PHY 611 - Homework 1

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(Dated: September 20, 2019)

SREDNICKI 27.1

Given

$$\frac{d\alpha}{d\ln\mu} = b_1\alpha^2 \qquad\qquad \frac{dm}{d\ln\mu} = c_1 m\alpha$$

we see that

$$d\ln\mu = \frac{d\alpha}{b_1\alpha^2} = \frac{dm}{c_1 m\alpha}$$

and so

$$\int \frac{d\alpha}{b_1\alpha} = \int \frac{dm}{c_1 m} \implies \ln m = \frac{c_1}{b_1}\ln\alpha + d$$

where d is an integration constant. Then, we can write

$$\ln m(\mu_2) - \ln m(\mu_1) = \frac{c_1}{b_1}\ln\alpha(\mu_2) - \frac{c_1}{b_1}\ln\alpha(\mu_1) \implies \ln\left(\frac{m(\mu_2)}{m(\mu_1)}\right) = \ln\left[\left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{\frac{c_1}{b_1}}\right] \implies m(\mu_2) = \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{\frac{c_1}{b_1}} m(\mu_1)$$

SREDNICKI 28.1

First we need to compute the required one-loop diagrams. Some useful identities are

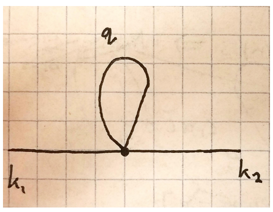
$$\int \frac{d^d p}{(p^2 + 2pq - m^2)^\alpha} = \frac{i\pi^{d/2}\Gamma(\alpha - d/2)}{\Gamma(\alpha)\,(-q^2 - m^2)^{\alpha - d/2}}$$

$$\Gamma(-n + x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n}\frac{1}{k} + \dots\right]$$

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\left[xA + (1-x)B\right]^2}$$

$$A^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln A + \dots$$

The first diagram is



Calculating the integral:

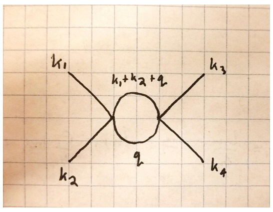
$$\int \frac{d^dq}{(2\pi)^d}\frac{-i\lambda\mu^\epsilon}{q^2+m^2} = -\frac{\lambda\mu^\epsilon}{(2\pi)^{4-\epsilon}}\frac{i\pi^{d/2}\Gamma(-1+\epsilon/2)}{(m^2)^{\epsilon/2-1}} = \frac{-\lambda m^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2}$$

$$= \frac{-\lambda m^2}{16\pi^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+\dots\right)$$

where $\dots$ indicates terms that are finite as $\epsilon \to 0$. Following the conventions of Srednicki Ch.14, this calculation indicates that

$$Z_\phi = 1 + O(\lambda^2) \qquad\qquad Z_m = 1 + \frac{\lambda}{16\pi^2\epsilon} + O(\lambda^2)$$

The next diagram to calculate is



This diagram gives

$$3\int \frac{d^dq}{(2\pi)^d}\frac{(-i\lambda)^2\mu^\epsilon}{\left[(\ell^2+q^2)^2+m^2\right]\left[q^2+m^2\right]}$$

where $\ell = k_1 + k_2$. Combining the denominators we find

$$-3\lambda^2\mu^\epsilon \int \frac{d^dq}{(2\pi)^2}\int_0^1 dx\left[x\ell^2+2x\ell q+q^2+m^2\right]^{-2} = -3\lambda^2\mu^\epsilon\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\left[(q+x\ell)^2+x(1-x)\ell^2+m^2\right]$$

Defining $p = q + x\ell$ and $D = x(1-x)\ell^2 + m^2$ we can now evaluate the integral over p:

$$\frac{-3\lambda^2}{(2\pi)^2}\int_0^1 dx\frac{i\mu^\epsilon\pi^{d/2}\Gamma(2-d/2)}{\Gamma(2)D^{2-d/2}} = \frac{-3i\lambda^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma\right)\int_0^1 dx\left(\frac{4\pi\mu^2}{D}\right)^{\epsilon/2}$$

$$= \frac{-3i\lambda^2}{16\pi^2}\left(\frac{2}{\epsilon}+\int_0^1 dx\ln\left(\frac{4\pi\mu^2}{D}\right)+\dots\right)$$

So we see that

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}$$

Now we define

$$Z_\phi = 1 + \sum_n \frac{a_n}{\epsilon^n} \qquad\qquad Z_m = 1 + \sum_n \frac{b_n}{\epsilon^n} \qquad\qquad Z_\lambda = \sum_n \frac{c_n}{\epsilon^n}$$

Comparing the renormalized Lagrangian to the original, we see that (now following the conventions of Srednicki Ch.28)

$$G(\lambda, \epsilon) = \ln\left(Z_\lambda Z_\phi^{-2}\right) = \sum_n \frac{G_n}{\epsilon^n}$$

and by (28.21), $\beta(\lambda)$ should be defined as

$$\beta(\lambda) = \lambda^2 \frac{dG_1}{d\lambda} = \lambda^2 \left(c_1 - 2a_1\right) = \lambda^2 \frac{d}{d\lambda}\left(\frac{3\lambda}{16\pi^2} + 0\right) = \frac{3\lambda^2}{16\pi^2}$$

Next, define

$$M(\lambda, \epsilon) = \ln\left(Z_\phi^{-1/2} Z_m^{1/2}\right) = \sum_n \frac{M_n}{\epsilon^n}$$

By (28.27), we see that

$$\gamma_m = \lambda \frac{dM_1}{d\lambda} = \lambda \frac{d}{d\lambda}\left(\frac{b_1}{2} - \frac{a_1}{2}\right) = \lambda \frac{d}{d\lambda}\left(\frac{\lambda}{32\pi^2}\right) = \frac{\lambda}{32\pi^2}$$

Finally, by (28.36) and (28.37), we see that

$$\gamma_\phi = -\frac{\lambda}{2}\frac{d}{d\lambda}\left[O(\lambda)^2\right]$$

so $\gamma_\phi \sim O(\lambda^2)$ and has no contribution of order λ.