Group Theory - Homework 4

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PROBLEM 1

The proper normal subgroups of S_N are (I looked this up)

N	Proper Normal Subgroups
1	None
2	None
3	S_2, A_3
4	V, A_4
$N \geq 5$	A_N

Since S_2, A_3, and V are abelian, the first semi-simple symmetric group is S_5 and all $N > 5$ are semi-simple as well.

PROBLEM 2

G_D is defined as

$$G_D = \{\pm e, \pm\gamma_1, \pm\gamma_2 \pm \gamma_1\gamma_2\} \qquad \gamma_i^2 = e \qquad \gamma_1\gamma_2 = -\gamma_2\gamma_1$$

It is clear that the $2d$ Dirac group is nonabelian, so it can only be isomorphic to D_4 or Q. The order of each element is

$$|-e| = |\pm\gamma_1| = |\pm\gamma_2| = 2 \qquad |\pm\gamma_1\gamma_2| = 4$$

This matches D_4, where

$$\left|a^2\right| = |b| = |ab| = \left|a^2 b\right| = \left|a^3 b\right| = 2 \qquad\qquad |a| = \left|a^3\right| = 4$$

On the other hand, Q has more than two elements of order 4, so it cannot be isomorphic to G_D.

PROBLEM 3

A three-dimensional rep is

$$(12)_A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (23)_A = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (123)_A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (132)_A = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying gives

$$(12)_A(23)_A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (123)_A$$

Reading the cycles the other way gives

$$(12)_P = (12)_A \quad (23)_P = (23)_A \quad (123)_P = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (132)_A \quad (132)_P = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (123)_A$$

So we see that $(12)_P(23)_P = (132)_P$.

A four-dimensional rep is

$$(12)_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (23)_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (123)_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (132)_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Multiplying gives

$$(12)_A(23)_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (123)_A$$

As before, reading the cycles the other way exchanges $(123) \leftrightarrow (132)$:

$$(12)_P = (12)_A \qquad (23)_P = (23)_A \qquad (123)_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (132)_A \qquad (132)_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (123)_A$$

so $(12)_P(23)_P = (132)_P$.