Solutions to Calin's *An Informal Introduction to Stochastic Calculus*

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EXERCISE 2.9.4

Show that

(a) $Cov(X,Y) = \mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$, *where* $\mu_X = \mathbb{E}[X]$ *and* $\mu_Y = \mathbb{E}[Y]$.

(b) $Var(X) = \mathbb{E}\left[(X-\mu_X)^2\right]$

SOLUTION 2.9.4

$$\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}[XY] - 2\mathbb{E}(x)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Part (b) follows trivially from part (a).

EXERCISE 2.9.5

Let μ *and* σ *denote the mean and standard deviation of the random variable* X*. Show that*

$$\mathbb{E}[X^2] = \mu^2 + \sigma^2$$

SOLUTION 2.9.5

For the random variable X, the mean squared μ^2 is given by

$$\mu^2 = \left[\int xp(x)dx\right]^2$$

and the standard deviation σ^2 is given by

$$\sigma^2 = Var(X) = \mathbb{E}(X^2) - \left(\mathbb{E}(X)\right)^2 = \int x^2p(x)dx - \left[\int xp(x)dx\right]^2$$

Therefore, we have

$$\mathbb{E}(X^2) = \int x^2p(x)dx = \int x^2p(x)dx - \left[\int xp(x)dx\right]^2 + \left[\int xp(x)dx\right]^2 = \mu^2 + \sigma^2$$

EXERCISE 2.9.7

(a) *Prove that for any random variables* A *and* B*, we have*

$$\mathbb{E}[AB]^2 \leq \mathbb{E}[A^2]\mathbb{E}[B^2]$$

(b) *Use part (a)* to show that for any random variables X and Y we have

$$-1 \leq \rho(X,Y) \leq 1$$

(c) *What can you say about the random variables* X *and* Y *if* $\rho(X,Y) = 1$*?*

SOLUTION 2.9.7

Consider the quantity

$$A - B\frac{\mathbb{E}(AB)}{\mathbb{E}(B^2)}$$

with $\mathbb{E}(B^2) \neq 0$. By the basic properties of expectations, we have that

$$\mathbb{E}\left[\left(A - B\frac{\mathbb{E}(AB)}{\mathbb{E}(B^2)}\right)^2\right] \geq 0$$

Expanding the parentheses and using linearity of the expectation gives

$$0 \leq \mathbb{E}(A^2) - \frac{\mathbb{E}^2(AB)}{\mathbb{E}(B^2)}$$

which implies $\mathbb{E}^2(AB) \leq \mathbb{E}(A^2)\mathbb{E}(B^2)$. Now, using the results of Exercise 2.9.4, we have

$$\rho^2(X,Y) = \frac{Cov^2(X,Y)}{Cov(X,X)Cov(Y,Y)} = \frac{\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]}$$

By the result of part (a):

$$\frac{\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]} \leq \frac{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]}{\mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]} = 1$$

so we find $-1 \leq \rho(X,Y) \leq 1$. If $\rho(X,Y) = 1$, then we have

$$\mathbb{E}^2\left[(X-\mu_X)(Y-\mu_Y)\right] = \mathbb{E}\left[(X-\mu_X)^2\right]\mathbb{E}\left[(Y-\mu_Y)^2\right]$$

which implies that $Y = aX + b$, where $a, b \in \mathbb{R}$.

EXERCISE 2.9.8

Let $g : [0,1] \to [0,\infty)$ be an integrable function with

$$\int_0^1 g(x)dx = 1.$$

Consider $Q : \mathcal{B}([0,1]) \to \mathbb{R}$ given by

$$Q(A) = \int_A g(x)dx$$

Show that Q is a probability measure on $\big(\Omega = [0,1], \mathcal{B}([0,1])\big)$

SOLUTION 2.9.8

Recall the definition of a probability measure:

A probability measure P on the space $(\Omega, \mathcal{F})$, where Ω is the set of outcomes and $\mathcal{F}$ is the σ-algebra corresponding to the collection of events, is a real-valued function defined on $\mathcal{F}$ that satisfies the following axioms:

(a) $P(A) \geq 0 \ \forall \ A \in \mathcal{F}$

(b) $P(\Omega) = 1$

(c) *If $\{E_i : i \in I\}$ is a countable, pairwise disjoint set of events, then*

$$P\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} P(E_i)$$

By the definition of $g(x)$, requirement 2 is automatically satisfied.
Now let $A \in \mathcal{B}([0,1])$. If A is a set of measure zero, then $g(x) = 0$, by the basic properties of the Lebesgue integral. For all other measurable A, the Lebesgue integral of a strictly positive function is also strictly positive. Combining these two statements, requirement 1 is satisfied.
Finally, consider a countable union of disjoint subintervals of $[0,1]$. By the additivity of the Lebesgue integral (since $E_i \cap E_j = \emptyset$ for $i \neq j$), we have

$$\int_{\bigcup_{i \in I} E_i} g(x)dx = \sum_{i \in I} \int_{E_i} g(x)dx$$

from which requirement 3 follows.

EXERCISE 2.10.4

Show that if X and Y are two independent random variables, then $m_{X+Y}(t) = m_X(t)m_Y(t)$.

SOLUTION 2.10.4

By Proposition 2.8.1, the density function of independent random variables factorizes. Then, we have

$$m_{X+Y}(t) = \int e^{t(x+y)} p_{XY}(x,y)dxdy = \int e^{tx} p_X(x) e^{ty} p_Y(y)dxdy = m_X(t)m_Y(t)$$

EXERCISE 2.10.5

Given that the moment generating function of a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is $m(t) = \mathbb{E}[e^{tX}] = e^{\mu t + t^2\sigma^2/2}$, show that

(a) $\mathbb{E}[Y^n] = e^{n\mu + n^2\sigma^2/2}$, *where $Y = e^X$.*

(b) *Show that the mean and variance of the log-normal random variable $Y = e^X$* are

$$\mathbb{E}[Y] = e^{\mu+\sigma^2/2}, \qquad Var[Y] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

SOLUTION 2.10.5

Part (a):

$$\mathbb{E}[Y^n] = \mathbb{E}[e^{nX}] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{(nX)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{n^k}{k!}\mathbb{E}[X^k] = \sum_{k=0}^{\infty} \frac{n^k}{k!}\frac{d^k}{dt^k}m_X(t)\Big|_{t=0} = m_X(n) = e^{n\mu+n^2\sigma^2/2}$$

$\mathbb{E}[Y]$ is just a special case of part (a). To calculate $Var(Y)$:

$$Var(Y) = \mathbb{E}[Y^2] - \mathbb{E}^2[Y] = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

EXERCISE 2.11.4

Consider the independent, exponentially distributed random variables $X \sim \lambda_1 e^{-\lambda_1 t}$,*and* $Y \sim \lambda_2 e^{-\lambda_2 t}$, *with* $\lambda_1 \neq \lambda_2$. Show that the sum is distributed as $X + Y \sim \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$.

SOLUTION 2.11.4

This result follows immediately from Theorem 2.11.1. The probability density p_{X+Y} of the sum is

$$p_{X+Y} = \int_0^t p_X(t-\tau)p_Y(\tau)d\tau = \lambda_1\lambda_2 e^{-\lambda_1 t}\int_0^t e^{(\lambda_1-\lambda_2)\tau}d\tau = \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

EXERCISE 2.12.1

EXERCISE 2.12.7

Prove that, if $\mathcal{H} \subset \mathcal{G}$,

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}\right] = \mathbb{E}\left[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}\right] = \mathbb{E}\left[X|\mathcal{H}\right]$$

SOLUTION 2.12.7

EXERCISE 2.12.9

SOLUTION 2.12.9

Part (a) follows immediately from the definition of $\mathcal{G}$-measurability.

