Group Theory - Homework 11

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(Dated: April 12, 2020)

PROBLEM 1

Part (a)

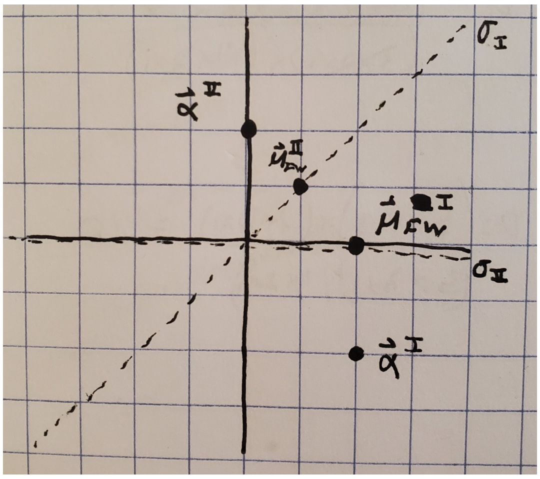
The simple roots of $SO(5)$ are

$$\vec{\alpha}^{I} = (1, -1) \qquad \vec{\alpha}^{II} = (0, 1)$$

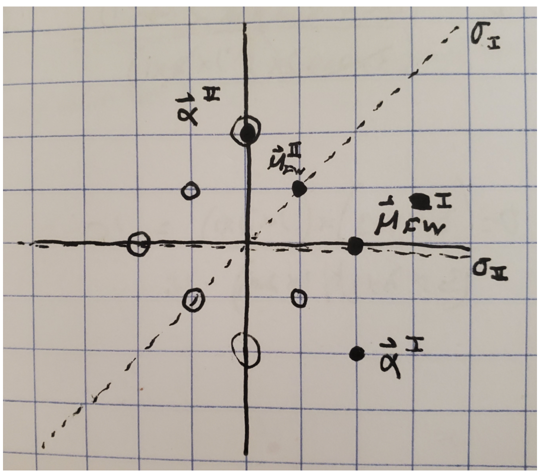
and the fundamental weights are

$$\vec{\mu}^{I}_{FW} = (1, 0) \qquad \vec{\mu}^{II}_{FW} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

These are plotted below, along with dashed lines to represent σ_I and σ_{II}:



By acting with combinations of σ_I and σ_{II} on $\vec{\mu}^{I}_{FW}$ and $\vec{\mu}^{II}_{FW}$, we can find other weights, denoted by open circles:



As equations:

$$(0,1) = \sigma_I \vec{\mu}^I_{FW} \qquad \left(\frac{1}{2}, -\frac{1}{2}\right) = \sigma_{II}\vec{\mu}^{II}_{FW} \qquad (0,-1) = \sigma_{II}\sigma_I\vec{\mu}^I_{FW}$$

$$\left(-\frac{1}{2}, \frac{1}{2}\right) = \sigma_I\sigma_{II}\vec{\mu}^{II}_{FW} \qquad \left(-\frac{1}{2}, -\frac{1}{2}\right) = \sigma_{II}\sigma_I\sigma_{II}\vec{\mu}^{II}_{FW} \qquad (-1,0) = \sigma_I\sigma_{II}\sigma_I\vec{\mu}^I_{FW}$$

Part (b)

We can now write a vector $\vec{v}$ as $\vec{v} = \lambda_I \vec{\mu}^I_{FW} + \lambda_{II}\vec{\mu}^{II}_{FW} = \left(\lambda_I + \frac{1}{2}\lambda_{II}, \frac{1}{2}\lambda_{II}\right)$. Then, if we transform v under σ_I we find

$$\sigma_I \vec{v} = \left(\frac{1}{2}\lambda_{II}, \lambda_I + \frac{1}{2}\lambda_{II}\right) = -\lambda_I(1,0) + (2\lambda_I + \lambda_{II})\left(\frac{1}{2}, \frac{1}{2}\right) = -\lambda_I\vec{\mu}^I_{FW} + (2\lambda_I + \lambda_{II})\,\vec{\mu}^{II}_{FW}$$

so the action of σ_I on $(\lambda_I, \lambda_{II})$ is $(\lambda_I, \lambda_{II}) \to (-\lambda_I, 2\lambda_I + \lambda_{II})$. Similarly, acting with σ_{II} gives

$$\sigma_{II}\vec{v} = \left(\lambda_I + \frac{1}{2}\lambda_{II}, -\frac{1}{2}\lambda_{II}\right) = (\lambda_I + \lambda_{II})\vec{\mu}^I_{FW} - \lambda_{II}\vec{\mu}^{II}_{FW}$$

so the action of σ_{II} on $(\lambda_I, \lambda_{II})$ is $(\lambda_I, \lambda_{II}) \to (\lambda_I + \lambda_{II}, -\lambda_{II})$.

