Pufu Lecture Problems

M. Ross Tagaras
(Dated: November 17, 2019)

PROBLEM 1

Part (a)

Define $\vec{\phi}$ as the column vector composed of the ϕ_i's. First consider the term $\vec{\phi}^T\vec{\phi}$ under the transformation $\vec{\phi} \to R\vec{\phi}$. The transformed term is $\tilde{\vec{\phi}}^T\tilde{\vec{\phi}} = \left(R\vec{\phi}\right)^T R\vec{\phi} = \vec{\phi}^T R^T R\vec{\phi}$. Taking $R \in O(N)$ we find $\tilde{\vec{\phi}}^T\tilde{\vec{\phi}} = \vec{\phi}^T \mathbb{1}\vec{\phi} = \vec{\phi}^T\vec{\phi}$, so this term is invariant under an $O(N)$ transformation, as is the entire sum of such terms. Similarly, we see that $\sum_i (\nabla\phi_i)^2$ will be invariant, as well as the sum of $(\phi_i\phi_i)^2$. The only term not of this form is $\sum_i \phi_i^4$.

Part (b)

The general form of the beta function is $\beta(\alpha) = a\frac{\partial}{\partial a}\left[\alpha_0 Z_\alpha^m Z_\phi^p \mu^q\right]$ for some m, n, q. The Z's are all of the form 1+something, so when multiplied out, the very first term will always be 1. Then, when we differentiate, we see that the first term in the beta function will be $\alpha_0 q$. The actual relations between the renormalized and bare constants are

$$t = t_0 a^2 Z_\phi^{-1} Z_t \qquad u = u_0 a^\epsilon Z_\phi^{-2} Z_u \qquad v = v_0 a^\epsilon Z_\phi^{-2} Z_v$$

so the needed constants are

$$c_1 = 2 \qquad c_2 = \epsilon \qquad c_3 = \epsilon$$

Part (c)

Using Mathematica, I found the following solutions to the system $\beta_t = \beta_u = \beta_v = 0$:

- $t = u = v = 0$
- $t = u = 0$, $v = \frac{\epsilon}{72}$
- $t = 0$, $u = \frac{\epsilon}{24N}$, $v = \frac{\epsilon}{72}\left(1 - \frac{4}{N}\right)$
- $t = 0$, $u = \frac{\epsilon}{8(8+N)}$, $v = 0$

Since we have two operators with the same naive scaling dimension, there can be mixing between them. The relationship between normalized and bare operators generalizes to (Peskin and Schroeder ch. 12.4)

$$\mathcal{O}_0^i = Z_\mathcal{O}^{ij} O^j$$

If we can diagonalize Z^{ij}, then we will have effectively "unmixed" the operators. Then, the dimensions of the operators in our theory minus the spacetime dimension are equal to the derivative of the beta function at the fixed point with respect to the coupling constant. However, I don't actually know how to determine what Z^{ij} is, so instead we can diagonalize $\partial\beta_i/\partial g_j$ at each fixed point. The general expression is

$$\frac{\partial\beta_i}{\partial g_j} = \begin{pmatrix} 2 - 8(N+2)u - 24v & -8t(N+2) & -24t \\ 0 & \epsilon - 16u(N+8) - 48v & -48u \\ 0 & -96v & \epsilon - 96u - 144v \end{pmatrix}$$

At the first fixed point, this becomes

$$\frac{\partial\beta_i}{\partial g_j} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$$

which gives $\Delta_{\phi^2} = d + 2$ and $\Delta_{(\phi^2)^2} = \Delta_{\phi^4} = d + \epsilon$. At the second fixed point, we find

$$\frac{\partial\beta_i}{\partial g_j} = \begin{pmatrix} 2 - \frac{\epsilon}{3} & 0 & 0 \\ 0 & \frac{\epsilon}{3} & 0 \\ 0 & -\frac{4\epsilon}{3} & -\epsilon \end{pmatrix}$$

This can be brought to a diagonal form:

$$\frac{\widetilde{\partial\beta_i}}{\partial g_j} = \begin{pmatrix} 2 - \frac{\epsilon}{3} & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & \frac{\epsilon}{3} \end{pmatrix}$$

which tells us that $\Delta_{\phi^2} = d - 2 - \epsilon/3$, $\Delta_{(\phi^2)^2} = d - \epsilon$, $\Delta_{\phi^4} = d + \epsilon/3$. At the third fixed point:

$$\frac{\widetilde{\partial\beta_i}}{\partial g_j} = \begin{pmatrix} -2 - \frac{2\epsilon}{3}\left(1 - \frac{1}{N}\right) & 0 & 0 \\ 0 & -\frac{\epsilon}{3}\left(1 + \frac{8}{N}\right) & -\frac{2\epsilon}{N} \\ 0 & -\frac{4\epsilon}{3}\left(1 - \frac{4}{N}\right) & -\epsilon\left(1 - \frac{4}{N}\right) - 4N \end{pmatrix}$$

This tells us that $\Delta_{\phi^2} = d + \epsilon(4 - N)/(3N)$, $\Delta_{(\phi^2)^2} = d - \epsilon$, and $\Delta_{\phi^4} = d + 2 - \frac{2\epsilon}{2}(1 - 1/N)$
At the fourth fixed point:

$$\frac{\widetilde{\partial\beta_i}}{\partial g_j} = \begin{pmatrix} 2 - \frac{\epsilon(N+2)}{N+8} & 0 & 0 \\ 0 & -\epsilon & -\frac{6\epsilon}{N+8} \\ 0 & 0 & \epsilon\left(1 - \frac{12}{N+8}\right) \end{pmatrix}$$

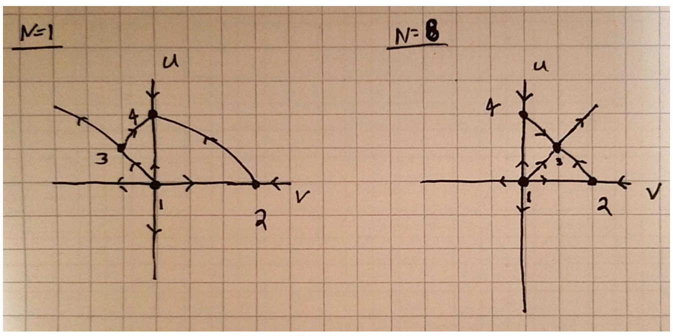
which gives $\Delta_{\phi^2} = d + \frac{\epsilon(N-4)}{N+8}$, $\Delta_{(\phi^2)^2} = d - \epsilon$, and $\Delta_{\phi^4} = d + 2 - \frac{\epsilon(N+2)}{N+8}$

Part (d)

The most stable fixed point should be the one with the smallest ratio of relevant to irrelevant directions, as determined by part (c). For $N < 4$, this is point 3, for $N > 4$, this is point 4.

Part (e)

Below are RG flow diagrams for $N = 1$ and $N = 8$:



Part (f)

The $O(N)$ fixed point is the one where $v \to 0$.