Statistical Mechanics - Homework 9

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PROBLEM 1

By Likharev equation (5.24) we have

$$\delta N = \sqrt{\langle N^2\rangle - \langle N\rangle^2} = \sqrt{T\frac{\partial \langle N\rangle}{\partial \mu}}$$

For fermions/bosons, we have

$$\langle N\rangle = \frac{1}{e^{(\varepsilon-\mu)/T} \pm 1}$$

and for classical particles,

$$\langle N\rangle = e^{-(\varepsilon-\mu)/T}$$

This gives

$$\delta N = \sqrt{\frac{e^{(\varepsilon-\mu)/T}}{\left(e^{(\varepsilon-\mu)/T} \pm 1\right)^2}} = \sqrt{\langle N\rangle^2 e^{(\varepsilon-\mu)/T}} = \sqrt{\langle N\rangle \mp \langle N\rangle^2}$$

for fermions/bosons and

$$\delta N = \sqrt{\frac{1}{e^{(\varepsilon-\mu)/T}}} = \sqrt{\langle N\rangle}$$

for classical particles. When $\langle N\rangle$ is small so that $\langle N\rangle^2$ is negligible, the three expressions become equal.

PROBLEM 2

The partition function is

$$Z = \sum_{k=0}^{N} \binom{N}{k} e^{\beta h(2k-N)} = e^{-\beta N h}\left(1 + e^{2\beta h}\right)^N$$

The magnetization is

$$M = \left\langle \sum_i s_i \right\rangle = \frac{1}{Z}\sum_{k=0}^{N}(2k-N)\binom{N}{k} e^{\beta h(2k-N)} = \frac{T}{Z}\frac{\partial Z}{\partial h} = N\tanh(\beta h)$$

The RMS fluctuation of M is

$$\delta M = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{N}\mathrm{sech}(\beta h)$$

Mathematica calculations for these quantities are below:

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In[132]:=
(*Partition Function*)
Z = ⅇ^(-h N β) (1 + ⅇ^(2 h β))^N;

(*Magnetization*)
FullSimplify[1/(β Z) D[Z, h]]
Out[133]=
N Tanh[h β]
In[134]:=
(*M is also*)
FullSimplify[1/Z Sum[(2 k - N) Binomial[N, k] ⅇ^(β h (2 k-N)), {k, 0, N}]]
Out[134]=
N Tanh[h β]

(*M^2*)
FullSimplify[1/Z Sum[(2 k - N)^2 Binomial[N, k] ⅇ^(β h (2 k-N)), {k, 0, N}]]
Out[129]=
N (N - (-1 + N) Sech[h β]^2)
In[135]:=

(*<M^2>-<M>^2*)
FullSimplify[(N - (N - 1) Sech[h β]^2) N - (N Tanh[h β])^2]
Out[135]=
N Sech[h β]^2
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PROBLEM 3

The partition function is

$$Z = \sum_i e^{-\beta E_i} = 2e^{3\beta J} + 6e^{-\beta J}$$

The energy and RMS fluctuation are

$$E = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \qquad (\delta E)^2 = -\frac{\partial E}{\partial \beta}$$

Calculating these in Mathematica:

In[136]:=

$$\texttt{(*Partition Function*)}$$
$$\texttt{Z} = 2\,e^{3\beta J} + 6\,e^{-\beta J};$$

In[140]:=

$$\texttt{(*Energy*)}$$
$$\texttt{e} = \frac{-1}{\texttt{Z}}\,\texttt{D[Z, }\beta\texttt{]};$$
$$\texttt{FullSimplify[e]}$$

Out[141]=

$$3\left(-1 + \frac{4}{3 + e^{4 J \beta}}\right) J$$

In[142]:=

$$\texttt{(*}\delta\texttt{E*)}$$
$$\texttt{FullSimplify[-D[e, }\beta\texttt{]]}$$

