AdS/CFT - Homework 3

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EXERCISE 2.5

For $\epsilon(x) = \lambda x^\mu$, the Ward identity is

$$-\left[x^\mu \frac{\partial}{\partial x^\mu} + \frac{\Delta}{d}\nabla_\alpha x^\alpha\right]\frac{1}{|y|^{2\Delta}} = \int_{\partial B} dS_\mu \; x_\nu \left[\frac{CH^{\mu\nu}(0,y,x)}{|y|^{2\Delta-d+2}|x|^{d-2}|y-x|^{d-2}} - \langle T^{\mu\nu}(x)\rangle \frac{1}{|y|^{2\Delta}}\right]$$

We also have

$$\nabla_\alpha x^\alpha = d + \frac{1}{2}x^\alpha g^{\rho\sigma}\partial_\alpha g_{\rho\sigma}$$

$$V_\alpha V^\alpha = \left(\frac{|x-y|^\alpha}{(x-y)^2} - \frac{x^\alpha}{x^2}\right)\left(\frac{|x-y|_\alpha}{(x-y)^2} - \frac{x_\alpha}{x^2}\right) = \frac{1}{(x-y)^2} - 2\frac{x\cdot|x-y|}{x^2(x-y)^2} + \frac{1}{x^2} = \frac{y^2}{x^2(x-y)^2}$$

$$x_\nu V^\mu V^\nu = \left(\frac{|x-y|^\mu}{(x-y)^2} - \frac{x^\mu}{x^2}\right)\left(\frac{x\cdot|x-y|}{(x-y)^2} - 1\right) = \left(\frac{|x-y|^\mu}{(x-y)^2} - \frac{x^\mu}{x^2}\right)\frac{y\cdot|x-y|}{(x-y)^2}$$

If $g_{\mu\nu}$ is Weyl equivalent to a flat metric, then $\langle T_{\mu\nu}(x)\rangle = 0$, and $\nabla_\alpha x^\alpha = d$. The Ward identity simplifies to

$$-\frac{\Delta}{y^{d-2}} = C\int_{\partial B} dS_\mu \left(\frac{|x-y|^\mu}{(x-y)^2} - \frac{x^\mu}{x^2}\right)\frac{y\cdot|x-y|}{|x-y|^d|x|^{d-2}} - \frac{Cy^2}{d}\int_{\partial B} dS_\mu \frac{x^\mu}{|x-y|^d|x|^d}$$

From here, we see that C is equal to

$$C = -\frac{\Delta d}{dy^{d-2}I_1 - y^d I_2}$$

where $I_{1,2}$ are

$$I_1 = \int_{\partial B} dS_\mu \left(\frac{|x-y|^\mu}{(x-y)^2} - \frac{x^\mu}{x^2}\right)\frac{y\cdot|x-y|}{|x-y|^d|x|^{d-2}} \qquad I_2 = \int_{\partial B} dS_\mu \frac{x^\mu}{|x-y|^d|x|^d}$$

To evaluate $I_{1,2}$ we can take an infinitesimal sphere containing x, so that $|x| \ll |y|$. Then we get

$$I_2 \approx \frac{1}{y^d}\int_{\partial B} dS_\mu \frac{x^\mu}{x^d} = \frac{S_d}{y^d}$$

$$I_1 \approx \int_{\partial B} dS_\mu \left(\frac{-x^\mu y^2 - y^\mu(x\cdot y) + y^\mu y^2}{(y^2 - 2x\cdot y)y^d x^{d-2}} + \frac{x^\mu y^2}{x^d y^d}\right) \approx \frac{1}{y^{d-2}}\int_{\partial B} dS_\mu \frac{x^\mu}{x^d} = \frac{S_d}{y^{d-2}}$$

so our final result for C is

$$C = -\frac{\Delta d}{d-1}\frac{1}{S_d}$$

EXERCISE 3.3

For simplicity, we set $Y = 0$ in all results. It can easily be restored in the final expression using translation invariance, if necessary. Using $2\Delta = d + \sqrt{d^2 + 4m^2}$ and the given expression for ∇^2, we can write $(\nabla^2 - m^2)\Pi$ as

$$(\nabla^2 - m^2)\Pi = -X^2 \partial_X^2 \Pi + X^\mu \frac{\partial}{\partial X^\mu}\left[d + X^\nu \frac{\partial}{\partial X^\nu}\right]\Pi - \Delta(\Delta - d)\Pi$$

