Statistical Mechanics - Homework 2

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PROBLEM 1

We have two kinds of work to consider: the work involved in the phase change and the work done to raise the temperature of the resulting water.

The phase change is isothermal, so $dW = -dQ$, and we can find the heat added from the latent heat: $L = Q/m$. This gives $\Delta W_1 = mL$. When raising the temperature of the water, the definition of Carnot efficiency gives

$$dW_2 = \left(1 - \frac{T}{T_H}\right) dQ_H$$

where T is the temperature of the cold reservoir (the ice) and T_H is the temperature of the heat bath. Using the definition of the specific heat $c = \frac{1}{m}\frac{\partial Q}{\partial T}$, we find

$$dW_2 = mc\left(1 - \frac{T}{T_H}\right) dT \implies \Delta W_2 = mc\int_{T_C}^{T_H} dT\left(1 - \frac{T}{T_H}\right) = mc\left(\frac{T_H}{2} + \frac{T_C^2}{2T_H} - T_C\right)$$

The total work is then

$$\Delta W = \Delta W_1 + \Delta W_2 = mc\left(\frac{T_H}{2} + \frac{T_C^2}{2T_H} - T_C\right) + mL \approx 8.1 \times 10^9 \ J$$

PROBLEM 2

For an ideal gas, we have $P = nT$, so using $n = \rho/m_0$, we find $P = \frac{T\rho}{m_0}$. This gives $u^2 = T/m_0$. For the Van der Waals gas, we have

$$\left(P + \frac{a}{v^2}\right)(v - b) = T \implies P = \frac{T}{v-b} - \frac{a}{v^2} = \frac{\rho T}{m_0 - b\rho} - \frac{a\rho^2}{m_0^2}$$

Then we find that the speed of sound is

$$u^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \frac{m_0 T}{(m_0 - b\rho)^2} - \frac{2a\rho}{m_0^2}$$

Note that this reduces to our result for the ideal gas when $a = b = 0$.

PROBLEM 3

EMF is defined as

$$\varepsilon = \frac{dW}{dq}$$

Since we know that $dW = -PdV$, it is easy to see that $\varepsilon = -P$ and $q = V$. The change in internal energy is

$$dU = TdS - PdV = TdS + \varepsilon dq$$

In a heat bath at constant T, we have $dU = -w(T)dq = TdS + \varepsilon dq$, which implies

$$w(T) = -\varepsilon - T\left(\frac{\partial S}{\partial q}\right)_T$$

Using a Maxwell relation, we find

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \implies \left(\frac{\partial S}{\partial q}\right)_T = -\left(\frac{\partial \varepsilon}{\partial q}\right)_T$$

so the final expression for $w(T)$ is

$$w(T) = \left(\frac{\partial \varepsilon}{\partial T}\right)_q - \varepsilon$$

The change in U when the battery is discharged by a finite amount is $\Delta U = -w(T)\Delta q$ and the work done is $\Delta W = \varepsilon \Delta q$. Using $\Delta U = \Delta Q + \Delta W$, we have

$$\Delta Q = -w(T)\Delta q \varepsilon \Delta q = -\left(\frac{\partial \varepsilon}{\partial T}\right)_q \Delta q$$

Since $\frac{\partial \varepsilon}{\partial T}$ is negative by assumption, ΔQ must be positive.

PROBLEM 4

The entropy of the system will be given by

$$S = \ln \binom{N}{N_0, N_+, N_-}$$

where $\binom{N}{k_1,\ldots,k_r} = \frac{N!}{k_1!\ldots k_r!}$ is a multinomial coefficient, which corresponds to distributing N distinguishable objects over r containers with k_i elements each. We can find $N_\pm$ in terms of known quantities κ, x_0, and N by using the definitions of N_0 and κ:

$$N = N_0 + N_+ + N_- \implies N(1 - x_0) = N_+ + N_-$$

$$\frac{N(1-x_0) - 2N_-}{N} = \frac{N_+ - N_-}{N} = \kappa \implies N_- = \frac{N(1 - x_0 - \kappa)}{2}$$

which in turn gives $N_+ = \frac{N(1-x_0+\kappa)}{2}$. Then the multinomial coefficient is

$$\binom{N}{N_0, N_+, N_-} = \frac{N!}{(Nx_0)!\left(\frac{N(1-x_0+\kappa)}{2}\right)!\left(\frac{N(1-x_0-\kappa)}{2}\right)!}$$

which gives us the entropy:

$$S = \ln\left(\frac{N!}{N_0!N_+!N_-!}\right) = \ln(N!) - \ln\left[(Nx_0)!\right] - \ln\left[\left(\frac{N(1-x_0+\kappa)}{2}\right)!\right] - \ln\left[\left(\frac{N(1-x_0-\kappa)}{2}\right)!\right]$$

Using Stirling's approximation and simplifying, this becomes

$$S = N\ln N - Nx_0\ln(Nx_0) - \frac{N}{2}\left[(1-x_0+\kappa)\ln\left(\frac{N}{2}(1-x_0+\kappa)\right) + (1-x_0-\kappa)\ln\left(\frac{N}{2}(1-x_0-\kappa)\right)\right]$$

We can find the equilibrium value of x_0 by extremizing S:

$$0 = \frac{\partial S}{\partial x_0} = -N\ln(Nx_0) + \frac{N}{2}\left[\ln\left[\frac{N}{2}(1-x_0+\kappa)\right] + \ln\left[\frac{N}{2}(1-x_0-\kappa)\right]\right] \implies x_0 = \frac{-1\pm\sqrt{4-3\kappa^2}}{3}$$

To find the entropy in terms of E and H, we can make the substitution $\kappa \to -\frac{E}{H\mu N}$, which gives

$$S(E,H) = \left(\frac{N(x_0-1)}{2} + \frac{E}{2H\mu}\right)\ln\left[-\frac{E}{2\mu H} - \frac{N(x_0-1)}{2}\right] + \left(\frac{N(x_0-1)}{2} - \frac{E}{2H\mu}\right)\ln\left[\frac{E}{2\mu H} - \frac{N(x_0-1)}{2}\right]$$
$$- Nx_0\ln(Nx_0) + N\ln N$$

The temperature is

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{2H\mu}\ln\left[\frac{N(1-x_0) - \frac{E}{\mu H}}{N(1-x_0) + \frac{E}{\mu H}}\right]$$

Solving for E, we find

$$E = \mu HN(1-x_0)\frac{1-e^{2H\mu/T}}{1+e^{2\mu H/T}} = \mu HN(x_0-1)\tanh\left(\frac{H\mu}{T}\right)$$

The total magnetization is

$$M = -\frac{E}{H} = \mu N(1-x_0)\tanh\left(\frac{H\mu}{T}\right)$$