Group Theory - Homework 15

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PROBLEM 1

Part (a)

Since K is discrete, there is an open cover of $\hat{G}$ such that each open subset of $\hat{G}$ corresponds to exactly one element of K. This immediately implies that $f_k(\hat{g}) = k$, so $k = \hat{g}k\hat{g}^{-1}$, which means that k is in the center of $\hat{G}$.

Part (b)

First, we will find the center of $SU(2)$. If an element $M \in SU(2)$ is in the center, then $MN = NM$ for all $N \in SU(2)$, in particular, $N = \sigma_2$, σ_3. We can write M as

$$M = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \qquad u^*u + v^*v = 1$$

Then, we have

$$\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \implies \begin{pmatrix} v & -u \\ u^* & v \end{pmatrix} = \begin{pmatrix} v^* & -u^* \\ u & v \end{pmatrix} \implies u = u^*,\ v = v^*$$

$$\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix} \implies \begin{pmatrix} u & -v \\ -v^* & -u^* \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & -u^* \end{pmatrix} \implies v = 0$$

We must also satisfy $u^2 = 1$ with $u \in \mathbb{R}$, which leaves $u = \pm 1$. The center of $SU(2)$ is therefore $\{\pm I_2\} = \mathbb{Z}_2$. Since $Spin(4) \simeq SU(2) \times SU(2)$, its center is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Part (c)

If N is a discrete normal subgroup of $\hat{G}$, then $\hat{G}/N$ has the same Lie algebra as $\hat{G}$[1]. The possible discrete normal subgroups are constructed from the elements of $\hat{G}$ that commute with all $\hat{g} \in \hat{G}$. We previously showed that the center of $Spin(4)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This group looks like

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \big\{(e_1, e_2), (e_1, k_2), (k_1.e_2), (k_1, k_2)\big\}$$

where $e_{1,2}$ are the identity elements and $k_{1,2}$ are the non-identity elements from each $\mathbb{Z}_2$. The possible subgroups are

$$\big\{(e_1, e_2), (e_1, k_2)\big\} \qquad \big\{(e_1, e_2), (k_1, e_2)\big\} \qquad \big\{(e_1, e_2), (k_1, k_2)\big\}$$

So we see that

$$Spin(4)/\big\{(e_1, e_2), (e_1, k_2)\big\} \qquad Spin(4)/\big\{(e_1, e_2), (k_1, e_2)\big\} \qquad Spin(4)/\big\{(e_1, e_2), (k_1, k_2)\big\}$$

will correspond to the three groups given in the problem statement. Since $\big\{(e_1, e_2), (e_1, k_2)\big\} \simeq \big\{(e_1, e_2), (k_1, e_2)\big\}$ we have

$$Spin(4)/\left\{(e_1,e_2),(e_1,k_2)\right\} \simeq SO(3)\times SU(2) \qquad Spin(4)/\left\{(e_1,e_2),(k_1,e_2)\right\} \simeq SU(2)\times SO(3)$$

$$Spin(4)/\left\{(e_1,e_2),(k_1,k_2)\right\} \simeq SO(3)\times SO(3)$$

Part (d)

Since $Spin(4)$ is the double cover of $SO(4)$ and $Spin(4) \simeq SU(2)\times SU(2)$, it is immediately obvious that $SO(4) \simeq \frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$. This is the standard way that we "undo" the two-to-one mapping from the double cover onto the main group.

Part (e)

Since $O(N)$ is not connected, $O(3) \neq SU(2)$. Locally however, since $O(N)$ and $SO(N)$ have the same Lie algebra and $SO(3)$ is the double cover of $SU(2)$, they are the same.

The volume of $O(N)$ is[2][3]

$$\mathrm{vol}\left[O(N)\right] = \frac{2^N \pi^{\frac{n^2+1}{4}}}{\prod_{k=1}^{N}\Gamma\left(\frac{n-k}{2}+1\right)}$$

which gives vol $\left[O(3)\right] = 16\pi^2$. This matches neither of the standard normalizations given in the notes for the volume of $SU(2)$, which are $\{2\pi^2, 4\pi^2\}$.

PROBLEM 2

Part (a)

We can write an element M of $U(2)$ as

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} e^{i\varphi/2}, \qquad a^*a + b^*b = 1$$

To check:

$$det(M) = e^{i\varphi}(a^*a + b^*b) = e^{i\varphi}$$

$$M^\dagger M = e^{i\varphi/2}e^{-i\varphi/2}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} a^*a+b^*b & -ab+ba \\ -a^*b^*+a^*b^* & a^*a+b^*b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant is an arbitrary complex number with magnitude 1 and $M^\dagger = M^{-1}$, as we would like.

