Homework 14

Due on: Monday, May 4

Problem 1. The Haar Measure for $SO(3)$.

We shall calculate this measure using two different parametrizations, and check that the volume of $SO(3)$ comes out to the same.

(a) **Euler angles**: Write the group elements of $SO(3)$ in the spin $\frac{1}{2}$ representation as follows:

$$g = e^{\alpha T_3} e^{\beta T_1} e^{\gamma T_3} \ , \quad \text{with } T_j = -i\sigma_j \ ; \alpha, \beta, \gamma \text{ real.} \tag{1.1}$$

What is the range of α, β, γ? (**Hint**: multiply the 2×2 matrices and obtain the most general $SU(2)$ group element.) Evaluate $g^{-1}dg$ and determine the group measure. What is the volume of $SO(3)$? If one uses as generators $T_j = -\frac{i}{2}\sigma_j$, what is then the volume of $SO(3)$?

(b) **Polar angles**: We take $g = e^{-i\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}}$ where $\vec{\boldsymbol{n}}$ is a unit vector, $\vec{\boldsymbol{n}} = \{\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta\}$, and ψ is the angle of rotation, so $0 \le \psi \le \pi$. Evaluate $g^{-1}dg$. This is a tedious calculation, so do not waste time on it if you feel not confident you can do such calculations.
Hint: Use the Pauli algebra $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ to simplify the expressions $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}\left(\frac{\partial}{\partial\theta}\vec{\boldsymbol{n}}\right)\cdot\vec{\boldsymbol{\sigma}}$ and $\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}\left(\frac{\partial}{\partial\varphi}\vec{\boldsymbol{n}}\right)\cdot\vec{\boldsymbol{\sigma}}$, and show that both of these expressions contain only terms with σ_k but no terms with the unit matrix. Then extract ${e_\mu}^m$ and proceed as before.

Problem 2. The involutary automorphisms of $SO(4)$.

There exist 3 real noncompact forms of $SO(4)$, namely $SO(2,2)$, $SO(3,1)$ and $SO^*(4)$. Associated with each is an involuntary ($\sigma^2 = 1$) automorphism σ of the compact Lie algebra $SO(4)$. Our aim is the find these 3 automorphisms.

One possible approach is to use that $SO(4)$ is semi-simple, namely

$$SO(4) = SO(3) \times SO(3) \quad \text{(for the Lie algebra)} \tag{2.1}$$

(a) Prove this relation. Express the generators of each $SO(3)$ in terms of the $SO(4)$ generators $M_{ij} = -M_{ij}$ with $i, j = 1, \ldots 4$.

Since $SO(3)$ has precisely one noncompact form $SO(2,1)$, we spot immediately two real noncompact forms of $SO(4)$

$$SO(2,1) \times SO(2,1) \quad , \qquad SO(2,1) \times SO(3) \tag{2.2}$$

The real noncompact Lie algebra $SO^*(4)$ was studied in homework 8, problem 5. The generators have the form

$$\begin{pmatrix} A & H \\ -H^T & -A^\dagger \end{pmatrix} \qquad \begin{array}{l} A\text{: complex antisymmetric } 2\times 2 \text{ matrix} \\ H\text{: hermitian } 2\times 2 \text{ matrix} \end{array} \tag{2.3}$$

Some of these generators are real, others are purely imaginary.

(b) Prove that the real generators form a subgroup H and the imaginary generators form the set K, such that (H, K) is a Cartan decomposition of $SO^*(4)$.

(c) Write the 6 generators of $SO^*(4)$ in direct product notation. For example, one of the generators due to A is written as

$$\left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \emptyset \\ \hline \emptyset & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) = i\sigma_2 \otimes \mathbb{I}\ . \tag{2.4}$$

Which 4 generators form H, and which form K?

The noncompact real forms of $SO(4)$ given by $SO(3,1)$ and $SO(2,2)$ can be written in rectangular block form as

$$\left(\begin{array}{ccc|c} & & & \\ & & & \\ & & & \\ \hline & & & \end{array}\right), \qquad \text{and} \qquad \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & \\ & & & \end{array}\right) \tag{2.5}$$

(d) Show that the block-diagonal parts form the set H, and the block off-diagonal parts form the set K of a Cartan decomposition.

We have now 3 noncompact forms of $SO(4)$: $SO(3,1)$, $SO(2,2)$ and $SO^*(4)$ and several Cartan decompositions: those in (2.1), the one in (2.4), and those in (2.5).

(e) Which noncompact form of $SO(4)$ corresponds to which involutary automorphism of the compact Lie algebra of $SO(4)$? Which Cartan decompositions are the same (isomorphic)?

Problem 3. Dynkin diagrams for $USp(2N)$.

Construct the extended Dynkin diagram of $USp(2N)$. We first write down all the roots $\alpha^1, \ldots, \alpha^N$ of $USp(2N)$ and then the simple roots. Next determine α^0, minus the highest weight. Finally follow the rules for constructing Dynkin diagrams, and apply them to the set α^0, α^1, …, α^N.

What are all maximal regular subalgebras of $USp(4)$? Compare with all maximal regular subalgebras of $SO(5)$.

Problem 4. Dirac matrices and spinor irreps in $d = 7, 8, 9$.

Using that a purely imaginary representation of the Dirac matrices in 7-dimensional Euclidean space exists,

(a) construct 8 real symmetric Dirac matrices in 8 Euclidean dimensions.

(b) What is C_+ and C_- in $d = 8$? Are they block-diagonal or block-antidiagonal? Are they symmetric or antisymmetric? Is the chiral matrix Γ_c (satisfying $\Gamma_c^2 = 1$) real?

(c) Now go to 9 Euclidean and then to 9 Minkowskian dimensions. Does a Majorana (real) representation exist in these cases?

(d) What are the reality properties of the spinor irreps in $d = 7$, and $d = 8$ Minkowski space?