

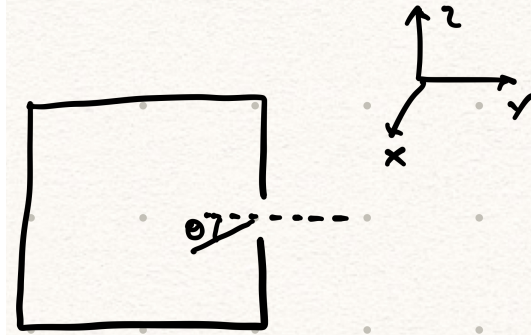
Statistical Mechanics - Homework 4

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(Dated: October 2, 2020)

PROBLEM 1

Part (a)

Only momentum that points directly towards the hole will contribute to a particle's escape, so a single particle approaching the hole has momentum $p \cos \theta$.



The partition function is

$$Z_1 = \int \frac{d^3p}{(2\pi\hbar)^3} p \cos \theta e^{-\frac{p^2}{2mT}} = \int_0^{2\pi} \cos \varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty dp p^3 \sin \theta \cos \theta e^{-\frac{p^2}{2mT}} = \frac{m^2 T^2}{4\pi^2 \hbar^3}$$

Notice that the θ integration goes to $\pi/2$ instead of π , since we only consider particles incident on the hole from the inside. The average energy is

$$\langle E \rangle = \frac{1}{Z_1} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^3}{2m} \cos \theta e^{-\frac{p^2}{2mT}} = \frac{1}{4m^3 T^2} \int_0^\infty dp p^5 e^{-\frac{p^2}{2mT}} = \frac{8m^3 T^3}{4m^3 T^2} = 2T$$

Part (b)

The thrust is given by

$$\left\langle \frac{dp}{dt} \right\rangle = m \langle v_y \rangle \frac{dN}{dt} = \langle p_y \rangle \frac{dN}{dt}$$

Using my result from part (c) (I did that part first), we get

$$\left\langle \frac{dp}{dt} \right\rangle = \frac{AN}{Vm} \left[\frac{\int \frac{d^3p}{(2\pi\hbar)^3} p \cos \theta e^{-\frac{p^2}{2mT}}}{\int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{p^2}{2mT}}} \right]^2 = \frac{NAT}{2V}$$

Part (c)

By the equipartition theorem, the total energy of the gas inside the box is $E = \frac{3}{2}NT$. We can imagine creating the system by adding a small amount of gas to the empty box, representing an energy contribution of $dE = \frac{3}{2}(TdN + NdT)$. If this gas is allowed to leave the box, energy must be conserved, so we should have $dE = \frac{3}{2}(TdN + NdT) - dE_{\text{leaving}} = 0$. From part (a), we know that a particle leaving the box has energy $E = 2T$, so dE becomes

$$dE = \frac{3}{2}(TdN + NdT) - 2TdN = 0 \implies TdN = 3NdT \implies \frac{\partial T}{\partial N} = \frac{T}{3N}$$

We can use this to find $\frac{\partial T}{\partial t} = \frac{\partial T}{\partial N} \frac{\partial N}{\partial t}$ as well. The rate at which particles leave the box is given by

$$\frac{\partial N}{\partial t} = \frac{\langle v_y \rangle AN}{V} = \frac{AN}{Vm} \langle p_y \rangle = \frac{AN}{Vm} \frac{\int \frac{d^3p}{(2\pi\hbar)^3} p \cos \theta e^{-\frac{p^2}{2mT}}}{\int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{p^2}{2mT}}} = \frac{AN}{V} \sqrt{\frac{T}{2\pi m}}$$

so the rate of temperature change is

$$\frac{\partial T}{\partial t} = \frac{A}{3V\sqrt{2\pi m}} T^{3/2}$$

PROBLEM 2

From class notes, we know that for monatomic ideal gases with masses $m_{1,2}$ and number densities $n_{1,2}$, μ is

$$\mu_{1,2} = T \left[\ln n_{1,2} - \frac{3}{2} \ln T - \frac{3}{2} \ln \frac{m_{1,2}}{2\pi\hbar^2} \right]$$

For the diatomic gas with mass m_3 and number density n_3 , μ_3 is

$$\mu_3 = T \left[\ln n_3 - \frac{5}{2} \ln T - \frac{3}{2} \ln \frac{m_3}{2\pi\hbar^2} - \ln \frac{2I_3}{\hbar^2} \right]$$

At chemical equilibrium, $\Delta G = 0$, so we have $\mu_3 = \mu_1 + \mu_2 + \varepsilon_B$. Using our expressions for μ_i , we find

$$\ln \frac{n_3}{n_1 n_2} = -\frac{1}{2} \ln T - \frac{3}{2} \ln \left(\frac{m_1 m_2}{(2\pi\hbar^2)^2} \right) + \frac{3}{2} \ln \left(\frac{m_3}{2\pi\hbar^2} \right) + \ln \frac{2I_3}{\hbar^2} + \frac{\varepsilon_B}{T} = \ln \left[\frac{2I_3}{\hbar^2 \sqrt{T}} \left(\frac{2\pi\hbar^2 m_3}{m_1 m_2} \right)^{3/2} e^{\varepsilon_B/T} \right]$$

The factor of ε_B/T is negligible, so we can drop it. This gives us

$$n_3 = n_1 n_2 f(T) \quad \text{with} \quad f(T) = \frac{2I_3}{\hbar^2 \sqrt{T}} \left(\frac{2\pi\hbar^2 m_3}{m_1 m_2} \right)^{3/2}$$

