Homework 2

Due on: February 5.

Problem 1. The Dirac Group.

The Dirac matrices in n-dimensional Euclidean space satisfy the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn}\mathbb{I} \ . \tag{1.1}$$

(In Minkowski space, one should replace δ^{mn} by η^{mn}. Then one of the Dirac matrices should be multiplied by $+i$ or $-i$.) This relation defines an algebra because in an algebra one can add and multiply. However, in groups one has only one operation, called group multiplication. We shall now construct a corresponding abstract group from the Dirac matrices with group multiplication given by matrix multiplication. So we view the γ^m as abstract group elements. The matrices $-\gamma^m$ are different group elements because in the group we cannot add or subtract, only multiply. Then we can take products of the γ^m, and products of products. In these products we use (1.1) to simplify the expressions. We go on until we get no new group elements, so we enforce closure. Then we have a group. The unit matrix $\mathbb{I}$ becomes the unit element e of the abstract group, and the relation $(\gamma^m)^2 = \mathbb{I}$ shows that the group elements γ^m have order 2. For example, the Dirac group in 2-dimensional Euclidean space consists of the following group elements:

$$G_{\text{Dirac}}(n = 2, \text{Euclidean}) = \{\pm e, \pm\gamma^1, \pm\gamma^2, \pm\gamma^1\gamma^2\} \ . \tag{1.2}$$

It has order 8. In 3 Euclidean dimensions, we get

$$G_{\text{Dirac}}(n = 3, \text{Euclidean}) = \{\pm e, \pm\gamma^m, \pm\gamma^m\gamma^n \text{ with } m < n, \pm\gamma^1\gamma^2\gamma^3\} \ . \tag{1.3}$$

(a) What is the order of the Dirac group in n dimensions?

(b) Is the set $(e, -e)$ a normal subgroup? Does the set of four group elements $(e, -e, \gamma_1 \cdots \gamma_n, -\gamma_1 \cdots \gamma_n)$ form a subgroup? A normal subgroup?

(c) Construct the **commutator subgroup $C(G)$** for the Dirac group in n dimensions. It is generated by all elements $aba^{-1}b^{-1}$. In the notes it is shown that $C(G)$ is a normal subgroup and $G/C(G)$ is abelian for any group G. Check this property for the Dirac groups.

Problem 2. Dihedral groups D_n.

The rotations and reflections of a regular n-gon form the dihedral group D_n. See the notes for a description of D_3 and D_4.

(a) What is the order of D_n?

(b) Consider the following presentation:

$$G = \{a, b \mid a^n = e, b^2 = e, bab^{-1} = a^{-1}\} \ . \tag{1.1}$$

Show that the elements of G are $\{e, a, \cdots, a^n, b, ab, \cdots, a^n b\}$. We want to show that $G = D_n$. If a regular n-gon has vertices $1, 2, \ldots, n$, give a representation of a and b in terms of cycles. Then prove that $G = D_n$.

(c) Construct the commutator subgroup $C(D_n)$ of D_n. Check that it is an abelian normal subgroup.