# Group Theory - Homework 7

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## PROBLEM 1

The elements $A$ in the kernel of $SL(2,\mathbb{C})$ should obey $A^\dagger x^\mu \sigma_\mu A = x^\mu \sigma_\mu$, or equivalently, $x^\mu \sigma_\mu A = (A^\dagger)^{-1} x^\mu \sigma_\mu$. This gives

$$\left[x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{a^* d^* - c^* b^*} \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix} \left[x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

which leads to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix} \qquad \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} -b^* & d^* \\ a^* & -c^* \end{pmatrix} \qquad \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} -b^* & -d^* \\ a^* & c^* \end{pmatrix} \qquad \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \begin{pmatrix} d^* & -b^* \\ c^* & -a^* \end{pmatrix}$$

From the second and third equations, we see that $a^* = a$ and $d^* = d$, and the first and fourth give us $a = d$. The first two equations give $b = c$ and the second two give $b = -c$, so $b = c = 0$. If we compare to $A^\dagger x^\mu \sigma_\mu A = x^\mu \sigma_\mu$, we see that we can have $a = \pm 1$, so the kernel is $\{\pm I\} = \mathbb{Z}_2$. This tells us that $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(3,1)$. $SL(2,\mathbb{C})$ is connected, simply connected (as shown in class), and noncompact.

$SO(3,1)$ is connected but not simply connected, since it contains rotations as a subgroup, which form a doubly-connected group. It is also noncompact.

## PROBLEM 2

The generators of $SL(2,\mathbb{R})$ are $\sigma_1, i\sigma_2, \sigma_3$. A group element in the exponential form is then $e^{A\sigma_1 + Bi\sigma_2 + C\sigma_3}$. Now define $\ell^2 = A^2 - B^2 + C^2$. We can write a general element as

$$M = e^{A\sigma_1 + Bi\sigma_2 + C\sigma_3} = \begin{cases} \begin{pmatrix} \cosh\ell + \frac{C}{\ell}\sinh\ell & \frac{1}{\ell}(A+B)\sinh\ell \\ \frac{1}{\ell}(A-B)\sinh\ell & \cosh\ell - \frac{C}{\ell}\sinh\ell \end{pmatrix}, & \ell^2 \geq 0 \\ \\ \begin{pmatrix} \cos\ell + \frac{C}{\ell}\sin\ell & \frac{1}{\ell}(A+B)\sin\ell \\ \frac{1}{\ell}(A-B)\sin\ell & \cos\ell - \frac{C}{\ell}\sin\ell \end{pmatrix}, & \ell^2 < 0 \end{cases}$$

First consider $\ell^2 \geq 0$. If $C > 0$, $M_{11} \geq 1$. Similarly, if $C < 0$, $M_{22} \geq 1$. If $C = 0$, then both $M_{11}, M_{22} \geq 1$. Therefore, we cannot have an exponential representation of an element with both diagonal entries less than one simultaneously.

When $\ell^2 < 0$, we can have

## PROBLEM 3

The elements of $SU(1,1)$ are

$$\left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \;\middle|\; a^* a - b^* b = 1 \right\}$$

We can decompose such an element as

$$\mathrm{Re}(a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{Re}(b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathrm{Im}(b) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathrm{Im}(a) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which indicates that the generators are $\sigma_1, i\sigma_2, \sigma_3$. These are also the generators for $SL(2, \mathbb{R})$, so (at least close to the identity), there should be an isomorphism between the two groups. This implies that the Lie algebras are the same.