AdS/CFT - Homework 6

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QUESTION 1

I have added the equivalence with the members of stress tensor multiplet to the table, as shown:

Spin	Field	Masses on S^5		Irred. reps.	
2	$h'_{\mu\nu}=H^{I_1}_{\mu\nu}Y^{I_1}$	$M^2=e^2k(k+4)$	$(k\geq 0)$	$1,6,20,\ldots$	$[2,2]_4^{(0,0,0)}$
1	$h_{\alpha\mu}=B^{I_5}_{\mu}Y^{I_5}_{\alpha}$ $a_{\mu\alpha\beta\gamma}=\phi^{I_5}_{\mu}\epsilon_{\alpha\beta\gamma}{}^{\delta\epsilon}D_\delta Y^{I_5}_{\epsilon}$	$M^2=e^2(k-1)(k+1)$ $M^2=e^2(k+3)(k+5)$	$(k\geq 1)$ $(k\geq 1)$	$15,64,175,\ldots$ $15,64,175,\ldots$	$[1,1]_3^{(1,0,1)}$
0	$h^{\alpha}_{\alpha}=\pi^{I_1}Y^{I_1}$ $a_{\alpha\beta\gamma\delta}=b^{I_1}\epsilon_{\alpha\beta\gamma\delta}{}^{\epsilon}D_\epsilon Y^{I_1}$	$M^2=e^2k(k-4)$ $M^2=e^2(k+4)(k+8)$	$(k\geq 2)$ $(k\geq 0)$	$20,50,\ldots$ $1,6,20,\ldots$	$B, \bar{B}, [0,0]_2^{(0,2,0)}$
0	$h_{(\alpha\beta)}=\phi^{I_{14}}Y^{I_{14}}_{(\alpha\beta)}$	$M^2=e^2k(k+4)$	$(k\geq 2)$	$84,300,\ldots$	
0	$B=B^{I_1}Y^{I_1}$	$M^2=e^2k(k+4)$	$(k\geq 0)$	$1_c,6_c,20_c,\ldots$	$[0,0]_4^{(0,0,0)}$
ant	$a_{\mu\nu\alpha\beta}=b^{I_{10},\pm}_{\mu\nu}Y^{I_{10},\pm}_{[\alpha\beta]}$	$M^2=e^2(k+2)^2$	$(k\geq 1)$	$10_c,45_c,\ldots$	
ant	$A_{\mu\nu}=a^{I_1}_{\mu\nu}Y^{I_1}$	$M^2=e^2k^2$ $M^2=e^2(k+4)^2$	$(k\geq 1)$ $(k\geq 0)$	$6_c,20_c,\ldots$ $1_c,6_c,\ldots$	$[0,2]_3^{(0,1,0)}$
1	$A_{\mu\alpha}=a^{I_5}_{\mu}Y^{I_5}_{\alpha}$	$M^2=e^2(k+1)(k+3)$	$(k\geq 1)$	$15_c,64_c,\ldots$	
0	$A_{\alpha\beta}=a^{I_{10},\pm}Y^{I_{10},\pm}_{[\alpha\beta]}$	$M^2=e^2(k-2)(k+2)$ $M^2=e^2(k+2)(k+6)$	$(k\geq 1)$ $(k\geq 1)$	$10_c,45_c,\ldots$ $10_c,45_c,\ldots$	$[0,0]_3^{(2,0,0)}$
$\frac{3}{2}$	$\psi_\mu=\psi^{I_L}_{\mu}\Xi^{I_L}$	$M=e(k+\frac{3}{2})$ $M=-e(k+\frac{7}{2})$	$(k\geq 0)$ $(k\geq 0)$	$4,20,\ldots$ $4^*,20^*,\ldots$	$[2,1]_{7/2}^{(1,0,0)}$
$\frac{1}{2}$	$\psi_{(\alpha)}=\psi^{I_T}\Xi^{I_T}_{\alpha}$	$M=e(k+\frac{5}{2})$ $M=-e(k+\frac{9}{2})$	$(k\geq 0)$ $(k\geq 0)$	$36^*,140^*,\ldots$ $36,140,\ldots$	
$\frac{1}{2}$	$\psi_{(\alpha)}=\psi^{I_L}D_{(\alpha)}\Xi^{I_L}+\chi\tau_\alpha\eta^+$	$M=e(k+\frac{11}{2})$ $M=-e(k-\frac{1}{2})$	$(k\geq 0)$ $(k\geq 1)$	$4,20,\ldots$ $20^*,\ldots$	$[0,1]_{5/2}^{(1,1,0)}$
$\frac{1}{2}$	$\lambda=\lambda^{I_L}\Xi^{I_L}$	$M=e(k+\frac{7}{2})$ $M=-e(k+\frac{3}{2})$	$(k\geq 0)$ $(k\geq 0)$	$4,20,\ldots$ $4^*,20^*,\ldots$	$[0,1]_{7/2}^{(1,0,0)}$

