2018 Bootstrap School - Mazac

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PART (A)

The 1d conformal blocks are

$$G_\Delta(z) = z^\Delta \ {}_2F_1(\Delta, \Delta; 2\Delta; z)$$

We can write the hypergeometric function in two ways:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dx \ x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a}$$

From the first, form, we see that for arbitrary Δ, we can have fractional powers of a negative value, so there will be a branch cut on the interval $(-\infty, 0]$, like for $\sqrt{z}$. Using the second form, we see that if $z \geq 1$ (and with no imaginary part), that the integral will obtain a pole, leading to a branch cut on the interval $[1, \infty)$.

PART (B)

The discontinuity of the hypergeometric function across the branch cut is given by

$$\delta F = \ {}_2F_1(\Delta, \Delta; 2\Delta; z + 0i) - \ {}_2F_1(\Delta, \Delta; 2\Delta; z - 0i) = \left(e^{2\pi i a} - 1\right) \frac{\Gamma(2\Delta)}{\left[\Gamma(\Delta)\right]^2} \int_{1/z}^1 dx \ x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a}$$

Making the change of variable $x = \left(1 - \frac{1}{z}\right) t + \frac{1}{z}$, we find

$$\delta F = \left(e^{2\pi i a} - 1\right) \frac{\Gamma(2\Delta)}{\left[\Gamma(\Delta)\right]^2} \left(1 - \frac{1}{z}\right)^\Delta (1-z)^{-\Delta} z^{1-\Delta} \int_0^1 dt \ t^{-\Delta} (1-t)^{\Delta-1} \left[1 - t(1-z)\right]^{\Delta-1}$$

$$= \frac{\left(e^{2\pi i a} - 1\right) \Gamma(2\Delta)\Gamma(1-\Delta)}{\Gamma(\Delta)^2} \left(1 - \frac{1}{z}\right)^\Delta (1-z)^{-\Delta} z^{1-\Delta} \ {}_2F_1(1-\Delta, 1-\Delta; 1; 1-z)$$

$$= \frac{2\pi i \Gamma(2\Delta)}{\left[\Gamma(\Delta)\right]^2} z^{-\Delta} \ {}_2F_1\left(\Delta, 1-\Delta; 1; \frac{z-1}{z}\right)$$

Thus, the discontinuity of the block is

$$G_\Delta(z + 0i) - G_\Delta(z - 0i) = \frac{2\pi i \Gamma(2\Delta)}{\left[\Gamma(\Delta)\right]^2} \ {}_2F_1\left(\Delta, 1-\Delta; 1; \frac{z-1}{z}\right)$$

PART (C)

In integral form, $p_\Delta(z)$ is

$$p_\Delta(z) = \int_0^1 dx \ \frac{(1-x)^{\Delta-1}}{x^\Delta (1-zx)^\Delta}$$

Making the change of variables $u = 4\sin^{-1}(x)$ gives

$$\frac{1}{4}\int_0^{2\pi} du \ \frac{\left[1-\sin\left(\frac{u}{4}\right)\right]^{\Delta-1}\cos\left(\frac{u}{4}\right)}{\sin^\Delta\left(\frac{u}{4}\right)\left[1-z\sin\left(\frac{u}{4}\right)\right]^\Delta}$$

To turn this into a contour integral, we make another change of variables $t = e^{iu/4}$ and use the unit circle as our contour of integration:

$$\frac{1}{(2i)^{\Delta+1}}\oint_{|z|=1} dt \ \frac{\left[1-\frac{1}{2i}\left(t-\frac{1}{t}\right)\right]^{\Delta-1}\left(t+\frac{1}{t}\right)}{t\left(t-\frac{1}{t}\right)^\Delta\left[1-\frac{z}{2i}\left(t-\frac{1}{t}\right)\right]}$$

This has a finite number of poles of finite order, so we will find a polynomial when we evaluate the sum of the residues.

