# Group Theory - Homework 10

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## PROBLEM 1

### Part (a)

If we use the representations

$$H_I = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \frac{1}{2}\sigma_3 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \qquad \frac{1}{2}\sigma_3 \text{ in } I^{th} \text{ position}$$

$$E_{JK}^{\eta\eta'} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^{\eta} \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma^{\eta'} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \qquad \sigma^{\eta} \text{ in } J^{th} \text{ position, } \sigma^{\eta'} \text{ in } K^{th} \text{ position, } J < K$$

then we can immediately see that the only nontrivial commutative behavior is when $I = J$ or $I = K$. If we take $\eta = +1$, $\eta' = -1$, then the usual commutation relations for Pauli matrices give

$$\left[\frac{1}{2}\sigma_3, \sigma^+\right] = \frac{1}{4}[\sigma_3, \sigma_1] + \frac{i}{4}[\sigma_3, \sigma_2] = \sigma^+$$

$$\left[\frac{1}{2}\sigma_3, \sigma^-\right] = \frac{1}{4}[\sigma_3, \sigma_1] - \frac{i}{4}[\sigma_3, \sigma_2] = -\sigma^-$$

and we find

$$\left[H_I, E_{JK}^{+-}\right] = (\delta_{IJ} - \delta_{JK})\, E_{JK}^{+-}$$

Generalizing to the other possible values of $\eta, \eta'$, we have

$$\left[H_I, E_{JK}^{\eta\eta'}\right] = \left(\eta\delta_{IJ} + \eta'\delta_{JK}\right) E_{JK}^{\eta\eta'}$$

### Part (b)

Since the raising/lowering operators should obey $[H, E_\alpha] = \alpha E_\alpha$, we see that $\vec{\alpha} = (0, \ldots, 0, \eta, 0, \ldots, 0, \eta', 0, \ldots, 0)$, where $\eta$ is in the $J^{th}$ position and $\eta'$ is in the $K^{th}$ position. For raising operators, this root should be positive, and since we define positive roots to have a positive first nonzero component, the raising operators should have $\eta = 1$.

### Part (c)

Since $(A \otimes B)^T = A^T \otimes B^T$, we have

$$E_{-\alpha} = \left(E_{JK}^{\eta\eta'}\right)^\dagger = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes (\sigma^\eta)^\dagger \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \left(\sigma^{\eta'}\right)^\dagger \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

$$= \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^{-\eta} \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma^{-\eta'} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} = E_{JK}^{(-\eta)(-\eta')}$$

These are lowering operators.

**Part (d)**

First, we calculate $\left[E_{IJ}^{+-}, E_{IJ}^{-+}\right]$:

$$\left[E_{IJ}^{+-}, E_{IJ}^{-+}\right] = \left[\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\sigma^{+}\otimes\sigma_3\otimes\cdots\otimes\sigma_3\otimes\sigma^{-}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}, \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\sigma^{-}\otimes\sigma_3\otimes\cdots\otimes\sigma_3\otimes\sigma^{+}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right]$$

$$\begin{aligned} &= \left(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\left(\sigma^{+}\sigma^{-}\right)\otimes\sigma_3^2\otimes\cdots\otimes\sigma_3^2\otimes\left(\sigma^{-}\sigma^{+}\right)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right) \\ &- \left(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\left(\sigma^{-}\sigma^{+}\right)\otimes\sigma_3^2\otimes\cdots\otimes\sigma_3^2\otimes\left(\sigma^{+}\sigma^{-}\right)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right) \end{aligned}$$

$$= \left(\mathbb{1}\otimes\ldots\mathbb{1}\otimes\frac{\mathbb{1}+\sigma_3}{2}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\frac{\mathbb{1}-\sigma_3}{2}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right) - \left(\mathbb{1}\otimes\ldots\mathbb{1}\otimes\frac{\mathbb{1}-\sigma_3}{2}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\frac{\mathbb{1}+\sigma_3}{2}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)$$

$$= \frac{1}{2}\left[\mathbb{1}^{\otimes^{I-1}}\otimes\sigma_3\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1} - I^{\otimes^{J-1}}\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}\right] = H_I - H_J$$

For the other values of $\eta, \eta'$ we find similar results, leading to

$$\left[E_{IJ}^{\eta\eta'}, E_{IJ}^{-\eta,-\eta'}\right] = \eta H_I + \eta' H_J = \alpha^K H_K g_{\alpha,-\alpha}$$

where $g_{\alpha,-\alpha}$ cancels a numerical factor from the inverse metric.

