Group Theory - Homework 14

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PROBLEM 1

Using the fact that $e^{i\theta\vec{v}\cdot\vec{\sigma}} = I_2\cos\theta + i\sin\theta\ \vec{v}\cdot\vec{\sigma}$ for any unit vector $\vec{v}$, we can write g as

$$g = \begin{pmatrix} \cos\beta\ e^{-i(\alpha+\gamma)} & -i\sin\beta\ e^{-i(\alpha+\gamma)} \\ -i\sin\beta\ e^{i(\alpha-\gamma)} & \cos\beta\ e^{i(\alpha+\gamma)} \end{pmatrix}$$

To preserve unitarity, take $\beta \in [0, \pi/2]$ and $\alpha, \gamma \in [0, 2\pi]$. Computing $g^{-1}dg$:

$$g^{-1}dg = g^{-1}dx^{\mu}\partial_{\mu}g(x) = g^{-1}d\alpha\frac{\partial}{\partial\alpha}\left(e^{\alpha T_3}e^{\beta T_1}e^{\gamma T_3}\right) + g^{-1}d\beta\frac{\partial}{\partial\beta}\left(e^{\alpha T_3}e^{\beta T_1}e^{\gamma T_3}\right) + g^{-1}d\gamma\frac{\partial}{\partial\gamma}\left(e^{\alpha T_3}e^{\beta T_1}e^{\gamma T_3}\right)$$

$$= \begin{pmatrix} -i(\mathrm{d}\gamma + \mathrm{d}\alpha(\cos(2\beta))) & (\sin(2\gamma) - i(\cos(2\gamma)))(\mathrm{d}\beta - i\mathrm{d}\alpha(\sin(2\beta))) \\ (\cos(2\gamma) - i(\sin(2\gamma)))(\mathrm{d}\alpha(\sin(2\beta)) - i\mathrm{d}\beta) & i(\mathrm{d}\gamma + \mathrm{d}\alpha(\cos(2\beta))) \end{pmatrix}$$

This can be rewritten as

$$g^{-1}dg = d\alpha\left[\cos(2\beta)T_3 + \sin(2\beta)\sin(2\gamma)T_1 + \sin(2\beta)\cos(2\gamma)T_2\right] + d\beta\left[-\sin(2\gamma)T_2 + \cos(2\gamma)T_1\right] + d\gamma T_3$$

This gives

$$e_{\mu}{}^{m} = \begin{pmatrix} \sin(2\beta)\sin(2\gamma) & \sin(2\beta)\cos(2\gamma) & \cos(2\beta) \\ \cos(2\gamma) & -\sin(2\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has determinant $\sin(2\beta)$. Then, the volume is

$$V = \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi/2} d\beta\ \sin(2\beta) = 4\pi^2$$

If we rescale the generators by a factor of 1/2, then our expression for $g^{-1}dg$ gets an overall factor of 2, as does $e_{\mu}{}^{m}$. This causes the determinant grows by a factor of 8, so V does as well.

PROBLEM 2

Part (a)

The generators of $SO(4)$ are

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

These satisfy $M_{ij} = -M_{ji}$ and $\left[M_{ij}, M_{k\ell}\right] = \delta_{i\ell} M_{jk} + \delta_{jk} M_{i\ell} - \delta_{ik} M_{j\ell} - \delta_{j\ell} M_{ik}$ where $i, j = 1, \ldots, 4$. If we define $N^{\pm}_{ij} = M_{ij} \pm \varepsilon_{ijk} M_{k4}$, where now $i, j = 1, 2, 3$, then with a bit of algebra, we find that these obey the relation $\left[N^{\pm}_{ij}, N^{\pm}_{k\ell}\right] = \delta_{i\ell} N^{\pm}_{jk} + \delta_{jk} N^{\pm}_{i\ell} - \delta_{ik} N^{\pm}_{j\ell} - \delta_{j\ell} N^{\pm}_{ik}$. For $SO(3)$, we expect 3 generators and indeed, there are 3 possible N^{+} and 3 possible N^{-}. Since none of the $\pm$ generators mix in the expression for $\left[N^{\pm}_{ij}, N^{\pm}_{k\ell}\right]$, we have two copies of $SO(3)$.

