AdS/CFT - Homework 1

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PROBLEM 1

We identify the generators of conformal transformations with $SO(d+1,1)$ generators as follows:

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \\ J_{\mu+} &= P_\mu \\ J_{\mu-} &= K_\mu \\ J_{+-} &= D \end{aligned}$$

We should have

$$[J_{MN}, J_{RS}] = -i\left(\eta_{MR}J_{NS} + \eta_{SM}J_{RN} - \eta_{NR}J_{MS} - \eta_{SN}J_{RM}\right)$$

where η_{MN} is the $\mathbb{R}^{d+1,1}$ metric. The d-dimensional generators obey the relations

$$\begin{aligned} \left[M_{\mu\nu}, M_{\rho\sigma}\right] &= -i\left(\delta_{\mu\rho}M_{\nu\sigma} + \text{permutations}\right) \\ \left[M_{\mu\nu}, P_\rho\right] &= i\left(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu\right) \\ \left[M_{\mu\nu}, K_\rho\right] &= -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \\ \left[D, P_\mu\right] &= -iP_\mu \\ \left[D, K_\mu\right] &= iK_\mu \\ \left[P_\mu, K_\nu\right] &= 2i\left(\delta_{\mu\nu}D - M_{\mu\nu}\right) \end{aligned}$$

Using these, we can check that the $\{J_{MN}\}$ obey the correct algebra. The $\left[J_{\mu\nu}, J_{\rho\sigma}\right]$ commutator is trivially correct. The others are

$$\begin{aligned} \left[J_{\mu\nu}, J_{\rho+}\right] &= \left[M_{\mu\nu}, P_\rho\right] = -i\left(\delta_{\mu\rho}J_{\nu+} - \delta_{\nu\rho}J_{\mu+}\right) = -i\left(\eta_{\mu\rho}J_{\nu+} - \eta_{\nu\rho}J_{\mu+} + \eta_{+\mu}J_{\rho\nu} - \eta_{+\nu}J_{\rho\mu}\right) \\ \left[J_{\mu\nu}, J_{\rho-}\right] &= \left[M_{\mu\nu}, K_\rho\right] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) = -i\left(\eta_{\mu\rho}J_{\nu-} + \eta_{-\mu}J_{\rho\nu} - \eta_{\nu\rho}J_{\mu-} - \eta_{-\nu}J_{\rho\mu}\right) \\ \left[J_{\mu\nu}, J_{+-}\right] &= \left[M_{\mu\nu}, D\right] = 0 = -i\left(\eta_{\mu+}J_{\nu-} + \eta_{\mu-}J_{\nu+} - \eta_{\nu+}J_{\mu-} - \eta_{-\nu}J_{\mu+}\right) \\ \left[J_{\mu+}, J_{\nu-}\right] &= \left[P_\mu, K_\nu\right] = i\left(\delta_{\mu\nu}D - M_{\mu\nu}\right) = i\left(\eta_{\mu\nu}J_{+-} + \eta_{+-}J_{\mu\nu}\right) \\ \left[J_{\mu+}, J_{+-}\right] &= \left[P_\mu, D\right] = iJ_{\mu+} = -i\left(\eta_{\mu+}J_{+-} + \eta_{-\mu}J_{++} - \eta_{++}J_{\mu-} - \eta_{-+}J_{+\mu}\right) \\ \left[J_{\mu-}, J_{+-}\right] &= \left[K_\mu, D\right] = -iJ_{\mu-} = -i\left(\eta_{\mu+}J_{--} + \eta_{-\mu}J_{+-} - \eta_{-+}J_{\mu-} - \eta_{--}J_{+\mu}\right) \end{aligned}$$

