Solutions to Calin's *An Informal Introduction to Stochastic Calculus*

M. Ross Tagaras
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EXERCISE 2.9.4

Show that

(a) $Cov(X,Y) = \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$, *where* $\mu_X = \mathbb{E}[X]$ *and* $\mu_Y = \mathbb{E}[Y]$.

(b) $Var(X) = \mathbb{E}\left[(X-\mu_X)^2\right]$

SOLUTION 2.9.4

$$\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}[XY] - 2\mathbb{E}(x)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Part (b) follows trivially from part (a).

EXERCISE 2.9.5

Let μ *and* σ *denote the mean and standard deviation of the random variable* X. *Show that*

$$\mathbb{E}[X^2] = \mu^2 + \sigma^2$$

SOLUTION 2.9.5

For the random variable X, the mean squared μ^2 is given by

$$\mu^2 = \left[\int xp(x)dx\right]^2$$

and the standard deviation σ^2 is given by

$$\sigma^2 = Var(X) = \mathbb{E}(X^2) - \big(\mathbb{E}(X)\big)^2 = \int x^2 p(x)dx - \left[\int xp(x)dx\right]^2$$

Therefore, we have

$$\mathbb{E}(X^2) = \int x^2 p(x)dx = \int x^2 p(x)dx - \left[\int xp(x)dx\right]^2 + \left[\int xp(x)dx\right]^2 = \mu^2 + \sigma^2$$

EXERCISE 2.9.7

(a) *Prove that for any random variables* A *and* B, *we have*

$$\mathbb{E}[AB]^2 \leq \mathbb{E}[A^2]\mathbb{E}[B^2]$$

(b) *Use part (a)* to show that for any random variables X and Y we have

$$-1 \leq \rho(X,Y) \leq 1$$

(c) *What can you say about the random variables* X *and* Y *if* $\rho(X,Y) = 1$*?*

SOLUTION 2.9.7

Consider the quantity

$$A - B\frac{\mathbb{E}(AB)}{\mathbb{E}(B^2)}$$

with $\mathbb{E}(B^2) \neq 0$. By the basic properties of expectations, we have that

$$\mathbb{E}\left[\left(A - B\frac{\mathbb{E}(AB)}{\mathbb{E}(B^2)}\right)^2\right] \geq 0$$

Expanding the parentheses and using linearity of the expectation gives

$$0 \leq \mathbb{E}(A^2) - \frac{\mathbb{E}^2(AB)}{\mathbb{E}(B^2)}$$

which implies $\mathbb{E}^2(AB) \leq \mathbb{E}(A^2)\mathbb{E}(B^2)$. Now, using the results of Exercise 2.9.4, we have

$$\rho^2(X,Y) = \frac{Cov^2(X,Y)}{Cov(X,X)Cov(Y,Y)} = \frac{\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]}$$

By the result of part (a):

$$\frac{\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]} \leq \frac{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]} = 1$$

so we find $-1 \leq \rho(X,Y) \leq 1$. If $\rho(X,Y) = 1$, then we have

$$\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]$$

which implies that $Y = aX + b$, where $a, b \in \mathbb{R}$.

EXERCISE 2.9.8

Let $g : [0,1] \to [0,\infty)$ be an integrable function with

$$\int_0^1 g(x)dx = 1.$$

Consider $Q : \mathcal{B}([0,1]) \to \mathbb{R}$ given by

$$Q(A) = \int_A g(x)dx$$

Show that Q is a probability measure on $\big(\Omega = [0,1], \mathcal{B}([0,1])\big)$

SOLUTION 2.9.8

Recall the definition of a probability measure:

A probability measure P on the space $(\Omega, \mathcal{F})$, where Ω is the set of outcomes and $\mathcal{F}$ is the σ-algebra corresponding to the collection of events, is a real-valued function defined on $\mathcal{F}$ that satisfies the following axioms:

1. $P(A) \geq 0 \;\forall\; A \in \mathcal{F}$
2. $P(\Omega) = 1$
3. *If $\{E_i : i \in I\}$ is a countable, pairwise disjoint set of events, then*

$$P\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I} P(E_i)$$

By the definition of $g(x)$, requirement 2 is automatically satisfied.
Now let $A \in \mathcal{B}([0,1])$. If A is a set of measure zero, then $g(x) = 0$, by the basic properties of the Lebesgue integral. For all other measurable A, the Lebesgue integral of a strictly positive function is also strictly positive. Combining these two statements, requirement 1 is satisfied.
Finally, consider a countable union of disjoint subintervals of $[0,1]$. By the additivity of the Lebesgue integral (since $E_i \cap E_j = \emptyset$ for $i \neq j$), we have

$$\int_{\bigcup_{i\in I} E_i} g(x)dx = \sum_{i\in I} \int_{E_i} g(x)dx$$

from which requirement 3 follows.

