Statistical Mechanics - Homework 5

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PROBLEM 1

Part (a)

n_e^{cond} is given by

$$n_e^{cond} = \frac{N_e^{cond}}{V} = \frac{g}{2\pi^2\hbar^3}\int_0^\infty dp \frac{p^2}{e^{\left(\frac{p^2}{2m_C}+\Delta-\mu\right)/T}+1} = \frac{g}{2\pi^2\hbar^3}\int_0^\infty dp \frac{p^2}{e^{\frac{\Delta-\mu}{T}}e^{\frac{p^2}{2m_C T}}+1} \approx \frac{ge^{\frac{\mu-\Delta}{T}}}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2 e^{-\frac{p^2}{2m_C T}}$$

$$= \frac{g}{\hbar^3}\left(\frac{m_C T}{2\pi}\right)^{3/2} e^{\frac{\mu-\Delta}{T}}$$

The number of holes is given by the total number of sites minus the number of electrons, so n_h^{val} is given by

$$n_h^{val} = \frac{N_h^{val}}{V} = \frac{g}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2\left(1 - \frac{1}{e^{-\frac{p^2}{2m_V T}-\frac{\mu}{T}}+1}\right) = \frac{g}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2\left(\frac{e^{-\frac{p^2}{2m_V T}-\frac{\mu}{T}}}{e^{-\frac{p^2}{2m_V T}-\frac{\mu}{T}}+1}\right)$$

$$= \frac{g}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2\left(\frac{1}{e^{\frac{p^2}{2m_V T}+\frac{\mu}{T}}+1}\right) \approx \frac{ge^{-\frac{\mu}{T}}}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2 e^{-\frac{p^2}{2m_V T}} = \frac{ge^{-\frac{\mu}{T}}}{\hbar^3}\left(\frac{m_V T}{2\pi}\right)^{3/2}$$

Part (b)

Equating the results from the previous part gives

$$e^{\frac{\Delta}{T}}\left(\frac{m_V}{m_C}\right)^{3/2} = e^{\frac{2\mu}{T}}$$

so solving for μ results in

$$\frac{2\mu}{T} = \ln\left[e^{\frac{\Delta}{T}}\left(\frac{m_V}{m_C}\right)^{3/2}\right] \implies \mu = \frac{\Delta}{2} + \frac{3T}{4}\ln\frac{m_V}{m_C}$$

PROBLEM 2

From class notes on the Sommerfeld expansion, we know that

$$\mu(T) = \varepsilon_F + \frac{\pi^2 T^2}{6}\frac{g'(\mu)}{g(\mu)} \approx \varepsilon_F + \frac{\pi^2 T^2}{6}\frac{g'(\varepsilon_F)}{g(\varepsilon_F)}$$

For a Fermi gas, $g(\varepsilon)$ is

$$g(\varepsilon) = \frac{gm^{3/2}V}{\sqrt{2}\pi^2\hbar^3}\varepsilon_F^{1/2}$$

so we find

$$\frac{g'(\varepsilon_F)}{g(\varepsilon_F)} = \frac{gm^{3/2}V}{2\sqrt{2}\pi^2\hbar^3}\varepsilon_F^{-1/2}\frac{\sqrt{2}\pi^2\hbar^3}{gm^{3/2}V}\varepsilon_F^{-1/2} = \frac{1}{2\varepsilon_F}$$

Thus,

$$\mu(T) = \varepsilon_F + \frac{\pi^2 T^2}{12\varepsilon_F}$$

When we move an electron, the Gibbs energy will change by one unit of charge. We get a difference in chemical potential, so the emf created by moving the charges will be $c\mathcal{E} = \Delta\mu$. This is

$$\Delta\mu = \varepsilon_F^{Al} - \varepsilon_F^{Cu} + \frac{\pi^2\left(T_m^2 - T_r^2\right)}{12}\left(\frac{1}{\varepsilon_F^{Al}} - \frac{1}{\varepsilon_F^{Cu}}\right)$$

Then at 293K we get (T_r is a constant)

$$\frac{\partial\mathcal{E}}{\partial T_m} = \frac{\pi^2 T_m}{6e}\left(\frac{1}{\varepsilon_F^{Al}} - \frac{1}{\varepsilon_F^{Cu}}\right) \approx -2.38\times 10^{-3} \quad \frac{V}{K}$$

