PHY 611 - Homework 5

M. Ross Tagaras
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SREDNICKI 32.1

Part (a)

We can find the Noether current by computing the variation of the Lagrangian under the infinitesimal $U(1)$ transformation $\delta\varphi = -i\alpha\varphi$:

$$\delta\mathcal{L} = -\partial^\mu \left(-i\alpha\varphi\right)^\dagger \partial_\mu\varphi - \partial^\mu\varphi^\dagger\partial_\mu \left(-i\alpha\varphi\right) - m^2 \left(-i\alpha\varphi\right)^\dagger \varphi - m^2\varphi^\dagger \left(-i\alpha\varphi\right) - \frac{\lambda}{2}\varphi^\dagger\varphi \left(-\alpha\varphi^\dagger\varphi - i\alpha\varphi^\dagger\varphi\right)$$

$$= \left(\partial^\mu\alpha\right) \left(-i\varphi^\dagger\partial_\mu\varphi + i\varphi\partial_\mu\varphi^\dagger\right)$$

So the Noether current is

$$J_\mu = i\left(\left(\partial_\mu\varphi^\dagger\right)\varphi - \varphi^\dagger\partial_\mu\varphi\right)$$

The charge is

$$Q = \int d^3x\ J_0 = i\int d^3x\ \left(\dot{\varphi}^\dagger\varphi - \varphi^\dagger\dot{\varphi}\right)$$

From basic quantum mechanics, we know that for operators A and B,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$$

so for our problem we have

$$e^{-i\alpha Q}\varphi e^{i\alpha Q} = \varphi + (-i\alpha)[Q, \varphi] + \frac{(-i\alpha)^2}{2!}[Q, [Q, \varphi]] + \ldots$$

Now we can calculate the commutator:

$$[\varphi, Q] = i\int d^3x\ \left\{\left[\varphi(y), \dot{\varphi}^\dagger(x)\right]\varphi(x) - \varphi^\dagger(x)\left[\varphi(y), \dot{\varphi}(x)\right]\right\} = i\int d^3x\ \left\{-\varphi(x)i\delta^3(y-x)\right\} = \varphi$$

Using this in the previous expression gives

$$e^{-i\alpha Q}\varphi e^{i\alpha Q} = \varphi - (-i\alpha)\varphi + \frac{(-i\alpha)^2}{2!}\varphi + \cdots = \varphi\left(1 + i\alpha + \frac{(i\alpha)^2}{2!} + \ldots\right) = \varphi e^{i\alpha}$$

Part (b)

We are given that

$$\frac{v}{\sqrt{2}}e^{-i\theta} = \langle\theta|\varphi|\theta\rangle$$

and using part (a), this can be rewritten as

$$\frac{v}{\sqrt{2}}e^{-i\theta} = e^{-i\alpha}\langle\theta|e^{-i\alpha Q}\varphi e^{i\alpha Q}|\theta\rangle \implies \langle\theta|e^{-i\alpha Q}\varphi e^{i\alpha Q}|\theta\rangle = \frac{v}{\sqrt{2}}e^{-i(\theta-\alpha)}$$

From this we see that

$$e^{i\alpha Q}|\theta\rangle = |\theta-\alpha\rangle$$

or, reversing the sign of α,

$$e^{-i\alpha Q}|\theta\rangle = |\theta+\alpha\rangle$$

Part (c)

The result from part (b) can be written as (for $\theta = 0$)

$$e^{-i\alpha Q}|0\rangle = |0\rangle + \left(-i\alpha + \frac{(-i\alpha)^2}{2}Q + \dots\right)Q|0\rangle = |\alpha\rangle$$

if $Q|0\rangle = 0$, then we see that $|0\rangle = |\alpha\rangle$, which is not true in general.

SREDNICKI 83.1

Part (a)

The **3** of $SO(3)$ must be a real representation. When the symmetry group was $SU(3)$, we had one Weyl spinor that transforms as **3** and one that transforms as $\bar{\mathbf{3}}$. Now, they should both transform as **3**, which results in a $U(2n_f)$ flavor symmetry, instead of a $U(n_f)\times U(n_f)$ symmetry. Removing the anomalous $U(1)_A$ symmetry from $U(2n_f)$ gives $SU(2n_f)$.

