PHY 611 - Homework 6

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PROBLEM 1

We were given in class that

$$\mathbf{5} = (3,1)_{-1/3} \oplus (1,2)_{1/2} \implies \bar{\mathbf{5}} = (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}$$

In $SU(5)$, $\bar{\mathbf{10}} = \bar{\mathbf{5}} \otimes \bar{\mathbf{5}}$ (see Peter Van Nieuewnhuizen's group theory notes). Calculating this:

$$\Big((\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}\Big) \otimes \Big((\bar{3},1)_{1/3} \oplus (1,2)_{-1/2}\Big) = (3,1)_{2/3} \oplus (\bar{3},2)_{1/6} \oplus (1,1)_{-1}$$

So this gives

$$\mathbf{5} \oplus \bar{\mathbf{10}} = (3,1)_{-1/3} \oplus (1,2)_{1/2} \oplus (3,1)_{2/3} \oplus (\bar{3},2)_{1/6} \oplus (1,1)_{-1}$$

which corresponds the usual right-handed fermions.

SREDNICKI 84.1

Part (a)

Substituting the expression for Φ into $V(\Phi)$ gives

$$V(\Phi) = \frac{1}{2}m^2v^2 + \frac{1}{4}\lambda_1 v^4 \sum_{i=1}^{n} a_i^4 + \frac{1}{4}\lambda_2 v^4$$

To extremize:

$$0 = \frac{\partial V}{\partial v} = m^2 v + v^3 \left(\lambda_1 \sum_{i=1}^{n} a_i^4 + \lambda_2\right) \implies v = \pm\sqrt{\frac{-m^2}{\lambda_1 \sum_{i=1}^{n} a_i^4 + \lambda_2}}$$

Substituting this back into $V(\Phi)$ gives

$$V(\Phi) = -\frac{1}{4}\frac{m^4}{\lambda_1 \sum_{i=1}^{n} a_i^4 + \lambda_2}$$

Part (b)

We found that the minimum for the potential is at

$$v = \pm\sqrt{\frac{-m^2}{\lambda_1 \sum_{i=1}^{n} a_i^4 + \lambda_2}}$$

Since $m^2 < 0$, if $\lambda_1 \sum_{i=1} a_i^4 + \lambda_2 < 0$, then the minimum is imaginary and so the potential is unbounded.

Part (c)

This is clear from the form of $V(\Phi)$ at its minimum. Making $\lambda_1 \sum_{i=1} a_i^4 + \lambda_2$ as small as possible will give $V(\Phi)$ its greatest possible magnitude (assuming m is fixed), and the overall minus sign ensures that this occurs at a minimum instead of a maximum.

Part (d)

To extremize the potential, we must extremize $\sum_{i=1}^n a_i^4$, subject to the constraints $\sum_{i=1}^n a_i^2 = 1$ and $\sum_{i=1}^n a_i = 0$. Introducing Lagrange multipliers, this gives us the function to optimize:

$$f(a) = \sum_{i=1}^{n} \left(a_i^4 + A a_i^2 + B a_i \right)$$

Taking the derivative with respect to a_i to optimize this, we will find a cubic equation to solve for each a_i, and cubics have either one or three real roots, so each a_i can have at most three distinct values, and these are still subject to the constraint that their sum should vanish.

Part (e)

SREDNICKI 86.1

Part (a)

We have two expressions for $\delta\varphi_i$:

$$\delta\varphi_i = -i\theta^a \left[\mathrm{Re}\,(T_R^a)_i^{\ j} + i\mathrm{Im}\,(T_R^a)_i^{\ j} \right] \varphi_j = -\frac{1}{\sqrt{2}} i\theta^a \left[\mathrm{Re}\,(T_R^a)_i^{\ j} + i\mathrm{Im}\,(T_R^a)_i^{\ j} \right] \left(\phi_j + i\phi_{j+d(R)} \right)$$

$$\delta\varphi_i = -\frac{1}{\sqrt{2}} i\theta^a \left[\delta\phi_i + i\delta\phi_{i+d(R)} \right]$$

These give

$$\delta\phi_i + i\delta\phi_{i+d(R)} = -i\theta^a \left[\mathrm{Re}\,(T_R^a)_i^{\ j} + i\mathrm{Im}\,(T_R^a)_i^{\ j} \right] \phi_j - i\theta^a \left[i\mathrm{Re}\,(T_R^a)_i^{\ j} - \mathrm{Im}\,(T_R^a)_i^{\ j} \right] \phi_{j+d(R)}$$

We don't want the transformations of the fields to be complex, so it helps to write this expression as a completely real term (corresponding to $\delta\phi_i$) and a completely imaginary term (corresponding to $i\delta\phi_{i+d(R)}$):

$$\delta\phi_i + i\delta\phi_{i+d(R)} = -i\theta^a \left[i\mathrm{Im}\,(T_R^a)_i^{\ j} \phi_j + i\mathrm{Re}\,(T_R^a)_i^{\ j} \phi_{j+d(R)} \right] + i(-i\theta^a) \left[i\mathrm{Im}\,(T_R^a)_i^{\ j} \phi_{j+d(R)} - i\mathrm{Re}\,(T_R^a)_i^{\ j} \phi_j \right]$$

