Homework 15 (Last homework)

Due on: Monday, May 11

Problem 1. The global structure of $SO(4)$.

In physics, one frequently writes $SO(4) \cong SU(2) \times SU(2)$ or $SO(4) \cong SO(3) \times SO(3)$. However, these isomorphisms do not hold at the level of Lie groups, and are strictly valid only "locally", i.e. at the level of the Lie algebras. In this problem, we will explore different local isomorphisms of $SO(4)$. These are $Spin(4) \cong SU(2) \times SU(2)$, $SO(3) \times SU(2)$, and $SO(3) \times SO(3)$. All these Lie groups have the property that their universal covering group is $Spin(4)$.

Recall the following fact about covering groups from the discussion in class: given a topological group G and its universal covering group $\widehat{G}$, there is a homomorphism of $\widehat{G}$ onto G. The kernel of this map π is defined by

$$K = \{\widehat{g} \in \widehat{G} \mid \pi(\widehat{g}) = e_G\} \ . \tag{1.1}$$

The covering group $\widehat{G}$ is always a connected and simply-connected Lie group. Since K is a normal subgroup of $\widehat{G}$, the quotient group $\widehat{G}/K$ is a Lie group. Note that this does not mean that K lies in the center of $\widehat{G}$. However, if additionally K is a *discrete* normal subgroup of $\widehat{G}$, then K lies in the center of $\widehat{G}$.

(a) As a warm-up exercise, prove that if K is a discrete normal subgroup of $\widehat{G}$, then K lies in the center of $\widehat{G}$. **Hint**: For a fixed element $k \in K$, consider the map $f_k : \widehat{G} \to K$, defined by $f_k(\widehat{g}) := \widehat{g} k (\widehat{g})^{-1}$, and consider what happens if K is discrete.

(b) The double-cover of $SO(4)$ is the group $Spin(4)$, which is isomorphic (not just locally isomorphic!) to $SU(2)_{\mathrm{a}} \times SU(2)_{\mathrm{b}}$. Show that the center of $Spin(4)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) Explain how the 3 locally isomorphic forms $SO(3)_{\mathrm{a}} \times SU(2)_{\mathrm{b}}$, $SU(2)_{\mathrm{a}} \times SO(3)_{\mathrm{b}}$ and $SO(3)_{\mathrm{a}} \times SO(3)_{\mathrm{b}}$ are obtained from Spin(4).

(d) Repeat the exercise for $SO(4)$ and show that $SO(4) \cong \frac{SU(2)_{\mathrm{a}} \times SU(2)_{\mathrm{b}}}{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ is a diagonal subgroup. This is the fourth (and perhaps most important) locally isomorphic form.

(e) Is $SU(2) = O(3)$ a correct statement? Is the volume of $SU(2)$ equal to the volume of $O(3)$?

Frequently one works at the level of the Lie algebra and these discrete quotients are not worrisome. However, in some applications in physics, e.g. anomalies, topological phases, partition functions in the presence of extended objects like line operators, etc., subtleties arise due to different discrete quotients.

Problem 2. The global structure of $U(N)$.

It is often stated that $U(N)$ consists of two kinds of group elements: (i) unitary matrices of the form $e^{i\theta}\mathbb{I}_N$ (where $\mathbb{I}_N$ is the $N \times N$ identity matrix) – this part is clearly isomorphic to $U(1)$, and (ii) the set of $N \times N$ unitary matrices with determinant equal to $+1$ – this is just $SU(N)$. It is therefore tempting to say that $U(N)$ is isomorphic to the direct product of $SU(N) \times U(1)$. However, this is only true "locally". We will prove that globally $U(N)$ is **not** isomorphic to $SU(N) \times U(1)$. This stems from the simple observation that $SU(N)$ and $U(1)$ have the group elements $e^{2i\pi k/N}\mathbb{I}$ in common, where $\mathbb{I}$ is the unit matrix. In fact, as we shall see,

$$U(N) \cong \frac{SU(N) \times U(1)}{\mathbb{Z}_N} \; . \tag{2.1}$$

(a) As a warm up, consider the $N = 2$ case, where

$$U(2) = \frac{SU(2) \times U(1)}{\mathbb{Z}_2} \; , \tag{2.2}$$

where $\mathbb{Z}_2 = (e, \tau)$, and τ acts on an element (g, x) of $SU(2) \times U(1)$ (where $g \in SU(2)$ and $x \in U(1)$) as $\tau : (g, x) \to (-g, -x)$. Prove (2.2) by writing an element of the group $U(2)$ as a 2×2 matrix, and exhibit the action of this $\mathbb{Z}_2$.

(b) Now prove (2.1).

(c) Can the group elements $e^{2i\pi k/N}\mathbb{I}$ be written as the group elements in the exponential form, (i) for SU(2), and (ii) for SU(3)?

Problem 3.

Apply equation (17.25) of the notes to the particular case of the ${}^L\overset{+}{\omega}$ and ${}^R\overset{-}{\omega}$ connections, and obtain equation (17.22).

Problem 4.

Consider left-multiplication in a group. In the notes we showed that a little vector $\widetilde{da}^m$ at the origin could be swept out by left-multiplication with $g(a)$ to a point P with coordinates a^μ. Denote this little vector at P by $dv^\mu(a)$. Then we used left-multiplication by $g(da)$; the point P was moved to a point P', with coordinates $a^\mu + \Delta a^\mu$, and the little vector $dv^\mu(a)$ was moved to P' and became a little vector $dv^\mu(a + \Delta a)$.

One can also go directly from the origin to P' by a group element $g(b)$. Show explicitly that the little vector one obtains in this one-step process is equal to the little vector we got from the two-step process. (Parallelizability of the group manifold.)