The particular form we have chosen immediately suggests that $U(2) \sim SU(2)\times U(1)$, but there is an ambiguity. We can multiply both the $SU(2)$ part and the $U(1)$ part of M by $e^{2\pi ik/2}$ for $k = 0, 1$ without changing M or the fundamental properties of either of its parts (since in the determinant, the -1 squares). This indicates that a map from $SU(2)\times U(1)$ to $U(2)$ is two-to-one. If we mod out a $\mathbb{Z}_2$ factor, then this is resolved, so the actual relation is $U(2) \simeq \frac{SU(2)\times U(1)}{\mathbb{Z}_2}$.

Part (b)

We can now generalize the result of part (a) to all N. An element of $U(N)$ can be written as $M = Se^{i\varphi}$, where $S \in SU(N)$. Now, if we transform $S \to e^{2\pi ik/N}S$ and $e^{i\varphi} \to e^{-2\pi ik/N}e^{i\varphi}$, M is unchanged as before. In the general case, instead of having the freedom to transform by ±1, we have N possible N^{th} roots of unity, which form the group $\mathbb{Z}_N$. Modding this out gives the final result:

$$U(N) \simeq \frac{SU(N) \times U(1)}{\mathbb{Z}_N}$$

Part (c)

In homework 7, we showed that every element of $SU(2)$ can be written in exponential form. Since $SU(3)$ is simply connected and compact, all of its elements can also be written in exponential form.

PROBLEM 3

For a general connection, we have

$${}^R\omega_\mu{}^m{}_n(a) = f_\mu{}^l(a)c_{nl}{}^m +{}^L \omega_{\mu s}{}^r(a)(A^{-1})_r{}^m(a)A_n{}^s(a)$$

Taking ${}^R\omega_\mu{}^m{}_n(a) = {}^R\overset{+}{\omega}{}_\mu{}^m{}_n = 0$, we find

$${}^L\overset{+}{\omega}{}_\mu{}^s{}_r(A^{-1})_s{}^m A_n{}^r = f_\mu{}^l c_{nl}{}^m \implies {}^L\overset{+}{\omega}{}_\mu{}^s{}_r = f_\mu{}^l c_{nl}{}^m A_m{}^s(A^{-1})_r{}^n$$

Using $f_\mu{}^m = e_\mu{}^n(A^{-1})_n{}^m$ and the fact that $c_{ij}{}^k$ are invariant tensors gives

$${}^L\overset{+}{\omega}{}_\mu{}^s{}_r = e_\mu{}^k c_{nl}{}^m A_m{}^s(A^{-1})_r{}^n(A^{-1})_k{}^l = -e_\mu{}^k c_{rk}{}^s$$

Getting the second part of (17.22) from (17.25) with ${}^R\omega = {}^R\overset{-}{\omega}$ is trivial, since ${}^L\overset{-}{\omega} = 0$. We could also obtain it from (17.23) in nearly the same way we got the first part from (17.25).

PROBLEM 4

Moving from the origin to P with $g(b)$ gives the vector $dv^\mu(b)$. We want to show that $dv^\mu(b) = dv^\mu(a + \Delta a)$ regardless of the path taken. If we first parallel transport from the origin to a, we get

$$dv^\mu(a) = dv^\mu(0) - \Delta a^\sigma \bar{\Gamma}_{\sigma\rho}{}^\mu(0)dv^\rho(0)$$

Going to $a + \Delta a$ gives

$$dv^\mu(a + \Delta a) = \Big(dv^\mu(0) - \Delta a^\sigma \bar{\Gamma}_{\sigma\rho}{}^\mu(0)dv^\rho(0)\Big) - \tilde{\Delta} a^\sigma \bar{\Gamma}_{\sigma\rho}{}^\mu(a)\left(dv^\rho(0) - \Delta a^\lambda \bar{\Gamma}_{\lambda\tau}{}^\rho(0)dv^\lambda(0)\right)$$

Now we need to combine the terms. The term quadratic in Δa vanish and using $\Delta a^\mu = da^r f_r{}^\sigma(a)$, we can combine the remaining terms into something of the form $\Delta a\ \Gamma(a)dv(0)$, plus a term proportional to the symmetric part of Γ, which vanishes. This is then the standard expression:

$$dv^\mu(a + \Delta a) = dv^\mu(0) - \hat{\Delta} a^\sigma \bar{\Gamma}_{\sigma\rho}{}^\mu(0)dv^\mu(0)$$

where now $\hat{\Delta}a^\mu$ includes the full transformation from $0 \to a + \Delta a$. Therefore, we have found that the composition of translations is path independent.

[1] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Dover (2002)
[2] J. Muirhead, *Aspects of Multivariate Statistical Theory*, Wiley (1982)
[3] L. Zhang, *Volumes of Orthogonal Groups and Unitary Groups*, arXiv:1509.00537v5 (2017)