EXERCISE 2.10.4

Show that if X and Y are two independent random variables, then $m_{X+Y}(t) = m_X(t)m_Y(t)$.

SOLUTION 2.10.4

By Proposition 2.8.1, the density function of independent random variables factorizes. Then, we have

$$m_{X+Y}(t) = \int e^{t(x+y)} p_{XY}(x,y)dxdy = \int e^{tx} p_X(x) e^{ty} p_Y(y)dxdy = m_X(t)m_Y(t)$$

EXERCISE 2.10.5

Given that the moment generating function of a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is $m(t) = \mathbb{E}[e^{tX}] = e^{\mu t + t^2\sigma^2/2}$, show that

(a) $\mathbb{E}[Y^n] = e^{n\mu + n^2\sigma^2/2}$, *where $Y = e^X$.*

(b) *Show that the mean and variance of the log-normal random variable $Y = e^X$* are

$$\mathbb{E}[Y] = e^{\mu+\sigma^2/2}, \qquad Var[Y] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

SOLUTION 2.10.5

Part (a):

$$\mathbb{E}[Y^n] = \mathbb{E}[e^{nX}] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(nX)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{n^k}{k!}\mathbb{E}[X^k] = \sum_{k=0}^{\infty} \frac{n^k}{k!}\frac{d^k}{dt^k} m_X(t)\Big|_{t=0} = m_X(n) = e^{n\mu + n^2\sigma^2/2}$$

$\mathbb{E}[Y]$ is just a special case of part (a). To calculate $Var(Y)$:

$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}^2[Y] = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

EXERCISE 2.11.4

Consider the independent, exponentially distributed random variables $X \sim \lambda_1 e^{-\lambda_1 t}$,*and* $Y \sim \lambda_2 e^{-\lambda_2 t}$, *with* $\lambda_1 \neq \lambda_2$. Show that the sum is distributed as $X + Y \sim \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$.

SOLUTION 2.11.4

This result follows immediately from Theorem 2.11.1. The probability density p_{X+Y} of the sum is

$$p_{X+Y} = \int_0^t p_X(t-\tau)p_Y(\tau)d\tau = \lambda_1\lambda_2 e^{-\lambda_1 t}\int_0^t e^{(\lambda_1-\lambda_2)\tau}d\tau = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

EXERCISE 2.12.1

EXERCISE 2.12.7

Prove that, if $\mathcal{H} \subset \mathcal{G}$,

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}\right] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}\right] = \mathbb{E}\left[X|\mathcal{H}\right]$$

SOLUTION 2.12.7

EXERCISE 2.12.9

SOLUTION 2.12.9

Part (a) follows immediately from the definition of $\mathcal{G}$-measurability.

By the definition of integration, we have

$$P(A) = \int_A dP$$

From the definition of χ_A, this is equal to

$$\int_A \chi_A(\omega)dP = \int_\Omega \chi_A(\omega)dP$$

but this is easily seen to be $\mathbb{E}\chi_A$.

Part (d) follows immediately from parts (b) and (c).

EXERCISE 3.1.9

For $t_0 \geq 0$, show that the process $X_t = W_{t+t_0} - W_{t_0}$ is a Brownian motion.

SOUTION 3.1.9

It is clear that $X_0 = 0$. Since X_t is a sum of continuous functions, it is continuous as well. Now consider the increments $X_A - X_B$ and $X_C - X_D$, with $A > B > 0$ and $C > D > 0$. We have

$$\mathbb{E}\left[(X_A - X_B)(X_C - X_D)\right] = \mathbb{E}[X_A X_C] - \mathbb{E}[X_A X_D] - \mathbb{E}[X_B X_C] + \mathbb{E}[X_B X_D]$$

$$\begin{aligned} &= \mathbb{E}\left[(W_{A+t_0} - W_{t_0})(W_{C+t_0} - W_{t_0})\right] - \mathbb{E}\left[(W_{A+t_0} - W_{t_0})(W_{D+t_0} - W_{t_0})\right] \\ &-\mathbb{E}\left[(W_{B+t_0} - W_{t_0})(W_{C+t_0} - W_{t_0})\right] + \mathbb{E}\left[(W_{B+t_0} - W_{t_0})(W_{D+t_0} - W_{t_0})\right] \end{aligned}$$

