PHY 611 - Homework 3

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SREDNICKI 62.2

In ch. 62, Srednicki does all the required calculations with $\xi = 1$, so we can adapt some of his results to save calculation time. The first thing to notice is that the one-loop correction to the photon propagator is independent of the choice of ξ, so we can reuse the result

$$Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon}$$

The fermion propagator gets an extra term:

$$(\xi - 1)e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\not\ell(-\not p - \not\ell + m)\not\ell}{\left[(p+\ell)^2 + m^2\right]\ell^4}$$

We can combine the denominators using the identity

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{\left[xA + yB + (1-x-y)C\right]^3}$$

which gives

$$2(\xi - 1)e^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{\not\ell(-\not p - \not\ell + m)\not\ell}{[xp^2 + 2x\ell p + m^2 x + \ell^2]^3}$$

Defining $q = \ell + xp$ and evaluating the integral over y gives

$$2(\xi - 1)e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{(\not q - x\not p)\left(-\not p - (\not q - x\not p) + m\right)(\not q - x\not p)(1-x)}{\left[q^2 + p^2(x - x^2) + m^2 x\right]^3}$$

Now we need to simplify the numerator a bit. If we neglect terms with odd powers of $\not q$, which will vanish upon integrating, and terms with no $\not q$, which are finite, we can expand the terms with gamma matrices as

$$(x-1)\gamma^\mu\gamma^\nu\gamma^\alpha q_\mu q_\alpha p_\nu - mq^2 + x\gamma^\mu\gamma^\nu\gamma^\alpha q_\nu q_\alpha p_\mu - xq^2\not p$$

Using the identity

$$q^\mu q^\nu = \frac{q^2}{d}\eta^{\mu\nu}$$

we can rearrange this expression to

$$q^2\left[\left(\frac{2}{d}(1-x) - (1+x)\right)\not p - m\right]$$

Evaluating the integral over q in $4 - \epsilon$ dimensions (finite terms are dropped):

$$\int \frac{d^d q}{(2\pi)^d} \frac{\mu^\epsilon q^2}{(q^2+D)^3} = \frac{\mu^\epsilon \Gamma(2-d/2\Gamma(1+d/2)}{2\Gamma(d/2)(4\pi)^{d/2}D^{2-d/2}} = \frac{\mu^\epsilon (4-\epsilon)\Gamma(\epsilon/2)}{4(4\pi)^{2-\epsilon/2}D^{\epsilon/2}} = \frac{(4-\epsilon)\left(\frac{2}{\epsilon}-\gamma\right)}{64\pi^2}\left(\frac{4\pi\mu^2}{D}\right)^{\epsilon/2} = \frac{1}{8\pi^2\epsilon}$$

Combining the last two results, our new term is

$$(1-\xi)e^2 \int_0^1 dx \frac{\left[\left(\frac{1}{2}(1-x)-(1+x)\right)\not{p}-m\right](1-x)}{4\pi^2\epsilon} = \frac{(1-\xi)e^2(m+\not{p})}{8\pi^2\epsilon}$$

Equation 62.33 tells us that the full expression for $\Sigma(p)$ is

$$\Sigma(p) = -\frac{e^2}{8\pi^2\epsilon}(\not{p}+4m) + \frac{(1-\xi)e^2(\not{p}+m)}{8\pi^2\epsilon} - (Z_2-1)\not{p} - (Z_m-1)m + O(e^4)$$

So the general results for Z_2 and Z_m are

$$Z_2 = 1 - \frac{\xi e^2}{8\pi^2\epsilon} \qquad Z_m = 1 - \frac{(\xi+3)e^2}{8\pi^2\epsilon}$$

Next we need to evaluate the additional term in the photon-fermion-fermion vertex. Comparing to Srednicki's result, we can see that the divergent part of this diagram will not depend on external momenta, so we can set them equal to zero to get

$$iV_\xi^\mu = (\xi-1)e^3 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho \frac{m-\not{\ell}}{\ell^2+m^2}\gamma^\mu \frac{m-\not{\ell}}{\ell^2+m^2}\gamma^\nu\right]\frac{\ell_\nu \ell_\rho}{\ell^4}$$

$$= 6(\xi-1)e^3 \int \frac{d^4\ell}{(2\pi)^4}\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{\gamma^\rho(m-\not{\ell})\gamma^\mu(m-\not{\ell})\gamma^\nu \ell_\nu \ell_\rho}{\left[x(\ell^2+m^2)+y(\ell^2+m^2)+z\ell^2+(1-x-y-z)\ell^2\right]^4}$$

$$= 6(\xi-1)e^3 \int \frac{d^4\ell}{(2\pi)^4}\int_0^1 dx \int_0^{1-x} dy \frac{\gamma^\rho(m-\not{\ell})\gamma^\mu(m-\not{\ell})\gamma^\nu \ell_\nu \ell_\rho(1-x-y)}{\left[\ell^2+(x+y)m^2\right]^4}$$

