Homework 5

Due on: Friday, February 21

Problem 1

Consider the following 5×5 Latin square which satisfies the postulates of closure, unit and inverse, but which violates associativity.

$$
\begin{array}{c|ccccc}
 & e & a & b & c & d \\
\hline
e & e & a & b & c & d \\
a & a & e & d & b & c \\
b & b & c & e & d & a \\
c & c & d & a & e & b \\
d & d & b & c & a & e
\end{array}
\tag{1.1}
$$

Construct the matrices $M(a)$, $M(b)$, $M(c)$ as discussed in class, namely

$$
(M(g_k))^{l}{}_{m} = \langle g_l | \widehat{M}(g_k) | g_m \rangle = \langle g_l | g_k g_m \rangle \tag{1.2}
$$

where the products $g_k g_m$ are given by the Latin square. Show that these matrices do not form a matrix representation of some 5-dimensional group. (The Latin square violates associativity: $(ab)c = a$ but $a(bc) = c$.)

Problem 2

(a) Construct the 6×6 matrices of the regular representation of the group S_3 for the group elements (12), (13), (123) and (132). Reading products of cycles from right to left, one get $(12)(13) = (321)$. Check that the matrices $M(12)$, $M(13)$ and $M(321)$ form a matrix representation of this group product.

Hint: To reduce the amount of algebra, you may use the following group multiplication table. First complete the table using the Latin square structure of a group multiplication table. (You should be able to fix the 13 open entries in the table without evaluating any product of cycles.

You may check a few entries to test your expertise.)

$$
\begin{array}{c|cccccc}
D^3 & e & (123) & (132) & (12) & (23) & (13) \\
\hline
e & e & (123) & (132) & (12) & (23) & (13) \\
(123) & (123) & (132) & e & (13) & & \\
(132) & (132) & e & (123) & & (13) & \\
(12) & (12) & (23) & & e & (123) & \\
(23) & (23) & & (12) & (132) & e & \\
(13) & (13) & & & & &
\end{array}
\tag{2.1}
$$

To even further reduce the amount of labour, we provide the answers for $M(12)$ and $M(13)$ (only the nonzero entries are indicated)

$$
M(12) = \left(\begin{array}{ccc|ccc}
 & & & 1 & & \\
 & & & & 1 & \\
 & & & & & 1 \\
\hline
1 & & & & & \\
 & 1 & & & & \\
 & & 1 & & &
\end{array}\right); \qquad
M(13) = \left(\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & 1 & & \\
 & & & & 1 & \\
\hline
 & 1 & & & & \\
 & & 1 & & & \\
1 & & & & &
\end{array}\right)
\tag{2.2}
$$

Construct $M(321)$ and show that it satisfies $M(12)M(13) = M(321)$.

(b) If we read products of cycles from left to right instead of from right to left, we get another multiplication table. We can still construct (different) 6×6 matrices $M(g)$. Do they still form a representation? (For example $(12)(13) = (123)$, so do the new matrices $M(g)$ satisfy $M(12)M(13) = M(123)$? No tedious explicit calculations, please.)

(c) Are the matrices $M(g)$ unitary? (The matrices $M(12)$ and $M(13)$ are unitary. Is $M(123)$ a unitary matrix?) If we order the group elements as follows: columns are labeled $g_0, g_1, \cdots, g_{n-1}$ as before, but rows are now labeled as $g_0^{-1} = g_0 = e, g_1^{-1}, \cdots, g_{n-1}^{-1}$, we find a group multiplication table with e along the diagonal. We can again construct the matrix elements as follows

$$
M(g)^k{}_l = \langle g_k^{-1} | M(g) | g_l \rangle \tag{2.3}
$$

Do these matrices form a representation? Are they unitary?

(d) Now answer the following question: is the regular representation faithful?, reducible?, unitary?

Problem 3

Using the three (not yet proven) theorems on matrix representations, answer the following questions for the group Z_3.

1. How many one-dimensional irreps are there?

2. How many higher-dimensional irreps are there?

3. Check the relation $|G| = \sum_i (\dim R^i)^2$.

4. Write down all matrix irreps for all group elements.

5. Which irreps are faithful, reducible, unitary?

6. Check the orthogonality relations of matrix elements in a few (two) cases.