PROBLEM 2

The induced metric $g_{\mu\nu}(x)$ is determined by

$$ds^2 = dx^2 - dX^+dX^-\Big|_{X^+=f(x^\mu),\ X^-=\frac{x^2}{X^+}} = g_{\mu\nu}(x)dx^\mu dx^\nu$$

where the x^μ are $\mathbb{R}^d$ coordinates. Evaluating dX^+ and dX^- for a general $f(x)$ gives

$$dX^+ = \frac{\partial f(x^\mu)}{\partial x^\nu} dx^\nu \qquad dX^- = \left(\frac{2x^\nu}{f(x)} - \frac{x^2}{f^2(x)} \frac{df}{dx^\nu} \right) dx^\nu$$

which in turn gives

$$ds^2 = \left[1 - \frac{2x}{f(x)} \frac{\partial f}{\partial x} + \left(\frac{x}{f(x)} \frac{\partial f}{\partial x} \right)^2 \right] dx^2 = \big(h(x) - 1\big)^2 dx^2$$

where $h(x) = \frac{x}{f(x)} \frac{\partial f}{\partial x}$. We will pick some $h(x)$ that gives our desired induced metric. Then we have to solve the differential equation given by the definition of $h(x)$:

$$\frac{\partial f(x)}{\partial x} = \frac{h(x)}{x} f(x) \implies f(x) = e^{\int dx \, \frac{h(x)}{x}}$$

The metric for $\mathbb{R} \times S_{d-1}$ is related to the metric of $\mathbb{R}^d$ by

$$r^{-2} ds^2_{\mathbb{R}^d} = ds^2_{\mathbb{R} \times S_{d-1}}$$

so if we choose $h(x) = 1 + \frac{1}{r}$, we find

$$f(x) = e^{\ln r - \frac{1}{r}} = r e^{-1/r} \implies ds^2 = \frac{1}{r^2} dx^2 = ds^2_{\mathbb{R} \times S_{d-1}}$$

Using Poincaré coordinates (with $R = 1$), the metric for $EAdS_2 \times S_2$ is

$$ds^2 = \frac{dt^2 + dr^2}{r^2} + ds^2_{S_2} = \frac{1}{r^2} \left(dt^2 + dr^2 + r^2 ds^2_{S_2} \right) = \frac{1}{r^2} \left(dt^2 + ds^2_{\mathbb{R}^3} \right) = \frac{1}{r^2} ds^2_{\mathbb{R}^4}$$

so we can choose $f(x) = re^{-1/r}$ and follow the same procedure to get the induced metric.

PROBLEM 3

On the null cone, a three-point function of two spin-one operators with gauge indices will take the form

$$\left\langle J^a_M(P_1) J^b_N(P_2) \phi(P_3) \right\rangle = \frac{C^{ab}}{(P_{12})^{\alpha_{123}} (P_{23})^{\alpha_{231}} (P_{13})^{\alpha_{132}}} \times (\text{tensor structure})_{MN}$$

where $2\alpha_{ijk} = \tau_i + \tau_j - \tau_k$ with $\tau_i = \Delta_i + \ell_i$ and $P_{ij} = -2P_i \cdot P_j$. Throughout, capital letters will denote quantities on the null cone, and lowercase letters will denote projected quantities.

The tensor structure must be transverse:

$$X^M (\dots)_{MN} = 0 \qquad Y^N (\dots)_{MN} = 0$$

To solve this problem, I will use the methods described in[1]. To summarize, we can take advantage of the one-to-one map between symmetric tensors $f_{a_1 a_2 \dots a_n}$ and d-dimensional polynomials $f(z) = f_{a_1 \dots a_n} z^{a_1} \dots z^{a_n}$. From here, it can be shown that the following two useful properties hold:

1. Any three-point function of operators of arbitrary spin can be constructed on the null cone using a simple set of identically transverse structures, defined below.
2. Conservation can be applied as needed, directly on the null cone.