**Part (e)**

We already know that $g_{IJ}^{def} = 2\delta_{IJ}$ and $g_{\alpha,-\alpha}^{def} = 2$. For the spinor representations, we have

$$g_{IJ}^s = \mathrm{tr} H_I^s H_J^s = \mathrm{tr}\left[\left(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\underbrace{\frac{1}{2}\sigma_3}_{\mathrm{I}}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)\left(\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\underbrace{\frac{1}{2}\sigma_3}_{J}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right)\right]$$

$$= \begin{cases} \mathrm{tr}\left[\mathbb{1}\otimes\ldots\mathbb{1}\otimes\frac{1}{2}\sigma_3\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\frac{1}{2}\sigma_3\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right] = 0 & I \neq J \\ \mathrm{tr}\left[\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\frac{1}{4}\sigma_3^2\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right] = \frac{N}{2} & I = J \end{cases}$$

So we have $g_{IJ}^s = \frac{N}{2}\delta_{IJ}$ and $g_s^{IJ} = \frac{2}{N}\delta^{IJ}$. For $g_{\alpha,-\alpha}^s$:

$$g_{\alpha,-\alpha}^s = \mathrm{tr} E_\alpha^s E_{-\alpha}^s = \mathrm{tr}\left[\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\left(\sigma^\eta\sigma^{-\eta}\right)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\left(\sigma^{\eta'}\sigma^{-\eta'}\right)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\right]$$

We have that

$$\sigma^\eta\sigma^{-\eta} = \frac{1}{4}(\sigma_1 \pm i\sigma_2)(\sigma_1 \mp i\sigma_2) = \frac{1\pm\sigma_3}{2}$$

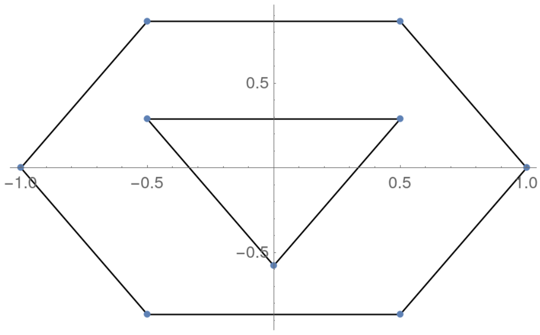
Therefore, $g_{\alpha,-\alpha}^s = \frac{N}{2}$. These two results give $g_s^{IJ} g_{\alpha,-\alpha}^s = \delta^{IJ}$, which matches the result for the defining representation.

**PROBLEM 2**

The weights for $SU(3)$ are

$$\vec{\mu}^1_{def} = \left(\frac{1}{2}, \frac{1}{\sqrt{12}}\right) \qquad \vec{\mu}^2_{def} = \left(-\frac{1}{2}, \frac{1}{\sqrt{12}}\right) \qquad \vec{\mu}^3_{def} = \left(0, -\frac{2}{\sqrt{12}}\right)$$

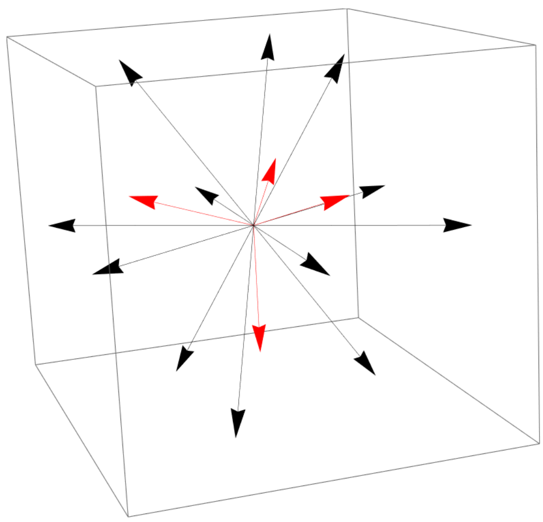
The roots are given by $\vec{\alpha}^{jk} = \vec{\mu}^j_{def} - \vec{\mu}^k_{def}$. The root/weight diagram for $SU(3)$ is




The vertices of the interior triangle are the weights. The weights for $SU(4)$ are

$$\vec{\mu}^1_{def} = \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^2_{def} = \left(-\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^3_{def} = \left(0, -\frac{2}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^4_{def} = \left(0, 0, -\frac{3}{\sqrt{24}}\right)$$

The root/weight diagram for $SU(4)$ is



where weights are shown in red. The roots are differences of two weights. If we treat the weights as vectors that point to each vertex of the triangle/tetrahedron, then the difference of any two gives the vector connecting the vertices, which are edges.

**PROBLEM 3**

The weights for the defining representation of $SO(4)$ are

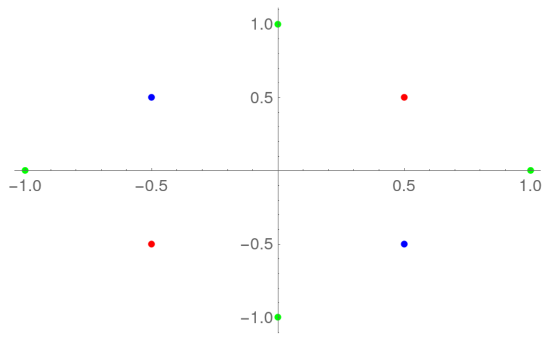
$$\vec{\mu}^1_{def} = (1, 0) \qquad \vec{\mu}^2_{def} = (-1, 0) \qquad \vec{\mu}^3_{def} = (0, 1) \qquad \vec{\mu}^4_{def} = (0, -1)$$

