Conformal Field Theory Problems

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TRANSFORMATION RULES OF A FIELD UNDER CONFORMAL GROUP

Given a field $\Phi(x)$ and defining the action of infinitesimal Lorentz transformations on $\Phi(0)$, we can find the action on the field by infinitesimal conformal transformations away from the origin by applying the Hausdorff formula

$$\Phi'(x) = \theta(x)\Phi(x) = e^{ix^\lambda P_\lambda}\theta(0)e^{-ix^\lambda P_\lambda}\Phi(x)$$

$$\begin{aligned} e^{ix^\lambda P_\lambda} L_{\mu\nu} e^{-ix^\lambda P_\lambda} &= L_{\mu\nu} + \left[L_{\mu\nu}, -ix^\lambda P_\lambda\right] + \frac{1}{2!}\left[\left[L_{\mu\nu}, -ix^\lambda P_\lambda\right], -ix^\lambda P_\lambda\right] + \dots \\ &= L_{\mu\nu} - i\left(\left[L_{\mu\nu}, x^\lambda\right] P_\lambda + x^\lambda \left[L_{\mu\nu}, P_\lambda\right]\right) + \dots \\ &= L_{\mu\nu} - i\left[0 + ix^\lambda(\eta_{\lambda\mu}P_\nu - \eta_{\lambda\nu}P_\mu)\right] + \dots \\ &= L_{\mu\nu} - x_\nu P_\mu + x_\mu P_\nu - \frac{i}{2}\left[x_\mu P_\nu - x_\nu P_\mu, x^\lambda P_\lambda\right] + \dots \end{aligned}$$

$$\left[x_\mu P_\nu - x_\nu P_\mu, x^\lambda P_\lambda\right] = \left(-ix_\mu \eta^\lambda_\nu + ix_\nu \eta^\lambda_\mu\right) P_\lambda + x^\lambda\left(i\eta_{\mu\lambda}P_\nu - i\eta_{\nu\lambda}P_\mu\right) = 0$$

so higher order terms vanish.

$$\begin{aligned} e^{ix^\lambda P_\lambda} L_{\mu\nu} e^{-ix^\lambda P_\lambda} &= L_{\mu\nu} - x_\nu P_\mu + x_\mu P_\nu \\ \Phi'(x) &= (L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu)\Phi(x) \\ &= i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \end{aligned} \tag{1}$$

$$\begin{aligned} e^{ixp} D e^{-ixp} &= D + [D, -ixp] + \frac{1}{2!}\left[[D, -ixp], -ixp\right] + \dots \\ [D, x^\mu P_\mu] &= x^\mu [D, P_\mu] + [D, x^\mu] P_\mu = ix^\mu P_\mu + 0 \\ [ix^\mu P_\mu, x^\mu P_\mu] &= 0 \end{aligned}$$

so higher order terms vanish.

$$e^{ixp} D e^{-ixp} = D + x^\mu P_\mu \tag{2}$$

$$\begin{aligned} e^{ixp} K_\mu e^{-ixp} &= K_\mu + [K_\mu, -ix^\nu P_\nu] + \frac{1}{2!}\left[\left[K_\mu, -ix^\nu P_\nu\right], -ix^\nu P_\nu\right] + \dots \\ [K_\mu, -ix^\nu P_\nu] &= -ix^\nu [K_\mu, P_\nu] = 2x^\mu D - 2x^\nu L_{\mu\nu} \\ [2x_\mu D - 2x^\nu L_{\mu\nu}, -ix^\rho P_\rho] &= -2ix_\mu x^\rho [D, P_\rho] + 2ix^\nu x^\rho [L_{\mu\nu}, P_\rho] = 4x_\mu x^\rho P_\rho - 2x^\nu x_\nu P_\mu \\ [4x_\mu x^\rho P_\rho - 2x^\nu x_\nu P_\mu, -ix^\sigma P_\sigma] &= 0 \end{aligned}$$

$$e^{ixp} K_\mu e^{-ixp} = K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu x^\rho P_\rho - x^\nu x_\nu P_\mu \tag{3}$$