Part (b)

Since our Weyl spinors no longer transform in different representations, there is only a single transformation matrix T that applies to both spinors:

$$\chi_{\alpha i} \to T_i{}^j\chi_{\alpha j} \qquad\qquad \xi^{\alpha i} \to T^i{}_j\,\xi^{\alpha j}$$

Using these transformations in (83.7) gives

$$\langle 0|\chi_{\alpha i}\xi^{\alpha j}|0\rangle \to \langle 0|T_i{}^k\chi_{\alpha k}T^j{}_\ell\,\xi^{\alpha\ell}|0\rangle = -v^3 T_i{}^k T^j{}_\ell\,\delta^\ell_k = -v^3 T_i{}^k T^j{}_k = -v^3\left(TT^T\right)_i{}^j$$

If we want this transformation to preserve as much of the $SU(2n_f)$ symmetry as possible, then we want $\left(TT^T\right)_i{}^j = \delta^j_i$, or equivalently $TT^T = \mathbb{1}$. Since we started with $SU(2n_f)$ symmetry and not $U(2n_f)$, the largest possible unbroken subgroup is $SO(2n_f)$.

Part (c)

Goldstone's theorem tells us that there should be a massless Goldstone boson for each broken generator. $SU(2n_f)$ has $(2n_f)^2 - 1$ generators and $SO(2n_f)$ has $\frac{2n_f(2n_f-1)}{2}$ generators, so there are a total of $2n_f^2 + n_f - 1$ Goldstone bosons. For the case $n_f = 2$, we have 9 bosons.

Part (d)

We still have $SU(2n_f)$ flavor symmetry, but an $SU(2)$ singlet looks like $\varepsilon^{ab}\chi_a\xi_b$, so the condensate takes a slightly different form:

$$\langle 0|\varepsilon^{ab}\chi_{ai}\xi_{bj}|0\rangle = -v^3\Omega_{ij}$$

where Ω is an antisymmetric tensor. Transforming the spinors gives

$$\langle 0|\varepsilon^{ab}\chi_{ak}\xi_{b\ell}|0\rangle = -v^3 L_i{}^k L_j{}^\ell \Omega_{k\ell}$$

To leave the largest possible subgroup unbroken, we see that $L \in Sp(2n_f)$. $Sp(2n_f)$ has $2n_f^2 + n_f$ generators, so there are a total of $2n_f^2 - n - 1$ Goldstone bosons. For $n_f = 2$, there are $8 - 2 - 1 = 5$.

SREDNICKI 83.3

Expanding U to third order in $1/f_\pi$, which incorporates all possible terms of up to order $1/f_\pi^4$, we find

$$U = 1 + \frac{2i}{f_\pi}\pi^a T^a - \frac{2}{f_\pi^2}\pi^a T^a \pi^b T^b - \frac{4i}{3f_\pi^3}\pi^a T^a \pi^b T^b \pi^c T^c + \dots$$

Substituting this into the Lagrangian and ignoring cross-terms of higher order gives

$$\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}\left\{\frac{4}{f_\pi^2}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)T^a T^b + \frac{16}{3f_\pi^4}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\pi^c\pi^d T^a T^c T^b T^d - \frac{16}{3f_\pi^4}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\pi^c\pi^d T^a T^b T^c T^d\right\}$$

$$= -\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\mathrm{Tr}\left(T^a T^b\right) + \frac{4}{3f_\pi^2}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\pi^c\pi^d\left[\mathrm{Tr}\left(T^a T^b T^c T^d\right) - \mathrm{Tr}\left(T^a T^c T^b T^d\right)\right]$$

Some identities for the Pauli matrices give

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}$$

$$\mathrm{Tr}\left(T^a T^b T^c T^d\right) = \frac{1}{8}\left(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right)$$

so the expression for the Lagrangian simplifies to

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^a\right) + \frac{1}{6f_\pi^2}\left[\left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^a\right)\pi^b\pi^b - \left(\partial_\mu \pi^a\right)\left(\partial^\mu \pi^b\right)\pi^a\pi^b\right]$$

SREDNICKI 83.6

The mass Lagrangian is

$$\mathcal{L}_m = -\frac{4v^3}{f_\pi^2}\mathrm{Tr}\left(MT^aT^b\right)\pi^a\pi^b = -4v^3\mathrm{Tr}\left(M\Pi^2\right) =$$

$$= -\frac{4v^3}{2f_\pi^2}\left((m_d+m_s)K^0\bar{K}^0 + (m_u+m_s)K^+K^- + \frac{2m_s}{3}\eta^2 + (m_u+m_d)\pi^+\pi^- + \frac{m_d}{2}\left(\frac{1}{\sqrt{3}}\eta - \pi^0\right)^2 + \frac{m_u}{2}\left(\frac{1}{\sqrt{3}}\eta + \pi^0\right)^2\right)$$

$$= -\frac{1}{2}\frac{4v^3}{f_\pi^2}\Bigg((m_d+m_s)K^0\bar{K}^0 + (m_u+m_s)K^+K^- + (m_u+m_d)\pi^+\pi^-$$

$$+\frac{1}{3}\left(2m_s + \frac{m_u+m_d}{2}\right)\eta^2 + \frac{1}{2}(m_u+m_d)\pi^0\pi^0 + \frac{1}{\sqrt{3}}(m_u-m_d)\eta\pi^0\Bigg)$$