We now need to check that this expression gives

$$(\nabla^2 - m^2)\Pi = -\delta(X) \qquad \text{with} \qquad \Pi = \frac{C_\Delta}{\zeta^\Delta}\ {}_2F_1\left[\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, \frac{-4}{\zeta}\right]$$

Taking the derivatives gives

$$\begin{aligned}\left(\nabla^2 - m^2\right)\Pi =& \frac{2C}{\zeta^{\Delta-2}}\frac{\partial^2 F}{\partial \zeta^2} - \frac{4C(\Delta - 1 - d)}{\zeta^\Delta}\frac{\partial F}{\partial \zeta} + \frac{8C(2\Delta + d - 1)}{\zeta^{\Delta+1}}F - \frac{C\Delta(d-\Delta)}{\zeta^\Delta}F + \frac{C(d - 4\Delta - 1)}{\zeta^\Delta}\frac{\partial F}{\partial \zeta} \\ &+ \frac{8C}{\zeta^{\Delta-1}}\frac{\partial^2 F}{\partial \zeta^2} - \Delta(\Delta - d)\frac{C}{\zeta^\Delta}F\end{aligned}$$

$$= \frac{8}{\zeta^{\Delta+1}}\left[\left(\frac{\zeta^2}{4}(-d + 2\Delta - 1) + \zeta\left(-\frac{d}{2} + 2\Delta - \frac{1}{2}\right)\right)\frac{\partial F}{\partial \zeta} - \Delta\left(-\frac{d}{2} + \Delta + \frac{1}{2}\right)F - \left(\frac{\zeta^3}{4} + \zeta^2\right)\frac{\partial^2 F}{\partial \zeta^2}\right]$$

Conveniently, the term in brackets is identically zero. To see this, note that ${}_2F_1[a, b, c, z]$ is defined as the solution to

$$z(1-z)\frac{d^2F}{dz^2} + \left[c - z(a + b + 1)\right]\frac{dF}{dz} - abF = 0$$

If we take $z = -4/\zeta$, we find

$$\frac{\partial F}{\partial z} = \frac{\zeta^2}{4}\frac{\partial F}{\partial \zeta} \qquad\qquad \frac{\partial^2 F}{\partial z^2} = \frac{\zeta^4}{16}\frac{\partial^2 F}{\partial \zeta^2} + \frac{\zeta^3}{8}\frac{\partial F}{\partial \zeta}$$

Substituting these into the hypergeometric equation gives the terms in parentheses, so we conclude that $(\nabla^2 - m^2)\Pi(X) = 0$ for $X \neq 0$. If we take $X \to 0$, we get $0/0$, and presumably this is finite if we take the limit in a precise way.

EXERCISE 3.5

The integral we would like to evaluate is

$$I = \frac{1}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\int_{AdS} dX \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty ds_3\ s_1^{\Delta_1 - 1} s_2^{\Delta_2 - 1} s_3^{\Delta_3 - 1} e^{2X\cdot(s_1P_1 + s_2P_2 + s_3P_3)}$$

Using the identity for the AdS integral, this is

$$I = \frac{\pi^{d/2}}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}\int_0^\infty dz \int_0^\infty ds_1 \int_0^\infty ds_2 \int_0^\infty ds_3\ z^{-1-\frac{d}{2}} s_1^{\Delta_1 - 1} s_2^{\Delta_2 - 1} s_3^{\Delta_3 - 1} e^{-z + (s_1P_1 + s_2P_2 + s_3P_3)^2/z}$$