Expanding the numerator as before, and dropping terms that are odd in ℓ, we find

$$m^2\gamma^\rho\gamma^\mu\not{\ell}\ell_\rho - \ell^2\gamma^\rho\not{\ell}\gamma^\mu\ell_\rho$$

and using the same identity as before and some gamma matrix manipulation, this can be rewritten as

$$\left(\frac{d-2}{d}m^2\ell^2+\ell^4\right)\gamma^\mu$$

The entire integral in $4-\epsilon$ dimensions is then

$$6(\xi-1)e^3\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d}\frac{\mu^\epsilon\left(\frac{d-2}{d}m^2\ell^2+\ell^4\right)\gamma^\mu(1-x-y)}{\left[\ell^2+(x+y)m^2\right]^4}$$

After using the usual expression to evaluate the integral over ℓ, simplifying the resulting gamma functions, and plugging in $d=4-\epsilon$, we find

$$6(\xi-1)e^3\int_0^1 dx\int_0^{1-x}dy\left(\frac{\mu^\epsilon m^2(d-2)\Gamma(1+\frac{\epsilon}{2})}{12(4\pi)^{2-\epsilon/2}\left[(x+y)m^2\right]^{1+\epsilon/2}}+\frac{\mu^\epsilon(4-\epsilon)(6-\epsilon)\Gamma(\epsilon/2)}{24(4\pi)^{2-\epsilon/2}\left[(x+y)m^2\right]^{\epsilon/2}}\right)$$

Now we notice that when $\epsilon\to 0$, $\Gamma(1+\epsilon/2)$ is finite, so the entire first term can be ignored. Making the usual approximations, the integral is

$$(\xi-1)e^2\int_0^1 dx\int_0^{1-x}dy\frac{(4-\epsilon)(6-\epsilon)\left(\frac{2}{\epsilon}-\gamma\right)}{64\pi^2}\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{(x+y)m^2}\right)\right)\gamma^\mu(1-x-y)$$

$$=(\xi-1)e^3\int_0^1 dx\int_0^{1-x}dy\frac{3(1-x-y)\gamma^\mu}{4\pi^2\epsilon}=\frac{(\xi-1)e^3\gamma^\mu}{8\pi^2\epsilon}$$

Finally, we can compare to equations (62.39) and (62.49) to find

$$Z_1=1-\frac{e^2\xi}{8\pi^2\epsilon}$$

As required, $Z_1=Z_2$ for any choice of ξ, and in particular, when $\xi=0$, we see that $Z_1=Z_2=1+O(e^4)$.

SREDNICKI 67.2

This follows immediately from Problem 74.1. In the S-matrix, we are replacing the state $a^\dagger a^\dagger|\Omega\rangle$ with $a^\dagger\tilde{a}^\dagger|\Omega\rangle$, where

$$\tilde{a}^\dagger=-i\left(0+k^\mu\right)\int d^3x e^{ikx}\overleftrightarrow{\partial_0}A_\mu(x)$$

We know that the second term creates a state that is BRST exact, and the first term gives us a zero matrix element, so the entire expression vanishes. The problem statement says to show this *explicitly* though, so I'll do the calculation too. First, we make the substitution $\varepsilon_1^\mu\to k_1^\mu$:

$$\tilde{\mathcal{T}}=e^2\bar{v}_2\left[\not{\varepsilon}_2\left(\frac{-\not{p}_1+\not{k}_1+m}{m^2-t}\right)\not{k}_1+\not{k}_1\left(\frac{-\not{p}_1+\not{k}_2+m}{m^2-u}\right)\not{\varepsilon}_2\right]u_1$$

