Group Theory - Homework 9

M. Ross Tagaras
(Dated: March 30, 2020)

PART (A)

The Cartan generators are

$$H_I^{def} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad H_{II}^{def} = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad H_{III}^{def} = \frac{1}{\sqrt{24}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

and the weights are

$$\vec{\mu}^1_{def} = \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^2_{def} = \left(-\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^3_{def} = \left(0, -\frac{2}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^4_{def} = \left(0, 0, -\frac{3}{\sqrt{24}}\right)$$

If we treat these vectors as points in $\mathbb{R}^3$ and shift them so that $\vec{\mu}^4_{def}$ is the origin, we can construct the following matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{6}} \\ 0 & -\frac{2}{\sqrt{12}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

which has determinant $\frac{1}{\sqrt{2}} \neq 0$. Therefore, the weight vectors form a tetrahedron.

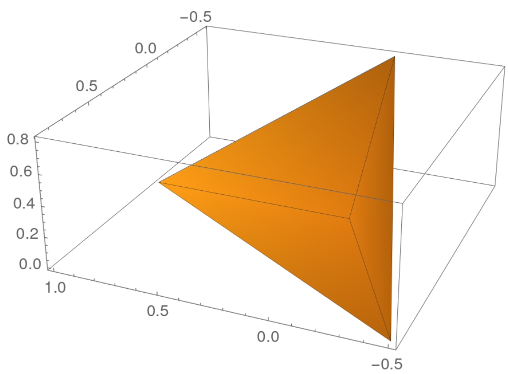
PART (B)

The roots are differences of two weights. If we treat the weights as vectors that point to each vertex of the tetrahedron, then the difference of any two gives the vector connecting the vertices, which are edges. The positive roots are

$$\vec{\alpha}^{12} = (1,0,0) \qquad \vec{\alpha}^{13} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \qquad \vec{\alpha}^{14} = \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{2}{\sqrt{6}}\right)$$

$$\vec{\alpha}^{23} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \qquad \vec{\alpha}^{24} = \left(-\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{2}{\sqrt{6}}\right) \qquad \vec{\alpha}^{34} = \left(0, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}\right)$$

The simple roots are $\vec{\alpha}^{12}, \vec{\alpha}^{23}, \vec{\alpha}^{34}$. Plotting these:



PART (C)

Using $\vec{\mu}^J_{FW} = \sum_{k=1}^J \vec{\mu}^k_{def}$, we find the fundamental weights:

$$\vec{\mu}^1_{FW} = \left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}\right) \qquad \vec{\mu}^2_{FW} = \left(0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \qquad \vec{\mu}^3_{FW} = \left(0, 0, \frac{3}{\sqrt{24}}\right)$$

$\vec{\mu}^1_{FW}$ is the highest weight of the defining representation, which contains vector states. $\vec{\mu}^2_{FW}$ is the highest weight of the two-index antisymmetric tensor representation, with states $|_i\rangle \otimes |_j\rangle - |_j\rangle \otimes |_i\rangle$, which corrensponď to the anti-symmetric tensors T^{ij}. $\vec{\mu}^3_{FW}$ is the highest weight of the three-index antisymmetric tensor representation, with states $\big(|_i\rangle \otimes |_j\rangle \otimes |_k\rangle + \text{antisymmetrized combinations}\big)$, which correspond to three index, totally antisymmetric tensors T^{ijk}.

PART (D)

The transformation of $\varepsilon^{\mu_1\cdots\mu_n}$ is

$$\delta_a \varepsilon^{\mu_1\cdots\mu_n} = (T_a)^{\mu_1}{}_{\nu_1}\varepsilon^{\nu_1\mu_2\cdots\mu_n} + \cdots + (T_a)^{\mu_n}{}_{\nu_n}\varepsilon^{\mu_1\cdots\mu_{n-1}\nu_n} \propto \mathrm{tr}(T_a) = 0$$

We can also exponentiate the transformation to find $v^i \to X^i{}_j v^j$, where $X \in SU(N)$. Then, the finite transformation of $\varepsilon^{\mu_1\cdots\mu_n}$ is $\varepsilon^{\mu_1\cdots\mu_n} \to X^{\mu_1}{}_{\nu_1}\ldots X^{\mu_n}{}_{\nu_n}\varepsilon^{\nu_1\cdots\nu_n} = \det(X)\varepsilon^{\mu_1\cdots\mu_n} = \varepsilon^{\mu_1\cdots\mu_n}$.

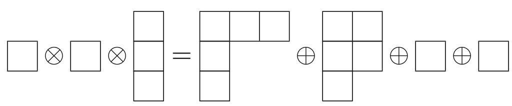
Therefore, $\varepsilon^{\mu_1\cdots\mu_n}$ is an invariant of $SU(N)$ and $\mathfrak{su}(N)$.

Since $\varepsilon^{\mu_1\cdots\mu_n}$ is invariant, and using $t_{\mu_1\ldots\mu_k} = \varepsilon_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_n}t^{\mu_{k+1}\cdots\mu_n}$, we can see that a k-dimensional conjugate irrep transforms the same way as an $N-k$ dimensional irrep.

PART (E)

The decomposition into irreps is similar to equation 58 in the notes. The difference is that in that case, we had $\delta^j_i v^i_j = 0$, but here $\delta^k_j v^j w_k \neq 0$. We can still decompose into a traceless symmetric tensor with dimension $\frac{1}{2}4^2(4+1) - 4 = 36$ and a traceless antisymmetric tensor with dimension $\frac{1}{2}4^2(4-1) - 4 = 20$, but this time we will need to subtract off one extra trace. Thus, we get two overall trace terms instead of one. The decomposition should then be $\mathbf{4} \otimes \mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{36} \oplus \mathbf{20} \oplus \mathbf{4} \oplus \mathbf{4}$.

As Young tableaux, this calculation is



$\mathbf{4} \quad \mathbf{4} \quad \bar{\mathbf{4}} \quad \mathbf{36} \quad \mathbf{20} \quad \mathbf{4} \quad \mathbf{4}$

These dimensions were all calculated using the standard factors-over-hooks rule.