Homework 8

Due on: Monday, March 23

Problem 1

We introduced the classical groups $SO(p,q;\mathbb{R})$ and $SO(p,q;\mathbb{C})$ as leaving $x^T H x$ invariant with $H = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}$. Then we introduced the classical groups $SU(p,q) \equiv SU(p,q;\mathbb{C})$ as leaving $x^\dagger H x$ invariant. Finally we defined the classical groups $Sp(2n,\mathbb{R})$ and $Sp(2n,\mathbb{C})$ as leaving $x^T \Omega y$ invariant where $\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$. Can one define a group by requiring that $x^\dagger \Omega x$ is invariant? If so, is it equivalent to one of the classical groups?

Problem 2

Consider the group $U(2)$. Is it connected? Simply connected? Compact? Consider the map $\varphi(g) = R$ from $g \in U(2)$ into $R \in SO(3)$, given by $U^\dagger x^i \sigma_i U = \sigma_i R^i{}_j x^j$. Is this a homomorphism? If so, what is its kernel? Is $U(2)$ a covering group of $SO(3)$?

Problem 3

The group $SO^*(2n)$ is defined as leaving both $x^T x$ and $x^\dagger \Omega x$ invariant. In physics it appears in 7 dimensions where $SO(6,2) = SO^*(8)$ is the anti-de Sitter group, or in $5+1$ dimensions where it is the conformal group.

(a) Show that this implies that both $x^T y$ and $x^\dagger \Omega y$ are invariant.

(b) If one writes M in block form as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, what are the conditions the $n \times n$ complex matrices A, B, C, D must satisfy?

Problem 4

The groups $USp(2p, 2q)$ are defined as leaving $x^T \Omega y$ and $x^\dagger H y$ invariant. What is the expression for the generators of $USp(n,n)$? Recall that $USp(2n) = Sp(2n,\mathbb{C}) \cap SU(2n)$.

Problem 5

Finally $SU^*(2n)$. Its generators satisfy $M^* \Omega = \Omega M$ and are traceless. Do matrices satisfying this condition form a Lie algebra? Why are these groups called $SU^*(2n)$ and not, for example, $Sp^*(2n)$? And why are the groups $SO^*(2n)$ in problem 3 called $SO^*(2n)$ and not, for example, $SU^*(2n)$ or

$Sp^*(2n)$?

Hint: count the number of generators.