AdS/CFT - Homework 2

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PROBLEM 1

To find the number of cross ratios, we start with the number of configurations of n points in d dimensions and subtract the number of configurations related by symmetry. The number related by symmetry will change depending on how large n is relative to d.

The points all need to be related by conformal symmetry, so for large n, we should find

$$nd - \text{Dim}\left[SO(d+1,1)\right] = dn - \frac{(d+2)(d+1)}{2}$$

When n is small enough, there may be a subgroup of $SO(d+1,1)$ that fixes all n points. This occurs when $n < d+1$. In that case, the number of cross ratios is

$$nd - \text{Dim}\left[SO(d+1,1)\right] + \text{Dim}\left[\text{Subgroup}\right]$$

For each point that we add, the size of the subgroup that fixes all points decreases; for $n < d+1$ it is $SO(d+1-n,1)$. That gives

$$nd - \frac{(d+1)(d+2)}{2} + \frac{(d-n+1)(d+2-n)}{2} = \frac{n^2-3n}{2}$$

so all together, the number of cross ratios is

$$\begin{array}{ll} dn - \frac{(d+2)(d+1)}{2} & \quad n \geq d+1 \\ \frac{n^2-3n}{2} & \quad n < d+1 \end{array}$$

PROBLEM 2

The stress tensor is the unique real, traceless, symmetric, primary operator of dimension d constructed from φ and $\varphi^\dagger$. Note that φ and $\varphi^\dagger$ are primary operators. φ has dimension $[\varphi] = \frac{d-2}{2}$, so we begin by writing every possible term with φ, $\varphi^\dagger$, and two derivatives:

$$T_{\mu\nu} = A\left(\partial_\mu\varphi^\dagger\partial_\nu\varphi + \partial_\mu\varphi\partial_\nu\varphi^\dagger\right) + C\left(\varphi^\dagger\partial_\mu\partial_\nu\varphi + \varphi\partial_\mu\partial_\nu\varphi^\dagger\right) + D\delta_{\mu\nu}\partial_\rho\varphi^\dagger\partial^\rho\varphi + E\delta_{\mu\nu}\left(\varphi^\dagger\partial^2\varphi + \varphi\partial^2\varphi^\dagger\right)$$

Imposing tracelessness and doing some algebra, we find

$$0 = \delta_{\mu\nu}T_{\mu\nu} = (2A + dD)\partial_\mu\varphi^\dagger\partial^\mu\varphi + (C + dE)\left(\varphi^\dagger\partial^2\varphi + \varphi\partial^2\varphi^\dagger\right)$$

which tells us that $(2A + dD) = 0 = (C + dE)$. To ensure that $T_{\mu\nu}$ is a primary, we will impose $\left[K_\rho, T_{\mu\nu}\right] = 0$. This gives (after more algebra)

$$
\begin{aligned}
0 = A &\left(\partial_\mu\varphi^\dagger\left[K,\partial_\nu\varphi\right] + \left[K,\partial_\mu\varphi^\dagger\right]\partial_\nu\varphi + \partial_\mu\varphi\left[K,\partial_\nu\varphi^\dagger\right] + \left[K,\partial_\mu\varphi\right]\partial_\nu\varphi^\dagger\right)\\
&+ C\left(\varphi^\dagger\left[K,\partial_\mu\partial_\nu\varphi\right] + \varphi\left[K,\partial_\mu\partial_\nu\varphi^\dagger\right]\right) + D\delta_{\mu\nu}\left(\partial_\rho\varphi^\dagger\left[K,\partial^\rho\varphi\right] + \left[K,\partial_\rho\varphi^\dagger\right]\partial^\rho\varphi\right)\\
&+ E\delta_{\mu\nu}\left(\varphi\left[K,\partial^2\varphi^\dagger\right] + \varphi^\dagger\left[K,\partial^2\varphi\right]\right)
\end{aligned}
$$

So we find $A = C = D = E = 0$. This clearly isn't right...
The most general object with three indices and dimension $d+1$ that we can write down is

$$
J_{\mu\nu\rho} = A\partial_\mu\partial_\nu\varphi^\dagger\partial_\rho\varphi + B\delta_{\mu\nu}\left(\partial^2\varphi^\dagger\right)\partial_\rho\varphi + C\varphi^\dagger\partial_\mu\partial_\nu\partial_\rho\varphi + D\delta_{\mu\nu}\varphi^\dagger\partial_\rho\partial^2\varphi + \text{perms} + \text{h.c.}
$$

Taking the trace in μ and ν gives

$$
0 = \delta^{\mu\nu}J_{\mu\nu\rho} = \left[A + (d+2)B\right]\left(\partial^2\varphi^\dagger\partial_\rho\varphi\right) + 2A\left(\partial^\mu\varphi^\dagger\partial_\mu\partial_\rho\varphi\right) + \left[C + (d+2)D\right]\left(\varphi^\dagger\partial^2\partial_\rho\varphi\right) + \text{h.c.}
$$

which implies $A = B = 0$ and $C = -(d+2)D$. Similar results are found from the other traces.