By the definition of integration, we have

$$P(A) = \int_A dP$$

From the definition of χ_A, this is equal to

$$\int_A \chi_A(\omega)dP = \int_\Omega \chi_A(\omega)dP$$

but this is easily seen to be $\mathbb{E}\chi_A$.

Part (d) follows immediately from parts (b) and (c).

EXERCISE 3.1.9

For $t_0 \geq 0$, show that the process $X_t = W_{t+t_0} - W_{t_0}$ is a Brownian motion.

SOUTION 3.1.9

It is clear that $X_0 = 0$. Since X_t is a sum of continuous functions, it is continuous as well. Now consider the increments $X_A - X_B$ and $X_C - X_D$, with $A > B > 0$ and $C > D > 0$. We have

$$\mathbb{E}\left[(X_A - X_B)(X_C - X_D)\right] = \mathbb{E}[X_A X_C] - \mathbb{E}[X_A X_D] - \mathbb{E}[X_B X_C] + \mathbb{E}[X_B X_D]$$

$$\begin{aligned} &= \mathbb{E}\left[(W_{A+t_0} - W_{t_0})(W_{C+t_0} - W_{t_0})\right] - \mathbb{E}\left[(W_{A+t_0} - W_{t_0})(W_{D+t_0} - W_{t_0})\right] \\ &- \mathbb{E}\left[(W_{B+t_0} - W_{t_0})(W_{C+t_0} - W_{t_0})\right] + \mathbb{E}\left[(W_{B+t_0} - W_{t_0})(W_{D+t_0} - W_{t_0})\right] \end{aligned}$$

By the independence of Brownian motion increments, this is equal to

$$\begin{aligned} &\mathbb{E}[W_{A+t_0} - W_{t_0}]\,\mathbb{E}[W_{C+t_0} - W_{t_0}] - \mathbb{E}[W_{A+t_0} - W_{t_0}]\,\mathbb{E}[W_{D+t_0} - W_{t_0}] \\ &- \mathbb{E}[W_{B+t_0} - W_{t_0}]\,\mathbb{E}[W_{C+t_0} - W_{t_0}] + \mathbb{E}[W_{B+t_0} - W_{t_0}]\,\mathbb{E}[W_{D+t_0} - W_{t_0}] \end{aligned}$$

$$= \mathbb{E}[X_A]\mathbb{E}[X_C] - \mathbb{E}[X_A]\mathbb{E}[X_D] - \mathbb{E}[X_B]\mathbb{E}[X_C] + \mathbb{E}[X_B]\mathbb{E}[X_D]$$

$$= \mathbb{E}[X_A - X_B]E[X_C - X_D]$$

Since the increments are independent, their sum is also normally distributed. The mean and variance are

$$\mathbb{E}[X_t - X_s] = \mathbb{E}[W_{t+t_0} - W_{t_0} - W_{s+t_0} + W_{t_0}] = \mathbb{E}[W_{t+t_0}] - \mathbb{E}[W_{s+t_0}] = 0$$

$$Var(X_t - X_s) = \mathbb{E}\left[(X_t - X_s)^2\right] - \mathbb{E}^2[X_t - X_s] = \mathbb{E}\left[X_t^2 + X_s^2 - 2X_t X_s\right] + 0$$

$$= \mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[X_s^2\right] - 2\mathbb{E}[X_t X_s] = Var(X_t) + \mathbb{E}^2[X_t] + Var(X_s) + \mathbb{E}^2[X_s] - 2\mathbb{E}[X_t X_s]$$

$$= t + 0 + s + 0 - 2s = t - s$$

EXERCISE 3.1.14

Consider the process $Y_t = tW_{1/t}$, $t > 0$ and define $Y_0 = 0$. Find the distribution of Y_t, the PDF of Y_t, $Cov(Y_s, Y_t)$, $\mathbb{E}[Y_t - Y_s]$, and $Var(Y_t - Y_s)$ for $t > s > 0$. Note that Y_t is a Brownian motion.

SOLUTION 3.1.14

Since Y_t is a Brownian motion, it is normally distributed. The mean and variance are

$$\mathbb{E}[Y_t] = t\mathbb{E}[W_{1/t}] = 0$$

$$Var(Y_t) = Var(tW_{1/t}) = t^2 Var(W_{1/t}) = t$$

The PDF is

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

The covariance is

$$Cov(Y_s, Y_t) = \mathbb{E}[Y_s Y_t] - \mathbb{E}[Y_s]\mathbb{E}[Y_t] = \mathbb{E}[Y_s Y_t] = \mathbb{E}[Y_s (Y_t - Y_s) + Y_s^2] = \mathbb{E}[Y_s^2] = s$$

We also have

$$\mathbb{E}[Y_t - Y_s] = \mathbb{E}[Y_t] - \mathbb{E}[Y_s] = 0$$

$$Var(Y_t - Y_s) = \mathbb{E}\left[(Y_t - Y_s)^2\right] - \mathbb{E}^2\left[Y_t - Y_s\right] = \mathbb{E}[Y_t^2] + \mathbb{E}[Y_s^2] - 2\mathbb{E}[Y_t Y_s] = t - s$$

EXERCISE 3.1.17

Let $0 < s < t$. Show that

$$\mathbb{E}[W_t^3 | \mathcal{F}_s] = W_s^3 + 3W_s(t-s)$$

$$\mathbb{E}[W_t^4 | \mathcal{F}_s] = W_s^4 + 6W_s^2(t-s) + 3(t-s)^2$$

SOLUTION 3.1.17

$$\mathbb{E}[(W_t - W_s)^3 | \mathcal{F}_s] = \mathbb{E}[W_t^3 | \mathcal{F}_s] - \mathbb{E}[W_s^3 | \mathcal{F}_s] + 3\mathbb{E}[W_t W_s^2 | \mathcal{F}_s] - 3\mathbb{E}[W_t^2 W_s | \mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^3] = \mathbb{E}[W_{t-s}^3] = 0$$

$$\mathbb{E}[W_t^3 | \mathcal{F}_s] = \mathbb{E}[W_s^3 | \mathcal{F}_s] + 3\mathbb{E}[W_t^2 W_s | \mathcal{F}_s] - 3\mathbb{E}[W_t W_s^2 | \mathcal{F}_s] = W_s^3 + 3W_s \mathbb{E}[W_t^2 | \mathcal{F}_s] - 3W_s^2 \mathbb{E}[W_t | \mathcal{F}_s]$$

$$= W_s^3 + 3W_s\left(W_s^2 + t - s\right) - 3W_s^3 = W_s^3 + 3W_s(t-s)$$

$$\mathbb{E}[(W_t - W_s)^4 | \mathcal{F}_s] = \mathbb{E}[W_{t-s}^4] = 3(t-s)^2 = \mathbb{E}[|W_t^4 \mathcal{F}_s] - 4\mathbb{E}[W_t^3 W_s | \mathcal{F}_s] + 6\mathbb{E}[W_t^2 W_s^2 | \mathcal{F}_s] - 4\mathbb{E}[W_t W_s^3 | \mathcal{F}_s] + \mathbb{E}[W_s^4 | \mathcal{F}_s]$$

$$= \mathbb{E}[W_t^4 | \mathcal{F}_s] - 4W_s \mathbb{E}[W_t^3 | \mathcal{F}_s] + 6W_s^2 \mathbb{E}[W_t^2 | \mathcal{F}_s] - 4W_s^3 \mathbb{E}[W_t | \mathcal{F}_s] + W_s^4$$

$$= \mathbb{E}[W_t^4 | \mathcal{F}_s] - 4W_s^4 - 12W_s^2(t-s) + 6W_s^4 + 6W_s^2(t-s) - 3W_s^4$$

