Group Theory - Homework 5

M. Ross Tagaras
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PROBLEM 1

Using $M(a)^i{}_j = \langle g_i | a g_j \rangle$, $M(b)^i{}_j = \langle g_i | b g_j \rangle$, $M(c)^i{}_j = \langle g_i | c g_j \rangle$ and $\langle g_i | g_j \rangle = \delta_{ij}$, we find

$$M(a) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad M(b) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad M(c) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A matrix representation should obey $M(g_1)M(g_2) = M(g_1 g_2)$. For our specific $M(a), M(b), M(c)$, we should have $M(b)M(a) = M(ba) = M(c)$, but we can see that

$$M(b)M(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \neq M(c)$$

so these matrices cannot form a representation.

PROBLEM 2

Part (a)

Using the definition of the fundamental representation, we have

$$M(132)^i{}_j = \langle g_i | \hat{M}(132) | g_j \rangle = \langle g_i | (132) g_j \rangle$$

A sample calculation for one matrix element is

$$M(132)^1{}_2 = \langle e | (132)(123) \rangle = \langle e | e \rangle = 1$$

This gives

$$M(132) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad M(123) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

To check that this is correct, we can calculate $M(12)M(13)$:

$$M(12)M(13) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = M(132)$$

Part (b)

Changing the order in which we evaluate the product of cycles exchanges the roles of (123) and (132). Now, since $(12)(13) = (123)$, we can see that $M_{new}(12)M_{new}(13) = M_{old}(132) = M_{new}(123)$ and we still have a representation.

Part (c)

Both of the matrices found in part (a) are unitary, since they are orthogonal and real.

If we use the new ordering of group elements, then the matrix for a product of elements is

$$M(g\tilde{g})^i{}_j = \langle g_i^{-1}|M(g\tilde{g})|g_j\rangle = \langle g_i^{-1}|M(g)M(\tilde{g})|g_j\rangle = \sum_{c\in D_3} \langle g_i^{-1}|M(g)|c\rangle\langle c|M(\tilde{g})|g_j\rangle$$

which is only equal to $M(g)M(\tilde{g})$ if $c = c^{-1}$, which is not the case for (123) and (132).

Part (d)

The regular representation is faithful (six elements for a six dimensional group) and unitary (which we have already seen), but not irreducible. There are three classes, which gives three irreducible representations, and the sum of the squares of their dimensions must sum to six. Clearly, it is not possible to have a six-dimensional irreducible representation.

PROBLEM 3

$\mathbb{Z}_3$ is defined as $\{e, a, b\}$, where $a^2 = b$, $b^2 = a$, and $ab = e$.

The number of one-dimensional irreps is given by $|G| / \big|C(G)\big|$. For $\mathbb{Z}_3$, the commutator subgroup is trivially $\{e\}$, so the number of irreps is $3/1 = 3$.

The classes of $\mathbb{Z}_3$ are

$$C_e = \{e\} \qquad C_a = \{eae^{-1}aaa^{-1}, bab^{-1}\} = \{a\} \qquad C_b = \{ebe^{-1}, aba^{-1}, bbb^{-1}\} = \{b\}$$

The total number of inequivalent irreps is equal to the number of classes, and since we have three one-dimensional irreps, there must be no higher dimensional irreps.

As desired, we see that $|G| = \sum_{i=1}^{3} \left(\dim(R^i)\right)^2 = 1 + 1 + 1 = 3$. This expression also allows us to check that our calculation of the number of one-dimensional irreps is correct. We know that there is always at least one one-dimensional irrep, the trivial irrep. The only possible sum of squares that adds to three is $1 + 1 + 1$, so there cannot be any irreps of dimension two or higher.

The trivial representation is $e = a = b = 1$. In the second irrep, we can write a and b as third roots of unity: $e = 1$, $a = e^{2\pi i/3}$, $b = e^{2\times 2\pi i/3} = e^{4\pi i/3}$. In the last irrep, we exchange a and b: $e = 1$, $a = e^{4\pi i/3}$, $b = e^{2\pi i/3}$.

The trivial irrep is clearly not faithful, but it is unitary. The other two irreps are both faithful and unitary.

We can check the orthogonality relations for a few cases:

$$\left(M^{(1)}(e)\right)^* \left(M^{(1)}(e)\right) + \left(M^{(1)}(a)\right)^* \left(M^{(1)}(a)\right) + \left(M^{(1)}(b)\right)^* \left(M^{(1)}(b)\right) = 1\times 1 + e^{-2\pi i/3}e^{2\pi i/3} + e^{-4\pi i/3}e^{4\pi i/3} = 3 = \frac{\dim(\mathbb{Z}_3)}{1}$$

$$\left(M^{(1)}(e)\right)^* \left(M^{(2)}(e)\right) + \left(M^{(1)}(a)\right)^* \left(M^{(2)}(a)\right) + \left(M^{(1)}(b)\right)^* \left(M^{(2)}(b)\right) = 1 + e^{2\pi i/3} + e^{-2\pi i/3} = 1 + (-1) = 0$$