Part (b)

We can write a real antisymmetric matrix as $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ and a real Hermitian matrix as $H = \begin{pmatrix} p & h \\ h & q \end{pmatrix}$ where a, h, p, q are real. Then, the commutator of the real generators has the form

$$\left[\begin{pmatrix} A & H \\ -H^T & -A^\dagger \end{pmatrix}, \begin{pmatrix} B & J \\ -J^T & -B^\dagger \end{pmatrix}\right] = \begin{pmatrix} 0 & hp - hq - jr + js & 2aj - 2bh & -ap + aq + br - bs \\ -hp + hq + jr - js & 0 & -ap + aq + br - bs & 2bh - 2aj \\ 2bh - 2aj & ap - aq - br + bs & 0 & hp - hq - jr + js \\ ap - aq - br + bs & 2aj - 2bh & -hp + hq + jr - js & 0 \end{pmatrix}$$

which has the correct block form. The other properties (antisymmetry, linearity, Jacobi identity) follow trivially when using the matrix commutator, so the real generators form a subalgebra. To have a Cartan decomposition, the commutator of two imaginary generators must produce a real generator and the commutator of a real generator with an imaginary generator must produce an imaginary generator. To see that this is the case, notice that we can write a purely imaginary antisymmetric matrix as $\tilde{A} = \begin{pmatrix} 0 & ia \\ -ia & 0 \end{pmatrix} = iA$ and a purely imaginary Hermitian matrix as $\tilde{H} = \begin{pmatrix} ip & ih \\ ih & iq \end{pmatrix} = iH$ where a, h, p, q are again real. When we construct the block matrix, we will have a result similar to the real case, but with an overall factor of i on each matrix (which will square to -1 in the commutator) and a few flipped minus signs. Doing the calculation explicitly as a check, we find

$$\left[\begin{pmatrix} \tilde{A} & \tilde{H} \\ -\tilde{H}^T & -\tilde{A}^\dagger \end{pmatrix}, \begin{pmatrix} \tilde{B} & \tilde{J} \\ -\tilde{J}^T & -\tilde{B}^\dagger \end{pmatrix}\right] = \begin{pmatrix} 0 & -hp + hq + jr - js & 2bh - 2aj & ap - aq - br + bs \\ hp - hq - jr + js & 0 & ap - aq - br + bs & 2aj - 2bh \\ 2aj - 2bh & -ap + aq + br - bs & 0 & -hp + hq + jr - js \\ -ap + aq + br - bs & 2bh - 2aj & hp - hq - jr + js & 0 \end{pmatrix}$$

which has the correct block form for the real generators. By similar logic, we also see that the commutator of a real generator and an imaginary generator will produce an imaginary generator.

Part (c)

The general form of a complex generator is

$$T = \left(\begin{array}{cc|cc} 0 & a + bi & c & d + ei \\ -a - bi & 0 & d - ei & f \\ \hline -c & -d + ei & 0 & a - bi \\ -d - ei & -f & -a + bi & 0 \end{array}\right)$$

where all $a, \ldots, f$ are real. This suggests that the individual generators are

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = I_2 \otimes i\sigma_2 \qquad H_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = i\sigma_2 \otimes \sigma_3$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = i\sigma_2 \otimes \sigma_1 \qquad H_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = i\sigma_2 \otimes I_2$$

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = -\sigma_3 \otimes \sigma_2 \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = -\sigma_1 \otimes \sigma_2$$

These are named in anticipation that $\{H_i\} = H$, $\{K_i\} = K$. From the generators we can construct the Killing metric:

$$g_{\alpha\beta} = \mathrm{Tr}\left(T_\alpha T_\beta\right) = -4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We can immediately see that any inner product $H^\alpha g_{\alpha\beta} K^\beta$ that would mix H and K is zero. We would also like for $g_{\alpha\beta}$ to be positive definite for K and negative definite for H. This is easy to check:

$$\left(H^\alpha\right)^\dagger g_{\alpha\beta} H^\beta = -4 I_4 \qquad \left(K^\alpha\right)^\dagger g_{\alpha\beta} K^\beta = 4 I_4$$

Therefore, our guess was correct, and the matrices in H and K satisfy the needed properties.