By the independence of Brownian motion increments, this is equal to

$$\begin{aligned} &\mathbb{E}[W_{A+t_0} - W_{t_0}]\,\mathbb{E}[W_{C+t_0} - W_{t_0}] - \mathbb{E}[W_{A+t_0} - W_{t_0}]\,\mathbb{E}[W_{D+t_0} - W_{t_0}] \\ -&\mathbb{E}[W_{B+t_0} - W_{t_0}]\,\mathbb{E}[W_{C+t_0} - W_{t_0}] + \mathbb{E}[W_{B+t_0} - W_{t_0}]\,\mathbb{E}[W_{D+t_0} - W_{t_0}] \end{aligned}$$

$$= \mathbb{E}[X_A]\mathbb{E}[X_C] - \mathbb{E}[X_A]\mathbb{E}[X_D] - \mathbb{E}[X_B]\mathbb{E}[X_C] + \mathbb{E}[X_B]\mathbb{E}[X_D]$$

$$= \mathbb{E}[X_A - X_B]E[X_C - X_D]$$

Since the increments are independent, their sum is also normally distributed. The mean and variance are

$$\mathbb{E}[X_t - X_s] = \mathbb{E}[W_{t+t_0} - W_{t_0} - W_{s+t_0} + W_{t_0}] = \mathbb{E}[W_{t+t_0}] - \mathbb{E}[W_{s+t_0}] = 0$$

$$Var(X_t - X_s) = \mathbb{E}\left[(X_t - X_s)^2\right] - \mathbb{E}^2[X_t - X_s] = \mathbb{E}\left[X_t^2 + X_s^2 - 2X_t X_s\right] + 0$$

$$= \mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[X_s^2\right] - 2\mathbb{E}[X_t X_s] = Var(X_t) + \mathbb{E}^2[X_t] + Var(X_s) + \mathbb{E}^2[X_s] - 2\mathbb{E}[X_t X_s]$$

$$= t + 0 + s + 0 - 2s = t - s$$

EXERCISE 3.1.14

Consider the process $Y_t = tW_{1/t}$, $t > 0$ and define $Y_0 = 0$. Find the distribution of Y_t, the PDF of Y_t, $Cov(Y_s, Y_t)$, $\mathbb{E}[Y_t - Y_s]$, and $Var(Y_t - Y_s)$ for $t > s > 0$. Note that Y_t is a Brownian motion.

SOLUTION 3.1.14

Since Y_t is a Brownian motion, it is normally distributed. The mean and variance are

$$\mathbb{E}[Y_t] = t\mathbb{E}[W_{1/t}] = 0$$

$$Var(Y_t) = Var(tW_{1/t}) = t^2 Var(W_{1/t}) = t$$

The PDF is

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

The covariance is

$$Cov(Y_s, Y_t) = \mathbb{E}[Y_s Y_t] - \mathbb{E}[Y_s]\mathbb{E}[Y_t] = \mathbb{E}[Y_s Y_t] = \mathbb{E}[Y_s\left(Y_t - Y_s\right) + Y_s^2] = \mathbb{E}[Y_s^2] = s$$

We also have

$$\mathbb{E}[Y_t - Y_s] = \mathbb{E}[Y_t] - \mathbb{E}[Y_s] = 0$$

$$Var(Y_t - Y_s) = \mathbb{E}\left[(Y_t - Y_s)^2\right] - \mathbb{E}^2\left[Y_t - Y_s\right] = \mathbb{E}[Y_t^2] + \mathbb{E}[Y_s^2] - 2\mathbb{E}[Y_t Y_s] = t - s$$

