PHY 622 - Homework 5

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PART (A)

We have previously seen that $X^\mu = f^\mu(t+\sigma) + g^\mu(t-\sigma)$ before boundary conditions are applied. First, we will consider the case of Dirichlet boundary conditions for all μ:

$$X^\mu(0,t) = a^\mu \qquad\qquad X^\mu(\pi,t) = b^\mu$$

The first condition gives

$$f^\mu(t) + g^\mu(t) = a^\mu \implies g^\mu(t) = a^\mu - f^\mu(t)$$

Substituting this back into X^μ:

$$X^\mu(\sigma,t) = f^\mu(t+\sigma) - f^\mu(t-\sigma) + a^\mu$$

Now, we apply the second condition:

$$f^\mu(t+\pi) - f^\mu(t-\pi) + a^\mu = b^\mu \implies f^\mu(t) = f^\mu(t+2\pi) + a^\mu - b^\mu$$

Since the derivative of f^μ is has a period of 2π, we can expand it as

$$\frac{df^\mu(t)}{dt} = \alpha_0^\mu + \sum_{i=1}^{\infty}\left[\alpha_n^\mu \cos(nt) + \beta_n^\mu \sin(nt)\right]$$

Integrating gives

$$f(t) = c^\mu + \alpha_0^\mu t + \sum_{i=1}^{\infty}\left(\frac{\alpha_n^\mu}{n}\sin(nt) - \frac{\beta_n^\mu}{n}\cos(nt)\right)$$

where c^μ is an integration constant. We can fix α_0^μ by using the periodicity condition for f:

$$\alpha_0^\mu(t+2\pi) + a^\mu - b^\mu = \alpha_0^\mu t \implies \alpha_0^\mu = \frac{b^\mu - a^\mu}{2\pi}$$

Substituting f into X^μ (with redefined α and β) gives

$$\begin{aligned}X^\mu(\sigma,t) =& c^\mu - c^\mu + a^\mu + \frac{b^\mu - a^\mu}{2\pi}(t+\sigma) - \frac{b^\mu - a^\mu}{2\pi}(t-\sigma) + \sum_{i=1}^{\infty}\Big(\alpha_n^\mu \cos\left[n(t+\sigma)\right] + \beta_n^\mu \sin\left[n(t+\sigma)\right]\Big)\\ &- \sum_{i=1}^{\infty}\Big(\alpha_n^\mu \cos\left[n(t-\sigma)\right] + \beta_n^\mu \sin\left[n(t-\sigma)\right]\Big)\end{aligned}$$

$$= a^\mu + \frac{b^\mu - a^\mu}{\pi}\sigma + \frac{1}{2}\sum_{i=1}^{\infty}\Big(\left(\alpha_n^\mu - i\beta_n^\mu\right)e^{in(t+\sigma)} + \left(\alpha_n^\mu + i\beta_n^\mu\right)e^{-in(t+\sigma)} + \left(-\alpha_n^\mu + i\beta_n^\mu\right)e^{in(t-\sigma)} + \left(-\alpha_n^\mu - i\beta_n^\mu\right)e^{-in(t-\sigma)}\Big)$$

Redefining our coefficients again gives

$$X^\mu(\sigma,t) = a^\mu + \frac{b^\mu - a^\mu}{\pi}\sigma + \frac{1}{2i}\sum_{i=1}^{\infty}\left(\alpha_n^\mu\left(e^{in(t+\sigma)} - e^{in(t-\sigma)}\right) + \beta_n^\mu\left(e^{-in(t+\sigma)} - e^{-in(t-\sigma)}\right)\right)$$

which simplifies to

$$X^\mu(\sigma,t) = a^\mu + \frac{b^\mu - a^\mu}{\pi}\sigma + \sum_{i=1}^{\infty}\left(\alpha_n^\mu e^{int} + \beta_n^\mu e^{-int}\right)\sin(n\sigma)$$

Requiring that $(X^\mu)^* = X^\mu$ implies that $(\alpha_n^\mu)^* = \beta_n^\mu$ and requiring that $(\alpha_n^\mu)^* - \alpha_{-n}^\mu$ implies that $\beta_n^\mu = \alpha_{-n}^\mu$. Then, we can rewrite X^μ as

$$X^\mu(\sigma,t) = a^\mu + \frac{b^\mu - a^\mu}{\pi}\sigma + c_1\sum_{n\neq 0}\alpha_n^\mu e^{-int}\sin(n\sigma)$$

where c_1 is a constant we have introduced that will be fixed when quantizing. For the case with mixed boundary conditions, we have

$$X^\mu(0,t) = a^\mu \qquad \partial_\sigma X^\mu(\sigma,t)\Big|_{\sigma=\pi} = 0$$

