Group Theory - Homework 13

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PROBLEM 1

If we have a representation $D(g)$, then $\sum_g \text{tr}\left(D^2(g)\right)$ is preserved under $D(g) \to MD(g)M^{-1}$ by cyclicity of the trace:

$$\sum_g \text{tr}\left(MD(g)M^{-1}MD(g)M^{-1}\right) = \sum \text{tr}\left((D(G))^2 MM^{-1}\right) = \sum_g \text{tr}\left(D^2(g)\right)$$

The Frobenius-Schur theorem tells us that this quantity determines whether a given irrep is real, pseudoreal, or complex, so since it is preserved under a similarity transformation, so are the reality properties of the irrep.

PROBLEM 2

An irrep $R(G)$ of a semi-simple Lie algebra G is self-conjugate if and only if there exists some matrix M such that $\bar{R} = MRM^{-1}$. Self-conjugate irreps can either be real or pseudoreal. If an irrep is not self-conjugate, then it is complex.

Using the Cartan matrix, defined as $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ where α_i are simple roots, the Dynkin diagram can be constructed by drawing a circle for each simple root and joining circles i and j with $A_{ij}A_{ji}$ lines. Is is then clear that any mapping $\alpha_i \to \tilde{\alpha_i}$ that preserves (α_i, α_j) will also preserve the Cartan matrix, and by extension, the form of the Dynkin diagram.

It can be shown[1] that we can write any automorphism of G as the composition $A_N \circ B$, where A_N represents conjugation by an algebra element N and B is one the the aforementioned automorphisms of G that correspond to an automorphism of the Dynkin diagram of G. It is clear that $R(A_N G) = R(G)$, since conjugation simply reshuffles the elements of G.

Now consider the automorphism C of G that maps each irrep to its conjugate. We have $\bar{R}(G) = R(CG) = R(A_N BG) = R(BG)$. Therefore, $\bar{R}(G) = R(G)$ if and only if B is trivial. To rephrase in a more useful way: if the only automorphism of a given Dynkin diagram is trivial, then any irrep of its corresponding algebra must be self-conjugate, and therefore real or pseudoreal.

Considering the classical Lie algebras, it is now easy to see that $SU(N)$ and $SO(2N)$ may have complex, real, and pseudoreal irreps, but $USp(2N)$ and $SO(2N+1)$ can only have (pseudo)real irreps. Incidentally, this answers part 1 of question 4, by showing why there can be no complex spinor irreps of $SO(2N+1)$.

We know that any irrep R of G is determined by its highest weight. A convenient basis for the components a_i of a weight λ, called the Dynkin basis, is $a_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$. There is a theorem, due to Dynkin[2], that states that the highest weight can be chosen such that the a_i are non-negative integers and that every irrep is uniquely identified by a set of integers $(a_1, \ldots a_n)$, where n is the rank of G. We can therefore identify an irrep by specifying $(a_1, \ldots a_n)$.

For irreps of $SU(N)$, conjugation is given by[2] $(a_1, \ldots, a_n)^* = (a_n, \ldots, a_1)$. Therefore, any complex irrep must obey $(a_1, \ldots, a_n) \neq (a_n, \ldots, a_1)$.

Now, we have to differentiate between real and pseudoreal representations for both $SU(N)$ and $USp(2N)$. To determine whether an irrep is real or pseudoreal, we calculate the height of the irrep[2] from the highest weight $\Lambda = (a_1, \ldots, a_n)$:

$$T(\Lambda) = 2\sum_i \sum_j \left(A^{-1}\right)_{ij} a_i$$

This quantity gives the largest number of simple roots that must be subtracted from the highest weight to obtain a given irrep. It was proved by Dynkin[3] that (for self-conjugate irreps) if $T(\Lambda)$ is even, its corresponding irrep must be real, and if $T(\Lambda)$ is odd, its irrep must be pseudoreal. All together, we have a systematic way to check whether an irrep is complex, real, or pseudoreal.

We can check these results with a few examples. We know from quantum mechanics that the spin 1/2 irreps of $SU(2)$ should be pseudoreal, so $USp(2) \simeq SU(2)$ suggests that the **2** of $USp(2)$ should be pseudoreal as well. We know that it cannot be complex. The Cartan matrix is $A = 2$, so $T(\Lambda) = 2 \times \frac{1}{2} \times 1 = 1$, so we indeed see that it is pseudoreal.

We can also use $USp(4) \simeq SO(5)$ as an example. We know that the **4** (the spinor irrep) of $SO(5)$ is pseudoreal. The Cartan matrix is $A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ and $\Lambda = (1, 0)$, so $T(\Lambda) = 2\left[1 \times 1 + \frac{1}{2} \times 1\right] = 3$. Again, we find that the reality properties match. Finally, we can check using $SU(4) \simeq SO(6)$. We know that $SO(6)$ has both a spinor and a conjugate spinor irrep, so we expect that the **4** of $SU(4)$ should be complex. In $SU(4)$, $\mathbf{4} = (1, 0, 0)$, which clearly is not equal to $(0, 0, 1)$, so our guess was correct.