PROBLEM 3

Using the projective null cone formalism, the most general expression for the 4-point function is

$$
F = \left\langle\mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{\Delta_3}(P_3)\mathcal{O}_{\Delta_4}(P_4)\right\rangle = CP_{12}^\alpha P_{12}^\beta P_{14}^\gamma P_{23}^\delta P_{24}^\epsilon P_{34}^\zeta
$$

where $P_{ij} = -2P_i\cdot P_j$ and C is an undetermined constant. Requiring homogeneity under $P_1 \to \lambda P_1$ implies

$$
\lambda^{-\Delta_1}F = \lambda^{\alpha+\beta+\gamma}F
$$

Similar results are found when transforming $P_{2,3,4}$. We find the following (underdetermined) linear system:

$$
\alpha+\beta+\gamma = -\Delta_1 \qquad \alpha+\delta+\epsilon = -\Delta_2 \qquad \beta+\delta+\zeta = -\Delta_3 \qquad \gamma+\epsilon+\zeta = -\Delta_4
$$

which has the solution

$$
\begin{aligned}
\alpha &= -\frac{\Delta_1+\Delta_2}{2} + x & \beta &= \frac{\Delta_4-\Delta_3}{2} - x - y & \gamma &= \frac{-\Delta_1+\Delta_2+\Delta_3-\Delta_4}{2} + y\\
\delta &= y & \epsilon &= \frac{\Delta_1-\Delta_2}{2} - x - y & \zeta &= -\frac{\Delta_3+\Delta_4}{2} + x
\end{aligned}
$$

where x and y cannot be determined. Plugging these values into F gives

$$
F = C\left(\frac{P_{12}P_{34}}{P_{13}P_{24}}\right)^x\left(\frac{P_{14}P_{23}}{P_{13}P_{24}}\right)^y \frac{1}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}}P_{34}^{\frac{\Delta_3+\Delta_4}{2}}} P_{24}^{\frac{\Delta_1-\Delta_2}{2}} P_{13}^{\frac{\Delta_4-\Delta_3}{2}} P_{14}^{\frac{\Delta_2-\Delta_1+\Delta_3-\Delta_4}{2}}
$$

Notice that the terms in parentheses are the usual cross-ratios u and v. When we project to the Euclidean section, we have $P_{ij} \to x_{ij}^2$, so F becomes

$$
\left\langle\mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{\Delta_3}(P_3)\mathcal{O}_{\Delta_4}(P_4)\right\rangle = \frac{f(u,v)}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}} x_{24}^{\Delta_1-\Delta_2} x_{13}^{\Delta_4-\Delta_3} x_{14}^{\Delta_2-\Delta_1+\Delta_3-\Delta_4}
$$

If we take $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4$, we recover the expected result

$$
\left\langle\mathcal{O}_{\Delta}(P_1)\mathcal{O}_{\Delta}(P_2)\mathcal{O}_{\Delta}(P_3)\mathcal{O}_{\Delta}(P_4)\right\rangle = \frac{f(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}
$$

PROBLEM 4

The correlation function on the cylinder and in flat space are related by

$$\left\langle \mathcal{O}(\tau_1, \vec{n}_1)\mathcal{O}(\tau_2, \vec{n}_2)\right\rangle_{cyl} = \frac{1}{r_1^{\Delta_1} r_2^{\Delta_2}} \left\langle \mathcal{O}(r_1, \vec{n}_1)\mathcal{O}(r_2, \vec{n}_2)\right\rangle_{flat}$$

Time translations on the cylinder correspond to dilations in flat space. The RHS is clearly invariant under dilations, so the LHS is invariant under time translations.

Now we show that $\left\langle \mathcal{O}(\tau_1, \vec{n}_1)\mathcal{O}(\tau_2, \vec{n}_2)\right\rangle_{cyl}$ admits the desired expansion. We know that $\mathcal{O}$ creates a sum of descendants and we can use (109) of DSD to find the action on the conjugate state. We have

$$\mathcal{O}(y) = e^{y\cdot P}|\mathcal{O}\rangle = \sum_{n=0}^{\infty} \frac{(y\cdot P)^n}{n!}|\mathcal{O}\rangle \qquad \langle 0|\mathcal{O}(x) = x^{-2\Delta}\langle \mathcal{O}|e^{-iK\cdot x} = x^{-2\Delta}\sum_{m=0}^{\infty}\langle \mathcal{O}|\frac{(\tilde{x}\cdot K)}{m!}$$

where the tilde on x denotes inversion. The flat-space inner product is

$$\left\langle \mathcal{O}(r_1, \vec{n}_1)\mathcal{O}(r_2, \vec{n}_2)\right\rangle = x^{-2\Delta}\sum_{n=0}^{\infty}\langle \mathcal{O}|\frac{(\tilde{x}\cdot K)^n (y\cdot P)^n}{(n!)^2}|\mathcal{O}\rangle$$