The following results are derived from the above expressions for dilitation and the special conformal boost acting at an arbitrary point

$$\begin{aligned} D\Phi(x) &= D(0)\Phi(x) + x^\mu P_\mu \Phi(x) \\ &= (\widetilde{\Delta} - ix^\mu \partial_\mu)\Phi(x) \end{aligned}$$

$$
\begin{aligned}
K_\mu \Phi(x) &= (K_\mu(0) + 2x_\mu D(0) - 2x^\nu L_{\nu\mu}(0) - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\nu)\Phi(x) \\
&= (\kappa_\mu + 2x_\mu \widetilde{\Delta} - 2x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\mu + ix^2 \partial_\mu)\Phi(x)
\end{aligned}
$$

CENTRAL EXTENSION OF THE WITT ALGEBRA

We want to find a central extension to the Witt algebra in the form of eq. (4)

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + cg(m,n); \qquad cg(m,n) \in \mathbb{C} \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + cg(m,n) \\
&[L_m, \bar{L}_n] = 0
\end{aligned}
\tag{4}
$$

$g(m,n)$ must be antisymmetric in its arguments by antisymmetry of the bracket. It is given that $g(n,0) = 0$ and $g(1,-1) = 0$. We also demand that the Jacobi identity holds for the extended algebra

$$
[L_m, [L_n, L_k]] + [L_n, [L_k, L_m]] + [L_k, [L_m, L_n]] = 0 \tag{5}
$$

Expanding the Jacobi identity in terms of the expression for the extended algebra we get

$$
\begin{aligned}
(n-k)((m-n-k)L_{m+n+k} + cg(m, n+k)) \\
+(k-m)((n-k-m)L_{n+k+m} + cg(n, k+m)) \\
+(m-n)((k-m-n)L_{k+m+n} + cg(k, m+n)) = 0
\end{aligned}
\tag{6}
$$

$$
(n-k)g(m, n+k) + (k-m)g(n, k+m) + (m-n)g(k, m+n) = 0 \tag{7}
$$

Setting $k = 0$ in (7) we find

$$
(n+m)g(m,n) + (m-n)g(0, m+n) = 0 \tag{8}
$$

By the prior restriction on $g(n,0)$, $g(m,n)$ must then vanish unless $n = -m$. Returning to (7) and setting $k = 1-n$ and $m = -1$ we get

$$
2ng(-1,1) + (2-n)g(n,-n) - (n+1)g(-(n-1), n-1) = 0 \tag{9}
$$

Using $g(-1,1) = 0$, a recursion formula for $g(n,-n)$ can be obtained

$$
g(n,-n) = \frac{n+1}{n-2} g(n-1, -(n-1)) \tag{10}
$$

Using the normalization $g(2,-2) = \frac{1}{2}$ and applying the recursion formula we find the result

$$
g(m,-m) = \frac{1}{2} \prod_{n=2}^{m-1} \frac{n+2}{n-1} = \frac{1}{12}(m^3 - m) \tag{11}
$$

OPE OF T WITH ϕ

From 3.24 we know that

$$
\delta_\epsilon \phi(z,\bar{z}) = (h\partial_z \epsilon + \epsilon \partial_z)\phi(z,\bar{z}) + \text{a.c.} \tag{12}
$$

Using 3.37, the variation can also be expressed as

$$
\delta_\epsilon \phi(z,\bar{z}) = \frac{1}{2\pi i} \oint dz \, \left[T(z)\epsilon(z), \phi(\omega,\bar{\omega})\right] + \text{a.c.} = \frac{1}{2\pi i} \oint dz \; \epsilon(z) RT(z)\phi(\omega,\bar{\omega}) + \text{a.c.} \tag{13}
$$

Writing (12) as a contour integral, we find

$$\epsilon(\omega)\partial_\omega\phi = \frac{1}{2\pi i}\oint da \frac{\epsilon(a)\partial_a\phi(a)}{a-\omega} \tag{14}$$

$$h\partial_\omega\epsilon(\omega)\phi(\omega) = \frac{1}{2\pi i}\oint da \frac{h\partial_a\epsilon(a)\phi(a)}{a-\omega} \tag{15}$$

We then have

$$\frac{1}{2\pi i}\oint dz \left[\frac{h\partial_z\epsilon(z)\phi(z)}{z-\omega} + \frac{\epsilon(z)\partial_z\phi(z)}{z-\omega}\right] = \frac{1}{2\pi i}\oint dz\ \epsilon(z)RT(z)\phi(\omega,\bar{\omega}) \tag{16}$$

$$\frac{h\partial_z\epsilon(z)\phi(z)}{z-\omega} + \frac{\epsilon(z)\partial_z\phi(z)}{z-\omega} = \epsilon(z)RT(z)\phi(\omega,\bar{\omega}) \tag{17}$$