PROBLEM 3

For a relativistic degenerate Fermi gas, we have

$$N = \frac{gV}{(2\pi\hbar)^3}\int\frac{d^3p}{e^{(\varepsilon-\mu)/T+1}} = \frac{4\pi gV}{(2\pi\hbar)^3}\int_0^\infty\frac{p^2}{e^{(\varepsilon-\mu)/T+1}} = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^\infty d\varepsilon\frac{\varepsilon^2}{e^{(\varepsilon-\mu)/T+1}}$$

As $T\to 0$, this is

$$N = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^{\varepsilon_F} d\varepsilon\ \varepsilon^2 = \frac{gV\varepsilon_F^3}{6\pi^2\hbar^3c^3}$$

The energy is

$$E = \langle\varepsilon\rangle = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^\infty d\varepsilon\frac{\varepsilon^3}{e^{(\varepsilon-\mu)/T+1}}$$

As $T\to 0$, this becomes

$$E = \frac{4\pi gV}{(2\pi\hbar)^3}\int_0^{\varepsilon_F} d\varepsilon\ \varepsilon^3 = \frac{gV}{8\pi^2\hbar^3c^3}\varepsilon_F^4$$

To find the relationship between P, V, and E, we use

$$PV = -\Omega = \frac{4\pi TgV}{(2\pi\hbar)^3}\int_0^\infty dp\ln\left(1+e^{(\mu-\varepsilon)/T}\right)p^2 = \frac{TgV}{2\pi^2\hbar^3c^3}\int_0^\infty d\varepsilon\ln\left(1+e^{(\mu-\varepsilon)/T}\right)\varepsilon^2$$

Integrating by parts, we find

$$\int_0^\infty d\varepsilon \ln\left(1+e^{(\mu-\varepsilon)/T}\right)\varepsilon^2 = \frac{\varepsilon^3}{3}\ln\left(1+e^{(\mu-\varepsilon)/T}\right)\Big|_0^\infty + \frac{1}{3T}\int_0^\infty d\varepsilon \frac{\varepsilon^3}{e^{(\mu-\varepsilon)/T}+1} = \frac{2\pi^2\hbar^3c^3}{3gVT}E$$

So our final relationship (at any T) is

$$P = \frac{E}{3V}$$

For our specific case of $T = 0$, this gives

$$P = \frac{g\varepsilon_F^4}{24\pi^2\hbar^3c^3}$$

PROBLEM 4

In homework 2, we defined $M = \mu(N_+ - N_-)$. From the definition of the density of states, we know that $g(E)\Delta E = K$, where K is the number of possible ways to commit the system to a particular state. Clearly, this corresponds to the difference between the number of spin-up particles and the number of spin-down particles. The difference in energy states is the difference in Fermi energy, which is just the difference in chemical potentials, so $\Delta E = 2\mu H$. All together, we see that $M = 2\mu^2 H g(\varepsilon_F)$.

For a nonrelativistic gas, N is

$$N = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\; p^2 = \frac{2^{3/2}m^2 g}{\pi^2\hbar^3}\int_0^{\varepsilon_F} d\varepsilon\; \varepsilon^{1/2} = \frac{2^{5/2}m^2 g}{3\pi^2\hbar^3}\varepsilon_F^{3/2}$$

so the density of states is

$$g(\varepsilon_F) = \frac{\partial N}{\partial \varepsilon} = \frac{2^{3/2}m^2 g}{\pi^2\hbar^3}\varepsilon_F^{1/2} = \frac{3N}{2\varepsilon_F}$$

This gives the magnetization:

$$M_{non-rel} = \frac{3N\mu^2 H}{\varepsilon_F}$$

For a relativistic gas, N is

$$N = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\; p^2 = \frac{g}{2\pi^2\hbar^3c^3}\int_0^{\varepsilon_F} d\varepsilon\; \varepsilon^2 = \frac{g}{6\pi^2\hbar^3c^3}\varepsilon_F^3$$

so the density of states is

$$g(\varepsilon_F) = \frac{\partial N}{\partial \varepsilon} = \frac{g}{2\pi^2\hbar^3c^3}\varepsilon_F^2 = \frac{3N}{\varepsilon_F}$$

This gives the magnetization:

$$M_{rel} = \frac{6N\mu^2 H}{\varepsilon_F} = 2M_{non-rel}$$