Changing variables to $s_i = \frac{\sqrt{zt_1t_2t_3}}{t_i}$, I becomes

$$I = \frac{\pi^{d/2}}{8\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \int_0^\infty dz \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 \, t_1^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}-1} t_2^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}-1} t_3^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}-1} z^{\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}-2} \times e^{-z} e^{2(P1P_2t_3+P_1P_3t_2+P_2P_3t_1)}$$

where terms with $P_i^2 = 0$ have been dropped. The integrals can now be directly evaluated, giving

$$I = \frac{\pi^{d/2}\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3}{2}-1\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)}{8\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\left(-2P_1\cdot P_2\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\left(-2P_1\cdot P_3\right)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}\left(-2P_2\cdot P_3\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}$$

which is of the desired form

$$\frac{\lambda_{123}}{\left(-2P_1\cdot P_2\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\left(-2P_1\cdot P_3\right)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}\left(-2P_2\cdot P_3\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}$$

EXERCISE 3.7

We want to compute

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{z^{2\Delta-d}\phi_b(y)}{\left(z^2+(x-y)^2\right)^\Delta}$$

If we take $x \neq y$, then this limit is easily seen to be zero. On the other hand, if we take $x = y$, then we find

$$\lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{z^{2\Delta-d}\phi_b(y)}{(z^2)^\Delta} = \lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{\phi_b(y)}{z^d} \to \infty$$

Therefore, we see that the final expression for this limit should be proportional to the integral of $\delta(x-y)$, which when evaluated, will give

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \alpha\phi_b(x)$$

To find the value of α, we can write

$$\lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{z^{2\Delta-d}\phi_b(y)}{\left(z^2+(x-y)^2\right)^\Delta} = \alpha\int d^dy\delta(x-y)\phi_b(y)$$

which gives

$$\lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{z^{2\Delta-d}}{\left(z^2+(x-y)^2\right)^\Delta} = \alpha\int d^dy\delta(x-y) = \alpha$$

Now we can temporarily set $x = 0$ and evaluate the integral:

$$\alpha = \lim_{z\to 0}\sqrt{C_\Delta}\int d^dy \frac{z^{2\Delta-d}}{(z^2+y^2)^\Delta} = \sqrt{C_\Delta}\int d\Omega \int_0^\infty dy \frac{y^{d-1}z^{2\Delta-d}}{(z^2+y^2)}$$

The integral over y has the following solution:

In[91]:=

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Assuming[{z > 0, 2 Δ > d, d > 0}, Integrate[y^(d-1) z^(2 Δ-d) / (z^2 + y^2)^Δ, {y, 0, ∞}]]
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Out[91]=

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Gamma[d/2] Gamma[-d/2 + Δ] / (2 Gamma[Δ])
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which gives

$$\alpha = \frac{\sqrt{C_\Delta}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\Delta-\frac{d}{2}\right)}{2\Gamma(\Delta)}\int d\Omega = \frac{\sqrt{C_\Delta}\pi^{d/2}\Gamma\left(\Delta-\frac{d}{2}\right)}{\Gamma(\Delta)}\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{2\Gamma\left(\Delta-\frac{d}{2}+1\right)} = \frac{\Gamma\left(\Delta-\frac{d}{2}\right)}{2\sqrt{C_\Delta}\Gamma\left(\Delta-\frac{d}{2}+1\right)} = \frac{1}{\sqrt{C_\Delta}(2\Delta-d)}$$

All together, we see that

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \frac{\phi_b(x)}{\sqrt{C_\Delta}(2\Delta-d)}$$

EXERCISE 3.8

The variation of the action is

$$0 = \int_{AdS}\sqrt{-g}\left[(2\beta+1)\nabla\phi\cdot\delta\nabla\phi + m^2\phi\delta\phi + \beta\delta\phi\nabla^2\phi + \beta\phi\delta\nabla^2\phi\right]$$

$$= \int_{AdS}\sqrt{-g}\left[\nabla_\alpha\left(\phi\nabla^\alpha\delta\phi\right) + 2\beta\nabla\phi\cdot\nabla\delta\phi + \beta\delta\phi\nabla^2\phi + \beta\phi\delta\nabla^2\phi\right]$$