Part (d)

We can decompose a block matrix into its diagonal and off-diagonal parts:

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

If we take the different possible commutators, we find

$$\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix}\right] = \begin{pmatrix} AA' - A'A & 0 \\ 0 & DD' - D'D \end{pmatrix} \qquad \left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix}\right] = \begin{pmatrix} BC' - B'C & 0 \\ 0 & CB' - C'B \end{pmatrix}$$

$$\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & AB - BD \\ DC - CA & 0 \end{pmatrix}$$

Decomposing into diagonal (H) and off-diagonal (K) parts obeys $[H, H] \subset H$, $[H, K] \subset K$, and $[K, K] \subset H$, just as we needed.

PROBLEM 3

The simple roots of $USp(2N)$ are

$$\vec{\alpha}^i = \left(\vec{\mu}^i - \vec{\mu}^{i+1}, 0\right) \quad , \quad 1 \le i < N \qquad\qquad \vec{\alpha}^N = \left(2\vec{\mu}^N, \sqrt{\frac{2}{N}}\right)$$

where the $\vec{\mu}^i$ are the weights of the defining representation of $SU(N)$. These satisfy $\vec{\mu}^j \cdot \vec{\mu}^j = \frac{N-1}{2N}$ and $\vec{\mu}^j \cdot \vec{\mu}^k = -\frac{1}{2N}$. The highest root is $\vec{\alpha}^0 = \left(-2\vec{\mu}^1, -\sqrt{\frac{2}{N}}\right)$. The inner products are

$$\vec{\alpha}^0 \cdot \vec{\alpha}^1 = \left(-2\vec{\mu}^1, -\sqrt{\frac{2}{N}}\right) \cdot \left(\vec{\mu}^1 - \vec{\mu}^2, 0\right) = -1$$

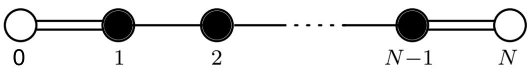
$$\vec{\alpha}^0 \cdot \vec{\alpha}^j = \left(-2\vec{\mu}^1, -\sqrt{\frac{2}{N}}\right) \cdot \left(\vec{\mu}^j - \vec{\mu}^{j+1}, 0\right) = 0 \quad , \quad 1 < j < N$$

$$\vec{\alpha}^0 \cdot \vec{\alpha}^N = \left(-2\vec{\mu}^1, -\sqrt{\frac{2}{N}}\right) \cdot \left(2\vec{\mu}^N, \sqrt{\frac{2}{N}}\right) = 0$$

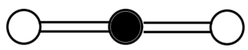
From these, we see that we will attach a single circle to the first circle of the usual $USp(2N)$ diagram. We have

$$\frac{\vec{\alpha}^0 \cdot \vec{\alpha}^1}{|\vec{\alpha}^0||\vec{\alpha}^1|} = -\frac{1}{\sqrt{2}}$$

so the extended Dynkin diagram is



The extended diagram for $USp(4)$ is



Removing a circle gives the diagram for a maximal regular subalgebra. The only one (other than the obvious $USp(4) \simeq SO(5)$) is $SU(2) \otimes SU(2)$. This is the same as for $SO(5)$.

