Homework 1

Due on: January 31.

Problem 1

(a) Show that the following Lorentz transformations form a group if $-c < v < c$:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \ , \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \ , \quad y' = y \ , \quad z' = z \ . \tag{1.1}$$

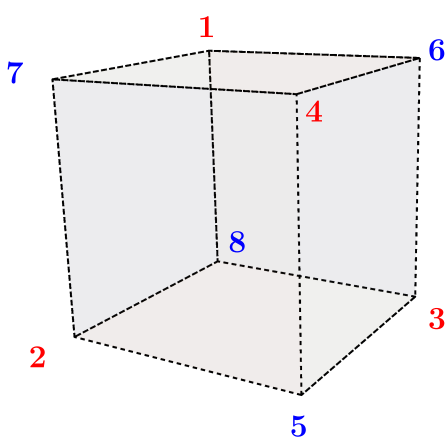
Check that the 4 group postulates are satisfied.

(b) Consider the (linear or nonlinear) transformations which leave the expression $(x-y)^2 \equiv (x-y)^\mu \eta_{\mu\nu} (x-y)^\nu$ invariant for any real x^μ and y^μ. Explain why these transformations form a group.

(c) Assuming the transformations are linear, we set $x'^\mu - y'^\mu = L^\mu{}_\nu (x^\nu - y^\nu)$. Show that the matrices L satisfy $(L^T)_\mu{}^\rho \eta_{\rho\sigma} L^\sigma{}_\nu = \eta_{\mu\nu}$. Show that this equation can be viewed in two ways: either as specifying the group $O(3,1)$, or the statement that $\eta_{\mu\nu}$ is a Lorentz-invariant tensor. (**Note**: to prove this matrix equation, you must show that it holds for all matrix entries. The equation for L which follows from $(x'-y')^2 = (x-y)^2$ contains only the diagonal entries.)

Historical note: The Dutch physicist Hendrik Lorentz studied these transformations in great detail in his studies of Maxwell theory. The French mathematician/physicist Henri Poincaré noted that they form a group. The Irish physicist George Fitzgerald proposed the equations in (1.1) to explain the null result of the Michelson-Morley experiment. The Danish physicist Ludvig Lorenz (without a 't') introduced the electromagnetic gauge choice $\partial^\mu A_\mu = 0$, but posterity has attributed it to Lorentz.

Problem 2



Consider a regular cube.

(a) Determine the isometry group. **Hint**: the cube consists of two tetrahedera with opposite orientation, one (T) with vertices $\{1,2,3,4\}$ and the other (T') with vertices $\{5,6,7,8\}$. Any isometry either maps T into T (and T' into T'), or it maps T into T' (and T' into T). One particular isometry of the cube is space inversion σ, under which $1 \leftrightarrow 5$, $5 \leftrightarrow 6$, $6 \leftrightarrow 7$, $4 \leftrightarrow 8$. We derived in class the isometry group of T (and of T').

(b) How many rotational isometries does the cube have? Write them in cycle notation.

(c) What is the subgroup of rotational isometries of the the tetrahedron? What is the subgroup of rotational isometries of the cube?