However, this is the relation between $n_{1,2,3}$ after equilibration. To find how n_3 changes as a result of the reaction, we must instead use

$$n_3 = f(T)(n_1 - n_3)(n_2 - n_3)$$

This gives a quadratic for n_3 , which we can solve:

$$n_3 = \frac{1 + f(T)(n_1 + n_2) \sqrt{(1 + f(T)(n_1 + n_2))^2 - 4f^2(T)n_1 n_2}}{2f(T)}$$

PROBLEM 3

The Debye temperature is defined as $T_D = \hbar\omega_D$. Using the known expression for the number of modes:

$$N_{modes} = \int_0^{\omega_D} d\omega \frac{dN_{modes}}{d\omega} \approx 3N$$

where $\frac{dN_{modes}}{d\omega}$ is given by

$$\frac{dN_{modes}}{d\omega} = \frac{V\omega^2}{2\pi^2} \left(\frac{2}{u_T^3} - \frac{1}{u_L^3} \right)$$

Together, these expressions give

$$3N = \frac{V}{2\pi^2} \left(\frac{2}{u_T^3} - \frac{1}{u_L^3} \right) \int_0^{T_D} d\omega \omega^2 = \frac{V\omega_D^3}{6\pi^2} \left(\frac{2}{u_T^3} - \frac{1}{u_L^3} \right)$$

Solving for ω_D , multiplying by \hbar and using $\frac{N}{V} = \frac{4}{a^3}$:

$$T_D = \frac{\hbar}{a} \left[72\pi^2 \left(\frac{2}{u_T^3} - \frac{1}{u_L^3} \right)^{-1} \right]^{1/3}$$

Using the information given in the table, we find $T_D^{Al} \approx 396K$, $T_D^{Cu} \approx 340K$, $T_D^{Pb} \approx 74K$.

Experimentally measured values at 298K are[1] $T_D^{Al} = 390K$, $T_D^{Cu} = 310K$, $T_D^{Pb} = 87K$.

The specific heat is

$$c = \frac{1}{\mu N} \frac{\partial E}{\partial T}$$

The energy is given by

$$E = \int_0^{\omega_D} d\omega \frac{dN_{modes}}{d\omega} \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} = \frac{V\hbar}{2\pi^2} \left(\frac{2}{u_T^3} - \frac{1}{u_L^3} \right) \int_0^{\omega_D} d\omega \frac{\omega^3}{e^{\hbar\omega/T} - 1}$$

which gives

$$c = \frac{\partial}{\partial T} \left[\frac{9\hbar^4}{\mu T_D^3} \int_0^{\omega_D} d\omega \frac{\omega^3}{e^{\hbar\omega/T} - 1} \right] = \frac{9}{\mu T_D^3 T^2} \int_0^{T_D} d\omega \frac{\omega^4 e^{\omega/T}}{(e^{\omega/T} - 1)^2}$$

Integrated numerically at 293K, we find $c_{Al} \approx 0.80$, $c_{Cu} = 0.34$, $c_{Pb} = 0.11$.

Experimentally measured values at 293K are[2] $c_{Al} = 0.9$, $c_{Cu} = 0.386$, $c_{Pb} = 0.128$.

PROBLEM 4

Part (a)

If the system is in equilibrium, absorption and emission should be equal. The emission rate is ϵ by definition, so the absorption rate is as well. Incident radiation can only be reflected or absorbed (there is no third option) so whatever is not absorbed must be reflected. Thus, the reflection rate is $1 - \epsilon$.

Part (b)

The heat emitted by surface 1 is

$$P_1 = \epsilon_1 \sigma T_1^4 + P_{reflected} = \epsilon_1 \sigma T_1^4 + P_2(1 - \epsilon_1)$$

Similarly, the heat emitted by surface 2 is

$$P_2 = \epsilon_2 \sigma T_2^4 + P_1(1 - \epsilon_2)$$

From these, we find two expressions for $P_1 - P_2$:

$$P_1 - P_2 = \epsilon_1 (\sigma T_1^4 - P_2) = \epsilon_2 (P_1 - \sigma T_2^4)$$

This is equivalent to

$$\frac{1}{\epsilon_2} (P_1 - P_2) + \frac{1}{\epsilon_1} (P_1 - P_2) = P_1 - \sigma T_2^4 - P_2 + \sigma T_1^4$$

Which can be solved for $P_1 - P_2$:

$$P_1 - P_2 = \frac{\sigma}{\frac{1}{\epsilon_2} + \frac{1}{\epsilon_1} - 1} (T_1^4 - T_2^4) = \frac{\sigma \epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} (T_1^4 - T_2^4)$$

Part (c)

From part (b), we find

$$P_L - P_S = \frac{\sigma \epsilon_1}{\epsilon_1 + 1 - \epsilon_1} (T_1^4 - T_S^4) = \sigma \epsilon_1 (T_1^4 - T_S^4)$$

$$P_R - P_S = \frac{\sigma \epsilon_2}{1 + \epsilon_2 - \epsilon_2} (T_S^4 - T_2^4) = \sigma \epsilon_2 (T_S^4 - T_2^4)$$

At equilibrium these should be the same, so we get

$$\epsilon_1 (T_1^4 - T_S^4) = \epsilon_2 (T_S^4 - T_2^4) \implies T_S = \left(\frac{\epsilon_1 T_1^4 + \epsilon_2 T_2^4}{\epsilon_1 + \epsilon_2} \right)^{1/4}$$

We can find the heat transfer from either of $P_L - P_S$ or $P_R - P_S$:

$$P_L - P_S = \sigma \epsilon_1 \left[T_1^4 - \frac{\epsilon_1 T_1^4 - \epsilon_2 T_2^4}{\epsilon_1 + \epsilon_2} \right] = \frac{\sigma \epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} (T_1^4 - T_2^4)$$

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- [1] http://www.knowledgedoor.com/2/elements.handbook/debye_temperature.html
 [2] <http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/sphtt.html>