Rearranging to solve for $\mathbb{E}[W_t^4 | \mathcal{F}_s]$ gives $\mathbb{E}[W_t^4 | \mathcal{F}_s] = W_s^4 + 6W_s^2(t-s) + 3(t-s)^2$

EXERCISE 3.1.18

Show that

$$\mathbb{E}\left[\int_s^t W_u \, du \middle| \mathcal{F}_s\right] = (t-s)W_s$$

SOLUTION 3.1.18

First, write the integral as a Riemann sum:

$$\mathbb{E}\left[\int_s^t W_u\ du \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\sum_i W_{u_i}(u_{i+1} - u_i) \Big| \mathcal{F}_s\right]$$

where $s \leq u_i < t$. By linearity of the expectation, this is equal to

$$\sum_i (u_{i+1} - u_i)\mathbb{E}\left[W_{u_i}|\mathcal{F}_s\right]$$

Since W_t is a martingale, $\mathbb{E}\left[W_{u_i}|\mathcal{F}_s\right] = W_s$, and passing back to the continuum limit, we have

$$E\left[W_{u_i}|\mathcal{F}_s\right] = W_s \int_s^t du = W_s(t-s)$$

EXERCISE 3.1.19

Show that the process

$$X_t = W_t^3 - 3\int_0^t W_s\ ds$$

is a martingale with respect to the information set $\mathcal{F}_t = \sigma\{W_s; s < t\}$

SOLUTION 3.1.19

It is clear that X_t is integrable for all t and that X_t is adapted to $\mathcal{F}_t$. We also need to have $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$ for all $s < t$. By the result of Exercise 3.1.17, this expectation is equal to

$$\mathbb{E}\left[W_t^3 - 3\int_0^t W_u\ du \Big| \mathcal{F}_s\right] = W_s^3 + 3W_s(t-s) - 3\left(\mathbb{E}\left[\int_s^t W_u\ du \Big| \mathcal{F}_s\right] + \mathbb{E}\left[\int_0^s W_u\ du \Big| \mathcal{F}_s\right]\right)$$

By the result of Exercise 3.1.18, we can evaluate the first integral, and the right side of the equality becomes

$$W_s^3 + 3W_s(t-s) - 3\left(W_s(t-s) + \mathbb{E}\left[\int_0^s W_u\ du \Big| \mathcal{F}_s\right]\right)$$

The second integral simplifies because W_t is a martingale, and after canceling terms, we are left with

$$\mathbb{E}\left[W_t^3 - 3\int_0^t W_u\ du \Big| \mathcal{F}_s\right] = W_s^3 - 3\int_0^s W_u\ du = X_s$$

so X_s is a martingale.

EXERCISE 3.1.21

Let W_t *and* $\tilde{W}_t$ *be independent Brownian motions and* ρ *be a constant with* $|\rho| \leq 1$. *Show that the process* $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$ *is continuous and has the distribution* $\mathcal{N}(0, t)$. *Is* X_t *a Brownian motion?*

SOLUTION 3.1.21

As a sum of continuous functions, X_t is continuous. The expectation is

$$\mathbb{E}[X_t] = \rho\mathbb{E}[W_t] + \sqrt{1-\rho^2}\mathbb{E}[\tilde{W}_t] = 0$$

The variance is

$$Var(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}^2[X_t] = \mathbb{E}\left[\rho^2 W_t^2 + (1-\rho^2)\tilde{W}_t^2 + 2\rho\sqrt{1-\rho^2}W_t\tilde{W}_t\right] = \rho^2 t + (1-\rho^2)t + 2\rho\sqrt{1-\rho^2}\mathbb{E}[W_t]\mathbb{E}[\tilde{W}_t] = t$$

Since $W_0 = \tilde{W}_0 = 0$, $X_0 = 0$. The mean and variance of the increments $X_t - X_s$ are

$$\mathbb{E}[X_t - X_s] = \mathbb{E}[\rho W_t + \sqrt{1-\rho^2}\tilde{W}_t - \rho W_s - \sqrt{1-\rho^2}\tilde{W}_s] = 0$$

$$Var(X_t - X_s) = \mathbb{E}[X_t^2 + X_s^2 - 2X_tX_s]$$

$$\begin{aligned}=&\mathbb{E}[\rho^2 W_t^2 + (1-\rho^2)\tilde{W}_t^2 + 2\rho\sqrt{1-\rho^2}W_t\tilde{W}_t] + \mathbb{E}[\rho^2 W_s^2 + (1-\rho^2)\tilde{W}_s^2 + 2\rho\sqrt{1-\rho^2}W_s\tilde{W}_s]\\ &- 2\mathbb{E}[\left(\rho W_t + \sqrt{1-\rho^2}\tilde{W}_t\right)\left(\rho W_s + \sqrt{1-\rho^2}\tilde{W}_s\right)]\end{aligned}$$

$$= \rho^2 t + (1-\rho^2)t + \rho^2 s + (1-\rho^2)s - 2[\rho^2 s + (1-\rho^2)s] = t - s$$

Finally, we determine the independence of the increments. Consider the increments $X_A - X_B$ with $A > B$ and $X_C - X_D$ with $C > D$. The correlation coefficient for these increments is

$$\rho_{\delta X_1, \delta X_2} = \frac{\min(A,C) - \min(A,D) - \min(B,C) + \min(B,D)}{\sqrt{(A-B)(C-D)}}$$

EXERCISE 3.2.4

Let $X_t = e^{W_t}$. Show that X_t is not a martingale, that $e^{-\frac{t}{2}}X_t$ is a martingale, and that for any constant $c \in \mathbb{R}$, the process $Y_t = e^{cW_t - \frac{1}{2}c^2 t}$ is a martingale.

SOLUTION 3.2.4

Part (a)

$$\mathbb{E}\left[e^{W_t}|\mathcal{F}_s\right] = \mathbb{E}\left[e^{W_t - W_s + W_s}\middle|\mathcal{F}_s\right] = e^{W_s}\left[e^{W_t - W_s}\middle|\mathcal{F}_s\right] = e^{W_s}\left(1 + \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \frac{1}{2!}\mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + \dots\right)$$

$$= e^{W_s}\left(1 + \mathbb{E}[W_t - W_s] + \frac{1}{2!}\mathbb{E}[(W_t - W_s)^2] + \dots\right) = e^{W_s}\left(1 + 0 + \frac{t-s}{2!} + 0 + \frac{3}{4!}(t-s)^2 + \dots\right)$$

$$= e^{W_s}e^{\frac{t-s}{2}} \neq e^{W_s}$$

Part (b)

Following the same logic as part (a), we find

$$\mathbb{E}\left[e^{-\frac{t}{2}}e^{W_t}\middle|\mathcal{F}_s\right] = e^{-\frac{t}{2}}e^{W_s}e^{\frac{t-s}{2}} = e^{-\frac{s}{2}}e^{W_t}$$

Part (c)

Again by similar logic to part (a), we find

$$\mathbb{E}\left[e^{cW_t-\frac{c^2t}{2}}\middle|\mathcal{F}_s\right] = e^{-\frac{c^2t}{2}}\mathbb{E}\left[e^{cW_t}\middle|\mathcal{F}_s\right] = e^{-\frac{c^2t}{2}}e^{cW_s}e^{\frac{c^2}{2}(t-s)} = e^{cW_s-\frac{c^2s}{2}}$$

EXERCISE 3.3.4

Show that the moment generating function of integrated Brownian motion is

$$m(u) = e^{u^2t^3/6}$$

and use it to find the mean and variance.

