Group Theory - Homework 8

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PROBLEM 1

If we take $x' = \mathcal{M}x$, then we must have that $\mathcal{M}^\dagger \Omega \mathcal{M} = \Omega$. Using $\det(\mathcal{M}^\dagger) = \det(\mathcal{M})^*$ and $\det(AB) = \det(A)\det(B)$, we see that $\det(\Omega) = \det(\Omega)\det(\mathcal{M})^*\det(\mathcal{M})$, which gives $|\det \mathcal{M}|^2 = 1$. If we take $\det(\mathcal{M}) = 1$, then we get a group.

This does not appear to be equivalent to any of the classical groups.

PROBLEM 2

Since we know that unitary matrices are generated by antihermitian matrices, $U(N)$ must be compact. Since unitary matrices can be diagonalized, we can write an element $M \in U(2)$ as

$$M(t) = D \begin{pmatrix} e^{itA} & 0 \\ 0 & e^{itB} \end{pmatrix} D^{-1}$$

It is clear that there is a continuous path parameterized by t from the identity to any choice of M, so $U(2)$ is connected. We can write $U(N) \cong U(1) \times SU(N)$. Since $U(1) \cong S_1$, which is not simply connected, $U(N)$ is also not simply connected. A covering group should be simply connected, so $U(2)$ is not the covering group for $SO(3)$.

To see that the given map is a homomorphism:

$$\sigma \cdot \varphi(gh) \cdot x = \sigma_i (RS)^i{}_j x^j = U^\dagger V^\dagger x^i \sigma_i V U = U^\dagger \sigma_i R^i{}_j x^j V = \sigma_i R^i{}_j R^j{}_k x^k = \sigma \cdot \varphi(g)\varphi(h) \cdot x$$

so we have $\varphi(gh) = \varphi(g)\varphi(h)$. To find the kernel, we need the matrices U that satisfy $U^\dagger \sigma_i x^i U = \sigma_i x^i$. If we define $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, apply unitarity, and expand the sum over i, we find three equations:

$$\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \sigma_1 = \sigma_1 \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \implies \begin{pmatrix} -B & D \\ A & -C \end{pmatrix} = \begin{pmatrix} -C & A \\ D & -B \end{pmatrix}$$

$$\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \sigma_2 = \sigma_2 \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \implies \begin{pmatrix} -B & -D \\ A & C \end{pmatrix} = \begin{pmatrix} C & -A \\ D & -B \end{pmatrix}$$

$$\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \sigma_3 = \sigma_3 \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \implies \begin{pmatrix} D & B \\ -C & -A \end{pmatrix} = \begin{pmatrix} D & -B \\ C & -A \end{pmatrix}$$

This is satisfied when $U = AI_2$, with $A^*A = 1$.

PROBLEM 3

If $x^T x = \tilde{x}^T \tilde{x}$, then $(x+y)^T(x+y)$ must also be invariant by closure. In terms of components,

$$(x^\alpha + y^\alpha)\,\delta_{\alpha\beta}\left(x^\beta + y^\beta\right) = (\tilde{x}^\gamma + \tilde{y}^\gamma)\,\delta_{\gamma\delta}\left(\tilde{x}^\delta + \tilde{y}^\delta\right)$$

$$x^\alpha \delta_{\alpha\beta} x^\beta + y^\alpha \delta_{\alpha\beta} y^\beta + 2x^\alpha \delta_{\alpha\beta} y^\beta = \tilde{x}^\gamma \delta_{\gamma\delta} \tilde{x} + \tilde{y}^\gamma \delta_{\gamma\delta} \tilde{y} + 2\tilde{x}^\gamma \delta_{\gamma\delta} \tilde{y} \implies x^T y = \tilde{x}^T \tilde{y}$$

Similarly, requiring $x^\dagger \Omega x = \tilde{x}^\dagger \Omega \tilde{x}$ implies that $x^\dagger \Omega y + y^\dagger \Omega x = \tilde{x}^\dagger \Omega \tilde{y} + \tilde{y}^\dagger \Omega \tilde{x}$, so $x^\dagger \Omega y$ is also invariant.

Now we consider the generators of the group. If we expand a group element as $\mathcal{M} = I + M + \ldots$, then $\mathcal{M}^T \mathcal{M} = I$ gives $I = (I + M^T + \ldots)(I + M + \ldots)$ which implies $M + M^T = 0$. The other condition $\mathcal{M}^\dagger \Omega \mathcal{M} = \Omega$ gives $(I + M^\dagger + \ldots)\Omega(I + M + \ldots) = \Omega$, which implies $\Omega M + M^\dagger \Omega = 0$.

Defining $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we find

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \implies A^T = -A, \qquad D^T = -D, \qquad B^T = -C, \qquad C^T = -B$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = 0 \implies C = C^\dagger, \qquad D = -A^\dagger, \qquad A = -D^\dagger, \qquad B = B^\dagger$$

PROBLEM 4

The constraints on our generators are $\Omega M + M^T \Omega = 0$ and $M^\dagger H + HM = 0$. Writing $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we find

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = 0 \implies C = C^T, \qquad D = -A^T, \qquad A = -D^T, \qquad B = B^T$$

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} + \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0 \implies A^\dagger = -A, \qquad B = C^\dagger, \qquad D = -D^\dagger$$

These conditions combine to give

$$M = \begin{pmatrix} A & S \\ S^* & A^* \end{pmatrix}$$

where A is antisymmetric and S is symmetric.

PROBLEM 5

Let M, N satisfy the two conditions given in the problem statement. Then,

$$\operatorname{tr}[M, N] = \operatorname{tr}(MN - NM) = \operatorname{tr}(MN) - \operatorname{tr}(NM) = \operatorname{tr}(MN) - \operatorname{tr}(MN) = 0$$

$$\Omega[M, N] = \Omega MN - \Omega NM = M^* N^* \Omega - N^* M^* \Omega = [M, N]^* \Omega$$

so we have closure. The matrix commutator trivially satisfies antisymmetry, linearity, and the Jacobi identity, so the given matrices form a Lie algebra.