Statistical Mechanics - Homework 10

M. Ross Tagaras
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PROBLEM 1

Part (a)

By the master equation,

$$\frac{\partial}{\partial t}\sum_i w_i = \sum_{i,j} P_{ij}(w_j - w_i)$$

This quantity vanishes by symmetry.

Part (b)

The entropy is

$$S = -\mathrm{Tr}(\rho \ln \rho) = -\sum_i w_i \ln w_i$$

The derivative of entropy is

$$\frac{dS}{dt} = -\sum_i \frac{dw_i}{dt}\ln w_i - \sum_i \frac{w_i}{w_i}\frac{\partial w_i}{\partial t} = \sum_{i,j} P_{ij}(w_i - w_j)\left(\ln w_i + 1\right)$$

Using the symmetry of P_{ij}:

$$\frac{dS}{dt} = \frac{1}{2}\sum_{i,j}(P_{ij} + P_{ji})(w_i - w_j)\left(\ln w_i + 1\right) = \frac{1}{2}\sum_{i,j} P_{ij}\left[(w_i - w_j)(\ln w_i + 1) - (w_i - w_j)(\ln w_j + 1)\right]$$

$$= \frac{1}{2}\sum_{i,j} P_{ij}(w_i - w_j)\left(\ln w_i - \ln w_j\right)$$

If $w_i > w_j$, $\ln w_i > \ln w_j$, so each term in the sum is positive. If $w_i < w_j$, then $\ln w_i < \ln w_j$, and the negative signs combine, again resulting in a sum of positive terms. Therefore, S is monotonically increasing.

Part (c)

The master equation gives

$$\frac{\partial w_1}{\partial t} = p(w_2 - w_1) \qquad \frac{\partial w_2}{\partial t} = p(w_1 - w_2) = -\frac{\partial w_1}{\partial t}$$

With the initial conditions $w_1(0) = 1$, $w_2(0) = 0$, we find

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DSolve[{w1'[t] == p (w2[t] - w1[t]), w2'[t] == p (w1[t] - w2[t]), w1[0] == 1, w2[0] == 0}, {w1, w2}, t]
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Out[100]=

$$\left\{\left\{\mathtt{w1} \to \mathtt{Function}\left[\{t\}, \frac{1}{2}e^{-2pt}\left(1 + e^{2pt}\right)\right], \mathtt{w2} \to \mathtt{Function}\left[\{t\}, \frac{1}{2}e^{-2pt}\left(-1 + e^{2pt}\right)\right]\right\}\right\}$$

so the time dependent w_i are

$$w_1(t) = \frac{1}{2}\left(1 + e^{-2pt}\right) \qquad w_2(t) = \frac{1}{2}\left(1 - e^{-2pt}\right)$$

PROBLEM 2

Adding a factor of $e^{i\omega t}$ for time dependence and calculating $\tilde{w}$ gives

$$-\frac{\tilde{w}}{\tau} = \left(\frac{\partial w}{\partial t} + q\vec{E}\cdot\frac{\vec{p}}{m}\frac{\partial w_0}{\partial \varepsilon}\right) \implies \tilde{w} = -\tau i\omega\tilde{w} - \tau\frac{q}{m}\vec{E}\cdot\vec{p}\frac{\partial w_0}{\partial \varepsilon}$$

so we find

$$\tilde{w} = -\frac{\tau}{1 - i\omega\tau}\frac{q}{m}\vec{E}\cdot\vec{p}\frac{\partial w_0}{\partial \varepsilon}$$

This is the same result for $\tilde{w}$ that we used in the time-independent case, except for the overall factor of $\frac{1}{1-i\omega\tau}$, which is independent of p. Therefore, the integration over $\vec{p}$ in the definition of $\vec{j}$ is the same, and we find

$$\sigma(\omega) = \frac{\sigma(0)}{1 - i\omega\tau}$$

In Joule heating, the power density is

$$P/V = \vec{j}\cdot E = |\sigma|\, E^2 = \left|\frac{\sigma(0)}{1 - i\omega\tau}\right| E^2 = \frac{\sigma(0)E^2}{\sqrt{1 + \omega^2\tau^2}}$$

This should be real-valued, so I took the absolute value. When $\omega\tau \gg 1$, this is approximately

$$P/V \approx \frac{\sigma(0)E^2}{\omega\tau}$$

PROBLEM 3

Part (a)

The electric conductivity is found from the current:

$$\vec{j} = q\int d^3p\ \vec{v}\tilde{w} = \sigma\vec{E}$$

The Liouville equation with the RTA gives (assuming spatial uniformity)

$$\tilde{w} = -\tau q E_i \frac{\partial w_0}{\partial p_i}$$

Then, we have

$$\sigma E_i = -\tau q \int d^3p \frac{p_i}{m}\left[qE_j\frac{\partial w_0}{\partial p_j}\right]$$