EXERCISE 3.3.4

The moment generating function is

$$m(u) = \mathbb{E}\left[e^{uZ_t}\right] = \mathbb{E}[e^{u\int_0^t W_s ds}] = 1 + u\mathbb{E}\left[\int_0^t W_s\ ds\right] + \frac{u^2}{2!}\mathbb{E}\left[\int_0^t\int_0^t W_rW_s\ drds\right] + \dots$$

By Fubini's theorem:

$$m(u) = 1 + u\int_0^t \mathbb{E}[W_s]ds + \frac{u^2}{2!}\int_0^t\int_0^t \mathbb{E}[W_rW_s]drds + \dots = 1 + \frac{u^2}{2}\frac{t^3}{3} + \dots = e^{\frac{u^2t^3}{6}}$$

The mean is

$$\mathbb{E}[Z_t] = \frac{\partial m}{\partial u}\bigg|_{u=0} = \frac{ut^3}{3}e^{\frac{u^2t^3}{6}}\bigg|_{u=0} = 0$$

and the variance is

$$Var(Z_t) = \mathbb{E}[Z_t^2] - \mathbb{E}^2[Z_t] = \frac{\partial^2 m}{\partial u^2}\bigg|_{u=0} + 0 = \left[\frac{t^3}{3}e^{\frac{u^2t^3}{6}} + \left(\frac{ut^3}{3}\right)^2 e^{\frac{u^2t^3}{2}}\right]_{u=0} = \frac{t^3}{3}$$

EXERCISE 3.7.2

Let $P(R_t \leq \rho)$ be the probability of a 2-dimensional Brownian motion being inside of the disk $D(0,\rho)$ at time $t > 0$. Show that

$$\frac{\rho^2}{2t}\left(1 - \frac{\rho^2}{4t}\right) < P(R_t \leq t) < \frac{\rho^2}{2t}$$

SOLUTION 3.7.2

$$P(R_t \le \rho) = \frac{2\pi}{2\pi t \Gamma(1/2)} \int_0^\rho r e^{-r^2/2t}\, dr = 1 - e^{-\frac{\rho^2}{2t}} = \frac{\rho^2}{2t} - \frac{\rho^4}{8t^2} + \dots$$

so the inequality is easily seen to be true.

EXERCISE 3.7.4

Let $X_t = R_t/t$ with $t > 0$ where R_t is a 2-dimensional Bessel process. Show that $X_t \to 0$ as $t \to \infty$ in mean-square.

SOLUTION 3.7.4

From chapter 2, we have that if $\mathbb{E}[X_n] \to k$ and $Var(X_n) \to 0$ as $n \to \infty$, then the mean-square limit of X_n is equal to k. The mean and variance of the X_t are

$$\mathbb{E}[R_t/t] = \sqrt{\frac{\pi}{2t}} \to 0$$

$$Var(R_t/t) = \frac{2}{t}\left(1 - \frac{\pi}{4}\right) \to 0$$

so $X_t \to 0$ as well.

EXERCISE 3.8.6

Compute $\mathbb{E}[N_t^2|\mathcal{F}_s]$ for $s < t$. Is the process N_t^2 an $\mathcal{F}_s$-martingale?

SOLUTION 3.8.6

$$\mathbb{E}[N_t^2|\mathcal{F}_s] = \mathbb{E}[(N_t - N_0)(N_t - N_s) + N_t N_s|\mathcal{F}_s] = \mathbb{E}[N_t|\mathcal{F}_s]\left(\mathbb{E}[N_t - N_s] + N_s\right) = \left(N_s + \lambda(t-s)\right)^2 \ne N_s$$

so N_t^2 cannot be a martingale.

EXERCISE 3.8.9

Show that the moment generating function of M_t is $m_{M_t}(x) = e^{\lambda t(e^x - x - 1)}$. Compute $\mathbb{E}[(M_t - M_s)^n]$ for $n = 1, \dots, 4$.

SOLUTION 3.8.9

$$m_{M_t}(x) = \mathbb{E}\left[e^{xM_t}\right] = \mathbb{E}\left[1 + xM_t + \frac{x^2}{2!}M_t^2 + \dots\right] = 1 + 0 + \frac{x^2}{2!}\lambda t + \frac{x^3}{3!}\lambda t + \frac{x^4}{4!}\left(3\lambda^2 t^2 + \lambda t\right) + \dots = e^{\lambda t(e^x - x - 1)}$$

The following Mathematica code confirms the needed calculations:

```
Table[FullSimplify[D[ⅇ^(λ (t-s) (ⅇ^x - x - 1)), {x, n}]] /. x → 0, {n, 1, 4}]
{0, (-s + t) λ, (-s + t) λ, (-s + t) λ (1 + 3 (-s + t) λ)}
```

EXERCISE 3.10.2

Using that the interarrival times $T_1, T_2, \ldots$ are independent and exponentially distributed, calculate $\mathbb{E}[S_n]$ and $Var(S_n)$.

SOLUTION 3.10.2

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[T_i] = \frac{n}{\lambda}$$

$$Var(S_n) = \mathbb{E}[S_n^2] - \mathbb{E}^2[S_n] = \mathbb{E}[S_n^2] - \frac{n^2}{\lambda^2}$$

To find $\mathbb{E}[S_n^2]$, notice that we can write S_n^2 as

$$S_n^2 = T_1^2 + \cdots + T_n^2 + \underbrace{T_1T_2 + \ldots}_{n^2 - n \text{ terms}}$$

Since the T_i are independent, we have

$$\mathbb{E}[T_iT_j] = \mathbb{E}[T_i]\mathbb{E}[T_j] = \frac{1}{\lambda^2}, \quad i \neq j$$

We can use the PDF to find

$$\mathbb{E}[T_i^2] = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = \frac{2}{\lambda^2}$$

Putting it all together, we find

$$\mathbb{E}[S_n^2] = \frac{2n}{\lambda^2} + \frac{n(n-1)}{\lambda^2} = \frac{n(n+1)}{\lambda^2}$$

so we can easily see that

$$Var(S_n) = \frac{n}{\lambda^2}$$

EXERCISE 3.11.9

Part (a)

Let T_k be the k^{th} interarrival time. Show that

$$\mathbb{E}\left[e^{-\sigma T_k}\right] = \frac{\lambda}{\lambda + \sigma}$$

Part (b)

Let $N_t = n$. Show that

$$U_t = nt - \big(nT_1 + (n-1)T_2 + \cdots + 2T_{n-1} + T_n\big)$$

Part (c)

Find the conditional expectation

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right]$$

Part (d)

Calculate

$$\mathbb{E}\left[e^{-\sigma U_t}\right]$$

SOLUTION 3.11.9

Part (a)

$$\mathbb{E}\left[e^{-\sigma T_k}\right] = \int_0^\infty \lambda e^{-(\sigma+\lambda)x} dx = \frac{\lambda}{\lambda+\sigma}$$

Part (b)

This immediately follows from Proposition 3.11.1 and part (a) of Exercise 3.11.4.