EXERCISE 3.1.17

Let $0 < s < t$. Show that

$$\mathbb{E}[W_t^3|\mathcal{F}_s] = W_s^3 + 3W_s(t-s)$$

$$\mathbb{E}[W_t^4|\mathcal{F}_s] = W_s^4 + 6W_s^2(t-s) + 3(t-s)^2$$

SOLUTION 3.1.17

$$\mathbb{E}[(W_t - W_s)^3|\mathcal{F}_s] = \mathbb{E}[W_t^3|\mathcal{F}_s] - \mathbb{E}[W_s^3|\mathcal{F}_s] + 3\mathbb{E}[W_t W_s^2|\mathcal{F}_s] - 3\mathbb{E}[W_t^2 W_s|\mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^3] = \mathbb{E}[W_{t-s}^3] = 0$$

$$\mathbb{E}[W_t^3|\mathcal{F}_s] = \mathbb{E}[W_s^3|\mathcal{F}_s] + 3\mathbb{E}[W_t^2 W_s|\mathcal{F}_s] - 3\mathbb{E}[W_t W_s^2|\mathcal{F}_s] = W_s^3 + 3W_s\mathbb{E}[W_t^2|\mathcal{F}_s] - 3W_s^2\mathbb{E}[W_t|\mathcal{F}_s]$$

$$= W_s^3 + 3W_s\left(W_s^2 + t - s\right) - 3W_s^3 = W_s^3 + 3W_s(t-s)$$

$$\mathbb{E}[(W_t - W_s)^4|\mathcal{F}_s] = \mathbb{E}[W_{t-s}^4] = 3(t-s)^2 = \mathbb{E}[|W_t^4\mathcal{F}_s] - 4\mathbb{E}[W_t^3 W_s|\mathcal{F}_s] + 6\mathbb{E}[W_t^2 W_s^2|\mathcal{F}_s] - 4\mathbb{E}[W_t W_s^3|\mathcal{F}_s] + \mathbb{E}[W_s^4|\mathcal{F}_s]$$

$$= \mathbb{E}[W_t^4|\mathcal{F}_s] - 4W_s\mathbb{E}[W_t^3|\mathcal{F}_s] + 6W_s^2\mathbb{E}[W_t^2|\mathcal{F}_s] - 4W_s^3\mathbb{E}[W_t|\mathcal{F}_s] + W_s^4$$

$$= \mathbb{E}[W_t^4|\mathcal{F}_s] - 4W_s^4 - 12W_s^2(t-s) + 6W_s^4 + 6W_s^2(t-s) - 3W_s^4$$

Rearranging to solve for $\mathbb{E}[W_t^4|\mathcal{F}_s]$ gives $\mathbb{E}[W_t^4|\mathcal{F}_s] = W_s^4 + 6W_s^2(t-s) + 3(t-s)^2$

EXERCISE 3.1.18

Show that

$$\mathbb{E}\left[\int_s^t W_u\ du \Big| \mathcal{F}_s\right] = (t-s)W_s$$

SOLUTION 3.1.18

First, write the integral as a Riemann sum:

$$\mathbb{E}\left[\int_s^t W_u \; du \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\sum_i W_{u_i}(u_{i+1} - u_i)\Big| \mathcal{F}_s\right]$$

where $s \leq u_i < t$. By linearity of the expectation, this is equal to

$$\sum_i (u_{i+1} - u_i)\mathbb{E}\left[W_{u_i}|\mathcal{F}_s\right]$$

Since W_t is a martingale, $\mathbb{E}\left[W_{u_i}|\mathcal{F}_s\right] = W_s$, and passing back to the continuum limit, we have

$$E\left[W_{u_i}|\mathcal{F}_s\right] = W_s \int_s^t du = W_s(t-s)$$

EXERCISE 3.1.19

Show that the process

$$X_t = W_t^3 - 3\int_0^t W_s \; ds$$

is a martingale with respect to the information set $\mathcal{F}_t = \sigma\{W_s; s < t\}$

SOLUTION 3.1.19

It is clear that X_t is integrable for all t and that X_t is adapted to $\mathcal{F}_t$. We also need to have $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ for all $s < t$. By the result of Exercise 3.1.17, this expectation is equal to

$$\mathbb{E}\left[W_t^3 - 3\int_0^t W_u \; du \Big| \mathcal{F}_s\right] = W_s^3 + 3W_s(t-s) - 3\left(\mathbb{E}\left[\int_s^t W_u \; du\Big|\mathcal{F}_s\right] + \mathbb{E}\left[\int_0^s W_u \; du \Big| \mathcal{F}_s\right]\right)$$

By the result of Exercise 3.1.18, we can evaluate the first integral, and the right side of the equality becomes

$$W_s^3 + 3W_s(t-s) - 3\left(W_s(t-s) + \mathbb{E}\left[\int_0^s W_u \; du\Big|\mathcal{F}_s\right]\right)$$

The second integral simplifies because W_t is a martingale, and after canceling terms, we are left with

$$\mathbb{E}\left[W_t^3 - 3\int_0^t W_u \; du\Big|\mathcal{F}_s\right] = W_s^3 - 3\int_0^s W_u \; du = X_s$$

so X_s is a martingale.