In equilibrium, w_0 is given by

$$w_0 = \frac{g}{(2\pi\hbar)^3}\left\langle N(\varepsilon)\right\rangle = \frac{g}{(2\pi\hbar)^3}e^{-(\varepsilon-\mu)/T} = \frac{g}{(2\pi\hbar)^3}e^{\frac{\mu}{T}-\frac{p^2}{2mT}}$$

so our expression for σ reduces to

$$\sigma E_i = \frac{\tau q^2 g E_j}{m^2(2\pi\hbar)^3 T}e^{\mu/T}\int d^3p\; p_i p_j e^{-\frac{p^2}{2mT}}$$

Integrating (for example, the $i=1$ component) gives

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$$\text{Integrate}\left[\text{Integrate}\left[\text{Integrate}\left[\text{p1}\,(\text{E1}\,\text{p1}+\text{E2}\,\text{p2}+\text{E3}\,\text{p3})\,e^{\frac{-1}{2\,\text{m}\,\text{T}}\left(\text{p1}^2+\text{p2}^2+\text{p3}^2\right)},\{\text{p1},-\infty,\infty\}\right],\{\text{p2},-\infty,\infty\}\right],\{\text{p3},-\infty,\infty\}\right]$$

Out[105]=

$$\text{ConditionalExpression}\left[2\sqrt{2}\,\text{E1}\,\text{m}^3\,\pi^{3/2}\sqrt{\frac{1}{\text{m}\,\text{T}}}\,\text{T}^3,\ \text{Re}\left[\frac{1}{\text{m}\,\text{T}}\right]>0\right]$$

After simplifying, we find

$$\sigma E_i = \frac{\tau q^2 g\sqrt{m}T^2}{(2\pi\hbar^2)^{3/2}}e^{\mu/T}E_i \implies \sigma = \frac{\tau q^2 g\sqrt{m}T^2}{(2\pi\hbar^2)^{3/2}}e^{\mu/T}$$

Part (b)

The thermal conductivity is defined by the heat flow density:

$$h_i = \int d^3p(\varepsilon-\mu)v_i\tilde{w} = -\kappa\frac{\partial T}{\partial x^i}$$

Likharev equation (6.55) gives an expression for $\tilde{w}$:

$$\tilde{w} = \tau\frac{\varepsilon-\mu}{T}v_i\frac{\partial T}{\partial x^i}\frac{\partial w}{\partial\varepsilon} = \frac{g\tau}{T^2(2\pi\hbar)^3}e^{\mu/T}v_i\frac{\partial T}{\partial x^i}\left[\mu-\frac{p^2}{2m}\right]e^{-\frac{p^2}{2mT}}$$

This gives

$$\kappa\frac{\partial T}{\partial x^i} = \frac{g\tau}{T^2m^2(2\pi\hbar)^3}e^{\mu/T}\int d^3p\; p_i p_j\frac{\partial T}{\partial x^j}\left(\frac{p^2}{2m}-\mu\right)^2 e^{-\frac{p^2}{2mT}}$$

Integrating as before, we find

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$$\text{Integrate}\left[\text{Integrate}\left[\text{Integrate}\left[\frac{-\tau\,\text{g}}{\text{m}^2\,\text{T}^2\,(2\,\pi\,\text{h})^3}\,e^{\frac{\mu}{\text{T}}}\,\text{p1}\,(\text{p1}\;\text{T1}+\text{p2}\;\text{T2}+\text{p3}\;\text{T3})\left(\frac{\text{p1}^2+\text{p2}^2+\text{p3}^2}{2\,\text{m}}-\mu\right)^2 e^{\frac{-1}{2\,\text{m}\,\text{T}}\left(\text{p1}^2+\text{p2}^2+\text{p3}^2\right)},\{\text{p1},-\infty,\infty\}\right],\{\text{p2},-\infty,\infty\}\right],\{\text{p3},-\infty,\infty\}\right]$$

Out[108]=

$$\text{ConditionalExpression}\left[-\frac{e^{\frac{\mu}{\text{T}}}\,\text{g}\,\text{T1}\left(35\,\text{T}^2-20\,\text{T}\,\mu+4\,\mu^2\right)\tau}{8\sqrt{2}\,\text{h}^3\,\pi^{3/2}\sqrt{\frac{1}{\text{m}\,\text{T}}}},\ \text{Re}\left[\frac{1}{\text{m}\,\text{T}}\right]>0\right]$$

which results in

$$\kappa = \frac{g\tau e^{\mu/T}\sqrt{mT}\left(4\mu^2+35T^2-20\mu T\right)}{8\sqrt{2}\pi^{3/2}\hbar^3}$$

Part (c)