Homework 12

Due on: Monday, April 20

Problem 1. Casimir operators and Dynkin indices.

Compare the values of the quadratic Casimir operator for the following multiplets of $SU(3)$: $\underline{\mathbf{1}}$, $\underline{\mathbf{3}}$, $\underline{\mathbf{3}}^*$, $\underline{\mathbf{6}}$, $\underline{\mathbf{8}}$, $\underline{\mathbf{10}}$.

Problem 2. The Euclidean group E_2.

The Poincaré group E_2 for Euclidean space with 2 dimensions contains two translation generators P_x, P_y and the rotation generator L. The Lie algebra reads

$$[P_x, P_y] = 0 \ , \quad [L, P_x] = P_y \ , \quad [L, P_y] = -P_x \ . \tag{2.1}$$

(a) Show that this Lie algebra is non-semisimple.

(b) Compute the Killing metric $g^K_{\mu\nu}$.

(c) Although we cannot construct the quadratic Casimir operator because the inverse of $g^K_{\mu\nu}$ does not exist, there are still quadratic operators that commute with all generators. Find one for E_2.

(d) The group E_2 acts on a point with coordinates (x, y) as follows:

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta + a \ , \\ y' &= x\sin\theta + y\cos\theta + b \ . \end{aligned} \tag{2.2}$$

Although this transformation is not linear in x, y, one can still find a matrix representation but in terms of 3×3 matrices. The vectors $v = \left(\begin{smallmatrix} x \\ y \\ 1 \end{smallmatrix}\right)$ transform under E_2 as $v' = gv$ with

$$g = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

Show that these matrices form a group.

(e) Construct the generators of the group. What Lie algebra do they generate?

(f) Consider now the Euclidean group in 3 dimensions with generators P_j and L_{jk}. Find two quadratic operators that commute with all six generators.

(g) **Group contractions**. One can contract a simple (or semisimple) group by rescaling some of the generators to obtain a non-semisimple group. Consider the group $SO(3)$, and produce E_2.

(h) Simple Lie algebras of rank r have r Casimir operators. How many Casimir operators does $SO(3)$ have? What are they? What do they become after the group contraction?

(i) Now consider $SO(2,1)$. First construct explicitly a set of 3×3 matrices for the defining rep of $SO(2,1)$. Next compute its Killing metric. Define $\mathrm{Tr}(T_i^{(R)} T_j^{(R)}) = -\eta_{ij} T(R)$ for $SO(2,1)$. What is the Dynkin index $T(\mathbf{3})$ for $SO(2,1)$? Then evaluate the quadratic Casimir operator $C_2(R) = -\eta^{ij} T_i^{(R)} T_j^{(R)}$ on the triplet representation of $SO(2,1)$.