The weights for the spinor representation (denoted by $s$) and the conjugate spinor representation (denoted by $c$) are

$$\vec{\mu}_s^1 = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\mu}_s^2 = \left(-\frac{1}{2}, -\frac{1}{2}\right)$$

$$\vec{\mu}_c^1 = \left(\frac{1}{2}, -\frac{1}{2}\right) \qquad \vec{\mu}_c^2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

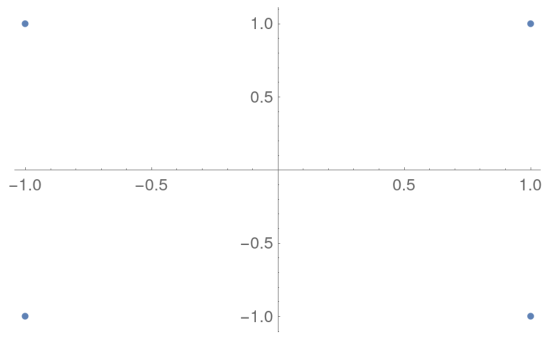
The weight diagram for $SO(4)$ is




The weights of the defining representation are green, the weights for the spinor representation are red, and the weights for the conjugate representation are blue. The roots are

$$\vec{\alpha}_1 = (1, 1) \qquad \vec{\alpha}_1 = (-1, 1) \qquad \vec{\alpha}_1 = (1, -1) \qquad \vec{\alpha}_1 = (-1, -1)$$

and the root diagram is




The weights for the defining representation of $SO(6)$ are

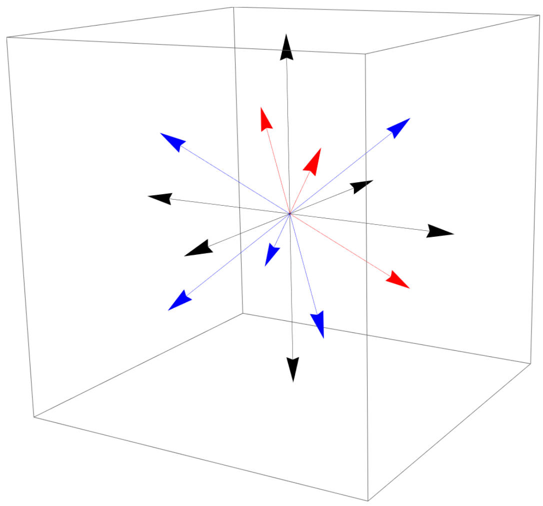
$$\vec{\mu}_{def}^1 = (1, 0, 0) \qquad \vec{\mu}_{def}^2 = (-1, 0, 0) \qquad \vec{\mu}_{def}^3 = (0, 1, 0) \qquad \vec{\mu}_{def}^4 = (0, -1, 0) \qquad \vec{\mu}_{def}^5 = (0, 0, 1) \qquad \vec{\mu}_{def}^6 = (0, 0, -1)$$

The weights for the spinor representation (denoted by $s$) and the conjugate spinor representation (denoted by $c$) are

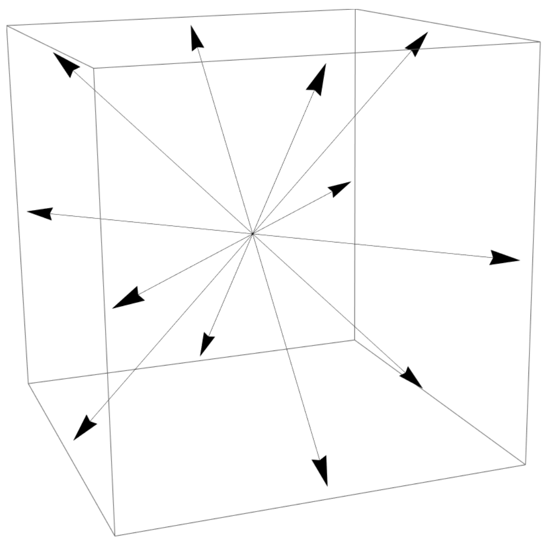
$$\vec{\mu}_s^1 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \qquad \vec{\mu}_s^2 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\mu}_s^3 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\vec{\mu}_c^1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\mu}_c^2 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\mu}_c^3 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \qquad \vec{\mu}_c^4 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \qquad \vec{\mu}_c^5 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

The weight diagram for $SO(6)$ is



The weights of the defining representation are black, the weights for the spinor representation are red, and the weights for the conjugate representation are blue. The root diagram is



When considering the square/cube, we see that the roots end at the vertices, and the fundamental weights end at the centers of the edges. The weights of the spinor irreps end along the diagonals, halfway from the origin to the vertices.

The fundamental weights of $SO(6)$ are

$$\vec{\mu}^1_{FW} = (1, 0, 0) \qquad \vec{\mu}^2_{FW} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\mu}^3_{FW} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

One ends on the surface of the cube, while two do not.

**PROBLEM 4**

The root systems of $SU(4)$ and $SO(6)$ are isomorphic. We should expect this, since their Dynkin diagrams are the same.