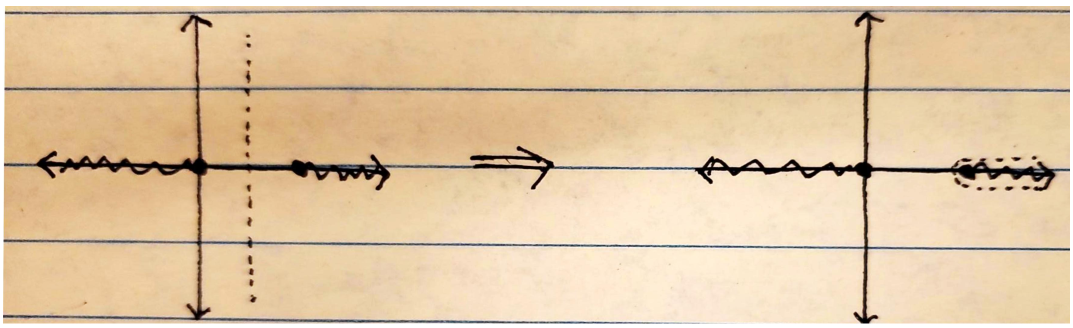
PART (D)

Shown below are the functions we wish to integrate when calculating the action of the functionals on the terms in the crossing equation:

Function	**Branch Cuts**	**Singularities**
$z^{-2}p_j\left(\frac{z-1}{z}\right)$	$(-\infty, 0]$	0
$z^{-2}p_j\left(\frac{z-1}{z}\right)G_k(z)$	$(-\infty,0],[1,\infty)$	0 (for $k<2$)
$\frac{1}{z(1-z)}p_j\left(\frac{z-1}{z}\right)$	$(-\infty, 0]$	0,1
$\frac{1}{z(1-z)}p_j\left(\frac{z-1}{z}\right)G_k(1-z)$	$(-\infty, 0]$	0

Note that $G_k(1-z)$ contains a factor of $(1-z)^k$, so since $k>0$, there is no singularity at $z=1$. When integrating the first and fourth cases, we see that we are allowed to deform the contour to move it to infinity, since we are not trapped by any singularities or branch cuts. Since there is no contribution to these functions at infinity, $\omega_j[z^{-1}] = \omega_j[(1-z)^{-1}G_k(1-z)] = 0$.

To evaluate the second integral, we deform the contour as follows:



This transformation allows us to write the action of the functional as

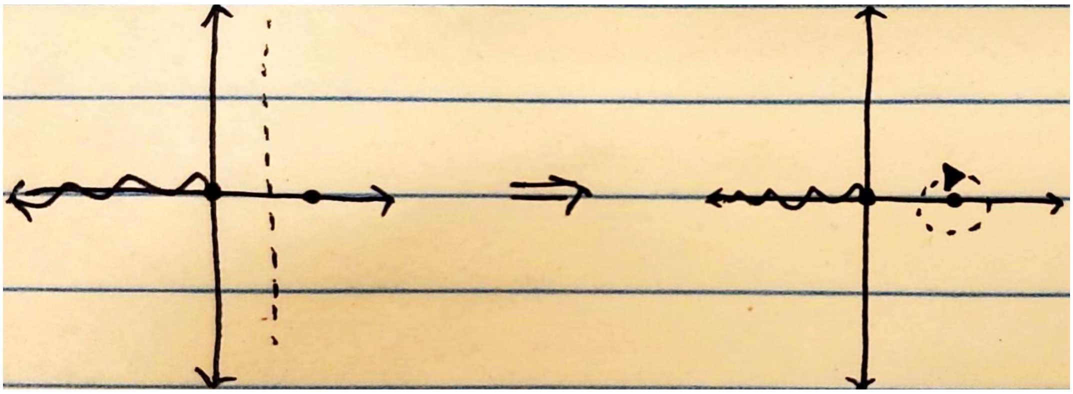
$$\omega_j\left[z^{-1}G_k(z)\right] = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\ z^{-2}p_j\left(\frac{z-1}{z}\right)G_k(z) = \frac{1}{2\pi i}\int_1^{\infty} dz\ z^{-2}p_j\left(\frac{z-1}{z}\right)\text{Disc}\left[G_k(z)\right]$$

$$= \frac{\Gamma(2k)}{\Gamma^2(k)}\int_1^{\infty} dz\ z^{-2}p_j\left(\frac{z-1}{z}\right)p_k\left(\frac{z-1}{z}\right)$$

