PHY 611 - Homework 8

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SREDNICKI 75.1

The anomaly coefficient (which should vanish) is given by

$$A(R)d^{abc} = \frac{1}{2}\text{Tr}(\{T^a_R, T^b_R\}T^c_R)$$

We have a theory with fields transforming under $U(1)$ with charge Q_i and also under a nonabelian symmetry in the representation $R = \bigoplus_i R_i$, so we should consider a combination of the generators of each symmetry in our expression for $A(R)$.

We also know that the nonabelian generator for R will be block-diagonal, and each individual generator of R_i will be traceless (see pg. 423). The generators of $U(1)$ will be $dim(R) \times dim(R)$ diagonal matrices.

If the 3 T's are all $U(1)$ generators, then we have

$$\text{Tr}(Q^3) = 2\sum_{i=1}^{n} dim(R_i)Q_i^3$$

This sum must vanish for $A(R)$ to vanish. If each T is a nonabelian generator, since the trace of a block-diagonal matrix is the sum of the traces of the blocks, we have

$$\text{Tr}\left(\{T^a_R, T^b_R\}T^c_R\right) = \sum_{i=1}^{n} \text{Tr}\left(\{T^a_{R_i}, T^b_{R_i}\}T^c_{R_i}\right) = \sum_{i=1}^{n} A(R_i)d^{abc}$$

so that for the entire anomaly coefficient to vanish, the sum of the coefficients for each R_i must also vanish. If we have two $U(1)$ generators and one nonabelian generator, then we have

$$\text{Tr}\left(T^a_R Q^2\right) = 2\sum_{i=1}^{n} Q_i^2 \text{Tr}(T^a_{R_i}) = 0$$

since the nonabelian generators are traceless. We find, therefore, that in this case, A always vanishes.

The case with two nonabelian generators and one $U(1)$ generator initially seems more complicated than the previous case. There, the anticommutator was always trivial, but now we must decide whether to put Q inside the anticommutator or have the nontrivial anticommutator of two nonabelian generators. However, by cyclicity of the trace, we have

$$\text{Tr}\left(\{T^a_R, T^b_R\}Q\right) = \text{Tr}\left(T^a_R T^b_R Q + T^b_R T^a_R Q\right) = \text{Tr}\left(Q T^a_R T^b_R + T^a_R Q T^b_R\right) = \text{Tr}\left(\{Q, T^a_R\}T^b_R\right)$$

It is more convenient to put both nonabelian generators in the anticommutator, because we can use the definition of $T(R)$ and the (block) diagonality of the generators:

$$\text{Tr}\left(\{T^a_R, T^b_R\}Q\right) = \sum_{i=1}^{n} Q_i T(R_i)\delta^{ab}$$

As before, this sum should vanish.

SREDNICKI 76.1

We want to calculate the following matrix element:

$$\langle p,q| - \frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}|0\rangle = -\frac{g^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\langle p,q|(\partial_\mu A_\nu)(\partial_\rho A_\sigma)|0\rangle$$

Using the expansion from chapter 55 gives

$$-\frac{g^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\sum_{\lambda,\lambda'}\int \widetilde{dk}\widetilde{dk'}\langle 0|a_\lambda(p)a_{\lambda'}(q)\partial_\mu\left(\varepsilon_{\lambda\nu}(k)a^\dagger_\lambda(k)e^{-ikx}\right)\partial_\rho\left(\varepsilon_{\lambda'\sigma}(k')a^\dagger_{\lambda'}(k')e^{-ik'x}\right)+\ldots|0\rangle$$

where ... denotes terms that will vanish when we compute their contribution to the complete matrix element. Taking the derivatives gives

$$-\frac{g^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\sum_{\lambda,\lambda'}\int \widetilde{dk}\widetilde{dk'}\langle 0|a_\lambda(p)a_{\lambda'}(q)\varepsilon_{\lambda\nu}(k)a^\dagger_\lambda(k)(-ik_\mu)\varepsilon_{\lambda'\sigma}(k')a^\dagger_{\lambda'}(-ik_\rho)e^{-ix(k+k')}|0\rangle$$

$$= -\frac{g^2}{2\pi^2}\varepsilon^{\mu\nu\rho\sigma}p_\mu q_\rho\varepsilon_\nu\varepsilon'_\sigma e^{-ix(p+q)} = \frac{g^2}{2\pi^2}\varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\varepsilon_\mu\varepsilon'_\nu e^{-ix(p+q)}$$

This matches equation (76.29), as expected.