Part (c)

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right] = \mathbb{E}\left[e^{-\sigma nt + \sigma \sum_{k=1}^n S_k}\right] = e^{-\sigma nt} \prod_{k=1}^n \mathbb{E}\left[e^{\sigma S_k}\right]$$

To calculate $\mathbb{E}\left[e^{\sigma S_k}\right]$, we need $\mathbb{E}[S_k^n]$:

$$\mathbb{E}[S_k^n] = \int_0^\infty \frac{t^{n+k-1}\lambda^k e^{-\lambda t}}{\Gamma(k)} = \frac{\Gamma(n+k)}{\lambda^n \Gamma(k)}$$

Then we have

$$\mathbb{E}\left[e^{\sigma S_k}\right] = \sum_{n=0}^\infty \left(\frac{\sigma}{\lambda}\right)^n \frac{\Gamma(k+n)}{n!\Gamma(k)} = \frac{1}{\left(1-\frac{\sigma}{\lambda}\right)^k}$$

so the final expectation is

$$\mathbb{E}\left[e^{-\sigma U_t}\middle| N_t = n\right] = e^{-\sigma nt} \prod_{k=1}^n \frac{1}{\left(1-\frac{\sigma}{\lambda}\right)^k} = e^{-\sigma nt}\left(1-\frac{\sigma}{\lambda}\right)^{\frac{-n(n+1)}{2}}$$

Part (d)

By Exercise 3.11.7, we have

$$\mathbb{E}\left[e^{-\sigma U_t}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{-\sigma U_t}\Big|N_t = n\right] P(N_t = n) = \sum_{n=0}^{\infty} e^{-\sigma n t}\left(1-\frac{\sigma}{\lambda}\right)^{\frac{-n(n+1)}{2}} \frac{\lambda^n t^n}{n!} e^{-\lambda t}$$

EXERCISE 5.2.3

Let W_t be a Brownian motion with $s < t$. Show that $\mathbb{E}\left[(W_t - W_s)^4\right] = 3(t-s)^2$ *and* $\mathbb{E}\left[(W_t - W_s)^6\right] = 15(t-s)^3$.

SOLUTION 5.2.3

Using the identity

$$\mathbb{E}[W_t^n] = \int_{-\infty}^{\infty} \frac{x^n}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = \frac{2^{\frac{n}{2}-1}\left[(-1)^n+1\right] t^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}$$

where $n > 0$, we find that

$$\mathbb{E}[W_{t-s}^n] = \mathbb{E}[(W_t - W_s)^n] = \frac{2^{\frac{n}{2}-1}\left[(-1)^n+1\right]\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}(t-s)^{n/2}$$

EXERCISE 5.4.1

Show the following:

$$\mathbb{E}\left[\int_0^T dW_t\right] = 0$$

$$\mathbb{E}\left[\int_0^T W_t dW_t\right] = 0$$

$$Var\left[\int_0^T W_t dW_t\right] = \frac{T^2}{2}$$

SOLUTION 5.4.1

$$\mathbb{E}\left[\int_0^T dW_t\right] = \mathbb{E}[W_t] = 0$$

$$\mathbb{E}\left[\int_0^T W_t dW_t\right] = \frac{1}{2}\mathbb{E}[W_T^2] - \frac{T}{2} = \frac{T}{2} - \frac{T}{2} = 0$$

$$Var\left[\int_0^T W_t dW_t\right] = \mathbb{E}\left[\left(\int_0^T W_t dW_t\right)^2\right] - \mathbb{E}^2\left[\int_0^T W_t dW_t\right] = \mathbb{E}\left[\left(\frac{1}{2}W_T^2 - \frac{T}{2}\right)^2\right] = \frac{3T^2}{4} + \frac{T^2}{4} - \frac{T^2}{2} = \frac{T^2}{2}$$

EXERCISE 5.6.2

Let $Z_t = \int_0^t W_s ds$.

(a) *Use integration by parts to show that*

$$Z_t = \int_0^t (t-s) dW_t$$

(b) *Use the properties of Weiner integrals to show that*

$$Var(z_t) = \frac{t^3}{3}$$

SOLUTION 5.6.2

Integrating by parts gives

$$Z_t = sW_s \Big|_{s=0}^{s=t} - \int_0^t s\ dW_s = tW_t - \int_0^t s\ dW_s = t\int_0^t dW_s - s\int_0^t dW_s = \int_0^t (t-s)\ dW_s$$

By Proposition 5.6.1,

$$Var(Z_t) = \int_0^t (t-s)^2\ ds = \frac{t^3}{3}$$

EXERCISE 5.6.7

Show that

$$\text{ms-}\lim_{t\to 0} \frac{1}{t}\int_0^t u\ dW_u = 0$$

SOLUTION 5.6.7

Making a change of variable:

$$\text{ms-}\lim_{t\to 0} \frac{1}{t}\int_0^t u\ dW_u = \text{ms-}\lim_{k\to\infty} k\int_0^{1/k} u\ dW_u$$

By Proposition 5.6.1, we have

$$\mathbb{E}\left[k\int_0^{1/k} u\ dW_u\right] = 0 \qquad Var\left(k\int_0^{1/k} u\ dW_u\right) = \int_0^{1/k} k^2u^2\ du = \frac{1}{3k}$$

The rest immediately follows from Proposition 4.9.1.

EXERCISE 5.6.9

Let n *be a positive integer. Show that*

$$Cov\left(W_t, \int_0^t u^n dW_u\right) = \frac{t^{n+1}}{n+1}$$

SOLUTION 5.6.9

$$Cov\left(W_t, \int_0^t u^n dW_u\right) = \mathbb{E}\left[W_t \int_0^t u^n dW_u\right] = \mathbb{E}\left[\int_0^t dW_s \int_0^t u^n dW_u\right] = \mathbb{E}\left[\int_0^t \int_0^t u^n dW_u^2\right]$$

Since $dW_u^2 = du$, we have

$$Cov\left(W_t, \int_0^t u^n dW_u\right) = \mathbb{E}\left[\int_0^t u^n du\right] = \int_0^t u^n du = \frac{t^{n+1}}{n+1}$$

EXERCISE 5.8.1

SOLUTION 5.8.1

EXERICSE 5.8.3

SOLUTION 5.8.3

EXERCISE 5.8.4

EXERCISE 5.8.4

EXERCISE 5.8.7

Show that

$$\mathbb{E}\left[\left(\int_0^t f(s)dM_s\right)^2\right] = \lambda \int_0^t f^2(s)ds$$

SOLUTION 5.8.7

$$\mathbb{E}\left[\left(\int_0^t f(s)dM_s\right)^2\right] = \mathbb{E}\left[\lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(s_i)\left(M_{s_{i+1}} - M_{s_i}\right)\right)^2\right]$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1} f^2(s_i)\mathbb{E}\left[\left(M_{s_{i+1}} - M_{s_i}\right)^2\right] + 2\lim_{n\to\infty}\sum_{i\neq j} f(s_i)f(s_j)\mathbb{E}\left[\left(M_{s_{i+1}} - M_{s_i}\right)\left(M_{s_{j+1}} - M_{s_j}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1} f^2(s_i)\lambda(S_{i+1} - s_i) + 0 = \lambda \int_0^t f^2(s)ds$$

EXERCISE 5.9.2

Compute the distribution function of $X_t = \int_0^t s\, dN_s$.