QUESTION 2

On the string side, $g_{YM}N=\frac{R^4}{\ell_s^4}$, so if $g_{YM}\sim 1$ and $N\gg 1$, $R\gg \ell_s$. Here the curvature is small, so IIB Sugra should be a good approximation to IIB string theory.

When $g^2_{YM}N\ll 1$ and N is large, the bulk theory is very strongly coupled and involves a large number of coincident D-branes.

EXERCISE 4.1

Close to the horizon, we can take $z=z_H+\epsilon$, where $\epsilon\ll 1$. Then we find

$$1-\left(\frac{z}{z_H}\right)^d=-\frac{\epsilon d}{z_H}+O(\epsilon^2)$$

so the metric can be rewritten as

$$ds^2=-\frac{R^2}{z^2}\left[\frac{z_H}{\epsilon d}d\epsilon^2+\frac{\epsilon d}{z_H}d\tau^2-\delta_{ij}dx^idx^j+\dots\right]$$

Now define the variables y and t by

$$y=2\sqrt{\frac{\epsilon z_H}{d}}\qquad\qquad t=\frac{\tau d}{2z_H}$$

This gives

$$\frac{z_H}{\epsilon d}d\epsilon^2=dy^2\qquad\qquad\frac{\epsilon d}{z_H}d\tau^2=\frac{4\epsilon z_H}{d}dt^2=y^2dt^2$$

so in the new coordinates, the metric is proportional to

$$ds^2\propto dy^2+y^2dt^2-\delta_{ij}dx^idx^j$$

Since the first two terms are now in polar coordinates, we should have $t\sim t+2\pi$. Transforming back:

$$\frac{\tau d}{2z_H}\sim\frac{\tau d}{2z_H}+2\pi=\frac{\left(\tau+\frac{4\pi z_h}{d}\right)d}{2z_H}$$

so we find $\tau\sim\tau+\frac{4\pi z_h}{d}$.

EXERCISE 4.2

To simplify slightly, we can consider the general metric

$$ds^2=R^2\left[\frac{dr^2}{g(r)}+g(r)d\tau^2+r^2d\Omega_{d-1}^2\right]$$

where $g(r)$ can be either $g(r)=1+r^2$ or $g(r)=1+r^2-\frac{m}{r^{d-2}}$. We have seen that regularity at the horizon imposes a particular periodicity on τ (which proceeds exactly as in Exercise 4.1), but AdS space has no such condition. However, in order to perform the difference of the path integrals for both metrics, we need to make sure that both spaces have the same asymptotic form. Therefore, we should take

$$\Delta\tilde{\tau}\sqrt{f(r_{max})}=\Delta\tau\sqrt{1+r_{max}^2}$$

For both metrics, the scalar curvature is $-\frac{d(d+1)}{2R}$, so the difference of the path integrals is

$$I_{AdS}-I_{BH}=\frac{1}{\ell_P^{d-1}}\int d^{d+1}x\sqrt{g_{AdS}}\left(-\frac{d(d+1)}{2R^2}+\frac{d(d-1)}{2R^2}\right)-\frac{1}{\ell_P^{d-1}}\int d^{d+1}x\sqrt{g_{BH}}\left(-\frac{d(d+1)}{2R^2}+\frac{d(d-1)}{2R^2}\right)$$