After performing our calculations on the null cone, we project to coordinate space as usual. The final step is to map back from the resulting polynomial to a symmetric tensor. This can be accomplished by repeatedly acting with a specific differential operator:

$$f_{a_1 \dots a_n} = \frac{1}{n! \left(\frac{d}{2} - 1\right)_n} D_{a_1} \dots D_{a_n} f(z)$$

where D_a is defined as

$$D_a = \left(\frac{d}{2} - 1 + z^b \frac{\partial}{\partial z^b}\right) \frac{\partial}{\partial z^a} - \frac{1}{2} z_a \frac{\partial^2}{\partial z^b \partial z_b}$$

A general three-point tensor structure will be a linear combination of terms of the form

$$\prod_i V_i^{m_i} \prod_{i<j} H_{ij}^{n_{ij}}$$

where

$$V_i \equiv V_{i,jk} = \frac{(Z_i \cdot P_j)(P_i \cdot P_k) - (Z_i \cdot P_k)(P_i \cdot P_j)}{P_j \cdot P_k}, \qquad i, j, k \in \{1, 2, 3\}$$

$$H_{ij} = 2(Z_i \cdot P_j)(Z_j \cdot P_i) - 2(Z_i \cdot Z_j)(P_i \cdot P_j)$$

and where the exponents must obey

$$m_i + \sum_{i \neq j} n_{ij} = \ell_i$$

Using our building blocks, the null cone expression for the correlation function is immediately obvious:

$$\langle J_M(X) J_N(Y) \phi(Z) \rangle = \frac{A V_1 V_2 + B H_{12}}{(P_{12})^{d - \frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}}}$$

To impose conservation, we act with

$$\frac{\partial}{\partial P_{1M}} D_M^{(Z_1)}$$

and demand that the result vanishes, allowing us to fix A and B. This is our equivalent of

$$\partial^\mu \left\langle J_\mu^a(x) J_\nu^b(y) \phi(z) \right\rangle = 0 \qquad \text{or} \qquad \partial^\nu \left\langle J_\mu^a(x) J_\nu^b(y) \phi(z) \right\rangle = 0$$

The relevant derivatives are (with individually vanishing terms dropped)

$$\frac{\partial}{\partial Z_{1M}} (V_1 V_2) = \frac{P_2^M (Z_2 \cdot P_3)(P_1 \cdot P_2)}{P_2 \cdot P_3} - P_2^M (Z_2 \cdot P_1) - \frac{P_3^M (P_1 \cdot P_2)^2 (Z_2 \cdot P_3)}{(P_2 \cdot P_3)(P_1 \cdot P_3)} + \frac{P_3^M (P_1 \cdot P_2)(Z_2 \cdot P_1)}{(P_1 \cdot P_3)}$$

$$\frac{\partial}{\partial P_{1M}} \frac{\partial}{\partial Z_1^M} (V_1 V_2) = -\frac{(P_1 \cdot P_2)(P_1 \cdot Z_2)}{P_1 \cdot P_3} + \frac{(P_3 \cdot Z_2)(P_1 \cdot P_2)}{P_1 \cdot P_3} = V_2$$

$$\frac{\partial}{\partial P_{1M}}\left(\frac{1}{(P_{12})^{d-\frac{\Delta}{2}}(P_{23})^{\frac{\Delta}{2}}(P_{13})^{\frac{\Delta}{2}}}\right)=\frac{\left(\frac{\Delta}{2}-d\right)P_{12}^{-1}P_{2M}-\frac{\Delta}{2}P_{13}^{-1}P_{3M}}{(P_{12})^{d-\frac{\Delta}{2}}(P_{23})^{\frac{\Delta}{2}}(P_{13})^{\frac{\Delta}{2}}}$$

$$\frac{\partial}{\partial Z_{1M}}H_{12}=2(P_1\cdot P_2)Z_{2M}-2(Z_2\cdot P_1)P_{2M}$$

$$\frac{\partial}{\partial P_{1M}}\frac{\partial}{\partial Z_1^M}H_{12}=0$$

Combining these gives

$$0=\frac{V_2\left[A(1-d+\Delta)-B\Delta\right]}{(P_{12})^{d-\frac{\Delta}{2}}(P_{23})^{\frac{\Delta}{2}}(P_{13})^{\frac{\Delta}{2}}}\implies B=\frac{A(1-d-\Delta)}{\Delta}$$

so the final null cone expression is

$$\left\langle J_M(X)J_N(Y)\phi(Z)\right\rangle=A\frac{V_1V_2+\frac{1-d-\Delta}{\Delta}H_{12}}{(P_{12})^{d-\frac{\Delta}{2}}(P_{23})^{\frac{\Delta}{2}}(P_{13})^{\frac{\Delta}{2}}}$$