Comparing the two individual transformation rules to the generic form gives

$$\mathcal{T}_{11}^a = i\mathrm{Im}(T_R^a) \qquad \mathcal{T}_{12}^a = i\mathrm{Re}\,(T_R^a) \qquad \mathcal{T}_{21}^a = -i\mathrm{Re}\,(T_R^a) \qquad \mathcal{T}_{22}^a = i\mathrm{Im}\,(T_R^a)$$

Part (b)

Defining $R^a := \mathrm{Re}\left(T_R^a\right)$ and $I^a := \mathrm{Im}\left(T_R^a\right)$, we can calculate $[\mathcal{T}^a, \mathcal{T}^b]$:

$$[\mathcal{T}^a, \mathcal{T}^b] = -\begin{pmatrix} I^a & R^a \\ -R^a & I^a \end{pmatrix}\begin{pmatrix} I^b & R^b \\ -R^b & I^b \end{pmatrix} - (a \leftrightarrow b) = \begin{pmatrix} [R^a, R^b] - [I^a, I^b] & -[R^a, I^b] - [I^a, R^b] \\ [R^a, I^b] + [I^a, R^b] & [R^a, R^b] - [I^a, I^b] \end{pmatrix}$$

The commutation relations for T_R^a are

$$\left[R^a + iI^a, R^b + iI^b\right] = if^{ab}{}_c\left(R^c + iI^c\right) \implies \left[R^a, R^b\right] + i\left[R^a, I^b\right] + i\left[I^a, R^b\right] - \left[I^a, I^b\right] = if^{ab}{}_c\left(R^c + iI^c\right)$$

which gives

$$\left[R^a, R^b\right] - \left[I^a, I^b\right] = -f^{ab}{}_c I^c \qquad \left[R^a, I^b\right] + \left[I^a, R^b\right] = f^{ab}{}_c R^c$$

Substituting these into our expression for $[\mathcal{T}^a, \mathcal{T}^b]$ gives

$$[\mathcal{T}^a, \mathcal{T}^b] = \begin{pmatrix} -f^{ab}{}_c I^c & -f^{ab}{}_c R^c \\ f^{ab}{}_c R^c & -f^{ab}{}_c I^c \end{pmatrix} = if^{ab}{}_c \begin{pmatrix} iI^c & iR^c \\ -iR^c & iI^c \end{pmatrix} = if^{ab}{}_c \mathcal{T}^c$$

SREDNICKI 87.3

Part (a)

The $SU(2)$ generators are $T^a = \frac{1}{2}\sigma^a$ and $Y = -\frac{1}{2}\mathbb{1}$, so using the previous problem, we find

$$\mathcal{T}^1 = \frac{i}{2}\begin{pmatrix} \mathrm{Im}(\sigma^1) & \mathrm{Re}(\sigma^1) \\ -\mathrm{Re}(\sigma^1) & \mathrm{Im}(\sigma^1) \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{T}^2 = \frac{i}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathcal{T}^3 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{Y} = \frac{i}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Part (b)

F is defined as $F^a{}_i = ig_a\left(\mathcal{T}^a\right)_{ij} v_j$. Equation (87.4) tells us that the only nonzero component of v_i is the first. Using $g_{1,2}$ as in equation (87.1), we have

$$F^1{}_i = ivg_2\left(\mathcal{T}^1\right)_{i1} = \frac{vg_2}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad F^2{}_i = ivg_2\left(\mathcal{T}^1\right)_{i1} = -\frac{vg_2}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$F^3{}_i = ivg_2\left(\mathcal{T}^3\right)_{i1} = \frac{vg_2}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \qquad F^4{}_i = ivg_1\left(\mathcal{T}^1\right)_{i1} = -\frac{vg_1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

Which give the matrix

$$F^a{}_i = \frac{v}{2}\begin{pmatrix} 0 & 0 & 0 & g_2 \\ 0 & -g_2 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & -g_1 & 0 \end{pmatrix}$$

Part (c)

The square of a matrix is (in components) $M^2_{ik} = M_{ij}M_{jk}$. Since we want to calculate ${F^a}_i {F^b}_i$, we need to calculate FF^T instead of F^2. The matrix is

$$(M^2)^{ab} = \frac{v^2}{4}\begin{pmatrix} g_2^2 & 0 & 0 & 0 \\ 0 & g_2^2 & 0 & 0 \\ 0 & 0 & g_2^2 & -g_1 g_2 \\ 0 & 0 & -g_1 g_2 & g_1^2 \end{pmatrix}$$

which has eigenvalues (by Mathematica) $(0, g_2^2, g_2^2, g_1^2 + g_2^2)$.