EXERCISE 3.1.21

Let W_t *and* $\tilde{W}_t$ *be independent Brownian motions and* ρ *be a constant with* $|\rho| \leq 1$. *Show that the process* $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$ *is continuous and has the distribution* $\mathcal{N}(0,t)$. *Is* X_t *a Brownian motion?*

SOLUTION 3.1.21

As a sum of continuous functions, X_t is continuous. The expectation is

$$\mathbb{E}[X_t] = \rho\mathbb{E}[W_t] + \sqrt{1-\rho^2}\mathbb{E}[\tilde{W}_t] = 0$$

The variance is

$$Var(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}^2[X_t] = \mathbb{E}\left[\rho^2 W_t^2 + (1-\rho^2)\tilde{W}_t^2 + 2\rho\sqrt{1-\rho^2}W_t\tilde{W}_t\right] = \rho^2 t + (1-\rho^2)t + 2\rho\sqrt{1-\rho^2}\mathbb{E}[W_t]\mathbb{E}[\tilde{W}_t] = t$$

Since $W_0 = \tilde{W}_0 = 0$, $X_0 = 0$. The mean and variance of the increments $X_t - X_s$ are

$$\mathbb{E}[X_t - X_s] = \mathbb{E}[\rho W_t + \sqrt{1-\rho^2}\tilde{W}_t - \rho W_s - \sqrt{1-\rho^2}\tilde{W}_s] = 0$$

$$Var(X_t - X_s) = \mathbb{E}[X_t^2 + X_s^2 - 2X_tX_s]$$

$$\begin{aligned}=&\mathbb{E}[\rho^2 W_t^2 + (1-\rho^2)\tilde{W}_t^2 + 2\rho\sqrt{1-\rho^2}W_t\tilde{W}_t] + \mathbb{E}[\rho^2 W_s^2 + (1-\rho^2)\tilde{W}_s^2 + 2\rho\sqrt{1-\rho^2}W_s\tilde{W}_s] \\ &- 2\mathbb{E}[\left(\rho W_t + \sqrt{1-\rho^2}\tilde{W}_t\right)\left(\rho W_s + \sqrt{1-\rho^2}\tilde{W}_s\right)]\end{aligned}$$

$$= \rho^2 t + (1-\rho^2)t + \rho^2 s + (1-\rho^2)s - 2[\rho^2 s + (1-\rho^2)s] = t - s$$

Finally, we determine the independence of the increments. Consider the increments $X_A - X_B$ with $A > B$ and $X_C - X_D$ with $C > D$. The correlation coefficient for these increments is

$$\rho_{\delta X_1, \delta X_2} = \frac{\min(A,C) - \min(A,D) - \min(B,C) + \min(B,D)}{\sqrt{(A-B)(C-D)}}$$

EXERCISE 3.2.4

Let $X_t = e^{W_t}$. Show that X_t is not a martingale, that $e^{-\frac{t}{2}}X_t$ is a martingale, and that for any constant $c \in \mathbb{R}$, the process $Y_t = e^{cW_t - \frac{1}{2}c^2 t}$ is a martingale.

SOLUTION 3.2.4

Part (a)

$$\mathbb{E}\left[e^{W_t}|\mathcal{F}_s\right] = \mathbb{E}\left[e^{W_t - W_s + W_s}\middle|\mathcal{F}_s\right] = e^{W_s}\left[e^{W_t - W_s}\middle|\mathcal{F}_s\right] = e^{W_s}\left(1 + \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \frac{1}{2!}\mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + \dots\right)$$

$$= e^{W_s}\left(1 + \mathbb{E}[W_t - W_s] + \frac{1}{2!}\mathbb{E}[(W_t - W_s)^2] + \dots\right) = e^{W_s}\left(1 + 0 + \frac{t-s}{2!} + 0 + \frac{3}{4!}(t-s)^2 + \dots\right)$$

$$= e^{W_s}e^{\frac{t-s}{2}} \neq e^{W_s}$$

Part (b)