PROBLEM 4

Part (a)

In D dimensions, the gamma matrices are $2^{[D/2]} \times 2^{[D/2]}$. Call the $D = 7$ matrices γ_i, where $i = 1, \ldots, 7$. To go from $D = 7$ to $D = 8$ matrices (called Γ), we will take the tensor product of the $D = 7$ representation with 2×2 matrices as follows:

$$\Gamma_i = \gamma_i \otimes \sigma_2 \qquad\qquad \Gamma_8 = \mathbb{1}_8 \otimes \sigma_1$$

For our $D = 7$ representation, we can use

$$\gamma_1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \qquad \gamma_2 = \sigma_2 \otimes \sigma_3 \otimes I_2 \qquad \gamma_3 = \sigma_2 \otimes \sigma_1 \otimes I_2$$

$$\gamma_4 = I_2 \otimes \sigma_2 \otimes \sigma_1 \qquad \gamma_5 = \sigma_1 \otimes I_2 \otimes \sigma_2 \qquad \gamma_6 = \sigma_3 \otimes I_2 \otimes \sigma_2$$

$$\gamma_7 = I_2 \otimes \sigma_2 \otimes \sigma_3$$

so the $D = 8$ matrices are

$$\Gamma_1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \qquad \Gamma_2 = \sigma_2 \otimes \sigma_3 \otimes I_2 \otimes \sigma_2 \qquad \Gamma_3 = \sigma_2 \otimes \sigma_1 \otimes I_2 \otimes \sigma_2 \qquad \Gamma_4 = I_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2$$

$$\Gamma_5 = \sigma_1 \otimes I_2 \otimes \sigma_2 \otimes \sigma_2 \qquad \Gamma_6 = \sigma_3 \otimes I_2 \otimes \sigma_2 \otimes \sigma_2 \qquad \Gamma_7 = I_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \qquad \Gamma_8 = I_8 \otimes \sigma_1$$

It is easy to see that they are all real and symmetric.

Part (b)

Since each Γ is symmetric, the charge conjugation matrices must obey

$$C_+\Gamma_\mu = \Gamma_\mu C_+ \qquad C_-\Gamma_\mu = -\Gamma_\mu C_-$$

C_- is easiest to find first. Since we know that $\{\Gamma_c, \Gamma_\mu\} = 0$, we can use $C_- = \Gamma_c$. The highest rank gamma (which is real) is

$$\Gamma_c = (-i)^4 \Gamma_1 \times \cdots \times \Gamma_8 = -I_8 \otimes \sigma_3$$

Since $C_- = C_+\Gamma_c$ and $\Gamma_c^2 = 1$, we find $C_+ = C_-\Gamma_c = \Gamma_c^2 = I_{16}$. Both $C_\pm$ are diagonal and symmetric.

Part (c)

To find a representation in Euclidean $D = 9$ (denoting matrices with a tilde), we simply take the $D = 8$ representation that we found previously and define $\tilde{\Gamma}_9 = \Gamma_c = -I_8 \otimes \sigma_3$. The other matrices are the same. To convert this representation to Minkowski $D = 9$, we choose one (it can be any) of our matrices and multiply by i.

In the Euclidean case, we clearly have a real representation, since all the matrices are already real. In the Lorentzian case, we have a single purely imaginary matrix, but all the others are real. To see that we can still have a real representation, we want to find an S such that $R^* = SRS^{-1}$. If we take M to the the timelike gamma, then this becomes $-M = SMS^{-1}$, or equivalently $SM + MS = 0$. We know that in odd dimensions, there are two inequivalent highest-rank gamma matrices. The one formed from the product $\tilde{\Gamma}_1 \times \cdots \times \tilde{\Gamma}_9$ is proportional to the identity, but the other still anticommutes with all $\tilde{\Gamma}$. Since we can use this matrix for S, we also have a real representation.

Part (d)

In the $D = 7$ case, since our matrices are all imaginary, the spinor representation is complex. For $D = 8$, just like in the $D = 9$ case, we can transform the single timelike gamma into a real matrix using Γ_c, so we can also have a real representation.