Statistical Mechanics - Homework 6

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PROBLEM 1

The energy of the individual particles is

$$\epsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

Assuming that they behave classically, we have

$$N_{BE}(T_c, \mu = 0) = \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \frac{g}{e^{\epsilon/T} - 1} = \frac{2g}{\pi\hbar^3} \int_0^\infty dp \int_0^\infty dr \frac{(pr)^2}{e^{\frac{p^2}{2m} + \frac{m\omega^2 r^2}{2T}} - 1}$$

The momentum integral can be evaluated in terms of the polylogarithm:

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In[1496]:=
Assuming[{a > 0, b > 0}, Integrate[x^2/(e^(a x^2+b) - 1), {x, 0, ∞}]]
Out[1496]=
(√π PolyLog[3/2, e^-b])/(4 a^(3/2))

In[1497]:=
Assuming[n > 0, Integrate[r^2 e^(-n r^2), {r, 0, ∞}]]
Out[1497]=
√π/(4 n^(3/2))
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$$N_{BE} = \frac{2g}{\pi\hbar^3} \int_0^\infty dr\ r^2 \left[\frac{\sqrt{\pi}}{4\left(\frac{1}{2mT}\right)^{3/2}} Li_{3/2}\left(e^{-\frac{m\omega^2 r^2}{2T}}\right) \right]$$

$Li_s(z)$ is defined as

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

so we get

$$N_{BE} = \frac{g\sqrt{\pi}(2mT)^{3/2}}{2\pi\hbar^3} \int_0^\infty dr\ r^2 \left[\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} e^{-\frac{m\omega^2 r^2}{2T}k} \right]$$

Exchanging the sum and integral and integrating gives

$$N_{BE} = \frac{gT^3}{\hbar^3\omega^3}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{gT^3\zeta(3)}{\hbar^3\omega^3}$$

If we now take $N_{total} = N_{BE}$ then $T = T_C$, so

$$T_C = \hbar\omega\left(\frac{N}{g\zeta(3)}\right)^{1/3}$$

PROBLEM 2

Part (a)

Using the Stefan-Boltzmann law $\frac{dP}{dA} = \sigma T^4$, the energy emitted by the earth is

$$E_{out}^{E} = 4\pi R_E^2 \sigma T_E^4$$

and the total energy emitted by the sun is

$$E_{out}^{S} = 4\pi R_S^2 \sigma T_S^4$$

However, the earth will only absorb some fraction of this total amount. Of the total solid angle 4π, the earth will absorb a fraction $A_E/d^2 = \frac{\pi R_E^2}{d^2}$. The total absorbed energy is

$$E_{in} = \pi\sigma\left(\frac{R_S R_E}{d}\right)^2 T_S^4$$

If the earth is in thermal equilibrium with the space around it, then $E_{in} = E_{out}^{E}$, giving

$$\pi\sigma\left(\frac{R_S R_E}{d}\right)^2 T_S^4 = 4\pi\sigma T_E^4 R_E^2 \implies T_E = \left(\frac{R_S^2 T_S^4}{4d^2}\right)^{1/4} \approx 288.998\ K$$

Part (b)

If the moon has no surface heat transfer, then the only heat emitted will be from the area that directly receives heat from the sun. The distance from the moon to the earth is approximately 3.84×10^5 km, so $d_M = 1.496 \times 10^8$ km. Assuming equilibrium, we have

$$\sigma T_M^4 A_M = \sigma T_S^4 4\pi R_S^2\left(\frac{A_M}{4\pi d_M^2}\right) \implies T_M = \left(\frac{R_S^2 T_S^4}{d_M^2}\right)^{1/4} \approx 409.251\ K$$

PROBLEM 3

The energy for an LC circuit is

$$E = \frac{1}{2}\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2}\frac{Q^2}{C}$$

so defining $P = L\frac{dQ}{dt}$, we immediately recognize the Hamiltonian for a harmonic oscillator:

$$H = \frac{P^2}{2L} + \frac{1}{2}L\omega^2 Q^2$$

Using $Q = CV$ and our analogy to a 1D quantum harmonic oscillator, we can identify $\frac{1}{C^2}\left\langle V^2 \right\rangle \leftrightarrow \left\langle x^2 \right\rangle$ and $L \leftrightarrow m$, so using this well-known result (see, for example, Likharev equation 2.78) we find

$$\sqrt{\langle V^2 \rangle} = \sqrt{\frac{\hbar\omega}{2C}\coth\frac{\hbar\omega}{2T}}$$