Following the same logic as part (a), we find

$$\mathbb{E}\left[e^{-\frac{t}{2}}e^{W_t}\middle|\mathcal{F}_s\right] = e^{-\frac{t}{2}}e^{W_s}e^{\frac{t-s}{2}} = e^{-\frac{s}{2}}e^{W_t}$$

Part (c)

Again by similar logic to part (a), we find

$$\mathbb{E}\left[e^{cW_t-\frac{c^2t}{2}}\middle|\mathcal{F}_s\right] = e^{-\frac{c^2t}{2}}\mathbb{E}\left[e^{cW_t}\middle|\mathcal{F}_s\right] = e^{-\frac{c^2t}{2}}e^{cW_s}e^{\frac{c^2}{2}(t-s)} = e^{cW_s-\frac{c^2s}{2}}$$

EXERCISE 3.3.4

Show that the moment generating function of integrated Brownian motion is

$$m(u) = e^{u^2t^3/6}$$

and use it to find the mean and variance.

EXERCISE 3.3.4

The moment generating function is

$$m(u) = \mathbb{E}\left[e^{uZ_t}\right] = \mathbb{E}[e^{u\int_0^t W_s ds}] = 1 + u\mathbb{E}\left[\int_0^t W_s\ ds\right] + \frac{u^2}{2!}\mathbb{E}\left[\int_0^t\int_0^t W_rW_s\ drds\right] + \dots$$

By Fubini's theorem:

$$m(u) = 1 + u\int_0^t \mathbb{E}[W_s]ds + \frac{u^2}{2!}\int_0^t\int_0^t \mathbb{E}[W_rW_s]drds + \dots = 1 + \frac{u^2}{2}\frac{t^3}{3} + \dots = e^{\frac{u^2t^3}{6}}$$

The mean is

$$\mathbb{E}[Z_t] = \left.\frac{\partial m}{\partial u}\right|_{u=0} = \left.\frac{ut^3}{3}e^{\frac{u^2t^3}{6}}\right|_{u=0} = 0$$

and the variance is

$$Var(Z_t) = \mathbb{E}[Z_t^2] - \mathbb{E}^2[Z_t] = \left.\frac{\partial^2 m}{\partial u^2}\right|_{u=0} + 0 = \left[\frac{t^3}{3}e^{\frac{u^2t^3}{6}} + \left(\frac{ut^3}{3}\right)^2 e^{\frac{u^2t^3}{2}}\right]_{u=0} = \frac{t^3}{3}$$

EXERCISE 3.7.2

Let $P(R_t \le \rho)$ be the probability of a 2-dimensional Brownian motion being inside of the disk $D(0,\rho)$ at time $t > 0$. Show that

$$\frac{\rho^2}{2t}\left(1 - \frac{\rho^2}{4t}\right) < P(R_t \le t) < \frac{\rho^2}{2t}$$

SOLUTION 3.7.2

$$P(R_t \leq \rho) = \frac{2\pi}{2\pi t\Gamma(1/2)} \int_0^\rho re^{-r^2/2t}\, dr = 1 - e^{-\frac{\rho^2}{2t}} = \frac{\rho^2}{2t} - \frac{\rho^4}{8t^2} + \dots$$

so the inequality is easily seen to be true.

EXERCISE 3.7.4

Let $X_t = R_t/t$ with $t > 0$ where R_t is a 2-dimensional Bessel process. Show that $X_t \to 0$ as $t \to \infty$ in mean-square.

SOLUTION 3.7.4

From chapter 2, we have that if $\mathbb{E}[X_n] \to k$ and $Var(X_n) \to 0$ as $n \to \infty$, then the mean-square limit of X_n is equal to k. The mean and variance of the X_t are

$$\mathbb{E}[R_t/t] = \sqrt{\frac{\pi}{2t}} \to 0$$

$$Var(R_t/t) = \frac{2}{t}\left(1 - \frac{\pi}{4}\right) \to 0$$

so $X_t \to 0$ as well.

EXERCISE 3.8.6

Compute $\mathbb{E}[N_t^2|\mathcal{F}_s]$ for $s < t$. Is the process N_t^2 an $\mathcal{F}_s$-martingale?

SOLUTION 3.8.6

$$\mathbb{E}[N_t^2|\mathcal{F}_s] = \mathbb{E}[(N_t - N_0)(N_t - N_s) + N_t N_s|\mathcal{F}_s] = \mathbb{E}[N_t|\mathcal{F}_s]\left(\mathbb{E}[N_t - N_s] + N_s\right) = \left(N_s + \lambda(t-s)\right)^2 \neq N_s$$

so N_t^2 cannot be a martingale.