Out[142]=

$$\frac{48\, e^{4 J \beta}\, J^2}{\left(3 + e^{4 J \beta}\right)^2}$$

PROBLEM 4

In class, we derived $S_F = \frac{T\eta}{\pi}$. We also know the relationship between S_q and S_F:

$$S_q = \frac{S_F}{m^2(\omega_0^2 - \omega^2)^2 + \eta^2\omega^2}$$

Using these two results, we find

$$K_q = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} S_q(\omega) = \frac{T\eta}{\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{m^2(\omega_0^2 - \omega^2)^2 + \eta^2\omega^2} = \frac{T\eta}{2\pi^2 i}\sum_i \lim_{\omega \to \omega_i} (\omega - \omega_i)\frac{e^{-i\omega t}}{m^2(\omega_0^2 - \omega^2)^2 + \eta^2\omega^2}$$

where the ω_i are the poles of S_q. Evaluating, we find

In[26]:=

$$\texttt{FullSimplify}\left[\texttt{Solve}\left[m^2\left(\omega 0^2 - \omega^2\right)^2 + \eta^2\omega^2 == 0,\ \omega\right]\right]$$

Out[26]=

$$\left\{\left\{\omega \to -\frac{\sqrt{-\frac{\eta^2 - 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2 \eta^2 \omega 0^2}}{m^2}}}{\sqrt{2}}\right\}, \left\{\omega \to \frac{\sqrt{-\frac{\eta^2 - 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2 \eta^2 \omega 0^2}}{m^2}}}{\sqrt{2}}\right\}, \left\{\omega \to -\sqrt{-\frac{\eta^2}{2 m^2} + \omega 0^2 + \frac{\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}{2 m^2}}\right\}, \left\{\omega \to \sqrt{-\frac{\eta^2}{2 m^2} + \omega 0^2 + \frac{\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}{2 m^2}}\right\}\right\}$$

In[42]:=

$$\texttt{zeros} = \left\{-\frac{\sqrt{-\frac{\eta^2 - 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2 \eta^2 \omega 0^2}}{m^2}}}{\sqrt{2}}, \frac{\sqrt{-\frac{\eta^2 - 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2 \eta^2 \omega 0^2}}{m^2}}}{\sqrt{2}}, -\sqrt{-\frac{\eta^2}{2 m^2} + \omega 0^2 + \frac{\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}{2 m^2}}, \sqrt{-\frac{\eta^2}{2 m^2} + \omega 0^2 + \frac{\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}{2 m^2}}\right\};$$

$$\texttt{FullSimplify}\left[\frac{T\eta}{2\pi^2 i}\,\texttt{Sum}\left[\texttt{Limit}\left[\frac{e^{-i\omega\tau}\left(\omega - \texttt{zeros[[i]]}\right)}{m^2\left(\omega 0^2 - \omega^2\right)^2 + \eta^2\omega^2},\ \omega \to \texttt{zeros[[i]]}\right],\ \{i, 1, 4\}\right]\right]$$

Out[43]=

$$\frac{m\,T\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}\left(\frac{\texttt{Sin}\left[\frac{\tau\sqrt{-\frac{\eta^2}{2} + m^2\omega 0^2 - \frac{1}{2}\sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}}{m}\right]}{\sqrt{-\eta^2 + 2 m^2\omega 0^2 - \sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}} - \frac{\texttt{Sin}\left[\frac{\tau\sqrt{-\eta^2 + 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}}{\sqrt{2}\,m}\right]}{\sqrt{-\eta^2 + 2 m^2\omega 0^2 + \sqrt{\eta^4 - 4 m^2\eta^2\omega 0^2}}}\right)}{\sqrt{2}\,\pi^2\left(\eta^3 - 4 m^2\eta\,\omega 0^2\right)}$$

PROBLEM 5

We can find the spectral density from

$$(\delta I)^2 = K_I(0) = \int_{-\infty}^{\infty} d\omega \; S_I(\omega)$$

The fluctuations in I are given by

$$(\delta I)^2 = \big(\delta[qN]\big)^2 = q^2(\delta N)^2 = q^2 \langle N \rangle = q\bar{I}$$

so we find

$$q\bar{I} = \int_{-\infty}^{\infty} d\omega S_I(\omega) \implies S_I(\omega) = q\bar{I}\delta(\omega)$$