# Beem Lecture Problems

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## PROBLEM I

### Part (i)

We can produce a special conformal transformation by inverting, translating, and the inverting again:

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + b^\mu \to \left(\frac{x^\mu}{x^2} + b^\mu\right)\left[\left(\frac{x^\nu}{x^2} + b^\nu\right)\left(\frac{x^\rho}{x^2} + b^\rho\right) g_{\nu\rho}\right]^{-1} \tag{1}$$

$$= \frac{x^\mu + x^2 b^\mu}{x^2}\left[\frac{1}{x^2} + \frac{2b\cdot x}{x^2} + b^2\right]^{-1} = \frac{x^\mu + x^2 b^\mu}{1 + 2b\cdot x + x^2 b^2} \tag{2}$$

### Part (ii)

The magnitude of a vector $x^\mu$ is $\sqrt{x^\mu x_\mu}$, so we can calculate $\left|x_1' - x_2'\right|$ as

$$\left|x_1' - x_2'\right| = \sqrt{\frac{x_1^\mu + b^\mu x_1^2}{\gamma_{x_1}}\frac{x_{1\mu} + b_\mu x_1^2}{\gamma_{x_1}} + \frac{x_2^\mu + b^\mu x_2^2}{\gamma_{x_2}}\frac{x_{2\mu} + b_\mu x_2^2}{\gamma_{x_2}} - \frac{2\left(x_1^\mu + b^\mu x_1^2\right)\left(x_{2\mu} + b_\mu x_2^2\right)}{\gamma_{x_1}\gamma_{x_2}}} \tag{3}$$

where $\gamma_x = 1 + 2b\cdot x + b^2 x^2$. This simplifies to

$$\sqrt{\frac{x_1^2}{\gamma_{x_1}} + \frac{x_2^2}{\gamma_{x_2}} - \frac{2\left(x_1\cdot x_2 + x_2^2(x_1\cdot b) + x_1^2(x_2\cdot b) + b^2 x_1^2 x_2^2\right)}{\gamma_{x_1}\gamma_{x_2}}} \tag{4}$$

After canceling terms in the numerator:

$$\left|x_1' - x_2'\right| = \sqrt{\frac{x_1^2 + x_2^2 - 2(x_1\cdot x_2)}{\gamma_{x_1}\gamma_{x_2}}} = \frac{\sqrt{\left(x_1 - x_2\right)^2}}{\gamma_{x_1}^{1/2}\gamma_{x_2}^{1/2}} = \frac{|x_1 - x_2|}{\gamma_{x_1}^{1/2}\gamma_{x_2}^{1/2}} \tag{5}$$

### Part (iv)

By rotation and translation symmetry, we can fix the two point function to be

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = f\left(|x_{12}|\right) \tag{6}$$

Then, since consistency under finite conformal transformations requires that the 2-point function transforms as

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \Omega_1^{\Delta_1}\Omega_2^{\Delta_2}\left\langle \mathcal{O}_1' \mathcal{O}_2' \right\rangle$$

we see that

$$f\left(|x_{12}|\right) = \Omega_1^{\Delta_1}\Omega_2^{\Delta_2} f\left(|x'_{12}|\right) \tag{7}$$

For the case $\Omega_i = \lambda$ (i.e. a rescaling), this implies

$$f\left(|x_{12}|\right) = \lambda^{\Delta_1+\Delta_2} f\left(|\lambda x_{12}|\right) \tag{8}$$

We can easily see that

$$f(|x_{12}|) = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{9}$$

is consistent with this. If we instead take $\Omega_i = 1/\gamma_i$ (a special conformal transformation), we find

$$\frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} = \frac{f\left(\frac{|x_{12}|}{\gamma_1^{1/2}\gamma_2^{1/2}}\right)}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}}\left(\gamma_1\gamma_2\right)^{\frac{\Delta_1+\Delta_2}{2}} \tag{10}$$

For this equality to hold, we must have

$$\gamma_1^{\frac{\Delta_1+\Delta_2}{2}}\gamma_2^{\frac{\Delta_1+\Delta_2}{2}} = \gamma_1^{\Delta_1}\gamma_2^{\Delta_2} \tag{11}$$

which implies that $\Delta_1 = \Delta_2$. The 2-point function is then

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle = \frac{C_{12}\delta_{\Delta_1,\Delta_2}}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{12}$$