AdS/CFT - Homework 5

M. Ross Tagaras
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EXERCISE 1

When we reach the unitarity bound, a generic long multiplet splits into a product of a short multiplet with the same Lorentz quantum numbers and a second short multiplet that contains the null states of the first. We first need to identify the null states for $[j, \bar{j}]$, $[j, 0]$, and $[0, 0]$. Generally, the representation $[j, \bar{j}]_{\tilde{\Delta}}$ has the null descendant $P^\mu |[j, \bar{j}]_{\tilde{\Delta}}\rangle$, where $\tilde{\Delta}$ is determined by the unitarity bound and the level. The exception is the case $j = \bar{j} = 0$, where an additional level two constraint gives the null descendant $P^\mu P_\mu |[0, 0]_1\rangle$.

The unitarity bounds are

$$\Delta \geq \begin{cases} \frac{j}{2} + \frac{\bar{j}}{2} + 2, & j > 0,\ \bar{j} > 0 \\ \frac{j}{2} + 1, & j > 0,\ \bar{j} = 0 \\ \frac{\bar{j}}{2} + 1, & j = 0,\ \bar{j} > 0 \\ 1, & j = \bar{j} = 0 \end{cases}$$

When $j, \bar{j} \neq 0$, the null state is

$$P^\mu |[j, \bar{j}]_{\frac{j}{2} + \frac{\bar{j}}{2} + 2}\rangle$$

and L' is $[j - 1, \bar{j} - 1]$. We find the rule

$$[j, \bar{j}]_{\frac{j}{2} + \frac{\bar{j}}{2} + 2} \oplus [j - 1, \bar{j} - 1]_{\frac{j}{2} + \frac{\bar{j}}{2} + 3} \rightarrow [j, \bar{j}]_{\frac{j}{2} + \frac{\bar{j}}{2} + 2 + \epsilon}$$

When $j \neq 0$, $\bar{j} = 0$, the null state is

$$P^\mu |[j, 0]_{\frac{j}{2} + 1}\rangle$$

and L' is $[j + 1, -1]$. We find the rule

$$[j, 0]_{\frac{j}{2} + 1} \oplus [j + 1, -1]_{\frac{j}{2} + 2} \rightarrow [j, 0]_{\frac{j}{2} + 1 + \epsilon}$$

When $j = \bar{j} = 0$, the null state is

$$P^\mu P_\mu |[0, 0]_1\rangle$$

and L' is $[,]$. We find the rule

$$[0, 0]_1 \oplus [,]_3 \rightarrow [0, 0]_{1 + \epsilon}$$

EXERCISE 2

The commutator of variations acts on a field $\mathcal{W}$ as

$$\begin{aligned}[\delta_1,\delta_2]\mathcal{W} =&\epsilon^{\alpha}{}_a\epsilon^{\beta}{}_b\left[Q^a{}_\alpha,Q^b{}_\beta\right]\mathcal{W}+\epsilon^{\alpha}{}_a\dot{\epsilon}^{\dot{\beta}b}\left[Q^a{}_\alpha,\dot{Q}_{\dot{\beta}_b}\right]\mathcal{W}+\epsilon^{\alpha}{}_a e^{\nu}\left[Q^a{}_\alpha,B_\nu\right]\mathcal{W}+\dot{\epsilon}^{\dot{\alpha}a}\epsilon^{\beta}{}_b\left[\dot{Q}_{\dot{\alpha}a},Q^b{}_\beta\right]\mathcal{W}\\ &+\dot{\epsilon}^{\dot{\alpha}a}\dot{\epsilon}^{\dot{\beta}b}\left[\dot{Q}_{\dot{\alpha}a},\dot{Q}_{\dot{\beta}b}\right]\mathcal{W}+\dot{\epsilon}^{\dot{\alpha}a}e^{\nu}\left[\dot{Q}_{\dot{\alpha}a},B_\nu\right]\mathcal{W}+e^{\mu}\epsilon^{\beta}{}_b\left[B_\mu,Q^b{}_\beta\right]\mathcal{W}+e^{\mu}\dot{\epsilon}^{\dot{\beta}b}\left[B_\mu,\dot{Q}_{\dot{\beta}b}\right]\mathcal{W}+e^{\mu}e^{\nu}\left[B_\mu,B_\nu\right]\mathcal{W}\end{aligned}$$

$$=\delta_1\left(\delta_2\mathcal{W}\right)-(1\leftrightarrow 2)$$

We find a similar expression for the anticommutator of variations. Performing the variations of $\mathcal{W}$ using the given expressions, we will find a sum of terms proportional to the various fields and combinations of ϵ and e. From these, we will be able to directly read off the expressions for the various (anti)commutators. For example, varying Φ_m gives

$$[\delta_1,\delta_2]\Phi_m=\delta_1(\delta_2\Phi_m)-(1\leftrightarrow 2)=\delta_1\left(\epsilon^{\beta}{}_b\sigma^{bc}_m\psi_{bc}+\dot{\epsilon}^{b\dot{\beta}}\sigma_{mbc}\dot{\psi}_{\dot{\beta}}{}^c+e^{\nu}D_\nu\Phi_m\right)-\dots$$