I have pulled out an overall factor of 1/2 so that the mass terms are normalized like $-\frac{m^2}{2}\pi^a\pi^a$. I'm not sure if this is the standard convention for mesons or not, so some intermediate answers may be off by a factor. This shouldn't affect the results of parts (c) and (d) though, since any normalization factors will fall out. The terms in the first line give

$$m_{K^0}^2 = \frac{4v^3}{f_\pi^2}(m_d+m_s) \qquad m_{K^\pm}^2 = \frac{4v^3}{f_\pi^2}(m_u+m_s) \qquad m_{\pi^\pm}^2 = \frac{4v^3}{f_\pi^2}(m_u+m_d)$$

To find the masses of η and π^0, we need to deal with the term proportional to $\eta\pi^0$. We can write the terms in the second line as

$$\begin{pmatrix}\eta & \pi^0\end{pmatrix}\begin{pmatrix}\frac{8v^3m_s}{3f_\pi^2} + \frac{2v^3(m_u+m_d)}{3f_\pi^2} & \frac{2v^3}{\sqrt{3}f_\pi^2}(m_u-m_d) \\ \frac{2v^3}{\sqrt{3}f_\pi^2}(m_u-m_d) & \frac{2v^3}{f_\pi^2}(m_u+m_d)\end{pmatrix}\begin{pmatrix}\eta \\ \pi^0\end{pmatrix}$$

The mass matrix can be brought to diagonal form (with Mathematica):

$$\begin{pmatrix}\eta & \pi^0\end{pmatrix}\left(\frac{8v^3}{3Af_\pi^2}\right)\begin{pmatrix}B+C+(m_u+m_d+m_s)A & 0 \\ 0 & -B-C+(m_u+m_d+m_s)A\end{pmatrix}\begin{pmatrix}\eta \\ \pi^0\end{pmatrix}$$

where $B = m_u^2+m_d^2+m_s^2$, $C = -m_um_s - m_dm_s - m_um_d$, and $A = \sqrt{B+C}$. This can be simplified a bit:

$$\begin{pmatrix}\eta & \pi^0\end{pmatrix}\left(\frac{8v^3}{3f_\pi^2}\right)\begin{pmatrix}\sqrt{B+C}+(m_u+m_d+m_s) & 0 \\ 0 & -\sqrt{B+C}+(m_u+m_d+m_s)\end{pmatrix}\begin{pmatrix}\eta \\ \pi^0\end{pmatrix}$$

From this, we can read off the last two masses:

$$m_\eta^2 = \left(\frac{8v^3}{3f_\pi^2}\right)\left[(m_u+m_d+m_s)+\sqrt{B+C}\right] \qquad m_{\pi^0}^2 = \left(\frac{8v^3}{3f_\pi^2}\right)\left[(m_u+m_d+m_s)-\sqrt{B+C}\right]$$

In the limit that $m_{u,d} \ll m_s$, the first three masses are unchanged, bu the remaining two will be affected. Expanding $\sqrt{B+C}$ gives

$$\sqrt{m_u^2+m_d^2+m_s^2-m_um_s-m_dm_s-m_um_d} = m_s - \frac{m_u+m_d}{2} + \ldots$$

Substituting this into $m_{\pi^0,\eta}^2$ gives

$$m_{\pi^0}^2 = \frac{4v^3}{f_\pi^2}(m_u+m_d) \qquad m_\eta^2 = \frac{4v^3}{3f_\pi^2}(m_u+m_d+4m_s)$$

Part (b)

Defining $\alpha = 4v^3/f_\pi^2$ and $i = m_i$, the expressions for the masses are

$$m_{K^0}^2 = \alpha(d+s) \qquad m_{K^\pm}^2 = \alpha(u+s)+2\Delta \qquad m_{\pi^\pm}^2 = \alpha(u+d)+\Delta \qquad m_{\pi^0}^2 = \alpha(u+d) \qquad m_\eta = \frac{\alpha}{3}(u+d+4s)$$

The solution to this system is

$$\Delta = m_{\pi^\pm}^2 - m_{\pi^0}^2 = (0.140 \text{ GeV})^2 - (0.135 \text{ GeV})^2 = 0.001375 \text{ GeV}^2$$

$$\frac{\alpha u}{4} = \frac{1}{8}(m_{\pi^0}^2 - m_{K^0}^2 + m_{K^\pm}^2 - 2\Delta) = 0.001438 \text{ GeV}^2$$

$$\frac{\alpha d}{4} = \frac{1}{8}(m_{\pi^0}^2 + m_{K^0}^2 - m_{K^\pm}^2 + 2\Delta) = 0.003118 \text{ GeV}^2$$

$$\frac{\alpha s}{4} = \frac{1}{8}(m_{K^\pm}^2 - \Delta + m_{K^0}^2 - m_{\pi^\pm}^2) = 0.058882 \text{ GeV}^2$$