SREDNICKI 77.1

Restoring the gauge indices and generators to equation(77.35) gives (up to an overall factor)

$$\mathrm{Tr}\left[\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A^b_\nu\right)\left(\partial_\rho A^c_\sigma\right)T^a_RT^b_RT^c_R - \frac{ig}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^b_\nu A^c_\rho A^d_\sigma\right)T^a_RT^b_RT^c_RT^d_R\right]$$

Using the results of chapter 70, we can write $\mathrm{Tr}(T^a_RT^b_RT^c_R)$ as

$$\mathrm{Tr}(T^a_RT^b_RT^c_R) = \frac{1}{2}\mathrm{Tr}(2T^a_RT^b_RT^c_R + T^a_RT^c_RT^b_R - T^a_RT^c_RT^b_R) = \frac{1}{2}\mathrm{Tr}\left(T^a_R\left\{T^b_R,T^c_R\right\} + T^a_R\left[T^b_R,T^c_R\right]\right)$$

$$= A(R)d^{abc} + \frac{i}{2}T(R)f^{abc}$$

We can also rewrite $\mathrm{Tr}(T^a_RT^b_RT^c_RT^d_R)$:

$$\mathrm{Tr}(T^a_RT^b_RT^c_RT^d_R) = \frac{1}{2}\mathrm{Tr}\left(T^a_RT^b_R\left\{T^c_R,T^d_R\right\}\right) + \frac{1}{2}\mathrm{Tr}\left(T^a_RT^b_R\left[T^c_R,T^d_R\right]\right)$$

Using the definition of the Lie algebra structure constants gives

$$\mathrm{Tr}(T^a_RT^b_RT^c_RT^d_R) = \frac{1}{2}\mathrm{Tr}\left(T^a_RT^b_R\left\{T^c_R,T^d_R\right\}\right) + \frac{1}{2}\mathrm{Tr}\left(T^a_RT^b_RT^e_Rf^{cde}\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left(T^a_RT^b_R\left\{T^c_R,T^d_R\right\}\right) + \frac{1}{2}A(R)f^{cde}d^{abe} + \frac{i}{4}T(R)f^{abe}f^{cde}$$

Now our complete expression is

$$\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu^b)(\partial_\rho A_\sigma^c)\left(A(R)d^{abc}+\frac{i}{2}T(R)f^{abc}\right)$$
$$-\frac{ig}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu^b A_\rho^c A_\sigma^d\right)\left[\frac{1}{2}\mathrm{Tr}\left(T_R^a T_R^b\left\{T_R^c,T_R^d\right\}\right)+\frac{1}{2}A(R)f^{cde}d^{abe}+\frac{i}{4}T(R)f^{abe}f^{cde}\right]$$

The second term on the first line and the first term on the second line vanish by symmetry and the last term on the second line vanishes by the Jacobi identity, so we are left with

$$\varepsilon^{\mu\nu\rho\sigma}A(R)\left[(\partial_\mu A_\nu^b)(\partial_\rho A_\sigma^c)d^{abc}-\frac{ig}{4}d^{abe}f^{cde}\partial_\mu\left(A_\nu^b A_\rho^c A_\sigma^d\right)\right]$$

which vanishes when $A(R)=0$. The term in brackets is nonzero, in general.

SREDNICKI 77.2

Absorbing a factor of ig into the gauge field allows us to rewrite the right side of equation (77.36) as

$$\frac{1}{2\pi^2}\mathrm{dTr}\left[A\wedge \mathrm{d}A-\frac{2}{3}A\wedge A\wedge A\right]$$

where d is the gauge covariant exterior derivative and A is the one-form gauge field. Similarly, we can rewrite the right side of equation (77.7) as

$$\frac{1}{2\pi^2}\mathrm{Tr}(F\wedge F)$$

where $F=\mathrm{d}A-A\wedge A$. Distributing the exterior derivative in the first expression gives

$$\frac{1}{2\pi^2}\mathrm{Tr}\left[\mathrm{d}A\wedge\mathrm{d}A-A\wedge\mathrm{d}^2A-\frac{2}{3}\left(\mathrm{d}A\wedge A\wedge A-A\wedge\mathrm{d}A\wedge A+A\wedge A\wedge\mathrm{d}A\right)\right]=\frac{1}{2\pi^2}\mathrm{Tr}\left[\mathrm{d}A\wedge\mathrm{d}A-2\mathrm{d}A\wedge A\wedge A\right]$$

$$=\frac{1}{2\pi^2}\mathrm{Tr}\left[(\mathrm{d}A-A\wedge A)\wedge(\mathrm{d}A-A\wedge A)-A\wedge A\wedge A\wedge A\right]$$

The trace of a p-form and a q-form acts as

$$\mathrm{Tr}(\omega_{(p)}\wedge\eta_{(q)})=(-1)^{pq}\mathrm{Tr}(\eta_{(q)}\wedge\omega_{(p)})$$

so the final term in our trace is

$$\mathrm{Tr}[(A)\wedge(A\wedge A\wedge A)]=(-1)^3\mathrm{Tr}[(A\wedge A\wedge A)\wedge(A)]$$

which must vanish. The definition of F then gives

$$\frac{1}{2\pi^2}\mathrm{Tr}(F\wedge F)$$

which is what we wanted to show.

SREDNICKI 89.3

This problem is an application of the results we found in Exercise 75.1. A useful fact is that the anomaly coefficient vanishes for real and pseudoreal reps. We also know that if we have pairs of fields in the representation $R \oplus \bar{R}$, then the anomaly will cancel. One generation of the fermions of the standard model are in the representation $(1, 2, -1/2) \oplus (1, 1, 1) \oplus (3, 2, 1/6) \oplus (\bar{3}, 1, -2/3) \oplus (\bar{3}, 1, 1/3)$.