We know by Cauchy's theorem that

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint dz\ \frac{f(z)}{(z-a)^{n+1}} \tag{18}$$

which gives

$$\partial_z\epsilon(z) = \frac{\epsilon(z)}{z-\omega} \tag{19}$$

All together, we have our final result:

$$RT(z)\phi(\omega,\bar{\omega}) = \frac{\partial_z\phi(\omega,\bar{\omega})}{z-\omega} - \frac{h\phi(\omega,\bar{\omega})}{(z-\omega)^2} \tag{20}$$

DERIVATION OF THE VIRASORO ALGEBRA FROM THE STRESS TENSOR OPE

We are given that

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{z-\omega} \tag{21}$$

and

$$T(z) = \sum_n z^{-n-2}L_n \tag{22}$$

It is known that

$$[A,B] = \oint_0 d\omega \oint_\omega dz\ a(z)b(\omega) \tag{23}$$

Using this identity with our given OPE, we find

$$[L_m,L_n] = \frac{1}{(2\pi i)^2}\oint_0 d\omega\ \omega^{n+1}\oint_\omega dz\ z^{n+1}\left[\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{z-\omega}\right] \tag{24}$$

$$= \frac{1}{2\pi i}\oint_0 d\omega\omega^{n+1}\left[\frac{c}{2}\frac{1}{3!}\partial_z^3 z^{n+1}\Big|_{z=\omega} + 2T(\omega)\partial_z z^{n+1}\Big|_{z=\omega} + \partial T(\omega) z^{n+1}\Big|_{z=\omega}\right] \tag{25}$$

$$= \frac{1}{2\pi i}\oint_0 d\omega\left[\frac{c}{12}(n+1)n(n-1)\omega^{m+n-1} + 2(n+1)T(\omega)\omega^{m+n+1} + \partial T(\omega)\omega^{m+n+2}\right] \tag{26}$$

$$= \frac{1}{2\pi i}\oint_0 d\omega\left[\frac{c}{12}(n+1)n(n-1)\omega^{m+n-1} + 2(n+1)\omega^{m+n+1}\sum_k \omega^{-k-2}L_k + \omega^{m+n+2}\sum_k(-k-2)\omega^{-k-3}L_k\right] \tag{27}$$

$$= \frac{1}{2\pi i}\oint_0 d\omega\left[\frac{c}{12}(n+1)n(n-1)\omega^{m+n-1} + \sum_k\big(2(n+1)-(k+2)\big)\,\omega^{m+n-k-1}L_k\right] \tag{28}$$

We want each of the terms here to be singular, or else they do not contribute to the integral. Furthermore, we demand that they each have a first order pole only - otherwise derivatives would be introduced when calculating residues. These derivatives would also cause terms to vanish. These two conditions tell us

$$m+n-1=-1 \qquad m+n-k-1=-1 \tag{29}$$

The first statement gives us a factor of $\delta_{0,m+n}$ and the second gives us $k=m+n$. All together we have

$$[L_m, L_n] = \frac{c}{12}(n+1)n(n-1)\delta_{0,m+n} + 2(n+1)L_{m+n} - (m+n+2)L_{m+n} \tag{30}$$

$$= \frac{c}{12}(n^3-n)\delta_{0,m+n} + (n-m)L_{n+m} \tag{31}$$

which is the correct form of the Virasoro algebra with central extension.

TWO RESULTS FROM THE OPE OF T

The Laurent expansion of the stress tensor $T(z)$ is

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \qquad\qquad L_n = \frac{1}{2\pi i}\oint dz\; z^{n+1}T(z) \tag{32}$$

The Laurent expansion of a field ϕ with conformal dimensions $(h,\bar{h})$ is

$$\phi(z,\bar{z}) = \sum_{m,\bar{n}\in\mathbb{Z}} z^{-m-h}\bar{z}^{\bar{n}-\bar{h}}\phi_{m,\bar{n}} \tag{33}$$

$$\big[L_n,\phi(z)\big] = \left[\frac{1}{2\pi i}\oint d\omega\;\omega^{n+1}T(\omega),\phi(z)\right] = \frac{1}{2\pi i}\oint d\omega\;\omega^{n+1}\,\big[T(\omega),\phi(z)\big] \tag{34}$$

