Group Theory - Homework 6

M. Ross Tagaras
(Dated: February 28, 2020)

The classes of S_4 are e, (12), (123), (12)(34), and (1234). The commutator subgroup is A_4. Since $S_4/A_4 = \mathbb{Z}_2$, there are two one-dimensional irreps, out of five total. The start of the character table is

S_4	e [1]	(12) [6]	(123) [8]	(12)(34) [3]	(1234) [6]
χ_1	1	1	1	1	1
χ_1'	1	-1	1	1	-1

To find the dimensions of the other irreps, we use $24 = \sum_{i=1}^{5}(d_i)^2 = 2 + a^2 + b^2 + c^2$, to which the solution is $a = 2$, $b = c = 3$. One of the three-dimensional irreps can be found by considering three-dimensional rotation matrices, for which the trace is $1 + 2\cos\theta$. A rotation by $2\pi/3$ corresponds to (123) and a rotation by π corresponds to (12)(34). The elements (12) and (1234) are generalized reflections, for which the trace is ± 1. This choice is what differentiates the two three-dimensional irreps. Now, the table is

S_4	e [1]	(12) [6]	(123) [8]	(12)(34) [3]	(1234) [6]
χ_1	1	1	1	1	1
χ_1'	1	-1	1	1	-1
χ_2	2	A	B	C	D
χ_3	3	1	0	-1	-1
χ_3'	3	-1	0	-1	1

We can now calculate the row for χ_2 by orthogonality. Column orthogonality gives

$$6(1^2 + (-1)^2 + 1^2 + (-1)^2 + A^2) = 24 \implies A = 0$$

$$8(2 + B^2) = 24 \implies B = \pm 1$$

$$3(4 + C^2) = 24 \implies C = \pm 2$$

$$6(4 + D^2) = 24 \implies D = 0$$

Orthogonality of χ_2 and χ_1 gives $2 + 8B + 3C = 0$, which is only satisfied by $B = -1$, $C = 2$. The updated table is

S_4	e [1]	(12) [6]	(123) [8]	(12)(34) [3]	(1234) [6]
χ_1	1	1	1	1	1
χ_1'	1	-1	1	1	-1
χ_2	2	0	-1	2	0
χ_3	3	1	0	-1	-1
χ_3'	3	-1	0	-1	1
c_g	4	2	1	0	0
χ_S	12	2	0	0	0
χ_{R^3}	3	1	0	-1	-1

Using $n_\alpha = (\chi_S, \chi_\alpha) = \frac{1}{24}\left[12 \times \chi_\alpha(e) + 12 \times \chi_\alpha(12)\right] = \frac{1}{2}\left[\chi_\alpha(e) + \chi_\alpha(12)\right]$ we can find χ_S in terms of the other characters:

$$n_1 = 1 \qquad n_{1'} = 0 \qquad n_2 = 1 \qquad n_3 = 2 \qquad n_{3'} = 1$$

$$\chi_S = \chi_1 + \chi_2 + 2\chi_3 + \chi_{3'}$$

which then gives

$$\chi_{gen} = \chi_S - \chi_{trans} - \chi_{rot} = \chi_1 + \chi_2 + \chi_{3'}$$

The normal modes are a singlet, a doublet, and a triplet, which matches the expected six total modes. The final table is

S_4	e [1]	(12) [6]	(123) [8]	(12)(34) [3]	(1234) [6]
χ_1	1	1	1	1	1
χ_1'	1	-1	1	1	-1
χ_2	2	0	-1	2	0
χ_3	3	1	0	-1	-1
χ_3'	3	-1	0	-1	1
c_g	4	2	1	0	0
χ_S	12	2	0	0	0
χ_{R^3}	3	1	0	-1	-1
χ_{rot}	3	1	0	-1	-1
χ_{trans}	3	1	0	-1	-1
χ_{gen}	6	0	0	-2	-2