The two sums have combined into one, since states with unequal numbers of P's and K's will vanish by orthogonality. Splitting $x = (r_1, \vec{n_1})$ (and similarly for y) and using the fact that $r = e^{\tau}$, this sum becomes

$$\left\langle \mathcal{O}(r_1, \vec{n}_1)\mathcal{O}(r_2, \vec{n}_2)\right\rangle = \frac{1}{(r_1 r_2)^{\Delta}}\sum_{n=0}^{\infty}\langle \mathcal{O}|\frac{\left(K^{\mu}\vec{n}_{1\mu}\right)^n \left(P^{\nu}\vec{n}_{2\nu}\right)^n}{(n!)^2}|\mathcal{O}\rangle e^{(\tau_2-\tau_1)(n+\Delta)}$$

The matrix elements correspond to descendants in the multiplet created by $\mathcal{O}$. Since the flat space correlator admits this expansion, the correlator on the cylinder does as well, modified by the factors we found previously.

PROBLEM 5

TO satisfy reflection positivity, time-symmetric configurations must have positive norm. The 2-point function for spin-1 operators with identical dimensions is

$$\left\langle J^{\mu}(x)J_{\nu}(y)\right\rangle = C_J\frac{I^{\mu}{}_{\nu}(x-y)}{(x-y)^{2\Delta}}, \qquad I^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - 2\frac{x^{\mu}x_{\nu}}{x^2}$$

The reflected version is

$$\left\langle J^{\dagger}_{\mu}(x)J^{\dagger}_{\nu}(y)\right\rangle = I_{\mu}{}^{\rho}(x)I_{\nu}{}^{\sigma}(y)(xy)^{-2\Delta}\left\langle J_{\rho}\left(\frac{x}{x^2}\right)J_{\sigma}\left(\frac{y}{y^2}\right)\right\rangle = C_J I_{\mu}{}^{\rho}(x)I_{\nu}{}^{\sigma}(y)I_{\rho\sigma}\left(\frac{xy^2 - yx^2}{x^2y^2}\right)\left(y^2 - x^2\right)^{-2\Delta}$$

PROBLEM 6

We want to calculate $\left|P^2|\mathcal{O}\rangle\right|^2 = \langle \mathcal{O}|K^2P^2|\mathcal{O}\rangle$:

$$\langle\mathcal{O}|K_\mu K^\mu P_\nu P^\nu|\mathcal{O}\rangle = \langle\mathcal{O}|K_\mu \left(P_\nu K^\mu + [K^\mu, P_\nu]\, P^\nu\right)|\mathcal{O}\rangle = \langle\mathcal{O}|\left[K_\mu, P_\nu\right] K^\mu P^\nu|\mathcal{O}\rangle + 2\langle\mathcal{O}|K_\nu D P^\nu|\mathcal{O}\rangle - 2\langle\mathcal{O}|K_\mu M^\mu{}_\nu P^\nu|\mathcal{O}\rangle$$

$$= 2\langle\mathcal{O}|D\delta_{\mu\nu} K^\mu P^\nu|\mathcal{O}\rangle + 2\langle\mathcal{O}|\left(DK_\nu + [K_\nu, D]\right) P^\nu|\mathcal{O}\rangle - 2\langle\mathcal{O}|K_\mu \left[M^\mu{}_\nu, P^\nu\right]|\mathcal{O}\rangle$$

$$= (4\Delta + 2)\langle\mathcal{O}|K_\mu P^\mu|\mathcal{O}\rangle - 2\langle\mathcal{O}|K_\mu \left(\delta^\nu_\nu P^\mu - \delta^{\mu\nu} P_\nu\right)|\mathcal{O}\rangle = (4\Delta + 4 - 2d)\langle\mathcal{O}|\left[K_\mu, P^\mu\right]|\mathcal{O}\rangle$$

$$= 2d\Delta(4\Delta + 4 - 2d)\langle\mathcal{O}|\mathcal{O}\rangle \geq 0$$

We normalize $|\mathcal{O}\rangle$ so that $\langle\mathcal{O}|\mathcal{O}\rangle = 1$, which gives $2d\Delta(4\Delta + 4 - 2d) \geq 0$. This is satisfied for $\Delta \geq 0$ or $\Delta \geq \frac{d-2}{2}$.