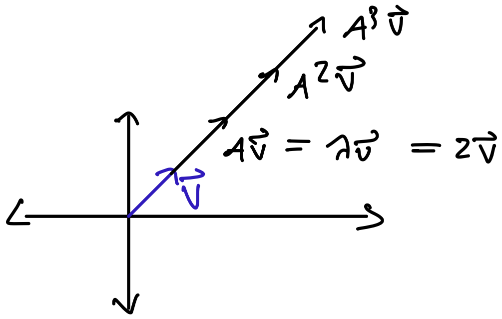
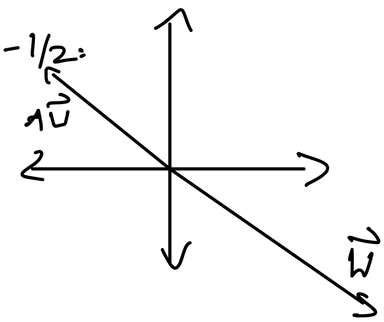
Lecture 8: Eigenvectors

given Square A, if $A\vec{v} = \lambda\vec{v}$ for some nonzero $\vec{v}$, then $\vec{v}$ is the eigenvector associated w/ A & λ is the eigenvalue

↳ $\vec{v}$ still points in same direction



Consider $\lambda = -1/2$:



Theorem: If $(\vec{v}, \lambda)$ is an evector/evalue pair of A, then

- $\vec{v}$ is an eigenvector for A^k, $k > 0$ $\square$

Proof: $A^2 v = A \cdot A\vec{v} = A \cdot \lambda v = \lambda A\vec{v} = \lambda^2 v$

Theorem: If A is invertible, then

- every $\vec{v}$ of A is also an eigenvector of A^{-1}, corresponding eigenvalue is $\frac{1}{\lambda}$

Proof: $A^{-1}\vec{v} = A^{-1} \cdot \dfrac{A\vec{v}}{\lambda} = \dfrac{\vec{v}}{\lambda} = \dfrac{1}{\lambda} \cdot \vec{v}$ $\square$

Spectral Theorem: every real, symmetric $A_{n \times n}$ has the following props:

1. all evalues are real
2. Set of n eigenvectors that are mutually orthogonal, i.e. $v_i^T v_j = 0$, $i \neq j$

We can use these n orthogonal evectors as a basis for $\mathbb{R}^n$

Building a matrix w/ specified eigenvectors: (reverse decomposition)

- choose n orthogonal vectors, unit norm: $\vec{v}_1 \dots \vec{v}_n$

Let $V = [\vec{v}_1 \dots \vec{v}_n]$ ⟵ $n \times n$ matrix

Observe $V^T V = I$ if V is orthonormal $\implies V^T = V^{-1}$, $VV^T = I$

Orthonormal $\implies$ rotates or reflects object

↳ if $\det(V) = -1$, then reflection occurs building matrix cont.

- choose scaling values: $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

def of eigenvector: $AV = V\Lambda \implies A = V\Lambda V^T$

Theorem: $A = V\Lambda V^T = \sum_{i=1}^{n} \lambda_i \underbrace{v_i v_i^T}_{\text{outter product, } n\times n \text{ matrix w/ rank 1}}$ has chosen evectors & evalues from above

eigendecomposition

example: above

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3/4 & 5/4 \\ 5/4 & 3/4 \end{bmatrix}$$

Observation: $A^2 = V\Lambda^2 V^T$, $A^{-2} = V\Lambda^{-2} V^T$

Given symmetric, PSD Σ, we can find a symmetric squareroot

$$A = \Sigma^{1/2} = V\Sigma^{1/2} V^T$$

how?

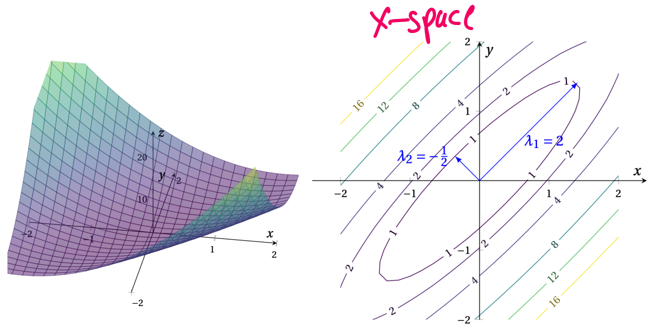
1) compute eigenvalues & eigenvectors of Σ
2) take square roots of eigenvalues
3) reassemble A

Visualizing quadratic forms

quadratic form of m: $f(x) = \vec{x}^T M \vec{x}$

isocontours of some matrix but which?

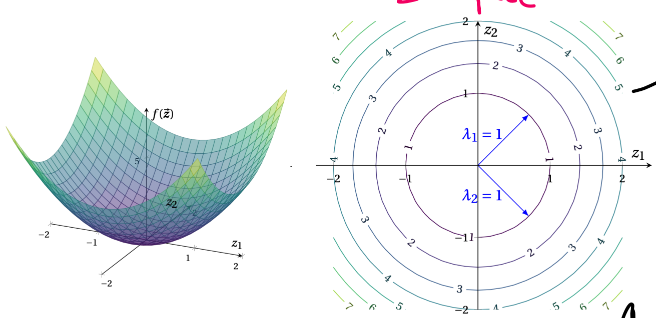
x-space



at high level, we'd be given normal distr. & need to find isocontours

start w/ bottom, transform into top plot

z-space



$q_1(z) = \|z\|_2$

↳ create A s.t. isocontours from bottom get mapped to top isocontours

↳ set $\vec{x} = A\vec{z}$

↳ $q_2(\vec{x}) = q_1(\vec{z}) = q_1(A^{-1}\vec{x})$

$= \|A^{-1}\vec{x}\|_2^2 = \vec{x}^T A^{-2}\vec{x}$

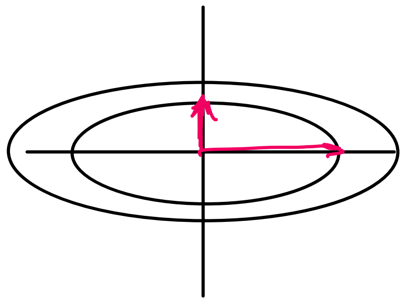
The isocontours of quadratic for $\vec{x}^T A^{-2} \vec{x}$ are ellipsoids determined by evalues/evectors of A:

$\{ \vec{x} : \vec{x}^T A^{-2} \vec{x} = 1 \}$ is an ellipsoid w/ axes $v_1 \dots v_n$ & $\lambda_1 \dots \lambda_n$

$\Longrightarrow$ contours of $x^T M x$ are ellipsoids determined by eigenvectors/eigenvalues of $M^{-1/2}$

$M = A^{-2} \Rightarrow A = M^{-1/2}$

Special Case: A is diagonal $\Longrightarrow$ eigenvectors are coordinate axes $\Longleftrightarrow$ ellipsoids are axis aligned

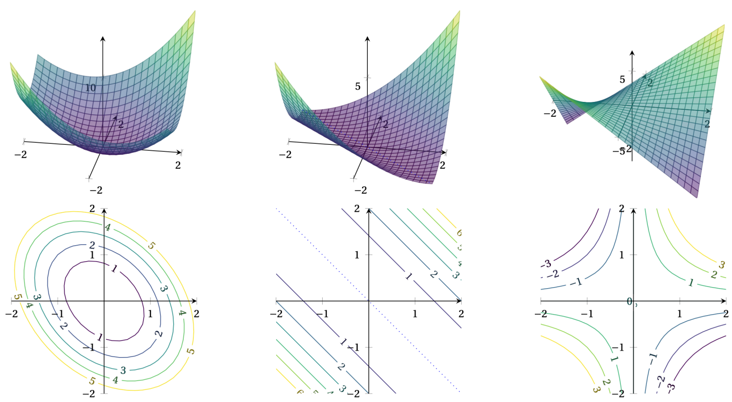


Positive Definite M: $w^T M w > 0 \Longrightarrow \lambda_i > 0 \ \forall \lambda_i$

PSD matrix M: $w^T M \vec{w} \geq 0 \Longrightarrow \lambda_i \geq 0 \ \forall \lambda_i$

Indefinite M: at least 1 positive & negative eigenvalue

Invertible M: no $\lambda_i = 0$



(a) Positive definite (b) Positive semidefinite (c) Indefinite

quadratic forms

might be multiple minima

hyperboloid

Every squared matrix is PSD, including A^{-2}

If A^{-2} exists, it is positive definite (no $\lambda_i = 0$)

ANISOTROPIC Gaussians

before, isotropic: variance same in every direction

$X \sim N(\mu, \Sigma)$

Σ: covariance matrix $\in \mathbb{R}^{d \times d}$

$$f(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2} \underbrace{(x-\mu)^T \Sigma^{-1} (x-\mu)}_{\text{quadratic form, } q(x)}}$$

Σ is a symmetric definite covariance matrix

Σ^{-1} is the $d \times d$ symmetric definite precision matrix

Write $f(x) = n(q(x))$, $q(x)$ is the quadratic form $(x-\mu)^T \Sigma^{-1} (x-\mu)$ min at μ

$n: \mathbb{R} \to \mathbb{R}$ $\quad q: \mathbb{R}^d \to \mathbb{R}$, quadratic

max at μ

Principle: given monotonic $n: \mathbb{R} \to \mathbb{R}$, isosurfaces of $n(q(x))$ = isosurfaces of $q(x)$

↳ different isovalues, same isocontours

The isocontours of $(x-\mu)^T \Sigma^{-1} (x-\mu)$ are determined by evectors/evalues of $\Sigma^{1/2}$

$\Rightarrow$ Sqrt of evalues of Σ are axis lengths of the isocontours

Nonzero covariance $\Longrightarrow$ ellipsoid isn't axis-aligned

Covariance

Let R, X be RVs, col vectors (or scalars)

$$\mathrm{Cov}(R,S) = E[\underbrace{(R-\mu_R)(S-\mu_S)^T}_{\text{Outer product}}] = E[RS^T] - \mu_R \mu_S$$

$\mathrm{Var}(R) = \mathrm{Cov}(R,R)$

If R is a vector, covariance matrix for R is

$$\mathrm{Var}(R) = \begin{bmatrix} \mathrm{Var}(R_1) & \cdots & \mathrm{Cov}(R_1, R_d) \\ \mathrm{Cov}(R_2, R_1) & & \vdots \\ \vdots & & \\ \mathrm{Cov}(R_d, R_1) & & \mathrm{Var}(R_d) \end{bmatrix}$$

For a Gaussian $R \sim N(\mu, \Sigma)$, one can show $\mathrm{Var}(R) = \Sigma$

- For indep $R_i, R_j \Longrightarrow \mathrm{Cov}(R_i, R_j) = 0$ ← reverse not necessarily true
- $\mathrm{Cov}(R_i, R_j) = 0$ & multivariate normal distr $\Longrightarrow$ independent
- all features pairwise indep $\Longrightarrow \mathrm{Var}(R)$ is diagonal
- $\mathrm{Var}(R)$ is diag. & joint normal $\Longleftrightarrow$ axis-aligned Gaussian

$$\Longleftrightarrow \underbrace{f(x)}_{\text{multivariate}} = \underbrace{f(x_1) \cdot \ldots \cdot f(x_d)}_{\text{univariate gaussians}}$$