PROBLEM 3

First, we notice that we can write a matrix M as

$$M = \sum_k c_k \mathrm{tr}\left[M\gamma_k\right]\gamma^k$$

where γ_k denotes the rank-k gamma matrix. To see that this equation is valid, multiply both sides by γ_n, take the trace, and use the fact that $\mathrm{tr}\left[\gamma_m\gamma^n\right] \sim \delta_m^n$. Then, if we take $M = \delta_\alpha^\beta \delta_\gamma^\delta$, we get

$$\delta_\alpha^\beta \delta_\gamma^\delta = \sum_k c_k \left(\gamma_k\right)_\alpha{}^\delta \left(\gamma^k\right)_\gamma{}^\beta$$

Multiplying by spinors ψ^ϵ and χ^η:

$$\delta_\alpha^\beta \delta_\gamma^\delta \psi^\epsilon \chi^\eta = \sum_k c_k \left(\gamma_k\right)_\alpha{}^\delta \left(\gamma^k\right)_\gamma{}^\beta \psi^\epsilon \chi^\eta = \sum_k c_k \left(\gamma_k\right)_\alpha{}^\delta \psi^\epsilon C^{\eta\kappa} \left(\gamma^k\right)_\gamma{}^\beta \chi_\kappa$$

Taking $\epsilon = \gamma$, $\beta = \kappa$, $\eta = \alpha$ gives

$$\psi^\delta \psi^\beta = \sum_k c_k \left(\gamma_k\right)_\alpha{}^\delta C^{\alpha\beta} \psi^\gamma \left(\gamma^k\right)_\gamma{}^\beta \chi_\beta = \sum_k c_k \left(\gamma_k C^{-1}\right)^{\delta\beta} \bar{\psi}\gamma^k \chi = \sum_k c_k \left(\gamma_k C^{-1}\right)^{\delta\beta} \psi^T C \gamma^k \chi$$

Now, we can use this result to decompose direct products of irreps into sums. For $SO(5)$, we have

$$\times \otimes \times = \psi^\alpha \chi^\beta = c_0 \left(C^{-1}\right)^{\alpha\beta} \psi^T C \chi + c_1 \left(\gamma^\mu C^{-1}\right)^{\alpha\beta} \psi^T C \gamma_\mu \chi + c_2 \left(\gamma^{\mu\nu} C^{-1}\right)^{\alpha\beta} \psi^T C \gamma_{\mu\nu} \chi$$

Since C and γ_k are both invariant tensors of $SO(N)$, we only need to look at the part in each term with $\psi^T C \gamma \chi$. μ, ν are 5-dimensional indices, so the first term in this sum corresponds to a singlet, the second to a **5**, and the last to a **10**, since $\gamma_{\mu\nu}$ is antisymmetric. Therefore, we find

$$\times \otimes \times = \mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10}$$

For $SO(6)$, things change slightly. We now consider $\times \otimes \times = \psi^\alpha \chi^\beta$ and $\times \otimes + = \psi^\alpha \chi_{\dot{\beta}}$. We know that in $d = 6$, C is block anti-diagonal, but $\psi^\alpha \chi^\beta$ should correspond to the diagonal components of a matrix. Therefore, we need to split our sum into a diagonal part (corresponding to $\psi^\alpha \chi^\beta$) and an off-diagonal part (corresponding to $\psi^\alpha \chi_{\dot{\beta}}$). If we use the off-diagonal representation for the γ's given in the notes, then, since the product of two block anti-diagonal matrices is block diagonal, $C\gamma_\mu$ and $C\gamma_{\mu\nu\rho}$ will contribute diagonal terms and C and $C\gamma_{\mu\nu}$ will contribute off-diagonal terms. The sums are then

$$\times \otimes + = \psi^\alpha \chi_{\dot\beta} = c_0 \left(C^{-1}\right)^{\alpha\beta} \psi^T C\chi + c_2 \left(\gamma_{\mu\nu} C^{-1}\right)^{\alpha\beta} \psi^T C\gamma_{\mu\nu}\chi \leftrightarrow \mathbf{4}_s \otimes \mathbf{4}_c = \mathbf{1} \oplus \mathbf{15}$$

$$\times \otimes \times = \psi^\alpha \chi^\beta = c_1 \left(\gamma^\mu C^{-1}\right)^{\alpha}_{\ \dot\beta} \psi C\gamma_\mu \chi + c_3 \left(\gamma^{\mu\nu\rho} C^{-1}\right)^{\alpha}_{\ \dot\beta} \psi^T C\gamma_{\mu\nu\rho}\chi \leftrightarrow \mathbf{4}_s \otimes \mathbf{4}_s = \mathbf{6} \oplus \mathbf{10}$$

PROBLEM 4

The proof that all spinor irreps of $SO(2N+1)$ are (pseudo)real was previously given in Problem (2). It is also not an accident that the s and c spinor irreps of $SO(2N)$ are similar to the conjugate of each other. We can see this by considering

[1] A. Bose and J. Patera, *Classification of Finite-Dimensional Irreducible Representations of Connected Complex Semisimple Lie Groups*, J. Math Phys. 11, (1970)

[2] R. Slansky, *Group Theory for Unified Model Building*, Physics Reports 79, (1981)

[3] E.B. Dynkin, *Maximal Subgroups of the Classical Groups*, Am. Math. Transl. 6, 245 (1957).