To project to the usual coordinate space, we use

$$P_1\cdot Z_j\to z_jx_{ij}\qquad P_i\cdot P_j\to-\frac{1}{2}x_{ij}^2\qquad Z_i\cdot Z_j\to z_iz_j$$

Substituting gives

$$\left\langle J_\mu^a(x_1)J_\nu^b(x_2)\phi(x_3)\right\rangle_{z_1,z_2}=\frac{C^{ab}}{x_{12}^{2d-\Delta}x_{23}^{\Delta}x_{13}^{\Delta}}\left[\frac{\left[(z_1\cdot x_{12})x_{13}^2-(z_1\cdot x_{31})x_{12}^2\right]\left[(z_2\cdot x_{32})x_{12}^2-(z_2\cdot x_{12})x_{23}^2\right]}{x_{23}^2x_{13}^2}\right.$$
$$\left.-\frac{2+2\Delta-2d}{\Delta}\left[-\frac{1}{2}(z_1\cdot z_2)x_{12}^2-(z_1\cdot x_{21})(z_2\cdot x_{12})\right]\right]$$

Finally, to remove $z_{1,2}$, we act with D:

$$\left\langle J^{\mu a}(x_1)J^{\nu b}(x_2)\phi(x_3)\right\rangle=D_\mu^{(z_1)}D_\nu^{(z_2)}\left\langle J^{\mu a}(x_1)J^{\nu b}(x_2)\phi(x_3)\right\rangle_{z_1,z_2}$$

$$=\frac{C^{ab}}{x_{12}^{2d-\Delta}x_{23}^{\Delta}x_{13}^{\Delta}}\left[\frac{\left[x_{21}^\mu x_{13}^2-x_{31}^\mu x_{12}^2\right]\left[x_{32}^\nu x_{12}^2-x_{12}^\nu x_{23}^2\right]}{x_{23}^2x_{13}^2}-\frac{2-2d+2\Delta}{\Delta}\left[x_{12}^\mu x_{12}^\nu-\frac{1}{2}\delta^{\mu\nu}x_{12}^2\right]\right]$$

We can use $I_{\mu\nu}(x)=\delta_{\mu\nu}-2\frac{x^\mu x^\nu}{x^2}$ to rewrite the last term slightly:

$$=\frac{C^{ab}}{x_{12}^{2d-\Delta}x_{23}^{\Delta}x_{13}^{\Delta}}\left[\frac{\left[x_{21}^\mu x_{13}^2-x_{31}^\mu x_{12}^2\right]\left[x_{32}^\nu x_{12}^2-x_{12}^\nu x_{23}^2\right]}{x_{23}^2x_{13}^2}+\frac{1-d+\Delta}{\Delta}x_{12}^2I^{\mu\nu}(x_{12})\right]$$

The null cone expression for the three-current correlation function is (with gauge indices suppressed)

$$\left\langle J_M(P_1)J_N(P_2)J_K(P_3)\right\rangle=\frac{AV_1V_2V_3+BH_{12}V_3+CH_{13}V_2+DH_{23}V_1}{(P_{12})^{\frac{d}{2}}(P_{13})^{\frac{d}{2}}(P_{23})^{\frac{d}{2}}}$$

