Group Theory - Homework 12

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PROBLEM 1

For $SU(N)$, we know that $T(def) = 1/2$ and $T(adj) = N$, so we get $T(3) = 1/2$, $T(8) = 3$. We also know that $T(1) = 0$. Using $C_2(R) = T(R)\frac{\dim(G)}{\dim(R)}$, we find

$$C_2(\mathbf{3}) = C_2(\bar{\mathbf{3}}) = \frac{4}{3} \qquad C_2(\mathbf{8}) = 3 \qquad C_2(\mathbf{1}) = 0$$

Now, we can use $T(R_1 \otimes R_2) = T(R_1)\dim(R_2) + T(R_2)\dim(R_1)$ and $T(R_1 \oplus R_2) = T(R_1) + T(R_2)$ with $\mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10}$ and $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$ to find

$$T(\mathbf{3} \otimes \mathbf{6}) = T(\mathbf{8} \oplus \mathbf{10}) \implies 6T(\mathbf{3}) + 3T(\mathbf{6}) = T(\mathbf{8}) + T(\mathbf{10}) \implies 3T(\mathbf{6}) = T(\mathbf{10})$$

$$T(\mathbf{3} \otimes \mathbf{3}) = T(\bar{\mathbf{3}} \oplus \mathbf{6}) \implies 6T(\mathbf{3}) = T(\bar{\mathbf{3}}) + T(\mathbf{6}) \implies T(\mathbf{6}) = \frac{5}{2} \implies T(\mathbf{10}) = \frac{15}{2}$$

Finally, we find

$$C_2(\mathbf{6}) = \frac{10}{3} \qquad C_2(\mathbf{10}) = 6$$

PROBLEM 2

Part (a)

$\{P_x, P_y\}$ is a nontrivial abelian ideal, so E_2 is not semisimple. We can also look at g_{ab} from Part (b), which has nonzero determinant. This again indicates that E_2 is not semisimple.

Part (b)

To compute the Killing metric, we use the definition $g_{ab} = f_{ap}{}^{q} f_{bq}{}^{p}$. To find the structure constants, we use $[T_a, T_b] = f_{ab}{}^{c} T_c$, which results in

$$[P_x, P_y] = f_{xy}{}^{c} T_c = f_{xy}{}^{x} P_x + f_{xy}{}^{y} P_y + f_{xy}{}^{L} L = 0 \implies f_{xy}{}^{x} = f_{xy}{}^{y} = f_{xy}{}^{L} = 0$$

$$[L, P_x] = f_{Lx}{}^{c} T_c = P_y \implies f_{Lx}{}^{x} = f_{Lx}{}^{L} = 0 \qquad f_{Lx}{}^{y} = 1$$

$$[L, P_y] = f_{Ly}{}^{c} T_c = -P_x \implies f_{Ly}{}^{y} = f_{Ly}{}^{L} = 0 \qquad f_{Ly}{}^{x} = -1$$

The components of the Killing metric are

$$g_{xx} = f_{xx}{}^{q} f_{xq}{}^{x} + f_{xy}{}^{q} f_{xq}{}^{y} + f_{xL}{}^{q} f_{xq}{}^{L} = 0$$

$$g_{xy} = g_{yx} = f_{xx}{}^{q} f_{yq}{}^{x} + f_{xy}{}^{q} f_{yq}{}^{y} + f_{xL}{}^{q} f_{yq}{}^{L} = f_{xL}{}^{y} f_{yy}{}^{L} = 0$$

$$g_{yy} = f_{yx}{}^{q} f_{yq}{}^{y} + f_{yy}{}^{q} f_{yq}{}^{y} + f_{yL}{}^{q} f_{yq}{}^{L} = 0$$

$$g_{xL} = g_{Lx} = f_{xx}{}^{q} f_{Lq}{}^{x} + f_{xy}{}^{q} f_{Lq}{}^{y} + f_{xL}{}^{q} f_{Lq}{}^{L} = 0$$

$$g_{yL} = g_{Ly} = f_{yx}{}^{q} f_{Lq}{}^{x} + f_{yy}{}^{q} f_{Lq}{}^{y} + f_{yL}{}^{q} f_{Lq}{}^{y} = 0$$

$$g_{LL} = f_{Lp}{}^{q} f_{Lq}{}^{p} = f_{Lx}{}^{q} f_{Lq}{}^{x} + f_{Ly}{}^{q} f_{Lq}{}^{y} = f_{Lx}{}^{y} f_{Ly}{}^{x} + f_{Ly}{}^{x} f_{Lx}{}^{y} = -2$$

Part (c)

