Group Theory - Homework 2

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PROBLEM 1

Part (a)

There are $2\binom{n}{m}$ possible elements of the form $\pm\gamma_{k_1}\dots\gamma_{k_m}$, along with $\pm e$. This gives a total of

$$2\left[1+\sum_{m=1}^{n}\binom{n}{m}\right]=2^{n+1}$$

elements in the n-dimensional Dirac group.

Part (b)

It is clear that $g(\pm e)g^{-1}=\pm e\in G_{Dirac}$ for all $g\in G_{Dirac}$. Thus, $\{\pm e\}\triangleleft G_{Dirac}$.

In the next set, we clearly have an identity, and associativity is inherited from G_{Dirac}. The element $\gamma_1\dots\gamma_n$ can be written as $(-1)^{\frac{n^2-n}{2}}\gamma_n\dots\gamma_1$, so we see that $(\pm\gamma_1\dots\gamma_n)(\pm\gamma_1\dots\gamma_n)=(-1)^{\frac{n^2-n}{2}}\gamma_n\dots\gamma_1\gamma_1\dots\gamma_n=(-1)^{\frac{n^2-n}{2}}e$ and $(-\gamma_1\dots\gamma_n)(\gamma_1\dots\gamma_n)=(\gamma_1\dots\gamma_n)(-\gamma_1\dots\gamma_n)=(-1)^{\frac{n^2-n}{2}+1}e$, so we have closure and inverses.

We already know that $\{\pm e\}$ forms a normal subgroup, so now we need to evaluate $g(\pm\gamma_1\dots\gamma_n)g^{-1}$. If $g=\pm e$, then $\pm\gamma_1\dots\gamma_n$ is trivially in $H=\{\pm e,\pm\gamma_1\dots\gamma_n\}$. Now, we need to take $g=\pm\gamma_1\dots\gamma_k$, for $k\le n$. We know that, up to a minus sign, $g^{-1}=\gamma_1\dots\gamma_k$, so we have (again up to some minus signs)

$$g(\pm\gamma_1\dots\gamma_n)g^{-1}=(\gamma_1\dots\gamma_k)(\gamma_1\dots\gamma_n)(\gamma_1\dots\gamma_k)=(\gamma_k\dots\gamma_1)(\gamma_1\dots\gamma_n)(\gamma_1\dots\gamma_k)=(\gamma_{k+1}\dots\gamma_n)(\gamma_1\dots\gamma_k)=\gamma_1\dots\gamma_n$$

Since H contains both of $\pm\gamma_1\dots\gamma_n$, we see that $g(\pm\gamma_1\dots\gamma_n)g^{-1}\in H$ for all $g\in G_{Dirac}$, so $H\triangleleft G_{Dirac}$.

Part (c)

If we take a,b to be products of distinct gamma matrices, then we know that a^{-1},b^{-1} will be equal to a,b, up to a potential minus sign. Then, in a product like $aba^{-1}b^{-1}$, we can commute a and b^{-1} to get $\pm aa^{-1}bb^{-1}=\pm e$. If we have a product like $(aba^{-1}b^{-1})(cdc^{-1}d^{-1})\dots$, then we can perform the same procedure on each part individually. The commutator subgroup is $C=\{\pm e\}$. We have already shown that this is a normal subgroup.

G_{Dirac}/C is given by $\{gC\mid g\in G_{Dirac}\}$ where $gC=\{gc\mid c\in C\}=\{\pm g\}$. Now consider two cosets gC and hC, where $g,h\in G_{Dirac}$. To see that G_{Dirac}/C is abelian:

$$(gC)(hC)=(gh)C=\{ghc\mid c\in C\}=\{\pm gh\}$$

$$(hC)(gC)=(hg)C=\{hgc\mid c\in C\}=\{\pm hg\}$$

Since g and h will either commute or anticommute depending on the number of individual gamma matrices in each, we can see that the two products are equal, so the group is abelian.

PROBLEM 2

Part (a)

For a regular n-gon, there are n reflections and n rotations, so the order of D_n is $2n$.

