Statistical Mechanics - Homework 8

M. Ross Tagaras
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PROBLEM 1

Part (a)

The entropy is

$$S = \ln\left[\binom{N_T}{N_b}\right] = \ln\left[\frac{N_T!}{N_b!(N_T-N_b)!}\right] = \ln\left[N_T!\right] - \ln\left[N_b!\right] - \ln\left[(N_T-N_b)!\right]$$

$$\approx N_T\ln[N_T] - N_T - N_b\ln[N_b] + N_b - (N_T-N_b)\ln[(N_T-N_b)] + (N_T-N_b)$$

$$= N_T\ln[N_T] - N_b\ln[N_b] - (N_T-N_b)\ln[N_T-N_b]$$

The free energy is $F = E - TS$. Binding energy is the minimum energy required to remove a particle from a system, so we can use $E = -\Delta$ for each of the gas/trap pairs. Then F is

$$F = -\Delta N_b - T\Bigg(N_T\ln[N_T] - N_b\ln[N_b] - (N_T-N_b)\ln[N_T-N_b]\Bigg)$$

Part (b)

The chemical potential is

$$\mu_i = \frac{\partial F}{\partial N_i}$$

which gives

$$\mu_b = -\Delta + T + T\ln N_b - \frac{TN_T}{N_T-N_b} - -T\ln(N_T-N_b) + \frac{TN_b}{N_T-N_b} = -\Delta + T\ln N_b - T\ln(N_t-N_b)$$

If the bound gas is in equilibrium with the free gas, then $\mu_b = \mu_{free}$. The chemical potential for the free gas is

$$\mu_{free} = -T\frac{\partial \ln Z}{\partial N} = -T\frac{\partial}{\partial N}\left[-\ln N! + \frac{3N}{2}\ln\left(\frac{mT}{2\pi\hbar^2}\right) + N\ln V\right] = T\ln n - \frac{3T}{2}\ln\left(\frac{mT}{2\pi\hbar^2}\right)$$

Part (c)

In equilibrium we have $\mu_b = \mu_{free}$, so we find

$$-\Delta + T\ln N_b - T\ln(N_T-N_b) = T\ln n - \frac{3T}{2}\ln\left(\frac{mT}{2\pi\hbar^2}\right)$$

In terms of x_b, the left hand side is

$$-\Delta - T\ln\left(\frac{N_T - N_b}{N_b}\right) = -\Delta - T\ln\left(\frac{1}{x_b} - 1\right)$$

which gives

$$\frac{1}{x_b} = 1 + \frac{1}{n}\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} e^{-\Delta/T}$$

PROBLEM 2

The known relation involving η simplifies when $\eta \ll 1$:

$$\eta = \tanh\left(\frac{H + 2Jd\eta}{T}\right) \approx \frac{H + 2Jd\eta}{T} \implies \eta = \frac{h}{T - 2Jd}$$

At the critical point, the derivative of the free energy is zero:

$$0 = \frac{\partial f}{\partial \eta} = 2a\left(\frac{T}{2Jd} - 1\right)\eta - h + O(\eta^3) \implies a = \frac{1}{\eta}\frac{h}{\frac{T}{Jd} - 2}$$

Substituting η gives

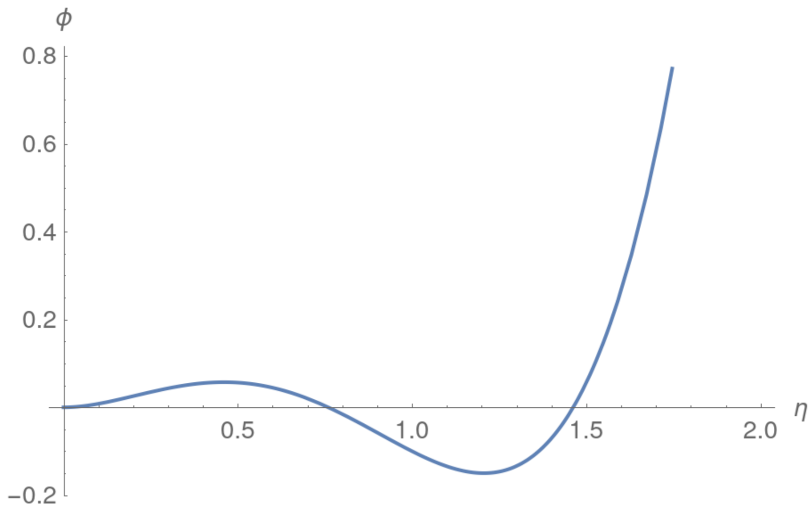
$$a = \frac{(T - 2Jd)}{T - 2Jd} Jd = Jd$$

Now we need the relationship between a and b:

PROBLEM 3

Part (a)

Schematically, $\phi(\eta)$ looks like



There are two local minima: one at $\eta = 0$ and one with positive η. The exact values of the minima are

$$0 = \frac{d\phi(\eta)}{d\eta} = \eta\left(2a\tau - 3c\eta + 2b\eta^2\right) \implies \eta_1 = 0, \quad \eta_2 = \frac{3c + \sqrt{9c^2 - 16ab\tau}}{4b}$$

Part (b)

At the phase transition, the free energy for both phases will be equal. We have $\phi(\eta_1) = \phi(\eta_2)$, but $\phi(\eta_1) = 0$, so $\phi(\eta_2) = 0$ as well. This gives

$$0 = a\tau\eta_2^2 - c\eta_2^3 + \frac{b}{2}\eta_2^4$$

which we can solve for T:

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In[313]:=
ClearAll[a, b, c, η, τ]
η = (3 c + Sqrt[9 c^2 - 16 a b (T/T0 - 1)])/(4 b);
FullSimplify[Solve[a (T/T0 - 1) η^2 - c η^3 + b/2 η^4 == 0, T]]
Out[315]=
{{T → T0}, {T → T0 + (c^2 T0)/(2 a b)}}
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Part (c)

For a first order phase transition, we have

$$\frac{L}{T_c} = \Delta S = -\Delta\left(\frac{\partial\phi}{\partial T}\right) = \left.\frac{\partial\phi(\eta_2)}{\partial T}\right|_{T=T_c} - \left.\frac{\partial\phi(0)}{\partial T}\right|_{T=T_c} = \left.\frac{\partial\phi(\eta_2)}{\partial T}\right|_{T=T_c}$$

The choice of sign in ΔS was fixed by requiring that $L > 0$. Evaluating with Mathematica:

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In[2]:=
ClearAll[a, b, c, η, T, T0]
η = 1/(4 b) (3 c + Sqrt[9 c^2 - 16 a b (T/T0 - 1)]);
In[4]:=
Assuming[{a > 0, b > 0, c > 0}, FullSimplify[T0 (1 + c^2/(2 a b)) D[a (T/T0 - 1) η^2 - c η^3 + b/2 η^4, T] /. T → T0 (1 + c^2/(2 a b))]]
Out[4]=
(2 a b c^2 + c^4)/(2 b^3)
```