Substituting $\delta\phi = z^\Delta f(x)$ gives

$$0 = \int_{AdS}\sqrt{-g}f(x)\left[\partial_z\left(\phi\partial_z z^\Delta\right) + 2\beta\partial_z\phi\partial_z z^\Delta + \beta z^\Delta\partial_z^2\phi + \beta\phi\partial_z^2 z^\Delta\right]$$

$$= \int_{AdS}\sqrt{-g}f(x)\left[\Delta z^{\Delta-1}\partial_z\phi + \Delta(\Delta-1)z^{\Delta-2}\phi + 2\beta\Delta z^{\Delta-1}\partial_z\phi + \beta z^\Delta\partial_z^2\phi + \beta\Delta(\Delta-1)z^{\Delta-2}\phi\right]$$

Since ϕ should obey the boundary condition we found in the previous problem, we can multiply by z^{2-d} and take $z\to 0$:

$$0 = \lim_{z\to 0}\int_{AdS}\sqrt{-g}f(x)\left[\Delta(2\beta+1)z^{\Delta-d+1}\partial_z\phi + \Delta(\Delta-1)(\beta+1)z^{\Delta-d}\phi + \beta z^{\Delta-d+2}\partial_z^2\phi\right]$$

We now integrate by parts to move powers of z inside the derivatives. This results in

$$0 = \lim_{z\to 0}\int_{AdS}\sqrt{-g}f(x)\left[(d-2)\left(\Delta+(\beta(d-1))\right)z^{\Delta-d}\phi + (\Delta+2\beta d-4\beta)\partial_z\left(z^{\Delta-d+1}\phi\right) + \beta\partial_z^2\left(z^{\Delta-d+2}\phi\right)\right]$$

$$= \int_{AdS}\frac{f(x)\phi_b(x)}{\sqrt{C_\Delta}(2\Delta-d)}\Big[(\beta(-(-d+\Delta+2))(-d+\Delta+1) + 2\beta(-d+\Delta+1)(-d+\Delta+2) + (2\beta+1)(-\Delta)(-d+\Delta+1)$$

$$+(\beta+1)(\Delta-1)\Delta) + ((2\beta+1)\Delta - 2\beta(-d+\Delta+2)) + 2\beta\Big]$$

Requiring the term in brackets to vanish gives $\beta = \frac{\Delta - d}{d}$. Using this value, S_2 becomes

$$S_2 = \int_{AdS} \sqrt{-g} \left[\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{2} m^2 \phi^2 + \frac{\Delta - d}{d} \nabla_\alpha (\phi \nabla^\alpha \phi) \right]$$

$$= \int_{AdS} \sqrt{-g} \left[\frac{1}{2} \nabla_\alpha (\phi \nabla^\alpha \phi) - \frac{1}{2} \phi (\nabla^2 - m^2) \phi + \frac{\Delta - d}{d} \nabla_\alpha (\phi \nabla^\alpha \phi) \right] = \frac{2\Delta - d}{2d} \int_{AdS} \sqrt{-g} \nabla_\alpha (\phi \nabla^\alpha \phi)$$

Now we can plug in our expression for ϕ:

$$S_2 = C_\Delta \frac{2\Delta - d}{2d} \int_{AdS} \sqrt{-g} \int d^d y_1 \int d^d y_2 \; \phi_b(y_1) \phi_b(y_2) \nabla_\alpha \left(\frac{z^\Delta}{\left(z^2 + (x - y_1)^2\right)^\Delta} \nabla^\alpha \frac{z^\Delta}{\left(z^2 + (x - y_2)^2\right)^\Delta} \right)$$

Partially performing the AdS integral gives

$$S_2 = C_\Delta \frac{2\Delta - d}{2d} \int d^d x d^d y_1 d^d y_2 \frac{1}{z^{d-1}} \left(\frac{z^\Delta}{\left(z^2 + (x - y_1)^2\right)^\Delta} \partial_z \frac{z^\Delta}{\left(z^2 + (x - y_2)^2\right)^\Delta} \right)$$

which matches equation (103) with (104) substituted.