EXERCISE 3.8.9

Show that the moment generating function of M_t is $m_{M_t}(x) = e^{\lambda t(e^x - x - 1)}$. Compute $\mathbb{E}[(M_t - M_s)^n]$ for $n = 1, \dots, 4$.

SOLUTION 3.8.9

$$m_{M_t}(x) = \mathbb{E}\left[e^{xM_t}\right] = \mathbb{E}\left[1 + xM_t + \frac{x^2}{2!}M_t^2 + \dots\right] = 1 + 0 + \frac{x^2}{2!}\lambda t + \frac{x^3}{3!}\lambda t + \frac{x^4}{4!}\left(3\lambda^2 t^2 + \lambda t\right) + \dots = e^{\lambda t(e^x - x - 1)}$$

The following Mathematica code confirms the needed calculations:

```
Table[FullSimplify[D[e^(λ (t-s) (e^x - x - 1)), {x, n}]] /. x → 0, {n, 1, 4}]

{0, (-s + t) λ, (-s + t) λ, (-s + t) λ (1 + 3 (-s + t) λ)}
```

EXERCISE 3.10.2

Using that the interarrival times $T_1, T_2, \ldots$ are independent and exponentially distributed, calculate $\mathbb{E}[S_n]$ and $Var(S_n)$.

SOLUTION 3.10.2

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[T_i] = \frac{n}{\lambda}$$

$$Var(S_n) = \mathbb{E}[S_n^2] - \mathbb{E}^2[S_n] = \mathbb{E}[S_n^2] - \frac{n^2}{\lambda^2}$$

To find $\mathbb{E}[S_n^2]$, notice that we can write S_n^2 as

$$S_n^2 = T_1^2 + \cdots + T_n^2 + \underbrace{T_1T_2 + \ldots}_{n^2 - n \text{ terms}}$$

Since the T_i are independent, we have

$$\mathbb{E}[T_iT_j] = \mathbb{E}[T_i]\mathbb{E}[T_j] = \frac{1}{\lambda^2}, \quad i \neq j$$

We can use the PDF to find

$$\mathbb{E}[T_i^2] = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = \frac{2}{\lambda^2}$$

Putting it all together, we find

$$\mathbb{E}[S_n^2] = \frac{2n}{\lambda^2} + \frac{n(n-1)}{\lambda^2} = \frac{n(n+1)}{\lambda^2}$$

so we can easily see that

$$Var(S_n) = \frac{n}{\lambda^2}$$

EXERCISE 3.11.9

Part (a)

Let T_k be the k^{th} interarrival time. Show that

$$\mathbb{E}\left[e^{-\sigma T_k}\right] = \frac{\lambda}{\lambda + \sigma}$$

Part (b)

Let $N_t = n$. Show that

$$U_t = nt - \big(nT_1 + (n-1)T_2 + \cdots + 2T_{n-1} + T_n\big)$$

Part (c)

Find the conditional expectation

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right]$$

Part (d)

Calculate

$$\mathbb{E}\left[e^{-\sigma U_t}\right]$$

SOLUTION 3.11.9

Part (a)

$$\mathbb{E}\left[e^{-\sigma T_k}\right] = \int_0^\infty \lambda e^{-(\sigma+\lambda)x} dx = \frac{\lambda}{\lambda+\sigma}$$

Part (b)

This immediately follows from Proposition 3.11.1 and part (a) of Exercise 3.11.4.

Part (c)

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right] = \mathbb{E}\left[e^{-\sigma nt + \sigma \sum_{k=1}^n S_k}\right] = e^{-\sigma nt} \prod_{k=1}^n \mathbb{E}\left[e^{\sigma S_k}\right]$$

To calculate $\mathbb{E}\left[e^{\sigma S_k}\right]$, we need $\mathbb{E}[S_k^n]$:

$$\mathbb{E}[S_k^n] = \int_0^\infty \frac{t^{n+k-1}\lambda^k e^{-\lambda t}}{\Gamma(k)} = \frac{\Gamma(n+k)}{\lambda^n \Gamma(k)}$$