Part (c)

Applying powers of $\sigma_I\sigma_{II}$ to $\vec{\mu}^I_{FW}$ generates the sequence

$$(1,0) \xrightarrow{\sigma_I\sigma_{II}} (0,1) \xrightarrow{\sigma_I\sigma_{II}} (-1,0) \xrightarrow{\sigma_I\sigma_{II}} (0,-1) \xrightarrow{\sigma_I\sigma_{II}} (1,0) \xrightarrow{\sigma_I\sigma_{II}} \dots$$

These transformations correspond to rotating the vertices of the tilted square shown in the second figure above. Applying powers of $\sigma_{II}\sigma_I$ rotates the square in the opposite direction:

$$(1,0) \xrightarrow{\sigma_{II}\sigma_I} (0,-1) \xrightarrow{\sigma_{II}\sigma_I} (-1,0) \xrightarrow{\sigma_{II}\sigma_I} (0,1) \xrightarrow{\sigma_{II}\sigma_I} (1,0) \xrightarrow{\sigma_{II}\sigma_I} \dots$$

Similarly, acting on $\vec{\mu}^{II}_{FW}$ generates

$$\left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_I\sigma_{II}} \left(-\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_I\sigma_{II}} \left(-\frac{1}{2}, -\frac{1}{2}\right) \xrightarrow{\sigma_I\sigma_{II}} \left(\frac{1}{2}, -\frac{1}{2}\right) \xrightarrow{\sigma_I\sigma_{II}} \left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_I\sigma_{II}} \dots$$

$$\left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_{II}\sigma_I} \left(\frac{1}{2}, -\frac{1}{2}\right) \xrightarrow{\sigma_{II}\sigma_I} \left(-\frac{1}{2}, -\frac{1}{2}\right) \xrightarrow{\sigma_{II}\sigma_I} \left(-\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_{II}\sigma_I} \left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\sigma_{II}\sigma_I}$$

These transformations rotate the edges of the square. This group has 8 total elements: 4 rotations and 4 reflections. It can be identified as D_4, the symmetry group of the square.

PROBLEM 2

$SU(4)$

$$\ydiagram{2,1,1} \implies D = \frac{5 \times 4 \times 3 \times 2}{4 \times 2} = 15$$

$$\ydiagram{2,1} \implies D = \frac{5 \times 4 \times 3}{3} = 20$$

$SO(6)$

$$\ydiagram{1,1,1} \sim \begin{array}{c|c|c} r_i & \lambda_i & R_i \\ \hline 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \implies D = \frac{(5 \times 4 \times 3) \times (1 \times 2 \times 1)}{(3 \times 2 \times 1) \times (1 \times 2 \times 1)} = 10$$

Since the conjugate of a rank-j antisymmetric tensor representation is a rank $2N - j$ antisymmetric tensor representation, this diagram corresponds to both the **10** and the $\overline{\mathbf{10}}$ of $SO(6)$. This is relevant in Problem 3.

$SO(5)$

$$\ydiagram{1,1} \sim \begin{array}{c|c|c} r_i & \lambda_i & R_i \\ \hline 3/2 & 1 & 5/2 \\ 1/2 & 1 & 3/2 \end{array} \implies D = \frac{\frac{5}{2} \times \frac{3}{2} \times \frac{8}{2} \times \frac{2}{2}}{\frac{3}{2} \times \frac{1}{2} \times \frac{4}{2} \times \frac{2}{2}} = 10$$

$$\ydiagram{3} \sim \begin{array}{c|c|c} r_i & \lambda_i & R_i \\ \hline 3/2 & 3 & 9/2 \\ 1/2 & 0 & 1/2 \end{array} \implies D = \frac{\frac{9}{2} \times \frac{1}{2} \times \frac{10}{2} \times \frac{8}{2}}{\frac{4}{2} \times \frac{2}{2} \times \frac{3}{2} \times \frac{1}{2}} = 30$$

PROBLEM 3

Part (a)

$$\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{3} \oplus \ydiagram{1} \leftrightarrow \mathbf{14} \otimes \mathbf{5} = \mathbf{35} \oplus \mathbf{30} \oplus \mathbf{5}$$

$$\ydiagram{2,1} \sim \begin{array}{c|c|c} r_i & \lambda_i & R_i \\ \hline 3/2 & 2 & 7/2 \\ 1/2 & 1 & 3/2 \end{array} \implies D = \frac{\frac{3}{2} \times \frac{7}{2} \times \frac{10}{2} \times \frac{4}{2}}{\frac{3}{2} \times \frac{1}{2} \times \frac{2}{2} \times \frac{4}{2}} = 35$$

$$\ydiagram{2} \sim \begin{array}{c|c|c} r_i & \lambda_i & R_i \\ \hline 3/2 & 2 & 7/2 \\ 1/2 & 1 & 1/2 \end{array} \implies D = \frac{\frac{1}{2} \times \frac{7}{2} \times \frac{8}{2} \times \frac{6}{2}}{\frac{3}{2} \times \frac{1}{2} \times \frac{4}{2} \times \frac{2}{2}} = 14$$

Part (b)

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1} \oplus \ydiagram{1} \leftrightarrow \mathbf{15} \otimes \mathbf{6} = \mathbf{64} \oplus \big(\mathbf{10} \oplus \overline{\mathbf{10}}\big) \oplus \mathbf{6}$$

$$\ydiagram{2,1} \quad \sim \quad \begin{array}{c|c|c} \mathrm{r}_i & \lambda_i & \mathrm{R}_i \\ \hline 2 & 2 & 4 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{array} \Longrightarrow D = \frac{(6 \times 4 \times 2) \times (2 \times 4 \times 2)}{(3 \times 2 \times 1) \times (1 \times 2 \times 1)} = 64$$