Making the change of variable $x = \frac{z-1}{z}$ gives

$$\frac{\Gamma(2k)}{\Gamma^2(k)}\int_0^1 dx\ p_j(x)p_k(x) = \frac{\Gamma(2k)}{\Gamma^2(k)}\frac{\delta_{jk}}{2j-1} = \frac{\Gamma(2k-1)}{\Gamma^2(k)}\delta_{jk}$$

The fourth integral is simpler to evaluate. We deform the contour as shown:



and then use the residue theorem to evaluate the result:

$$\omega_j[(1-z)^{-1}] = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\ \frac{1}{z(1-z)}p_j\left(\frac{z-1}{z}\right) = \frac{1}{2\pi i}\oint_C dz\ \frac{1}{z(1-z)}p_j\left(\frac{z-1}{z}\right) = \lim_{z\to 1}\left[\frac{1}{z}p_j\left(\frac{z-1}{z}\right)\right] = 1$$

PART (F)

Using the summation definition of the hypergeometric function, $p_j(x)$ can be written as

$$p_j\left(\frac{z-1}{z}\right) = \sum_{n=0}^{\infty}\frac{(j)_n(1-j)_n}{(n!)^2}\left(\frac{z-1}{z}\right)^n$$

In the large z limit, this goes to a constant, since $\lim_{z\to\infty}\left(\frac{z-1}{z}\right)^n = 1$. Thus, $h(z) \sim z^{-1}$ as $z \to \infty$.

PART (G)

Using the relationship between p_j and the Legendre polynomials $P_n(x)$, we can use

$$P_n(x) = 2^n\sum_{n=0}^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}x^k$$

along with the binomial theorem to write h_j as a power series:

$$h_j(z) = (-2)^{j-1} \sum_{k=0}^{j-1} \sum_{n=0}^{\infty} (-2)^n \binom{k}{n} \binom{j-1}{k} \binom{\frac{j+k}{2}-1}{j-1} \frac{1}{z^{n+1}}$$

We propose to add $h_2(z) = (-1)^{j-1}(\frac{2-z}{z^2})$ to each $h_j(z)$ to cancel the z^{-1} dependence. We see that if the coefficient of z^{-1} in the power series is $(-1)^{j-1}$, then this will be the case. The coefficient function is

$$c_j = (-2)^{j-1} \sum_{k=0}^{\infty} \binom{j-1}{k} \binom{\frac{j+k}{2}-1}{j-1} = (-1)^{j-1}$$

so we get the desired cancellation and $\tilde{h}_j(z) \sim z^{-2}$.

PART (H)

First, check that $\tilde{\omega}_{2j}$ does not lead to a contradiction:

$$\tilde{\omega}_{2j}\left[z^{-1}\right] + \tilde{\omega}_{2j}\left[\sum_{n=0}^{\infty} a_n z^{-1} G_{2n+1}(z)\right] = \tilde{\omega}_{2j}\left[(1-z)^{-1}\right] + \tilde{\omega}_{2j}\left[\sum_{n=0}^{\infty} a_n (1-z)^{-1} G_{2n+1}(1-z)\right] \implies 0+0 = 1-1+0$$

Now apply $\tilde{\omega}_{2j+1}$:

$$\sum_{n=0}^{\infty} a_n \omega_{2j+1}\left[z^{-1} G_{2n+1}(z)\right] = 2 \implies a_n = \frac{2\Gamma^2(2n+1)}{\Gamma(4n+1)}$$

PART (I)

Comparing with our result from part (f), it is easy to see that $h(z,\bar{z}) \sim z^{-1}\bar{z}^{-1}$ as $z, \bar{z} \to \infty$.

