Group Theory - Homework 1

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(Dated: January 31, 2020)

PROBLEM 1

Throughout, set $c = 1$ for convenience.

Part (a)

The identity transformation is the special case $v = 0$. To check closure, consider the composition of two transformations, one with parameter v and one with parameter w. Then,

$$x'' = \frac{x' - wt'}{\sqrt{1-w^2}} = \frac{1}{\sqrt{1-w^2}}\left[\frac{x-vt}{\sqrt{1-v^2}} - w\frac{t-vx}{\sqrt{1-v^2}}\right] = \frac{1+vw}{\sqrt{(1-v^2)(1-w^2)}}\left[x - \frac{v+w}{1+vw}t\right] = \frac{1}{\sqrt{1-\left(\frac{v+w}{1+vw}\right)^2}}\left[x - \frac{v+w}{1+vw}t\right]$$

$$t'' = \frac{t' - wx'}{\sqrt{1-w^2}} = \frac{1}{\sqrt{1-w^2}}\left[\frac{t-vx}{\sqrt{1-v^2}} - w\frac{x-vt}{\sqrt{1-v^2}}\right] = \frac{1+vw}{\sqrt{(1-v^2)(1-w^2)}}\left[t - \frac{v+w}{1+vw}x\right] = \frac{1}{\sqrt{1-\left(\frac{v+w}{1+vw}\right)^2}}\left[t - \frac{v+w}{1+vw}x\right]$$

The result is a set of new transformations with parameter $\frac{v+w}{1+vw}$. Thus, we have closure. From this expression, we can also see that composing transformations with parameters v and $-v$ will give $x'' = x$, so an inverse for each transformation exists.

Now consider three transformations a, b, c with parameters u, v, w. Using the result obtained when proving closure, we see that composing transformations like $(ab)c$ gives a transformation with parameter

$$\frac{\frac{u+v}{1+uv} + w}{1 + \frac{u+v}{1+uv}w} = \frac{u+v+w+uvw}{1+uv+uw+vw}$$

and composing transformations like $a(bc)$ gives

$$\frac{u + \frac{v+w}{1+vw}}{1 + u\frac{v+w}{1+vw}} = \frac{u+v+w+uvw}{1+uv+uw+vw}$$

so we have associativity.

Part (b)

These transformations obey the four group properties. The identity transformation is simply multiplying by 1. If we consider two separate transformations, since both leave $(x-y)^2$ invariant, so does their composition, so closure is satisfied.

Associativity follows from closure. If we take three transformations f, g, h, then

$$(fg)h(x-y)^2 = (fg)(x-y)^2 = (x-y)^2$$

$$f(gh)(x-y)^2 = f(x-y)^2 = (x-y)^2$$

It is clear that if we have some transformation f, we can find a transformation g such that $fg(x-y)^2 = (x-y)^2$.

Part (c)

Define $z^\mu = x^\mu - y^\mu$. Then,

$$z'^\mu = L^\mu{}_\nu z^\nu$$

Multiplying by $\eta_{\mu\rho}(z')^\rho$ gives

$$z'^2 = L^\mu{}_\nu z^\nu \eta_{\mu\rho} z'^\rho = L^\mu{}_\nu z^\nu \eta_{\mu\rho} L^\rho{}_\sigma z^\sigma$$

Multiplying by $\eta_{\nu\sigma}$ gives

$$z'^2 \eta_{\nu\sigma} = L^\mu{}_\nu L^\rho{}_\sigma \eta_{\mu\rho} z^2$$

Since z^2 is left invariant, $z^2 = z'^2$, so we can cancel it on both sides to find

$$\eta_{\nu\sigma} = L^\mu{}_\nu L^\rho{}_\sigma \eta_{\mu\rho}$$

If we take this constraint to be the definition of L, then it specifies the matrices that comprise $O(3,1)$. If we define L to be the matrix of a Lorentz transformation, then it is a statement that η is invariant under a similarity transformation.

PROBLEM 2

Part (a)

We know that the isometry group of the tetrahedron is S_4. Since the cube can be decomposed into two tetrahedra, its isometries should be those for an individual tetrahedron T combined a mapping between T and T'. Therefore, the isometry group of the cube is $S_4 \times \langle\sigma\rangle$, where $\langle\sigma\rangle$ is the group generated by spatial inversions.

Part (b)

In the first possible rotation, the axis of rotation passes through a face of the cube. There are three unique rotations and three ways to arrange the axis, so we have nine rotations of this type. In the next type of rotation, the axis is through two diagonally opposite corners. There are two unique rotations and four ways to choose the axis, so we have eight rotations of this type. Finally, we can let the axis pass through the center of a pair of opposite edges. There is a single unique rotation and six ways to choose the axis, so we have six possible rotations. Together with the identity rotation, there are $1 + 9 + 8 + 6 = 24$ rotations.

The nine rotations with axis through a pair of faces are

$$\begin{array}{ccc} (1647)(8352) & (14)(67)(85)(32) & (1746)(8253) \\ (4527)(6381) & (24)(57)(13)(68) & (4725)(6183) \\ (4635)(7182) & (34)(56)(12)(78) & (6453)(1728) \end{array}$$

The eight rotations with axis through opposite corners are

$$(678)(423) \qquad (432)(687)$$

$$(132)(567) \qquad (123)(576)$$

$$(142)(657) \qquad (124)(675)$$

$$(143)(758) \qquad (134)(785)$$

The six rotations with axis through a pair of opposite edges are

$$(18)(26)(37)(45) \qquad (15)(27)(36)(48) \qquad (16)(25)(37)(48)$$

$$(17)(26)(35)(48) \qquad (15)(28)(37)(46) \qquad (15)(26)(38)(47)$$

The identity is (1)(2)(3)(4)(5)(6)(7)(8).

Part (c)

Rotations of the tetrahedron correspond to even permutations, so the rotation group is A_4. Rotations of the cube correspond to permutations of the diagonals, so the rotation group is S_4.