From the OPE of T with ϕ, we find this is equal to

$$\frac{h}{2\pi i}\oint d\omega\;\frac{\omega^{n+1}}{(\omega-z)^2}\phi(z) + \frac{1}{2\pi i}\oint d\omega\;\frac{\omega^{n+1}}{\omega-z}\partial_z\phi(z) \tag{35}$$

Evaluating the contour integrals gives

$$h(n+1)z^n\phi(z) + z^{n+1}\partial_z\phi(z) \tag{36}$$

Using the Laurent expansion for ϕ, this is equivalent to

$$h(n+1)z^n\sum_m z^{-m-h}\phi_m + z^{n+1}\partial_z\sum_m z^{-m-h}\phi_m = \sum_m z^{n-m-h}(hn-m)\phi_m \tag{37}$$

We can also use the expansion of ϕ in the original commutator to find

$$\left[L_n, \phi(z)\right] = \sum_m \left[L_n, z^{-m-h}\phi_m\right] \tag{38}$$

so we see that

$$\sum_m z^{-m-h}\left[L_n, \phi_m\right] = \sum_m z^{n-m-h}(hn-m)\phi_m \tag{39}$$

Defining $m' = m - n$,

$$\sum_{m'+n} z^{-m'-h}\left(n(h-1) - m'\right)\phi_{m'+n} = \sum_m z^{-m-h}\left[L_n, \phi_m\right] \tag{40}$$

and so we see that

$$\left[L_n, \phi_m\right] = \left(n(h-1) - m\right)\phi_{m+n} \tag{41}$$

We can also calculate the transformation of T:

$$\delta_\epsilon T(z) = \frac{1}{2\pi i}\oint d\omega\ \epsilon(\omega)T(\omega)T(z) = \frac{1}{2\pi i}\oint d\omega\ \epsilon(\omega)\left[\frac{c/2}{(\omega-z)^4} + \frac{2T(z)}{(\omega-z)^2} + \frac{\partial_z T(z)}{\omega - z}\right] \tag{42}$$

After evaluating the integrals:

$$\delta_\epsilon T(z) = \frac{c}{12}\partial_z^3\epsilon(z) + 2\partial_z\left(T(z)\epsilon(z)\right) + \epsilon(z)\partial_z T(z) \tag{43}$$

$$= \frac{c}{12}\partial_z^3\epsilon(z) + 2T(z)\partial_z\epsilon(z) + 3\epsilon(z)\partial_z T(z) \tag{44}$$

EXPECTATION VALUE OF TT

Using the TT OPE, we can calculate

$$\langle 0|T(z)T(\omega)|0\rangle = \langle 0|\frac{c/2}{(\omega-z)^4} + \frac{2T(z)}{(\omega-z)^2} + \frac{\partial_z T(z)}{\omega - z}|0\rangle \tag{45}$$

Using the TT OPE, we see that the second and third terms are proportional to $\langle 0|L_n|0\rangle = 0$. Since

$$L_n|0\rangle = 0 \quad , \quad n \geq -1 \qquad\qquad \langle 0|L_n = 0 \quad , \quad n \leq 1 \tag{46}$$

we see that $\langle 0|L_n|0\rangle = 0$ for all n. Therefore, the second and third terms vanish and we are left with

$$\langle 0|T(z)T(\omega)|0\rangle = \frac{c/2}{(\omega-z)^4}\langle 0|0\rangle \tag{47}$$

POSITIVITY OF CENTRAL CHARGE IN UNITARY THEORIES

$$\langle 0| \left[L_2, L_{-2}\right] |0\rangle = \langle 0|4L_4 + \frac{c}{2}\delta_{00}|0\rangle = \frac{c}{2}\langle 0|0\rangle \tag{48}$$

Since expectation values should be non-negative in a unitary theory, we see that $c \geq 0$.

EIGENVALUES OF L_0

A previously found result is

$$\big[L_n, \phi(z)\big] = h(n+1)z^n\phi(z) + z^{n+1}\partial_z\phi(z) \tag{49}$$

Defining $|h\rangle = \phi(0)|0\rangle$, we find that

$$\big[L_0, \phi(0)\big] |0\rangle = h\phi(0)|0\rangle = h|h\rangle \tag{50}$$

Expanding the commutator, we find

$$h|h\rangle = L_0\phi(0)|0\rangle - \phi(0)L_0|0\rangle \tag{51}$$

The second term vanishes by (46), and we are left with

$$L_0|h\rangle = h|h\rangle \tag{52}$$