PART (J)

After some numerical experimentation, I found some functionals in terms of the $D=1$ functionals that satisfy the swapping requirement. In practice, we only need to worry about the case in which both indices on G_{ij} are even, since the spin sum in the crossing equation is over even ℓ only, and $2+2n+\ell$ is even. However, all possible cases are listed below:

$$\tilde{h}_{ij}(z,\bar{z}) = h_{ij}(z,\bar{z}) + h_1(\bar{z})h_i(z) + h_1(z)h_j(\bar{z}) + h_{11}(z,\bar{z}) \qquad \text{i, j even}$$

$$= h_i(z)h_j(\bar{z}) + h_1(\bar{z})h_i(z) + h_1(z)h_j(\bar{z}) + h_1(z)h_1(\bar{z})$$

$$\tilde{h}_{ij}(z,\bar{z}) = h_{ij}(z,\bar{z}) - h_1(\bar{z})h_i(z) - h_1(z)h_j(\bar{z}) + h_{11}(z,\bar{z}) \qquad \text{i, j odd}$$

$$\tilde{h}_{ij}(z,\bar{z}) = h_{ij}(z,\bar{z}) + (-1)^{i+1}h_1(\bar{z})h_i(z) + (-1)^{j+1}h_1(z)h_j(\bar{z}) + (-1)^{i+j}h_{11}(z,\bar{z}) \qquad \text{i + j odd}$$

Using these, it is now easy to see that

$$\omega_{ij}\left[(z\bar{z})^{-1}\right] = 0$$

$$\omega_{ij}\left[(1-z)^{-1}(1-\bar{z})^{-1}\right] = 4$$

$$\omega_{ij}\left[(1-z)^{-1}(1-\bar{z})^{-1}G^{2D}_{\Delta,\ell}(1-z,1-\bar{z})\right] = 0$$

Applying our functionals to $(z\bar{z})^{-1}G^{2D}_{\Delta,\ell}(z,\bar{z})$ is slightly more complicated, resulting in

$$\begin{aligned}\omega_{ij}\left[(z\bar{z})^{-1}G^{2D}_{\Delta,\ell}(z,\bar{z})\right] = \sum_{\ell=0,2,\ldots}\sum_{n=0}^{\infty}\frac{a_{n,\ell}}{\delta_{0,l}+1}\Bigg[&2\delta_{1,n+1}\delta_{1,\ell+n+1} + \frac{\Gamma(2k-1)}{\Gamma(k)^2}\Big(\delta_{k,n+1}\delta_{1,\ell+n+1}+\delta_{1,n+1}\delta_{k,\ell+n+1}\Big)\\ &+\frac{\Gamma(2k-1)\Gamma(2k+4m+3)}{\Gamma(k)^2\Gamma(k+2m+2)^2}\Big(\delta_{k,\ell+n+1}\delta_{k+2m+2,n+1}+\delta_{k,n+1}\delta_{k+2m+2,\ell+n+1}\Big)\\ &+\frac{\Gamma(2k+4m+3)}{\Gamma(k+2m+2)^2}\Big(\delta_{1,\ell+n+1}\delta_{k+2m+2,n+1}+\delta_{1,n+1}\delta_{k+2m+2,\ell+n+1}\Big)\Bigg]\end{aligned}$$

Resolving the Kronecker deltas and making the change of variables $b = k-1$, $c = 2+2m$, the crossing equation becomes

$$4 = a_{0,0} + \frac{\Gamma(2b+1)\Gamma(2b+2c+1)}{\left[\Gamma(b+1)\right]^2\left[\Gamma(b+c+1)\right]^2}\left(a_{b+c,-c}+a_{b,c}\right)$$

If we take $a_{b+c,-c} = 0$ (since the expansion of the four point function in conformal blocks is over positive integers only), this is consistent with

$$a_{n,\ell} = 2\frac{\left[\Gamma(n+1)\right]^2\left[\Gamma(n+\ell+1)\right]^2}{\Gamma(2n+1)\Gamma(2n+2\ell+1)}$$