Part (b)

We want to show that the elements of G are $\{a, \ldots, a^n, b, ab, \ldots, a^n b\}$. Since the presentation instructs us to consider products of powers of a and b, it is easy to see that the given elements are in G. What we need to check is that, using the relations given, we can transform an arbitrary sequence of a's and b's into one of these elements. Since $b^2 = a^n = 1$, we should consider products of the form $a^{k_1} b a^{k_2} b \ldots$ with $k_i < n$.

Using the relations $bab^{-1} = a^{-1}$ and $b^2 = 1$, we can see that $ab = ba^{-1}$. Then, we can apply this to an arbitrary sequence:

$$a^{k_1} b a^{k_2} b \cdots = a^{k_1 - 1} b a^{k_2 - 2} b \ldots$$

Repeating this process, we find that there are several possible forms for the simplified products:

$$a^q \qquad a^q b \qquad b a^q b \qquad b a^q$$

where $q < n$ is some unknown power. We have already accounted for the first two forms. The third form can be made into a power of a as follows:

$$b a^q b = b a^{q-1} b a^{-1} = \cdots = b^2 (a^{-1})^q = a^{q(n-1)}$$

The fourth form can be rewritten as

$$b a^q = b \underbrace{(ab^{-1}b) \ldots (ab^{-1}b)}_{q} = \underbrace{(bab^{-1}) \ldots (bab^{-1})}_{q} b = (a^{-1})^q b = a^{q(n-1)} b$$

In terms of cycles, $a = (1\ 2\ 3\ \ldots n)$ and one possible choice for b is $b = (1\ n)(2\ [n-1])(3\ [n-2]) \ldots$ for even n and $b = (2\ n)(3\ [n-1]) \ldots$ for odd n.

This choice of b corresponds to a reflection across the vertical axis when the vertices are labeled starting in the top right corner and increasing clockwise (for even n) or starting at the top vertex (for odd n). a gives a rotation by $2\pi/n$. Powers of a will generate the other rotations, and we can see that $a^n = 1$, since that corresponds to a rotation by 2π. Including a single power of b with any rotation gives the other possible configurations of the vertices.

Part (c)

We consider quantities like $(m_1 n_1 m_1^{-1} n_1^{-1})(m_2 n_2 m_2^{-1} n_2^{-1}) \ldots$. The elements m_i and n_i are of the form $a^{k_i} b^{j_i}$ where $k_i < n$ and $j_i \in \{0, 1\}$.

If we consider a product with all $j_i = 0$, then we trivially get the identity. The next possibility is all $j_i = 1$:

$$\left[\left(a^{k_1} b\right) \left(a^{k_2} b\right) \left(a^{k_1} b\right)^{-1} \left(a^{k_2} b\right)^{-1} \right] \times \cdots = \left[a^{k_1} \left(b a^{k_2} b\right) \left(b a^{n-k_2} b\right) a^{n-k_2} \right] \times \ldots$$

$$= \left[a^{k_1} b a^{n-k_1+k_2} b a^{n-k_2} \right] \times \cdots = a^{n-2k_2+2k_1}$$

Now take $j_1 = 1, j_2 = 0$:

$$\left[\left(a^{k_1}b\right)a^{k_2}\left(a^{k_1}b\right)^{-1}a^{n-k_2}\right] = a^{3n-2k_2}$$

Finally, $j_1 = 0, j_2 = 1$ gives

$$\left[\left(a^{k_1}\right)\left(a^{k_2}b\right)a^{n-k_1}ba^{n-k_2}\right] = a^{n+2k_1}$$

In each case, we see that we get an even power of a if n is even, and an odd power of a if n is odd. Therefore, the commutator subgroup of D_n is $\langle a \rangle$ if n is even and $\left\langle a^2 \right\rangle$ if n is odd. Here, $\langle x \rangle$ denotes the group generated by powers of x.

It is easy to see that $\langle a \rangle \cong \mathbb{Z}_n$ and that $\left\langle a^2 \right\rangle \cong \mathbb{Z}_{n/2}$, both of which are abelian.

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