AdS/CFT - Homework 4

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EXERCISE 1

From (140), we have

$$C_{ijk}(x_{12}, \partial_2) x_{23}^{-2\Delta} = f_{ijk} x_{12}^{\Delta - 2\Delta_\varphi} \left(1 + A x_{12}^\mu \partial_{2\mu} + B x_{12}^\mu x_{12}^\nu \partial_{2\mu}\partial_{2\nu} + C x_{12}^2 \partial_2^2\right) x_{23}^{-2\Delta}$$

The derivatives are, generally,

$$\partial_{2\mu} x_{23}^\alpha = \alpha x_{23}^{\alpha-1} \frac{x_{23\mu}}{x_{23}} = \alpha x_{23}^{\alpha-2} x_{23\mu}$$

$$\partial_{2\mu}\partial_{2\nu} x_{23}^\alpha = \alpha \partial_{2\mu}\left[x_{23}^{\alpha-2} x_{23\nu}\right] = \alpha(\alpha-2) x_{23}^{\alpha-4} x_{23\mu} x_{23\nu} + \alpha x_{23}^{\alpha-2}\delta_{\mu\nu}$$

which gives

$$f_{ijk} x_{12}^{\Delta-2\Delta_\varphi}\left(x_{23}^{-2\Delta} - 2\Delta A x_{23}^{-2\Delta-2} x_{12}^\mu x_{23\mu} + 2\Delta(2\Delta-2) B x_{23}^{-2\Delta-4}\left[x_{12}^\mu x_{23\mu}\right]^2 - 2\Delta B x_{23}^{-2\Delta-2} x_{12}^2 - 2C\Delta(d-2\Delta-2) x_{23}^{-2\Delta-2} x_{12}^2\right)$$

$$= f_{ijk} x_{12}^{\Delta-2\Delta_\varphi} x_{23}^{-2\Delta}\left[1 - 2A\Delta \frac{x_{12}^\mu x_{23\mu}}{x_{23}^2} + B\left(2\Delta(2\Delta-2)\left(\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2}\right)^2 - 2\Delta\left(\frac{x_{12}}{x_{23}}\right)^2\right) + 2C\Delta(2+2\Delta-d)\left(\frac{x_{12}}{x_{23}}\right)^2\right]$$

The left side of (142) is

$$f_{ijk} x_{12}^{\Delta-2\Delta_\varphi} x_{23}^{-\Delta} x_{31}^{-\Delta}$$

so to find A, B, C, we should expand $x_{23}^{-\Delta} x_{31}^{-\Delta}$ at small x_{12}/x_{23}:

$$x_{23}^{-\Delta} x_{31}^{-\Delta} = x_{23}^{-2\Delta}\left(\frac{x_{13}}{x_{23}}\right)^{-\Delta} = x_{23}^{-2\Delta}\left(1 + \left(\frac{x_{12}}{x_{23}}\right)^2 + \frac{x_{12}^\mu x_{23\mu}}{x_{23}^2}\right)^{-\Delta/2}$$

$$= x_{23}^{-2\Delta}\left(1 - \Delta\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2} - \frac{\Delta}{2}\left(\frac{x_{12}}{x_{23}}\right)^2 + \frac{\Delta(\Delta+2)}{2}\left(\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2}\right)^2\right)$$

Therefore, we have

$$-\Delta\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2} - \frac{\Delta}{2}\left(\frac{x_{12}}{x_{23}}\right)^2 + \frac{\Delta(\Delta+2)}{2}\left(\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2}\right)^2 = -2A\Delta\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2} + B\left(2\Delta(2\Delta-2)\left(\frac{x_{12}^\mu x_{23\mu}}{x_{23}^2}\right)^2 - 2\Delta\left(\frac{x_{12}}{x_{23}}\right)^2\right)$$
$$+ 2C\Delta(2+2\Delta-d)\left(\frac{x_{12}}{x_{23}}\right)^2$$

and from here we see that

$$A = \frac{1}{2} \qquad B = \frac{\Delta+2}{8(\Delta+1)} \qquad C = -\frac{\Delta}{16\left(\Delta - \frac{d-2}{2}\right)(\Delta+1)}$$

EXERCISE 2

9.1

For a descendant $\langle\psi| = \langle\mathcal{O}|K\dots K$, we can use

$$\big[K_\mu, \phi(x)\big] = (k_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu})\phi(x) \qquad\qquad \big[K_\mu, \phi(0)\big] = 0$$

to find

$$\langle\psi|\phi(x)|\phi\rangle = \langle\mathcal{O}|K^n\phi(x)|\phi\rangle = \langle\mathcal{O}|\left[K^n, \phi(x)\right]\phi(0)|0\rangle = \langle\mathcal{O}|K^{n-1}\left[K, \phi(x)\right] + \left[K^{n-1}, \phi(x)\right]K|\phi\rangle$$

$$= (k_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu})\langle\mathcal{O}|K^{n-1}\phi(x)|\phi\rangle$$

so by induction, we see that $\langle\psi|\phi(x)|\phi\rangle \propto \langle\mathcal{O}|\phi(x)|\phi\rangle$.