EXERCISE 3.9

The action with a cubic term is

$$S = \int_{AdS} \sqrt{-g} \left[\frac{1}{2} \nabla_a \phi \nabla^a \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi^3 \right] + \frac{\Delta - d}{d} \int_{AdS} \sqrt{-g} \nabla_\alpha (\phi \nabla^\alpha \phi)$$

Writing $\phi = \phi_0 + g\phi_1$, we first notice that

$$g(\phi_0 + \phi_1)^3 = g\phi_0^3 + O(g^2)$$

The remaining second order terms in ϕ_0 will reproduce our expression for S_2 from the previous problem, and terms of second order in ϕ_1 are $O(g^2)$. This leaves terms that are $O(g)$, containing one ϕ_0 and one ϕ_1, as the only terms unaccounted for. These terms are

$$g \nabla \phi_0 \nabla \phi_1 + g m^2 \phi_0 \phi_1 + g \frac{\Delta - d}{d} \left(\phi_0 \nabla^2 \phi_1 + \phi_1 \nabla^2 \phi_0 + 2 \nabla \phi_0 \nabla \phi_1 \right)$$

$$= g \frac{2\Delta - d}{d} \nabla \phi_0 \nabla \phi_1 + g m^2 \phi_0 \phi_1 + \frac{\Delta - d}{d} \left(\phi_0 \nabla^2 \phi_1 + \phi_1 \nabla^2 \phi_0 \right)$$

$$= g \frac{2\Delta - d}{d} \left(\nabla \phi_0 \nabla \phi_1 - m^2 \phi_0 \phi_1 \right) = 0$$

Therefore, the full action is

$$S = -\frac{1}{2} \int d^d x d^d y_1 d^d y_2 \phi_b(y_1) \phi_b(y_2) K(y_1, y_2) + \frac{g}{3!} \int d^d x \phi_0^3 + \dots$$

The three-point function is

$$\left\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\right\rangle = \frac{\delta}{\delta\phi_b(P_1)}\frac{\delta}{\delta\phi_b(P_2)}\frac{\delta}{\delta\phi_b(P_3)}W(\phi_b)\Big|_{\phi_b=0} = \frac{\delta}{\delta\phi_b(P_1)}\frac{\delta}{\delta\phi_b(P_2)}\frac{\delta}{\delta\phi_b(P_3)}\frac{1}{Z_0}\int \mathcal{D}\phi e^{-S[\phi]}\Big|_{\phi_b=0}$$

We can uplift our expression for S to the null cone and Taylor expand the interaction term to get

$$\left\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\right\rangle = \frac{\delta}{\delta\phi_b(P_1)}\frac{\delta}{\delta\phi_b(P_2)}\frac{\delta}{\delta\phi_b(P_3)}\frac{1}{Z_0}e^{\frac{1}{2}\int dXdY\ \ \phi_b(X)\Pi(X,Y)\phi_b(Y)}\left[1-\frac{gC_\Delta^{3/2}}{3!}\left(\int dXdP\frac{\phi_b(P)}{(-2P\cdot X)^\Delta}\right)^3\right]\Big|_{\phi_b=0}$$

which immediately gives

$$-gC_\Delta^{3/2}\int dX\frac{1}{(-2P_1\cdot X)^\Delta}\frac{1}{(-2P_2\cdot X)^\Delta}\frac{1}{(-2P_3\cdot X)^\Delta} = -gC_\Delta^{-3/2}\int dX\ \Pi(X,P_1)\Pi(X,P_2)\Pi(X,P_3)$$