Part (c)

$$m_u/m_d = \frac{0.001438}{0.003118} = 0.461 \qquad m_s/m_d = \frac{0.058882}{0.003118} = 18.885$$

Part (d)

$$m_\eta^2 = \frac{\alpha}{3}(u+d+4s) = \frac{4}{3}(0.001438 + 0.003118 + 4 \times 0.058882) \text{ GeV}^2 = 0.320112 \text{ GeV}^2 \implies m_\eta = 0.5657 \text{ GeV}$$

The measured value is 547.86 MeV, so this prediction is only off by 3.25%.

SREDNICKI 83.7

Part (a)

To find the kinetic term for $\pi^9(x)$, we expand U to first order and ignore terms with $\pi^a(x)$, since they will cancel when we multiply by the hermitian conjugate.

$$U(x) = 1 + \frac{i}{f_9}\pi^9(x) + O(1/f_9^2)$$

The Lagrangian for $\pi^9(x)$ is

$$\mathcal{L}_9 = -\frac{f_\pi^2}{4}\text{Tr}\left[\partial^\mu\left(1 - \frac{i}{f_9}\pi^9\right)\partial_\mu\left(1 + \frac{i}{f_9}\pi^9\right)\right] - \frac{F^2}{4}\partial^\mu\left[\det\left(1 - \frac{i}{f_9}\pi^9\right)\right]\partial^\mu\left[\det\left(1 + \frac{i}{f_9}\pi^9\right)\right] + O(1/f_9^4)$$

$$= \left[-\frac{f_\pi^2 \mathrm{Tr}(\mathbb{1})}{4f_9^2} - \frac{F^2 \mathrm{Tr}^2(\mathbb{1})}{4f_9^2} \right] \left(\partial_\mu \pi^9 \right) \left(\partial^\mu \pi^9 \right) + O(1/f_9^4)$$

To have the standard normalization, we should have

$$-\frac{1}{2} = -\frac{f_\pi^2}{f_9^2} - \frac{4F^2}{f_9^2} \implies F^2 = \frac{f_9^2 - 2f_\pi^2}{8}$$

Part (b)

The mass term is

$$\mathcal{L}_m = v^3 \mathrm{Tr} \left(MU + M^\dagger U^\dagger \right)$$

$$= v^3 \mathrm{Tr} \left[M \left(1 + \frac{2i}{f_\pi} \pi^a T^a + \frac{i}{f_9} \pi^9 - \frac{2}{f_\pi^2} \pi^a \pi^b T^a T^b - \frac{1}{2f_9^2} \left(\pi^9 \right)^2 \right) + M \left(1 - \frac{2i}{f_\pi} \pi^a T^a - \frac{i}{f_9} \pi^9 - \frac{2}{f_\pi^2} \pi^a \pi^b T^a T^b - \frac{1}{2f_9^2} \left(\pi^9 \right)^2 \right) \right]$$

$$= -\frac{4v^3}{f_\pi^2} \pi^a \pi^b \mathrm{Tr} \left(M T^a T^b \right) - \frac{v^3}{f_9^2} \left(\pi^9 \right)^2 \mathrm{Tr}(M)$$

Using the matrix Π from the previous problem, the mass Lagrangian becomes

$$\mathcal{L}_m = -4v^3 \mathrm{Tr} \left(M \Pi^2 \right) - \frac{v^3}{f_9^2} \mathrm{Tr}(M) \pi^9 \pi^9$$

Comparing to the previous mass Lagrangian, it is clear that only the terms on the diagonal of Π can be affected by the introduction of π^9. The trace is

$$-v^3 \left[\frac{(2m + m_s)\pi^9 \pi^9}{f_9^2} + \frac{4}{3f_\pi^2} \eta^2 + \frac{m}{f_\pi^2} \left(\frac{1}{\sqrt{3}} \eta - \pi^0 \right)^2 + \frac{m}{f_\pi^2} \left(\frac{1}{\sqrt{3}} \eta + \pi^0 \right)^2 \right]$$

which gives masses

$$m_{\pi^0}^2 = \frac{4v^3 m}{f_\pi^2} \qquad m_{\pi^9}^2 = \frac{2v^3 (2m + m_s)}{f_9^2} \qquad m_\eta^2 = \frac{4v^3}{3f_\pi^2} (2m_s + m)$$