SOLUTION 5.9.2

By equation 5.9.6, we know that

$$P(X_T \leq u) = e^{-\lambda T} \sum_{k \geq 0} \frac{\lambda^k Vol(D_k)}{k!}$$

where $D_k = \left\{ \sum_{i=1}^k g(x_i) \leq u \right\} \cap_{i=1}^k \{0 \leq x_i \leq T\}$. This is the intersection between a k-simplex and a k-dimensional hypercube. When calculating $Vol(D_k)$, there are three cases we need to consider. When $T \geq u$, the hypercube contains the entire simplex, so

$$Vol(D_k) = Vol(k\text{-simplex}) = \int_0^u dx_k \int_0^{u-x_k} dx_{k-1} \cdots \int_0^{u-\sum_{i=3}^k x_i} dx_2 \int_0^{u-\sum_{i=2}^k x_i} dx_1 = \frac{u^k}{k!}$$

When the diagonal of the hypercube is less than the height of the simplex (this is when $kT \leq u$), then the hypercube is completely inside the simplex, so

$$Vol(D_k) = Vol(\text{cube}) = T^k$$

The third case is the most complicated. When $T < u < kT$, the two volumes have a nontrivial intersection. The volume is calculated by subtracting the value of the "corner" of the hypercube that passes beyond the boundary of the simplex from the total volume of the hypercube:

$$Vol(D_k) = T^k - \int_{u-(k-1)T}^{T} dx_1 \int_{u-(k-2)T-x_1} dx_2 \cdots \int_{u-T-\sum_{i=1}^{k-2} x_i} dx_{k-1} \int_{u-\sum_{i=1}^{k-1} x_i} dx_k = T^k - \frac{(-1)^k}{k!}(u-kT)^k$$

EXERCISE 6.1.6

Let $G_t = \frac{1}{t}\int_0^t e^{W_u} du$ *be the average of the geometric Brownian motion on* $[0,t]$. *Find* dG_t.

SOLUTION 6.1.6

$$dG_t = d\left(\frac{1}{t}\right)\int_0^t e^{W_u} du + \frac{1}{t} d\left(\int_0^t e^{W_u} du\right) + d\left(\frac{1}{t}\right) d\left(\int_0^t e^{W_u} du\right)$$

$$d\left(\int_0^t e^{W_u} du\right) = \int_t^{t+dt} e^{W_u} du = \int_t^{dt}\left(1 + W_u + \frac{1}{2!}W_u^2 + \dots\right) du = dt\left(1 + W_t + \frac{1}{2!}W_t^2 + \dots\right) = e^{W_t} dt$$

$$dG_t = -\frac{dt}{t^2}\int_0^t e^{W_u} du + \frac{1}{t} e^{W_t} dt - \frac{dt}{t^2} e^{W_t} dt = \frac{1}{t}\left(e^{W_t} - G_t\right) dt$$

EXERCISE 6.2.4

Use Ito's formula to find the following differentials:

(a) $d\left(W_t e^{W_t}\right)$

(b) $d\left(3W_t^2 + 2e^{5W_t}\right)$

(c) $d\left(e^{t+W_t^2}\right)$

(d) $d\left((t+W_t)^n\right)$

(e) $d\left(\frac{1}{t}\int_0^t W_u du\right)$

(f) $d\left(\frac{1}{t^\alpha}\int_0^t e^{W_u} du\right)$

SOLUTION 6.2.4

$$d\left(W_t e^{W_t}\right) = \frac{1}{2}\frac{d^2}{dW_t^2}\left(W_t e^{W_t}\right)dt + \frac{d}{dW_t}\left(W_t e^{W_t}\right)dW_t = \frac{1}{2}\left(e^{W_t} + W_t e^{W_t} + e^{W_t}\right)dt + \left(W_t e^{W_t} + e^{W_t}\right)dW_t$$

$$= e^{W_t}\left(1 + \frac{W_t}{2}\right)dt + e^{W_t}(1+W_t)dW_t$$

$$d\left(3W_t^2 + 2e^{5W_t}\right) = 3d\left(W_t^2\right) + 2d\left(e^{5W_t}\right) = \left(3 + 25e^{5W_t}\right)dt + \left(6W_t + 10e^{5W_t}\right)dW_t$$

$$d\left(e^{t+W_t^2}\right) = \frac{1}{2}\frac{d^2}{dW_t^2}\left(e^{t+W_t^2}\right)dt + \frac{d}{dW_t}\left(e^{t+W_t^2}\right)dW_t = e^{t+W_t^2}(1+2W_t)dt + 2W_t e^{t+W_t^2}dW_t$$

$$d\left((t+W_t)^n\right) = \left[\frac{dX_t^n}{dX_t} + \frac{1}{2}\frac{d^2X_t^n}{dX_t^2}\right]dt + \frac{dX_t^n}{dX_t}dW_t = nX_t^{n-2}\left[X_t + \frac{n-1}{2}\right]dt + nX_t^{n-1}dW_t$$

$$= n(t+W_t)^{n-2}\left[t + W_t + \frac{n-1}{2}\right]dt + n(t+W_t)^{n-1}dW_t$$

$$d\left(\frac{1}{t}\int_0^t W_u du\right) = \frac{1}{t}\left(W_t - \frac{1}{t}\int_0^t W_u\ du\right)dt \qquad \text{as in Example 6.1.5.}$$

$$d\left(\frac{1}{t^\alpha}\int_0^t e^{W_u}du\right) = d\left(t^{-\alpha}\right)\int_0^t e^{W_u}du + t^{a-\alpha}d\left(\int_0^t e^{W_u}du\right) + d(t^{-\alpha})d\left(\int_0^t e^{W_u}du\right)$$

$$= -\alpha t^{-\alpha-1}dt\int_0^t e^{W_u}du + t^{-\alpha}e^{W_t}dt + 0 = \frac{1}{t^\alpha}\left(e^{W_t} - \frac{\alpha}{t}\int_0^t e^{W_u}du\right)dt$$

EXERCISE 6.2.5

Find $d(tW_t^2)$.

SOLUTION 6.2.5

Define $X_t = W_t^2$. Then,

$$d(tW_t^2) = \left[\frac{\partial}{\partial t}(tX_t) + \frac{\partial}{\partial X_t}(tX_t) + \frac{(2W_t)^2}{2}\frac{\partial^2}{\partial X_t^2}(tX_t)\right]dt + 2W_t\frac{\partial}{\partial X_t}(tX_t)dW_t = [X_t + t + 0]dt + 2W_t t dW_t$$

$$= [t + W_t^2]dt + 2tW_t dW_t$$

EXERCISE 7.1.4(D)

Show that

$$\int_0^t e^{W_s - \frac{s}{2}} dW_s = e^{W_t - \frac{t}{2}} - 1$$

SOLUTION 7.1.4(D)

$$d\left(\int_0^t e^{W_s} - \frac{s}{2} dW_s\right) = e^{W_t - \frac{t}{2}} dW_t$$

Define $X_t = W_t - \frac{t}{2}$. Then, $dX_t = dW_t - \frac{1}{2}dt$ and

$$d\left(e^{W_t - \frac{t}{2}} - 1\right) = d\left(e^{X_t}\right) = \left[-\frac{1}{2}\frac{d}{dX_t}e^{X_t} + \frac{1}{2}\frac{d^2}{dX_t^2}e^{X_t}\right]dt + dW_t\frac{d}{dX_t}e^{X_t} = e^{W_t - \frac{t}{2}} dW_t$$

Since we also have

$$\int_0^0 e^{W_s - \frac{s}{2}} dW_s = e^{W_0 - 0} - 1 = 0$$

then equality is satisfied.

EXERCISE 7.2.5

Show that

$$\int_0^T e^{W_t} dW_t = e^{W_t} - 1 - \frac{1}{2}\int_0^T e^{W_t} dt$$

and use this result to find $\mathbb{E}[e^{W_t}]$.