9.2

The matrix element $\langle\mathcal{O}^a|\phi(x)|\phi\rangle$ is proportional to the usual scalar 3-point function (which includes $f_{\phi\phi\mathcal{O}}$) times a tensor structure that carries the spin degrees of freedom. By rotational symmetry, we must have

$$\langle\mathcal{O}^a|\phi(x)|\phi\rangle = \langle\mathcal{O}^a|\phi(-x)|\phi\rangle$$

We know that a is a symmetric tensor index, so this matrix element will be proportional to a product of x's with individual free vector indices. Because of the rotational symmetry, there must be an even number of x's, which corresponds to even spin only. Therefore, for any odd spin, we must take $f_{\phi\phi\mathcal{O}} = 0$.

EXERCISE 3

In the spin-zero case, the conformal block is

$$g_{\Delta,0} = x_{12}^{\Delta_\varphi} x_{34}^{\Delta_\varphi} C(x_{12}, \partial_2) C(x_{34}, \partial_4) x_{24}^{-2\Delta}$$

We know the action of $C(x, \partial)$ on a single x_{ij}:

$$g_{\Delta,0} = x_{12}^{\Delta_\varphi} x_{34}^{\Delta_\varphi} C(x_{12}, \partial_2) x_{34}^{\Delta - 2\Delta_\varphi} x_{24}^{-\Delta} x_{23}^{-\Delta} = x_{12}^{\Delta_\varphi} x_{34}^{\Delta-\Delta_\varphi} C(x_{12}, \partial_2) x_{24}^{-\Delta} x_{23}^{-\Delta}$$

The action of $C(x, \partial)$ on more than one x_{ij} is more complicated, but to lowest order in x_{12}/x_{23}, $C(x, \partial) \sim 1$, so

$$= x_{12}^{\Delta-\Delta_\varphi} x_{34}^{\Delta-\Delta_\varphi} (1 + \dots) x_{24}^{-\Delta} x_{23}^{-\Delta} \approx \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^\Delta \frac{x_{12}^{-\Delta_\varphi} x_{34}^{-\Delta_\varphi} x_{13}^{\Delta}}{x_{23}^{\Delta}}$$

$$= u^{\Delta/2}\left(1 + \frac{x_{12}}{x_{23}}\right)^\Delta (x_{12}x_{34})^{-\Delta_\varphi} \approx u^{\Delta/2}$$

EXERCISE 4

Using the definition of the OPE, we have

$$\langle 0|R\left\{\phi(x_3)\phi(x_4)\right\}|\tilde{\mathcal{O}}|R\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle = \sum_{\mathcal{O}\mathcal{O}'} f_{\phi\phi\mathcal{O}} f_{\phi\phi\mathcal{O}'} \langle 0|C_a(x_{34},\partial_4)\mathcal{O}^a(x_4)|\tilde{\mathcal{O}}|C_b(x_{12},\partial_2)\mathcal{O}'^b(x_2)|0\rangle$$

$$= \sum_{\mathcal{O}\mathcal{O}'} \sum_{\substack{\alpha,\beta= \\ \tilde{\mathcal{O}},P\tilde{\mathcal{O}},PP\tilde{\mathcal{O}},\ldots}} f_{\phi\phi\mathcal{O}} f_{\phi\phi\mathcal{O}'} C_a(x_{34},\partial_4) C_b(x_{12},\partial_2)\langle 0|\mathcal{O}^a(x_4)|\alpha\rangle \mathcal{N}_{\alpha\beta}\langle\beta|\mathcal{O}'^b(x_2)|0\rangle$$

Now consider $\langle\beta|\mathcal{O}'^b(x_2)|0\rangle$. If the operator that creates $\langle\beta|$ does not have a nonzero two-point function with $\mathcal{O}'$ (which is primary), then $\langle\beta|\mathcal{O}'^b(x_2)|0\rangle = 0$. Therefore, we can take $\tilde{\mathcal{O}} = \mathcal{O}'$, and the sum over α and β must only include primary operators. By similar logic, we can also see that $\tilde{\mathcal{O}} = \mathcal{O}$. Our expression becomes

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} C_a(x_{34},\partial_4) C_b(x_{12},\partial_2) \sum_{\mathcal{O}} \langle 0|\mathcal{O}^a(x_4)|\mathcal{O}|\mathcal{O}^b(x_2)|0\rangle = \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} C_a(x_{34},\partial_4) C_b(x_{12},\partial_2)\langle 0|\mathcal{O}^a(x_4)\mathcal{O}^b(x_2)|0\rangle$$

By (149), this is equal to

$$\frac{1}{x_{12}^{\Delta} x_{34}^{\Delta}} \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} g_{\Delta_\phi,\ell_\phi}(x_i)$$

EXERCISE 5