Some useful derivatives are

$$\frac{\partial V_1}{\partial Z_{1M}} = \frac{P_2^M (P_1 \cdot P_3) - P_3^M (P_1 \cdot P_2)}{P_2 \cdot P_3}$$

$$\frac{\partial V_2}{\partial P_1^M} = \frac{(Z_2 \cdot P_3) P_2^M - Z_2^M (P_2 \cdot P_3)}{P_1 \cdot P_3} - \frac{(Z_2 \cdot P_3)(P_2 \cdot P_1) - (Z_2 \cdot P_1)(P_2 \cdot P_3)}{(P_1 \cdot P_3)^2} P_3^M$$

$$\frac{\partial V_3}{\partial P_1^M} = \frac{Z_3^M (P_2 \cdot P_3) - (Z_3 \cdot P_2) P_3^M}{P_1 \cdot P_2} - \frac{(Z_1 \cdot P_1)(P_2 \cdot P_3) - (Z_3 \cdot P_2)(P_1 \cdot P_3)}{(P_1 \cdot P_2)^2} P_2^M$$

$$\frac{\partial H_{12}}{\partial Z_{1M}} = 2(Z_2 \cdot P_1) P_2^M - 2(P_1 \cdot P_2) Z_2^M$$

$$\frac{\partial H_{13}}{\partial Z_{1M}} = 2(Z_3 \cdot P_1) P_3^M - 2(P_1 \cdot P_3) Z_3^M$$

$$\frac{\partial}{\partial P_{1M}} \left(\frac{1}{(P_{12})^{\frac{d}{2}} (P_{13})^{\frac{d}{2}} (P_{23})^{\frac{d}{2}}} \right) = -\frac{d}{2} \frac{(P_{12})^{-1} P_2^M + (P_{13})^{-1} P_3^M}{(P_{12})^{\frac{d}{2}} (P_{13})^{\frac{d}{2}} (P_{23})^{\frac{d}{2}}}$$

$$\frac{\partial V_2}{\partial Z_{1M}} = \frac{\partial V_3}{\partial Z_{1M}} = \frac{\partial^2 V_1}{\partial Z_{1M} \partial P_1^M} = \frac{\partial H_{23}}{\partial Z_{1M}} = \frac{\partial^2 H_{ij}}{\partial Z_{1M} \partial P_1^M} = 0$$

Imposing $\partial \cdot D \langle \ldots \rangle = 0$ again, we find $B = C = D$ (after a lot of algebra that I don't want to write out). The final null cone expression is

$$\langle J_M(P_1) J_N(P_2) J_K(P_3) \rangle = \frac{A V_1 V_2 V_3 + B \left(H_{12} V_3 + H_{13} V_2 + H_{23} V_1 \right)}{(P_{12})^{\frac{d}{2}} (P_{13})^{\frac{d}{2}} (P_{23})^{\frac{d}{2}}}$$

After projection and removing factors of z, $V_{i,jk}$ and H_{ij} become

$$V_{i,jk} \to \frac{x_{ik}^\mu x_{ij}^2 - x_{ij}^\mu x_{ik}^2}{x_{jk}^2} \qquad\qquad H_{ij} \to x_{ij}^2 I_{\mu\nu}(x_{ij})$$

Our final result is

$$\left\langle J^{a\mu}(x_1) J^{b\nu}(x_2) J^{c\rho}(x_3) \right\rangle = \frac{C^{abc}}{(x_{12})^d (x_{13})^d (x_{23})^d} \Bigg[A \left(\frac{\left(x_{13} x_{12}^2 - x_{12} x_{13}^2\right)^\mu \left(x_{12} x_{23}^2 - x_{23} x_{12}^2\right)^\nu \left(x_{13} x_{23}^2 - x_{23} x_{13}^2\right)^\rho}{x_{12}^2 x_{13}^2 x_{23}^2} \right)$$
$$+ B \left(\left(x_{13}^\mu x_{12}^2 - x_{12}^\mu x_{13}^2 \right) I^{\nu\rho}(x_{23}) + \left(x_{21}^\nu x_{23}^2 - x_{23}^\nu x_{21}^2 \right) I^{\mu\rho}(x_{13}) + \left(x_{13}^\rho x_{23}^2 - x_{23}^\rho x_{13}^2 \right) I^{\mu\nu}(x_{12}) \right) \Bigg]$$

where C^{abc} is a group theoretic factor and A and B are independent constants.