SOLUTION 7.2.5

By equation 7.2.4,

$$\int_0^T e^{W_t} dW_t = e^{W_t}\Big|_0^T - 0 - \frac{1}{2}\int_0^T e^{W_t} dt = e^{W_T} - 1 - \frac{1}{2}\int_0^T e^{W_t} dt$$

Since the expectation of the Ito integral is zero, we have

$$0 = \mathbb{E}\left[e^{W_T}\right] - 1 - \frac{1}{2}\int_0^T \mathbb{E}\left[e^{W_t}\right] dt$$

Differentiating both sides gives

$$\frac{d}{dt}\mathbb{E}\left[e^{W_T}\right] = \frac{1}{2}\frac{d}{dt}\left[\int_0^T \mathbb{E}\left[e^{W_T}\right] dt\right] = \frac{1}{2}\mathbb{E}\left[e^{W_T}\right]$$

Solving the differential equation, we find

$$\mathbb{E}\left[e^{W_T}\right] = ce^{T/2}$$

Since $\mathbb{E}\left[e^{W_0}\right] = 1$, $c = 1$.

EXERCISE 7.3.15

Find the value of the following stochastic integrals using the heat equation method:

(a) $\int_0^1 e^t \cos(\sqrt{2}W_t)\, dW_t$

(b) $\int_0^3 e^{2t} \cos(2W_t)\, dW_t$

(c) $\int_0^4 e^{-t+\sqrt{2}W_t}$

SOLUTION 7.3.15

Let $\varphi(x,t) = \frac{1}{\sqrt{2}}e^t \sin(\sqrt{2}x)$. By theorem 7.3.8,

$$\int_0^1 e^t \cos(\sqrt{2}W_t)\, dW_t = \varphi(W_1, 1) - \varphi(0,0) = \frac{e}{\sqrt{2}}\sin(\sqrt{2}W_1)$$

Let $\varphi(x,t) = \frac{1}{2}e^{2t}\sin(2x)$. By theorem 7.3.8,

$$\int_0^3 e^{2t}\cos(2W_t)\, dW_t = \varphi(W_3, 3) - \varphi(0,0) = \frac{e^6}{2}\sin(\sqrt{2}W_3)$$

Let $\varphi(x,t) = \frac{1}{\sqrt{2}}e^{\sqrt{2}x-t}$. By theorem 7.3.8,

$$\int_0^4 e^{-t+\sqrt{2}W_t} = \varphi(W_4, 4) - \varphi(0,0) = \frac{1}{\sqrt{2}}\left(e^{\sqrt{2}W_4-4} - 1\right)$$

EXERCISE 8.2.8

Solve the following stochastic differential equations by the method of integration:

(a) $dX_t = \left(t - \frac{1}{2}\sin W_t\right) dt + (\cos W_t)\, dW_t,\ X_0 = 0$

(b) $dX_t = \left(\frac{1}{2}\cos W_t - 1\right) dt + (\sin W_t)\, dW_t,\ X_0 = 0$

(c) $dX_t = \frac{1}{2}\left(\sin W_t + W_t \cos W_t\right) dt + (W_t \sin W_t)\, dW_t,\ X_0 = 0$

SOLUTION 8.2.8

Part (a)

$$X_t = \int_0^t s - \frac{1}{2}\sin W_s \; ds + \int_0^t \cos W_s \; dW_s = \frac{t^2}{2} - \frac{1}{2}\int_0^t \sin W_s \; ds + \sin W_t + \frac{1}{2}\int_0^t \sin W_s \; ds = \frac{t^2}{2} + \sin W_t$$

Part (b)

$$X_t = \frac{1}{2}\int_0^t \cos W_s \; ds - t + \int_0^t \sin W_s \; dW_s = \frac{1}{2}\int_0^t \cos W_s \; ds - t + 1 - \cos W_t - \frac{1}{2}\int_0^t \cos W_s \; ds = 1 - t - \cos W_t$$

Part (c)

$$X_t = \frac{1}{2}\int_0^t \sin W_s \; ds + \frac{1}{2}\int_0^t W_s \cos W_s \; ds + \int_0^t W_s \sin W_s \; dW_s$$

Integrate the third term by parts to find

$$\int_0^t W_s \sin W_s \; dW_s = \sin W_t - W_t \cos W_t - \frac{1}{2}\int_0^t \sin W_s \; ds - \frac{1}{2}\int_0^t W_s \cos W_s \; ds$$

The solution is then

$$X_t = \sin W_t - W_t \cos W_t$$

EXERCISE 8.3.6

Solve the following exact stochastic differential equations:

(a) $dX_t = e^t dt + \left(W_t^2 - t\right) dW_t, \; X_0 = 1$

(b) $dX_t = \sin t \; dt + \left(W_t^2 - t\right) dW_t, \; X_0 = -1$

(c) $dX_t = t^2 \; dt + e^{W_t - \frac{t}{2}} dW_t, \; X_0 = 0$

(d) $dX_t = t \; dt + e^{t/2} \cos W_t \; dW_t, \; X_0 = 1$

SOLUTION 8.3.6

Part (a)

First, we need to find $f(x,t)$. Solving $b(x,t) = \partial_x f(x,t)$ gives

$$f(x,t) = \int (x^2 - t) dx = \frac{x^3}{3} - xt + c(t)$$

$a(x,t) = \partial_t f(x,t) + \frac{1}{2}\partial_x^2 f(x,t)$ gives

$$e^t = \partial_t c(t) - x + \frac{1}{2}(2x + 0 - 0)$$

which we can solve to find

$$c(t) = e^t + c_2$$

where c_2 is a constant. The solution to the differential equation is then

$$X_t = \frac{W_t^3}{3} - tW_t + e^t + C$$

The condition $X_0 = 1$ fixes $C = 0$.

Part (b)

As in part (a), we find

$$f(x,t) = \frac{x^3}{3} - xt + c(t)$$

This time, the PDE for $a(x,t)$ gives

$$\sin(t) = \partial_t(t) \implies c(t) = -\cos t + c_2$$

Using $X_0 = -1$, the solution to the SDE is

$$X_t = \frac{W_t^3}{3} - tW_t - \cos t$$

Part (c)

The PDE for $b(x,t)$ gives

$$e^{x-\frac{t}{2}} = \partial_x f(x,t) \implies f(x,t) = e^{x-\frac{t}{2}} + c(t)$$

The PDE for $a(x,t)$ gives

$$t^2 = \partial_t c(t) \implies c(t) = \frac{t^3}{3} + c_2$$

Using $X_0 = 0$, the solution to the SDE is

$$X_t = e^{W_t - \frac{t}{2}} + \frac{t^3}{3} - 1$$

Part (d)

The PDE for $b(x,t)$ gives

$$e^{t/2}\cos x = \partial_x f(x,t) \implies f(x,t) = e^{t/2}\sin x + c(t)$$

The PDE for $a(x,t)$ gives

$$t = \partial_t c(t) \implies c(t) = \frac{t^2}{2} + c_2$$

Using $X_0 = 1$, the solution to the SDE is

$$X_t = e^{t/2}\sin W_t + \frac{t^2}{2} + 1$$

EXERCISE 8.5.4(B,F)

Solve the following linear stochastic differential equations:

(b) $dX_t = (3X_t - 2)dt + e^{3t}dW_t$

(f) $dX_t = -X_t\ dt + e^{-t}dW_t$

SOLUTION 8.5.4(B,F)

Part (b)

Multiplying by the integrating factor e^{-3t} gives

$$d\left(e^{-3t}X_t\right) = -2e^{-3t}dt + dW_t$$

Integrating both sides, we find

$$X_t = X_0e^{3t} - 2e^{3t}\int_0^t e^{-3s}ds + e^{3t}\int_0^t dW_s = X_0e^{3t} + \frac{2}{3}\left(1 - e^{3t}\right) + e^{3t}dW_t$$

Part (f)