To find $[B_\mu,B_\nu]$, we need to pick out the terms proportional to $e^\mu e^\nu\Phi_m$. Clearly, such terms can only come from varying $e^\nu D_\nu\Phi_m$:

$$[\delta_1,\delta_2]\Phi_m=2ige^\mu e^\nu\mathcal{F}_{\nu\mu}\Phi_m+e^\mu e^\nu\mathcal{D}_\nu\mathcal{D}_\mu\Phi_m-e^\mu e^\nu\mathcal{D}_\mu\mathcal{D}_\nu\Phi_m+\dots=e^\mu e^\nu\left(2ig\mathcal{F}_{\nu\mu}+\left[\mathcal{D}_\nu,\mathcal{D}_\mu\right]\right)\Phi_m+\dots$$

$$=e^\mu e^\nu\left[-ig\mathcal{F}_{\mu\nu}\right]\Phi_m+\dots$$

Comparing to the expression for $[\delta_1,\delta_2]$ in terms of commutators, we find

$$\left[B_\mu,B_\nu\right]=-ig\mathcal{F}_{\mu\nu}$$

The rest of the algebra follows similar logic.

EXERCISE 3

We want to compute the composite operators associated with

$$[0,1]^{(1,1,0)}_{5/2}\qquad [1,1]^{(1,0,1)}_{3}\qquad [0,0]^{(2,0,0)}_{3}$$

The case of $[0,2]^{(0,1,0)}_3$ was done in class as an example, and it is simple to relate the three cases above to operators created by exchanging $Q\leftrightarrow\tilde{Q}$ as needed. Using the notation of Beisert, the action of Q on the fields is (up to numerical factors)

$$[Q^a_\alpha,\phi_m]=\sigma^{ab}_m\psi_{\alpha b}$$

$$\left\{Q^a_\alpha,\psi_{b\beta}\right\}=\sigma^\mu_{\alpha\dot{\gamma}}\sigma^\nu_{\beta\dot{\delta}}\epsilon^{\dot{\gamma}\dot{\delta}}\delta^a_bF_{\mu\nu}+g\sigma^m_{ac}\sigma^{cd}_n\delta_{db}\epsilon_{\alpha\beta}\left[\phi_m,\phi^n\right]$$

$$\left\{Q^a_\alpha,\bar{\psi}^b_{\dot{\beta}}\right\}=\sigma^{ab}_n\sigma^\mu_{\alpha\dot{\beta}}D_\mu\phi^n$$

$$\left[\tilde{Q}_{\dot{\alpha}a},\phi^n\right]=\sigma_{m,ab}\bar{\psi}^b_{\dot{\alpha}}$$

$$\left\{\tilde{Q}_{\dot{\alpha}a}, \psi_{\beta b}\right\} = \sigma^{n}_{ab}\sigma^{\mu}_{\dot{\alpha}\beta}D_{\mu}\phi_{n}$$

The primary associated with $B_1\bar{B}_1[0,0]_2^{(0,2,0)}$ is

$$\mathrm{Tr}\left(\phi_m\phi_n + \delta_{mn}\phi_k\phi^k\right)$$

Applying a single Q, we find (again dropping coefficients)

$$[1,0]_{5/2}^{(0,1,1)} \leftrightarrow \mathrm{Tr}\left[\left(\sigma_m^{ab}\phi_n + \phi_m\sigma_n^{ab} + \delta_{mn}\sigma_k^{ab}\phi^k\right)\psi_{\alpha b}\right]$$

Applying QQ, we find

$$\begin{aligned}[0,0]_3^{(0,0,2)} \leftrightarrow \mathrm{Tr}\Bigg[& \left(\sigma_m^{ab}\sigma_n^{cd} + \sigma_n^{ab}\sigma_m^{cd} + \delta_{mn}\sigma_k^{ab}\sigma^{k,cd}\right)\psi_{\alpha b}\psi_{\gamma d} \\ & + \left(\sigma_m^{ab}\phi_n + \sigma_n^{ab}\phi_m + \delta_{mn}\sigma_k^{ab}\phi^k\right)\left(\sigma^{\mu}_{\gamma\dot{\delta}}\sigma^{\nu}_{\alpha\dot{\epsilon}}\epsilon^{\dot{\delta}\dot{\epsilon}}\delta_b^c F_{\mu\nu} + g\sigma^r_{cd}\sigma_s^{de}\delta_{eb}\epsilon_{\gamma\alpha}\left[\phi_r,\phi^s\right]\right)\Bigg]\end{aligned}$$

Applying $Q\tilde{Q}$ gives

$$[1,1]_3^{(1,0,1)} \leftrightarrow \mathrm{Tr}\left[\left(\sigma_m^{ab}\sigma_{n,cd} + \sigma_n^{ab}\sigma_{m,cd} + \delta_{mn}\sigma_k^{ab}\sigma^k_{cd}\right)\bar{\psi}^d_{\dot{\gamma}}\psi_{\alpha b} + \left(\sigma_m^{ab}\phi_n + \sigma_n^{ab}\phi_m + \delta_{mn}\sigma_k^{ab}\phi^k\right)\sigma^r_{cb}\sigma^{\mu}_{\dot{\gamma}\alpha}D_{\mu}\phi_r\right]$$