As before, the boundary condition at $\sigma = 0$ implies

$$X^\mu(\sigma,t) = f^\mu(t+\sigma) - f^\mu(t-\sigma) + a^\mu$$

Taking the σ derivative:

$$\partial_\sigma X^\mu = \partial_\sigma f^\mu(t+\sigma) + \partial_\sigma f^\mu(t-\sigma)$$

This needs to be evaluated at $\sigma = \pi$:

$$\frac{df^\mu}{d\sigma}(t+\pi) = -\frac{df^\mu}{d\sigma}(t-\pi)$$

This time the derivative of f is anti-periodic, so we can expand it as

$$\frac{df^\mu}{d\sigma} = \sum_{i=0}^{\infty}\left(\alpha_n^\mu\cos\left(\frac{(2n+1)t}{2}\right) + \beta_n^\mu\sin\left(\frac{(2n+1)t}{2}\right)\right)$$

Integrating, redefining our constants, and substituting into X^μ gives

$$\begin{aligned} X^\mu(\sigma,t) = a^\mu + \sum_{\text{odd } n>0} &\left(\alpha_n^\mu\left[\cos\left(\frac{n(t+\sigma)}{2}\right) - \cos\left(\frac{n(t-\sigma)}{2}\right)\right]\right. \\ &\left. + \beta_n^\mu\left[\sin\left(\frac{n(t+\sigma)}{2}\right) - \sin\left(\frac{n(t+\sigma)}{2}\right)\right]\right) \end{aligned}$$

After expanding and another coordinate redefinition (as before), this becomes

$$X^\mu(\sigma,t) = a^\mu + \sum_{\text{odd } n>0} \left(\alpha_n^\mu e^{\frac{int}{2}} + \beta_n^\mu e^{-\frac{int}{2}}\right) \sin\left(\frac{n\sigma}{2}\right)$$

As before, we can simplify this using the reality of X^μ, which results in

$$X^\mu(\sigma,t) = a^\mu + c_2 \sum_{\text{odd } n} \alpha_n^\mu e^{-\frac{int}{2}} \sin\left(\frac{n\sigma}{2}\right)$$

Both these results hold whether σ is increasing when moving from brane 1 to brane 2, or vice-versa, as long as we also take $a^\mu \leftrightarrow b^\mu$ when needed. These expressions could be condensed slightly by fixing coordinates such that one of a^μ or b^μ is zero, as convenient.

PART (B)

Using equation (2.48) to define our momentum, we find

$$P^\mu = T\int_0^\pi d\sigma\ \dot{X}^\mu(\sigma,t) = 2iTc \sum_{\text{odd } n} \alpha_n^\mu e^{-int}$$

This equation holds for both the DD and ND cases. In the DD case, $\sin(n\sigma)$ integrates to $((-1)^n - 1)$, which kills all even terms. In the ND case, the sum is already over odd terms only.

For the mixed case:

$$-i\hbar\delta^{\mu\nu}\delta(\sigma-\sigma') = \left[2iTc_2 \sum \alpha_n^\mu e^{-int}, \frac{c_2}{2i}\sum \alpha_m^\nu \left(e^{\frac{im\sigma'}{2}} - e^{-\frac{im\sigma'}{2}}\right)\right]$$

$$= Tc_2^2 \sum_{n,m} e^{-i(n+m)t}\left(e^{\frac{im\sigma'}{2}} - e^{-\frac{im\sigma'}{2}}\right)[\alpha_n^\mu, \alpha_m^\nu]$$

After several applications of Fourier's trick, we find

$$[\alpha_n^\mu, \alpha_m^\nu] = \frac{\hbar}{Tc_2^2}\delta^{\mu\nu}\delta_{m+n,0}$$

which tells us to fix $c_2 = \sqrt{1/T} = \sqrt{\pi}\ell$. The DD case proceeds nearly the same way.

I think the factor of π must have come from an extra/missed factor from one of the times I did Fourier's trick, but I couldn't find it. It should probably have canceled.