The determinant of the metric for general $g(r)$ is

$$g = \left(\frac{R^2}{g(r)}\right)\left(R^2 g(r)\right)\left(\prod_{j=1}^{d-1} r^2 R^2\right)\prod_{k=2}^{d-1}\prod_{i=1}^{k-1}\sin^2\theta_i = R^{2d+2} r^{2d-2}\prod_{k=2}^{d-1}\prod_{i=1}^{k-1}\sin^2\theta_i$$

where the θ_i are spherical coordinates that parameterize the $d-1$ sphere. The path integrals become

$$I_{AdS} - I_{BH} = -\frac{dR^{d+1}}{R^2\ell_P^{d-1}}\int_0^{\Delta\tau}\int_0^{r_{max}}\int d^{d-1}x\sqrt{g} + \frac{dR^{d+1}}{R^2\ell_P^{d-1}}\int_0^{\Delta\tilde{\tau}}\int_{r_H}^{r_{max}}\int d^{d-1}x\sqrt{g}$$

$$= -\frac{dR^{d+1}}{R^2\ell_P^{d-1}}\int_0^{\Delta\tau}\int_0^{r_{max}}\int_{S_{d-1}} r^{d-1} + \frac{dR^{d+1}}{R^2\ell_P^{d-1}}\int_0^{\Delta\tilde{\tau}}\int_{r_H}^{r_{max}}\int_{S_{d-1}} r^{d-1}$$

$$= -\frac{S_{d-1}R^{d-1}}{\ell_P^{d-1}}\left(\Delta\tau r_{max}^d - \Delta\tilde{\tau}\left(r_{max}^d - r_H^d\right)\right) = -\frac{S_{d-1}R^{d-1}}{\ell_P^{d-1}TL}\left(\frac{r_{max}^{d+1}}{\sqrt{1+r_{max}^2}} - \frac{r_{max}\left(r_{max}^d - r_H^d\right)}{\sqrt{1+r_{max}^2 - \frac{m}{r_{max}^{d-2}}}}\right)$$

When $r_{max} \to \infty$, this is

$$\frac{S_{d-1}R^{d-1}}{\ell_P^{d-1}TL}\left(r_H^{d-2} - r_H^d\right)$$

which is positive when $r_H < 1$.

EXERCISE 4.3

Substituting $A = z^{-1}$ gives

$$0 = z^{d+1}\partial_z\left(z^{1-d}\partial_z\psi\right) - k^2 z^2 - m^2R^2\psi = z^{d+1}\partial_z\left(z^{1-d}\partial_z\left[z^{d/2}h\right]\right) - k^2 z^{2+\frac{d}{2}}h - m^2R^2 z^{d/2}h$$

$$= \frac{d}{2}z^{d+1}\left[-\frac{d}{2}z^{-\frac{d}{2}-1}h + z^{-d/2}\partial_z h\right] + z^{d+1}\left[\left(1-\frac{d}{2}z^{-d/2}\partial_z h + z^{1-d/2}\partial_z^2 h\right)\right] - k^2 z^{2+d/2}h - \left(\alpha^2 - \frac{d^2}{4}\right)z^{d/2}h$$

$$= z^{d/2}\left[\frac{d}{2}z\partial_z h + \left(1-\frac{d}{2}\right)z\partial_z h + z^2\partial_z^2 h - k^2z^2h - \alpha^2 h\right]$$

Assuming that $z \neq 0$, this becomes

$$\left(z\partial_z + z^2\partial_z^2 - k^2z^2 - \alpha^2\right)h = 0$$

This has the general solution $h(z) = AJ_\alpha(-ikz) + BY_\alpha(-ikz)$. The boundary condition $h(0) = 0$ fixes

$$B = -\frac{AJ_\alpha(0)}{Y_\alpha(0)} = 0$$

The other boundary condition gives

$$0 = AJ_\alpha(-ikz_\star)$$

Since we don't want $A = 0$, this fixes $-ikz_\star = u_{\alpha,n}$, and squaring gives $-k^2 = \frac{u_{\alpha,n}^2}{z_\star^2}$. Substituting into $h(z)$, we find

$$h(z) = AJ_\alpha\left(\frac{z}{z_\star}u_{\alpha,n}\right)$$