Homework 10

Due on: Monday, April 6

Problem 1

The raising and lowering generators of $SO(2N)$ in the CW basis are $E^{\eta\eta'}_{IJ}$. Use the spinor representation[1] to derive the following commutator

$$[H_I, E^{\eta\eta'}_{JK}] = \eta\delta_{IJ}E^{\eta\eta'}_{JK} + \eta'\delta_{IK}E^{\eta\eta'}_{JK} \quad \text{for } J < K \ . \tag{1.1}$$

Show that the raising generators have $\eta = +1$. If $E^{\eta\eta'}_{IJ}$ is denoted by E_α, and α is a positive root, what is then $E_{-\alpha}$ in the spinor representation. (Recall that $E_{-\alpha}$ is defined by $(E_\alpha)^\dagger$). Use the spinor representation to derive

$$[E_\alpha, E_{-\alpha}] = \alpha_I g^{IJ} H_I g_{\alpha,-\alpha} \ . \tag{1.2}$$

Finally evaluate g_{IJ} and $g_{\alpha,-\alpha}$ for the defining representation, the spinor representations, and the adjoint representation, and check that $g^{IJ}g_{\alpha,-\alpha}$ is representation-independent.

Problem 2

Construct the weight diagrams for the defining representation and the root diagrams, first for $SU(3)$ and then $SU(4)$. Using that the weight diagram for the defining representation of $SU(3)$ is an equilateral triangle, and that of $SU(4)$ is a tetrahedron, find a relation between the root vectors and the edges of the triangle and the tetrahedron. What can you write about the representations whose highest weight is a fundamental weight of $SU(4)$?

Problem 3

Finally construct the weight diagrams for the defining irrep, the two spinor irreps and the root diagram, first for $SO(4)$, and then for $SO(6)$. Consider a square in 2 dimensions, and a cube in 3 dimensions, whose sides are of length 2. Where on the cube do the root vectors end, and where do the weight vectors of the defining irrep end, and where inside the cube do the weight vectors of the two spinor irreps lie? Which weight vectors form the s-irrep, and which form the c-irrep? What are the three fundamental weights of $SO(6)$, and do they end on the surface of the cube?

Problem 4

The Lie algebras of $SU(4)$ and $SO(6)$ are isomorphic. Is this reflected in the results for problems 2 and 3?

[1] There are two spinorial carrier spaces on which the generators act, denoted by s (for spinor) and c (for conjugate spinor), but the generators acting on these spaces have the same form.