Statistical Mechanics - Homework 7

M. Ross Tagaras
(Dated: February 14, 2021)

PROBLEM 1

Part (a)

Since the $3d$ gas loses particles to the $2d$ gas, we need to use the grand partition function for our calculations. The pressure of the $3d$ gas is

$$P = -\frac{\partial F}{\partial V} = -\frac{\partial}{\partial V}\left(\Omega - \mu N\right) = T\frac{\partial \ln Z_G^{3d}}{\partial V}$$

The single particle partition function is

$$Z_1^{3d} = \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dp\; p^2 e^{-\frac{p^2}{2mT}} = \left(\frac{mT}{2\pi}\right)^{3/2}\frac{V}{\hbar^3}$$

so Z_G^{3d} is

$$Z_G^{3d} = \sum_{N=0}^{\infty}\frac{1}{N!}\left(Z_1^{3d}\right)^N e^{\frac{\mu N}{T}} = e^{\left(\frac{mT}{2\pi}\right)^{3/2}\frac{V}{\hbar^3}e^{\frac{\mu}{T}}}$$

Using this result, the pressure is

$$P = \left(\frac{m}{2\pi}\right)^{3/2}\frac{T^{5/2}}{\hbar^3}e^{\mu/T}$$

For the $2d$ gas:

$$Z_1^{2d} = \frac{1}{(2\pi\hbar)^2}\int d^2x d^2p\; e^{-\frac{p^2}{2mT}+\frac{\Delta}{T}} = \frac{AmT}{2\pi\hbar}e^{\frac{\Delta}{T}}$$

Then, like before, the grand partition function is

$$Z_G^{2d} = e^{\frac{AmT}{2\pi\hbar}e^{\frac{\Delta+\mu}{T}}}$$

The particle number is

$$N_{2d} = -\frac{\partial\Omega}{\partial\mu} = T\frac{\partial\ln Z_G^{2d}}{\partial\mu} = \frac{AmT}{2\pi\hbar}e^{\frac{\Delta+\mu}{T}}$$

which gives

$$n = \frac{mT}{2\pi\hbar}e^{\frac{\Delta+\mu}{T}}$$

Therefore, we find

$$\frac{P}{n} = \sqrt{\frac{m}{2\pi}}\frac{T^{3/2}}{\hbar}e^{-\frac{\Delta}{T}}$$

Part (b)

The Clausius-Clapeyron equation can be written as

$$\frac{d\ln P}{dT} = \frac{l}{T^2}$$

so using our result for P/n, we find

$$\frac{l}{T^2} = \frac{d}{dT}\left[-\frac{\Delta}{T} + \frac{3}{2}\ln T\right] = \frac{\Delta}{T^2} + \frac{3}{2T}$$

which gives

$$l = \Delta + \frac{3T}{2}$$

PROBLEM 2

We have

$$p(h) = p_0 e^{-\frac{mgh}{T_a}} \qquad\qquad \Lambda = T(v_g - v_l)\frac{dp}{dT}$$

The volume of the liquid is very small compared to the atmosphere, so we can take $v_g - v_l \approx v_g$. Then, solving the differential equation for $T(p)$, we find

$$\frac{dT}{dp} = \frac{T^2}{\Lambda p} \implies T(p) = -\frac{\Lambda}{\ln p + \Lambda c} \longrightarrow -\frac{\Lambda}{\ln p_0 - \frac{mgh}{T_a} + \Lambda c}$$

Using $T(0) = 373$ K, we find c:

$$T(h) = \frac{373 T_a \Lambda}{373 mgh + \Lambda T_a}$$

which gives $T(1.6\text{ km}) \approx 368$ K. The difference is then approximately 5 K.

PROBLEM 3

There is a single state with all spins up, three states with a single spin down, three states with a single spin up, and one state with all spins down. The energies and degeneracies of the states are

State	Degeneracy	Energy
↑↑↑	1	$-3(J+h)$
↓↑↑	3	$J-h$
↓↓↑	3	$J+h$
↓↓↓	1	$-3(J-h)$

The partition function is

$$Z = \sum_i e^{-\beta E_i} = e^{3\beta(J+h)} + 3e^{-\beta(J-h)} + 3e^{-\beta(J+h)} + e^{3\beta(J-h)}$$

The magnetization is

$$M = \left\langle \sum_i s_i \right\rangle = \frac{T}{Z}\frac{\partial Z}{\partial h} = \frac{3}{Z}\left(e^{3\beta(J+h)} + e^{-\beta(J-h)} - e^{-\beta(J+h)} - e^{3\beta(J-h)}\right)$$

At $h = 0$, this becomes

$$M|_{h=0} = \frac{3}{Z}\left(e^{3\beta J} + e^{-\beta J} - e^{-\beta J} - e^{3\beta J}\right) = 0$$

The susceptibility and heat capacity are

$$\chi(T) = \frac{\partial M}{\partial h} \qquad C = \frac{\partial E}{\partial T} = -\beta^2\frac{\partial E}{\partial \beta} = \beta^2\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)$$

These quantities were calculated with Mathematica:

```
ClearAll[Z, M, h, J, β, χ, c]
(*Partition Function*)
Z = e^(3 β (J+h)) + e^(3 β (J-h)) + 3 e^(-β (J+h)) + 3 e^(-β (J-h));
(*Magnetization*)
M = 1/(β Z) D[Z, h];
(*Susceptibility*)
χ = D[M, h];
FullSimplify[χ]
```

$$\frac{24\, e^{6\,(h+J)\,\beta}\,\beta\,(4\,\mathrm{Cosh}[2\,h\,\beta] + \mathrm{Cosh}[4\,h\,\beta] + 3\,\mathrm{Cosh}[4\,J\,\beta])}{\left(e^{5\,J\,\beta} + 3\,e^{(2\,h+J)\,\beta} + 3\,e^{(4\,h+J)\,\beta} + e^{6\,h\,\beta+5\,J\,\beta}\right)^2}$$

```
(*High temp limit for susceptibility*)
Normal@Series[χ, {β, 0, 1}]
```

$3\,\beta$

```
(*Heat capacity*)
c = β^2 D[1/Z D[Z, β], β];
FullSimplify[c]
```

$$\frac{24\, e^{6\,(h+J)\,\beta}\,\beta^2\,\left(4\,(h^2+J^2)\,\mathrm{Cosh}[2\,h\,\beta] + (h^2+4\,J^2)\,\mathrm{Cosh}[4\,h\,\beta] + h\,\left(3\,h\,\mathrm{Cosh}[4\,J\,\beta] + 32\,J\,\mathrm{Cosh}[h\,\beta]^3\,\mathrm{Sinh}[h\,\beta]\right)\right)}{\left(e^{5\,J\,\beta} + 3\,e^{(2\,h+J)\,\beta} + 3\,e^{(4\,h+J)\,\beta} + e^{6\,h\,\beta+5\,J\,\beta}\right)^2}$$

```
(*High temp limit for heat capacity*)
Normal@Series[c, {β, 0, 2}]
```

$3\,(h^2 + J^2)\,\beta^2$