm{Alg}_{\mathbb{Z},V,\mathbb{C}}$. Taking the standard basis of $\mathbb{C}^2$, $v_0 = (1, 0)$ and $v_1 = (0, 1)$, we use the isomorphism $\varphi : V \xrightarrow{\sim} V^*$ defined by $\varphi(v_0) = v_1^*$, $\varphi(v_1) = -v_0^*$. We want to describe the spaces $B_n = \mathrm{Hom}(V^{\otimes n}, \mathbb{1} \simeq V_1)$.

3.1 Presentation of $\mathrm{Diag}_{(\mathbb{Z},V,\mathbb{C})}$ and map into $\mathrm{Alg}_{(\mathbb{Z},V,\mathbb{C})}$

We will include all the generators and the relations from (Ex 1) for all p in our presentation. We need one new generator and one new relation which are dependent on p.

Definition 5. *We present $D_{\mathbb{Z}_p,V,\mathbb{F}_p}$ by the same generators and relations of Definition 1, along with the new generator G_4 and relation R_5*

G_4 :