In the first term in parenthesis, the mass-shell condition allows us to set $\not{k}_1\not{k}_1=-k_1^2=0$. In the second, conservation of momentum tells us (see fig. 59.1) that $p_1-k_2=-p_2+k_1$. Multiplying by k_1 and rearranging, we find $-k_1p_1+k_1k_2=k_1p_2$. Now, we can write $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}}=e^2\bar{v}_2\left[\not{\varepsilon}_2\left(\frac{-\not{p}_1+m}{m^2-t}\right)\not{k}_1+\not{k}_1\left(\frac{\not{p}_2+m}{m^2-u}\right)\not{\varepsilon}_2\right]u_1$$

Chapter 46 now motivates us to use the Dirac equation:

$$(\not{p}+m)u=\bar{v}(-\not{p}+m)=0$$

However, we need to do some rearranging first. We want to move $\not{k}_1$ to second position in the first term, and to the second to last position of the third term. The first term is

$$-\not{\epsilon}_2 \not{p}_1 \not{k}_1 = -\not{\epsilon}_2 \gamma^\nu \gamma^\rho k_\rho p_\nu = \not{\epsilon}_2 \left(2\eta^{\rho\nu} + \gamma^\rho \gamma^\nu\right) k_\rho p_\nu = 2\not{\epsilon}_2 (k \cdot p) + \not{\epsilon}_2 \not{k}_1 \not{p}_1$$

and the second term is

$$\not{k}_1 \not{p}_2 \not{\epsilon}_2 = -2(k_1 \cdot p_2)\not{\epsilon}_2 - \not{p}_2 \not{k}_1 \not{\epsilon}_2$$

Now,

$$\frac{\tilde{\mathcal{T}}}{e^2} = \bar{v}_2 \left[\not{\epsilon}_2 \left(\frac{\not{k}_1(\not{p}_1 + m) + 2(k_1 \cdot p_1)}{m^2 - t}\right) + \left(\frac{(-\not{p}_2 + m)\not{k}_1 - 2(k_1 \cdot p_2)}{m^2 - u}\right) \not{\epsilon}_2\right] u_1 = \bar{v}_2 \left[\not{\epsilon}_2 \left(\frac{2(k_1 \cdot p_1)}{m^2 - t}\right) - \left(\frac{2(k_1 \cdot p_2)}{m^2 - u}\right) \not{\epsilon}_2\right] u_1$$

Using equation (59.2), we see that

$$\tilde{\mathcal{T}} = e^2 \bar{v}_2 \left[\not{\epsilon}_2 \left(\frac{2(k_1 \cdot p_1)}{m^2 + p_1^2 + k_1^2 - 2(k_1 \cdot p_1)}\right) - \left(\frac{2(k_1 \cdot p_2)}{m^2 + p_2^2 + k_1^2 - 2(p_2 \cdot k_1)}\right) \not{\epsilon}_2\right] u_1$$

Again using the mass-shell condition, the first three terms in each denominator cancel or vanish, and so the $(k \cdot p_i)$ dependence drops out from each term, leaving us with

$$\tilde{\mathcal{T}} = e^2 \bar{v}_2 \not{\epsilon}_2 \left[-1 + 1\right] u_1 = 0$$

SREDNICKI 70.6

We can write the covariant derivative of F as

$$\left(D_\rho F_{\mu\nu}\right)^a = \partial_\rho \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) + g f^{cba} \left(\partial_\rho \left(A^c_\mu A^c_\nu\right) + A^c_\rho \left(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu\right)\right) + g^2 f^{cba} f^{deb} A^c_\rho A^d_\mu A^e_\nu$$

The Bianchi identity (expanded) is then

$$\begin{aligned}
&\partial_\rho \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) + \partial_\mu \left(\partial_\nu A^a_\rho - \partial_\rho A^a_\nu\right) + \partial_\nu \left(\partial_\rho A^a_\mu - \partial_\mu A^a_\rho\right) \\
&+ g f^{cba} \left[\partial_\rho \left(A^c_\mu A^b_\nu\right) + \partial_\mu \left(A^c_\nu A^b_\rho\right) + \partial_\nu \left(A^c_\rho A^b_\mu\right) + A^c_\rho \left(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu\right) + A^c_\mu \left(\partial_\nu A^b_\rho - \partial_\rho A^b_\nu\right) + A^c_\nu \left(\partial_\rho A^b_\mu - \partial_\mu A^b_\rho\right)\right] \\
&+ g^2 f^{cba} f^{deb} \left[A^c_\rho A^d_\mu A^e_\nu + A^c_\mu A^d_\nu A^e_\rho + A^c_\nu A^d_\rho A^e_\mu\right]
\end{aligned}$$

