Group Theory - Homework 3

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PROBLEM 1

If $Z_a \triangleleft G$, then $gzg^{-1} \in Z_a$ for all $z \in Z_a$ and all $g \in G$. For gzg^{-1} to be in Z_a, we need $agzg^{-1} = gzg^{-1}a$ for all $z \in Z_a$ and all $g \in G$. However, this is not true in general, since there can be elements of G which do not commute with a, which prevents us from moving g through z.

PROBLEM 2

$D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = b^2 = e$ and $ba = a^{-1}b$. The inverses for the group elements are

$$a^{-1} = a^3 \qquad (a^2)^{-1} = a^2 \qquad (a^3)^{-1} = a$$

$$(ab)^{-1} = ba^3 = ab \qquad (a^2b)^{-1} = ba^2 = a^2b \qquad (a^3b)^{-1} = ba = a^3b$$

Now we can construct the classes, defined as $C_x = \{gxg^{-1} \;\; \forall g \in G\}$. They are

$$C_e = \{e\} \qquad C_a = C_{a^3} = \{a, a^3\} \qquad C_{a^2} = \{a^2\} \qquad C_b = C_{a^2b} = \{b, a^2b\} \qquad C_{ab} = C_{a^3b} = \{ab, a^3b\}$$

It is clear that the order of each class divides the order of D_4.

We can take $(1234) = a$ and $(12)(34) = b$, for which the centralizers are $Z_a = \{e, a, a^2, a^3\}$ and $Z_b = \{e, a^2, b, a^2b\}$. We can easily see that the desired properties are satisfied.

PROBLEM 3

We know that a subgroup is normal if and only if it is a union of classes. We also know that any normal subgroups of D_4 have to have $1, 2, 4$, or 8 elements, by Lagrange's theorem. The cases with 1 and 8 elements are trivial and correspond to $\{e\}$ and D_4 itself. To find the others, we should look for subgroups with 2 or 4 elements that are unions of classes.

The only possible subgroup with two elements is $\{e, a^2\} = C_e \cup C_{a^2}$. Since subgroups must contain e, any other normal subgroup must be a union of $\{e, a^2\}$ and exactly one other class with two elements.

The other normal subgroups are

$$\{e, a, a^2, a^3\} = C_e \cup C_a \cup C_{a^2} \qquad \{e, a^2, b, a^2b\} = C_e \cup C_{a^2} \cup C_b \qquad \{e, a^2, ab, a^3b\} = C_e \cup C_{a^2} \cup C_{ab}$$