We have a doublet of fields in the $\mathbf{3}$ and two singlets in the $\bar{\mathbf{3}}$, so the $(3, 3, 3)$ anomaly will cancel. The $(2, 2, 2)$ anomaly vanishes since the $\mathbf{2}$ of $SU(2)$ is pseudoreal. For the $(3, 3, 1)$ and the $(2, 2, 1)$ anomalies, in general we need $\sum_{i=1} Q_i T(R_i) = 0$, but since $T(\mathbf{3}) = T(\bar{\mathbf{3}}) = T(\mathbf{2}) = 1/2$, we just need $\sum_i Q_i = 0$. For the $(3, 3, 1)$, we have $3 \times 1/6 + 1/6 - 2/3 + 1/3 = 0$, and for the $(2, 2, 1)$ we have $-1/2 + 3 \times (1/6) = 0$ so both of these anomalies cancel. For the $(1, 1, 1)$ anomaly, we want $\sum_i dim(R_i) Q_i^3 = 0$. In this case, we have $2 \times (-1/2)^3 + 1 + 6 \times (1/6)^3 + 3 \times (-2/3)^3 + 3 \times (1/3)^3 = 0$.

The other possibilities are $(3, 3, 2), (3, 2, 2), (3, 2, 1), (3, 1, 1)$, and $(2, 1, 1)$. $(3, 1, 1)$ and $(2, 1, 1)$ vanish by our previous results for two abelian generators. $(3, 3, 2)$ and $(2, 2, 3)$ vanish in a similar manner. For $(3, 2, 1)$, no matter how we arrange the terms in the trace, we will end up with a trace of either an $S(2)$ or an $SU(3)$ generator, both of which are zero.

PROBLEM 6

We consider the massless $2d$ QED action:

$$S = \int d^2x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\psi} \not{D} \psi \right]$$

Under an axial transformation $\psi \to e^{-i\alpha\gamma_*}\psi$, the measure in the path integral transforms as

$$[D\bar{\psi}][D\psi] \to \exp\left[2i \int d^2x \; \alpha \mathrm{Tr}(\delta^2(0)\gamma_*) \right] [D\bar{\psi}][D\psi]$$

To regularize the integral over the trace, we introduce a parameter r that will go to zero after calculations are complete and multiply the delta function by $e^{(ir\not{D})^2}$, as Srednicki does in chapter 77:

$$\delta^2(x-y) \to e^{(ir\not{D})^2} \delta^2(x-y) = \int \frac{d^2k}{(2\pi)^2} e^{(ir\not{D})^2} e^{-k(x-y)}$$

Expanding the square in the first exponential gives

$$(i\not{D})^2 = -\frac{1}{2} \left(\{\gamma^\mu, \gamma^\nu\} + 2\gamma^{\mu\nu} \right) D_\mu D_\nu = -\frac{1}{2} D^2 + ig\gamma^{\mu\nu} F_{\mu\nu}$$

Now our expression for the delta function is

$$\int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} e^{-\frac{r^2}{2}(D-k)^2} e^{ir^2 g\gamma^{\mu\nu} F_{\mu\nu}}$$

Taking $k \to k/r$ gives

$$\frac{1}{r^2} \int \frac{d^2k}{(2\pi)^2} e^{i\frac{k}{r}(x-y)} e^{-\frac{r^2}{2}(D-\frac{k}{r})^2} e^{ir^2 g\gamma^{\mu\nu} F_{\mu\nu}}$$

This transformation gives a new expression for $\mathrm{Tr}(\delta^2(0)\gamma_*)$:

$$\mathrm{Tr}\left\{\frac{1}{r^2}\left[1+\left(-\frac{r^2}{2}\left(D-\frac{k}{r}\right)^2-ir^2g\gamma^{\mu\nu}F_{\mu\nu}\right)+\frac{1}{2!}\left(-\frac{r^2}{2}\left(D-\frac{k}{r}\right)^2-ir^2g\gamma^{\mu\nu}F_{\mu\nu}\right)^2+\ldots\right]\gamma_*\right\}$$

The first and second terms, as well as the term proportional to k^4 are proportional to $\mathrm{Tr}\gamma_*=0$. Most of the other terms are of order r or greater, and will vanish when we take $r\to 0$. The only remaining term is

$$-igF_{\mu\nu}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma_*\right)$$

To calculate this trace, we can use the representation $\gamma^0=\sigma_1$, $\gamma^1=\sigma_3$, which gives $\gamma_*=i\sigma_1\sigma_3=\sigma_2$. The trace is then

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\sigma_2)=\epsilon^{\mu\nu}$$

Using this in the transformation of the measure, we find

$$\exp\left[i\int\frac{d^2x}{(2\pi)^2}e^{-k^2/2}(-ig)\epsilon^{\mu\nu}F_{\mu\nu}\right]=\exp\left[\frac{g}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\right]$$

Comparing this to the usual expression for the path integral tells us that

$$\langle\partial_\mu j_A^\mu\rangle=\frac{g}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$