It is easy to see that the first line vanishes, thanks to the commutativity of partial derivatives. In the second line, we can expand the derivatives of A to cancel some terms, which gives

$$\begin{aligned}
&g f^{cba} \left[\left(\partial_\rho A^c_\mu\right) A^b_\nu + \left(\partial_\mu A^c_\nu\right) A^b_\rho + \left(\partial_\nu A^c_\rho\right) A^b_\mu + A^c_\rho \partial_\mu A^b_\nu + A^c_\mu \partial_\nu A^b_\rho + A^c_\nu \partial_\rho A^b_\mu\right] \\
&+ g^2 f^{cba} f^{deb} \left[A^c_\rho A^d_\mu A^e_\nu + A^c_\mu A^d_\nu A^e_\rho + A^c_\nu A^d_\rho A^e_\mu\right]
\end{aligned}$$

After relabeling indices on both lines and moving some terms around, this can be rewritten as

$$\begin{aligned}
&g \left(f^{cba} + f^{bca}\right) \left[\left(\partial_\rho A^c_\mu\right) A^b_\nu + \left(\partial_\mu A^c_\nu\right) A^b_\rho + \left(\partial_\nu A^c_\rho\right) A^b_\mu\right] \\
&+ g^2 \left[A^c_\rho A^d_\mu A^e_\nu\right] \left(f^{cba} f^{deb} + f^{eba} f^{cdb} + f^{dba} f^{ecb}\right)
\end{aligned}$$

The term in parenthesis on the first line vanishes by the antisymmetry of f, and the term in parenthesis on the second line vanishes by the Jacobi identity.

SREDNICKI 74.1

$$\tilde{a}_+^\dagger|\psi\rangle = \left(\varepsilon_+^{*\mu} + ck^\mu\right) I|\psi\rangle$$

Where

$$I = -i\int d^3x e^{ikx}\overleftrightarrow{\partial_0} A_\mu(x)$$

Writing I as

$$I = \sum_\lambda \varepsilon_\lambda^\mu a_\lambda^\dagger$$

where a factor of 1/4 has been absorbed, this becomes

$$\tilde{a}_+^\dagger|\psi\rangle = \sum_\lambda \left(\varepsilon_{+\mu}^* + ck_\mu\right)\varepsilon_\lambda^\mu a_\lambda^\dagger|\psi\rangle$$

Using equation 74.37, we see that all terms in the first sum vanish except ($\lambda = +$) and all terms in the second sum vanish except ($\lambda =<$). This gives

$$\tilde{a}_+^\dagger|\psi\rangle = \left(a_+^\dagger - \frac{2c\omega}{\sqrt{2}}a_<^\dagger\right)|\psi\rangle$$

As Srednicki notes, equation 74.40 tells us that $a_<^\dagger|\psi\rangle \propto Q_B b^\dagger|\psi\rangle$, so we see that

$$\tilde{a}_+^\dagger|\psi\rangle = a_+^\dagger|\psi\rangle - \xi c Q_B b^\dagger|\psi\rangle$$

so we can define $-\xi c b^\dagger|\psi\rangle = |\chi\rangle$

WEINBERG 15-2

Here, I'm going to use the notation of Srednicki instead of Weinberg.

The gauge fixing function is $G^a(x) = \partial^i A_i^a(x) - \omega^a(x)$. We can follow the same steps as Srednicki chapter 71 to find that this leads to

$$\det\frac{\delta G^a(x)}{\delta\theta(y)} = -\det\left(\partial^i D_i^{ab}\delta^4(x-y)\right)$$

which gives the ghost Lagrangian

$$\mathcal{L}_{gh} = \bar{c}^a\partial^i D_i^{ab}c^b = -\partial^i\bar{c}^a\partial_i c^a + gf^{abc}A_i^c\partial^i\bar{c}^a c^b$$

Then, (Weinberg 15.6), the propagator then satisfies

$$-\partial^i\partial_i\Delta^{ab}(x-y) = \delta^4(x-y)$$

so we see that $\Delta^{ab}(k) = \frac{\delta^{ab}}{k^i k_i - i\epsilon}$