Then we have

$$\mathbb{E}\left[e^{\sigma S_k}\right] = \sum_{n=0}^\infty \left(\frac{\sigma}{\lambda}\right)^n \frac{\Gamma(k+n)}{n!\Gamma(k)} = \frac{1}{\left(1-\frac{\sigma}{\lambda}\right)^k}$$

so the final expectation is

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right] = e^{-\sigma nt} \prod_{k=1}^n \frac{1}{\left(1-\frac{\sigma}{\lambda}\right)^k} = e^{-\sigma nt}\left(1-\frac{\sigma}{\lambda}\right)^{\frac{-n(n+1)}{2}}$$

Part (d)

By Exercise 3.11.7, we have

$$\mathbb{E}\left[e^{-\sigma U_t}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{-\sigma U_t} \middle| N_t = n\right] P(N_t = n) = \sum_{n=0}^{\infty} e^{-\sigma n t}\left(1 - \frac{\sigma}{\lambda}\right)^{\frac{-n(n+1)}{2}} \frac{\lambda^n t^n}{n!} e^{-\lambda t}$$

EXERCISE 5.2.3

Let W_t be a Brownian motion with $s < t$. Show that $\mathbb{E}\left[(W_t - W_s)^4\right] = 3(t-s)^2$ *and* $\mathbb{E}\left[(W_t - W_s)^6\right] = 15(t-s)^3$.

SOLUTION 5.2.3

Using the identity

$$\mathbb{E}[W_t^n] = \int_{-\infty}^{\infty} \frac{x^n}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{2^{\frac{n}{2}-1}\left[(-1)^n + 1\right] t^{n/2} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}$$

where $n > 0$, we find that

$$\mathbb{E}[W_{t-s}^n] = \mathbb{E}[(W_t - W_s)^n] = \frac{2^{\frac{n}{2}-1}\left[(-1)^n + 1\right] \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} (t-s)^{n/2}$$

EXERCISE 5.4.2

Show the following:

$$\mathbb{E}\left[\int_0^T dW_t\right] = 0$$

$$\mathbb{E}\left[\int_0^T W_t dW_t\right] = 0$$

$$Var\left[\int_0^T W_t dW_t\right] = \frac{T^2}{2}$$

SOLUTION 5.4.2

$$\mathbb{E}\left[\int_0^T dW_t\right] = \mathbb{E}[W_t] = 0$$

$$\mathbb{E}\left[\int_0^T W_t dW_t\right] = \frac{1}{2}\mathbb{E}[W_T^2] - \frac{T}{2} = \frac{T}{2} - \frac{T}{2} = 0$$

$$Var\left[\int_0^T W_t dW_t\right] = \mathbb{E}\left[\left(\int_0^T W_t dW_t\right)^2\right] - \mathbb{E}^2\left[\int_0^T W_t dW_t\right] = \mathbb{E}\left[\left(\frac{1}{2}W_T^2 - \frac{T}{2}\right)^2\right] = \frac{3T^2}{4} + \frac{T^2}{4} - \frac{T^2}{2} = \frac{T^2}{2}$$

EXERCISE 5.6.2

Let $Z_t = \int_0^t W_s ds$.

(a) *Use integration by parts to show that*

$$Z_t = \int_0^t (t-s) dW_t$$

(b) *Use the properties of Weiner integrals to show that*

$$Var(z_t) = \frac{t^3}{3}$$

SOLUTION 5.6.2

Integrating by parts gives

$$Z_t = sW_s\Big|_{s=0}^{s=t} - \int_0^t s\ dW_s = tW_t - \int_0^t s\ dW_s = t\int_0^t dW_s - s\int_0^t dW_s = \int_0^t (t-s)\ dW_s$$

By Proposition 5.6.1,

$$Var(Z_t) = \int_0^t (t-s)^2\ ds = \frac{t^3}{3}$$

EXERCISE 5.6.7

Show that

$$\underset{t\to 0}{\text{ms-lim}}\ \frac{1}{t}\int_0^t u\ dW_u = 0$$

SOLUTION 5.6.7

Making a change of variable:

$$\underset{t\to 0}{\text{ms-lim}}\ \frac{1}{t}\int_0^t u\ dW_u = \text{ms-}\lim_{k\to\infty} k\int_0^{1/k} u\ dW_u$$

By Proposition 5.6.1, we have

$$\mathbb{E}\left[k\int_0^{1/k} u\ dW_u\right] = 0 \qquad Var\left(k\int_0^{1/k} u\ dW_u\right) = \int_0^{1/k} k^2u^2\ du = \frac{1}{3k}$$

The rest immediately follows from Proposition 4.9.1.