Multiplying by the integrating factor e^t gives

$$d\left(e^tX_t\right) = dW_t \implies X_t = e^{-t}\left(X_0 + W_t\right)$$

EXERCISE 8.7.2

Use the method of variation of parameters to solve the following stochastic differential equation:

$$dX_t = X_tW_tdW_t$$

SOLUTION 8.7.2

Dividing by X_t, we find

$$\int \frac{dX_t}{X_t} = \int W_t \, dW_t \implies X_t = e^{\frac{1}{2}W_t^2 - \frac{t}{2} + c(t)}$$

By Ito's formula,

$$dX_t = \left[\frac{dc}{dt} + \frac{1}{2}W_t^2\right] X_t \, dt + X_t W_t \, dW_t$$

which gives a formula for $c(t)$:

$$\frac{dc}{dt} + \frac{1}{2}W_t^2 = 0 \implies c(t) = -\frac{1}{2}\int W_t^2 dt + c_1$$

so the final expression for X_t is

$$X_t = e^{\frac{1}{2}W_t^2 - \frac{t}{2} + -\frac{1}{2}\int W_t^2 dt + C}$$

EXERCISE 8.7.4

Use variation of parameters to solve

$$dX_t = \lambda(\mu - X_t)dt + \sigma dW_t$$

SOLUTION 8.7.4

First, consider

$$dX_t = \lambda(\mu - X_t)dt$$

This has the solution

$$X_t = \mu - ce^{-\lambda t}$$

Promoting the constant to a function of time and calculating the differential of this solution gives

$$dX_t = -e^{-\lambda t}dc + \lambda c(t)e^{-\lambda t}dt = -e^{-\lambda t}dc + dX_t - \sigma dW_t$$

$c(t)$ must satisfy

$$dc = -\sigma e^{\lambda t} dW_t$$

which has the solution

$$c(t) = c(0) - \sigma \int_0^t e^{\lambda s} dW_s$$

Plugging this into our solution for X_t and solving for $c(0)$ in terms of X_0 gives

$$X_t = \mu - e^{-\lambda t}\left(\mu - X_0 - \sigma \int_0^t e^{\lambda s} dW_s\right)$$

EXERCISE 8.8.3

Let X_t be the solution of the stochastic differential equation $dX_t = \sigma X_t dW_t$, with σ constant. Let $A_t = \frac{1}{t}\int_0^t X_s dW_s$. Find the stochastic differential equation satisfied by A_t and the mean and variance of A_t.

SOLUTION 8.8.3

Integrating both sides of the SDE gives

$$\frac{\sigma}{t}\int_0^t X_s dW_s = \frac{1}{t}\int_0^t dX_s \implies \sigma t A_t = \int_0^t dX_s$$

Taking the differential of both sides, we find

$$\sigma t \ dA_t + \sigma A_t \ dt = dX_t$$

The mean of A_t is zero, since it is defined by an Ito integral. The variance is

$$Var(A_t) = \mathbb{E}[A_t^2] - \mathbb{E}^2[A_t] = \mathbb{E}\left[\frac{1}{t^2}\left(\int_0^t X_s \ dW_s\right)^2\right] = \frac{1}{t^2}\mathbb{E}\left[\int_0^t X_s^2 ds\right] = \frac{1}{t^2}\int_0^t \mathbb{E}[X_s^2]ds = \frac{1}{t}\mathbb{E}[X_t^2]$$

EXERCISE 10.1.10

Let $\mathcal{F}_t = \sigma\{W_u; u \leq t\}$. Show that the following processes are $\mathcal{F}_t$ martingales:

(a) $e^{t/2}\cos W_t$

(b) $e^{t/2}\sin W_t$

SOLUTION 10.1.10

By equation 7 of Table 7.4, we see that

$$e^{t/2}\cos W_t = 1 - \int_0^t e^{t/2}\sin W_s dW_s$$

Using proposition 5.5.7, $e^{t/2}\cos W_t$ is an $\mathcal{F}_t$ martingale. A similar result holds for the second case.

EXERCISE 10.2.6

If W_t and $\tilde{W}_t$ are independent Brownian motions and $\rho \in [-1, 1]$ is a constant, use Lévy's theorem to show that the process $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$ is a Brownian motion.

SOLUTION 10.2.6

First we calculate the quadratic variation of X_t. For this, we need dX_t:

$$dX_t = \rho dW_t + \sqrt{1-\rho^2}d\tilde{W}_t$$

The quadratic variation is

$$\langle X, X\rangle_t = \int_0^t (dX_s)^2 = \int_0^t \left(\rho^2 dW_s^2 + (1-\rho^2)d\tilde{W}_s^2\right) = \int_0^t ds = t$$

We also have

$$\mathbb{E}\left[X_t \middle| \mathcal{F}_s\right] = \rho\mathbb{E}\left[W_t \middle| \mathcal{F}_s\right] + \sqrt{1-\rho^2}\mathbb{E}\left[\tilde{W}_t \middle| \mathcal{F}_s\right] = \rho W_s + \sqrt{1-\rho^2}\tilde{W}_s = X_s$$

so X_t is a martingale.

EXERCISE 10.3.11

Consider the equation $dX_t = -\lambda X_t dt + \sigma dW_t$ *with* $X_0 = 0$, λ *and* σ *constants, with* $\lambda > 0$. *Show that the solution is given by*

$$X_t = \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dW_s$$

Show that there is a Brownian motion W_t *such that*

$$X_t = \sigma e^{-\lambda t} W_{(e^{2\lambda t}-1)/(2\lambda)}$$

SOLUTION 10.3.11

The solution to the given stochastic differential equation is a special case of the result of Exercise 8.7.4, with $\mu = X_0 = 0$. Define $Y_t = \int_0^t e^{\lambda s} dW_s$. The quadratic variation of Y_t is

$$\langle Y, Y \rangle_t = \int_0^t (dY_s)^2 = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t} - 1}{2\lambda}$$

By theorem 10.3.1, there exists Brownian motion W_t such that

$$Y_t = W_{\frac{e^{2\lambda t}-1}{2\lambda}}$$

Then, since $X_t = \sigma e^{-\lambda t}$,

$$X_t = \sigma e^{-\lambda t} W_{\frac{e^{2\lambda t}-1}{2\lambda}}$$

EXERCISE 10.4.14

Use the reduction of drift formulas to show

(a) $\mathbb{E}\left[W_t e^{-\lambda W_t}\right] = -\lambda t e^{\frac{\lambda^2 t}{2}}$

(b) $\mathbb{E}\left[W_t^2 e^{-\lambda W_t}\right] = (t + \lambda^2 t^2) e^{\frac{\lambda^2 t}{2}}$

SOLUTION 10.4.14

Part (a)

$$\mathbb{E}\left[W_t e^{-\lambda W_t}\right] = e^{-\frac{\lambda^2 t}{2}} \mathbb{E}\left[(\lambda t + W_t) e^{-\lambda(\lambda t + W_t)} e^{-\lambda W_t}\right] = e^{-\frac{3\lambda^2 t}{2}} \left(\lambda t \mathbb{E}\left[e^{-2\lambda W_t}\right] + \mathbb{E}\left[W_t e^{-2\lambda W_t}\right]\right)$$

The individual expectations are

$$\mathbb{E}[e^{-2\lambda W_t}] = e^{2\lambda^2 t}$$

$$\mathbb{E}[W_t e^{-2\lambda W_t}] = \int_{-\infty}^{\infty} x e^{-2\lambda x} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = -2\lambda t e^{2\lambda^2 t}$$

Plugging these in, we find

$$\mathbb{E}\left[W_t e^{-\lambda W_t}\right] = -\lambda t e^{\frac{\lambda^2 t}{2}}$$

Similar logic gives the result for part (b).