The operator $P_x^2 + P_y^2$ commutes with all generators:

$$\left[P_x^2 + P_y^2, P_x\right] = P_x \left[P_x, P_x\right] + \left[P_x, P_x\right] P_x + P_y \left[P_y, P_x\right] + \left[P_y, P_x\right] P_y = 0$$

$$\left[P_x^2 + P_y^2, P_y\right] = P_x \left[P_x, P_y\right] + \left[P_x, P_x\right] P_y + P_y \left[P_y, P_y\right] + \left[P_y, P_y\right] P_y = 0$$

$$\left[P_x^2 + P_y^2, L\right] = P_x \left[P_x, L\right] + \left[P_x, L\right] P_x + P_y \left[P_y, L\right] + \left[P_y, L\right] P_y = -P_x P_y - P_y P_x + P_y P_x + P_x P_y = 0$$

Part (d)

The identity element is the matrix with $\theta = \pi$, $a = b = 0$. Matrix multiplication is associative by definition. The inverse of an arbitrary element exists and is given by

$$g^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & -a\cos\theta - b\sin\theta \\ -\sin\theta & \cos\theta & a\sin\theta - b\cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

The product of two elements is

$$gh = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & a_1 \\ \sin\theta_1 & \cos\theta_1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & a_2 \\ \sin\theta_2 & \cos\theta_2 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_1 + a_2\cos\theta_1 - b_2\sin\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & b_1 + a_2\sin\theta_1 + b_2\sin\theta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

so we have closure.

Part (e)

To construct the generators, we should consider group elements near the identity. Then for a vector of parameters α, an element in a representation R is $g(\alpha, R) = I + \alpha^{\mu} T_{\mu}^{(R)} + \ldots$. For infinitesimal θ, a, b, we can write an element as

$$\begin{pmatrix} 1 & -\theta & a \\ \theta & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we can define the generators as

$$L = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that these obey the same algebra as given in the problem statement. This explicit form also acts as a check for the Killing metric calculated in Part (b), using the definition $g_{ab} = \mathrm{Tr}\,(T_a T_b)$.

Part (f)

In the same way that we found a 3×3 matrix representation for E_2, we can find a 4×4 representation for E_3. Schematically, this looks like

$$g(\theta, a, b, c) = \begin{pmatrix} R_i(\theta) & v \\ 0 & 1 \end{pmatrix}$$

where R_i is one of the usual 3×3 rotation matrices and $v \in \mathbb{R}^3$. Performing a calculation similar to the one in Part (e) gives the generators:

$$L_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_z = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These generators obey

$$\left[L_i, L_j\right] = \varepsilon_{ijk} L_k \qquad \left[L_i, P_j\right] = \varepsilon_{ijk} P_k \qquad \left[P_i, P_j\right] = 0$$

Since $P_i^2 = 0$ for each i, then the P_i^2 trivially commute with each generator.

Part (g)

The $SO(3)$ generators are

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we rescale L_x and L_y by factors x and y respectively, then we find the following commutation relations:

$$\left[\tilde{L}_x, \tilde{L}_y\right] = xy L_z \qquad \left[\tilde{L}_x, L_z\right] = -\frac{x}{y}\tilde{L}_y \qquad \left[\tilde{L}_y, L_z\right] = \frac{y}{x}\tilde{L}_x$$

where a tilde denotes a scaled generator. If we take $x, y \to 0$ simultaneously, then these relations become

$$\left[\tilde{L}_x, \tilde{L}_y\right] = 0 \qquad \left[\tilde{L}_x, L_z\right] = -\tilde{L}_y \qquad \left[\tilde{L}_y, L_z\right] = \tilde{L}_x$$

which match the relations for E_2 if we rename $\tilde{L}_x \to P_x$, $\tilde{L}_y \to P_y$, $L_z \to L$.

Part (h)

The rank of $SO(3)$ is 1, so there is a single Casimir operator. Using the definition $C_2(R) = -\delta^{ab} T_a^{(R)} T_b^{(R)}$, we find that the Casimir operator is $C_2 = -L_x^2 - L_y^2 - L_z^2$. After contraction, this becomes $-L^2$, which no longer commutes with P_x and P_y.

Part (i)

Using the fact that $\mathfrak{so}(2,1) \simeq \mathfrak{sl}(2,\mathbb{R})$, we can find generators

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then using $g_{ab} = \mathrm{tr} T_a T_b$, we find

$$g_{ab} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

This is invertible, so we can use $C_2 = g^{ab} T_a T_b$ to find $C_2 = g^{11} T_1^2 + g^{23} T_2 T_3 + g^{32} T_3 T_2 = 3$.