PROBLEM 4

As in Kaplan's review, we can view a free particle in AdS_{d+1} as a particle in $d+2$ dimensions constrained to live on the surface $X_A X^A = -X_0^2 + X_i^2 + X_{d+1}^2 = R^2$. The geodesics for such a particle follow from the equations of motion of the action

$$S = \int d\xi \left[\frac{dX_A}{d\xi} \frac{dX^A}{d\xi} + \lambda(R^2 - X_A X^A) \right]$$

where ξ is some worldline parameterization. From this action we find an equation for the ξ-evolution of X_A and a constraint:

$$\frac{d^2 X_A}{d\xi^2} + \lambda X_A = 0 \qquad X_A X^A - R^2 = 0$$

The choice of $\lambda = 1, 0, -1$ corresponds to timelike, null, or spacelike geodesics, respectively. We find:

$\lambda = 1$:

$$\frac{d^2 X_A}{d\xi^2} + X_A = 0 \implies \begin{cases} \boxed{X_A(\xi) = \alpha_A \cos\xi + \beta_A \sin\xi} \\ R^2 = \alpha_A \alpha^A \cos^2\xi + \beta_A \beta^A \sin^2\xi + \alpha_A \beta^A \sin(2\xi) \implies \boxed{\alpha_A \alpha^A = \beta_A \beta^A = R^2, \qquad \alpha_A \beta^A = 0} \end{cases}$$

$\lambda = 0$:

$$\frac{d^2 X_A}{d\xi^2} = 0 \implies \begin{cases} \boxed{X_A(\xi) = \alpha_A + \beta_A \xi} \\ R^2 = \alpha_A \alpha^A + \beta_A \beta^A \xi^2 + 2\xi \alpha_A \beta^A \implies \boxed{\alpha_A \alpha^A = R^2, \quad \beta_A \beta^A = \alpha_A \beta^A = 0} \end{cases}$$

$\lambda = -1$:

$$\frac{d^2 X_A}{d\xi^2} - X_A = 0 \implies \begin{cases} \boxed{X_A(\xi) = \alpha_A e^{\xi} + \beta_A e^{-\xi}} \\ R^2 = \alpha_A \alpha^A e^{2\xi} + \beta_A \beta^A e^{-2\xi} + 2\alpha_A \beta^A \implies \boxed{\alpha_A \alpha^A = \beta_A \beta^A = 0, \qquad 2\alpha_A \beta^A = R^2} \end{cases}$$

To find the geodesics in global/Poincaré coordinates, we use

$$X_0 = R\frac{\cosh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2 + \vec{r}^2 + R^2}{z}\right)$$

$$X_i = R\tan\rho\,\Omega_i = \frac{R}{z} r_i$$

$$X_{d+1} = R\frac{\sinh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2 + \vec{r}^2 - R^2}{z}\right)$$

These relations can be solved to find global/Poincaré coordinates in terms of the embedding coordinates:

$$\rho = \cos^{-1}\left[\frac{R}{\sqrt{X_0^2 - X_{d+1}^2}}\right] \qquad \Omega_i = \frac{X_i}{\sqrt{X_0^2 - X_{d+1}^2 - R^2}} \qquad \tau = \cosh^{-1}\left[\frac{1}{\sqrt{X_0^2 - X_{d+1}^2}}\right]$$

$$z = \frac{R^2}{X_0 - X_{d+1}} \qquad r_i = \frac{RX_i}{X_0 - X_{d+1}}$$

[1] "Spinning Conformal Correlators", M. Costa, J. Penedones, D. Poland, S. Rychkov, arxiv:1107.3554v3, 2015