PHY 622 - Homework 4

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(Dated: September 17, 2019)

PART (A)

Starting from the constraint $\left(\dot{X}^- \pm X'^-\right) = \left(\dot{X}^i \pm X'^i\right)^2 \left(\ell^2 p^+\right)^{-1}$ and using the standard definitions for X^- and X^i, we easily find

$$\ell^2 p^- + \ell \sum_{n\neq 0} \alpha_n^- e^{-int \mp in\sigma} = \left(\ell^2 p^i + \ell \sum_{n\neq 0} \alpha_n^i e^{-int\mp in\sigma}\right)^2 \left(p^+\ell^2\right)^{-1}$$

which can be solved for p^- if needed. Using the relation

$$\alpha_n^- = \frac{1}{\ell p^+} \sum_m \alpha_{n-m}^i \alpha_m^i$$

we can eliminate the dependence on α_n^- from our expression for p^-:

$$\ell^2 p^- = \left(\ell^2 p^i + \ell \sum_{n\neq 0} \alpha_n^i e^{-int\mp in\sigma}\right)^2 \left(p^+\ell^2\right)^{-1} - \frac{1}{p^+} \sum_{n\neq 0} \sum_m \alpha_{n-m}^i \alpha_m^i e^{-int\mp in\sigma}$$

so we see that p^- can be expressed in terms of p^+, p^i, and the α^i's. We can also see that the α^i's have dependence on p^-, p^i, and p^+.

PART (B)

As long as we are careful to eliminate the $\tilde{\alpha}$ oscillators using their respective identity, the general ideas of part (a) should still hold for the closed string, so the operator dependencies should not change.

PART (C)

We begin with the equal-time commutation relations:

$$\left[P_\mu(\sigma,t), X^\nu(\sigma',t)\right] = -i\hbar\eta_\mu^\nu \delta(\sigma-\sigma')$$

For the open string in the light-cone gauge, P^μ is

$$P^\mu = T\int_0^\pi d\sigma\ \dot{X}^\mu = T\int_0^\pi d\sigma\ \ell^2 p^+ = T\ell^2\pi p^+ = p^+$$

Then, we have

$$2i\hbar\delta(\sigma-\sigma') = \left[p^+, x^- + \ell^2 p^- t + -\ell \sum_{n\neq 0} \frac{\alpha_n^-}{n} e^{-int}\cos(n\sigma)\right] = \left[p^+, x^-\right] + 0 + 0$$

so we see that $[p^+, x^-] = 2i\hbar$.

Next, we can argue that $[p^-, x^+] = 0$. Since we have previously seen that we can express p^- as a function of p^+, the commutator becomes $[f(p^+), x^+] \propto \eta^{++} = 0$. Similarly, this reasoning implies that $[p^+, x^+] = 0$. These results will also hold for the closed string.

The remaining two commutators can be found as follows:

$$0 = \left[p^-, x^- + \ell^2 p^- t + i\ell \sum_{n\neq 0} \frac{\alpha_n^-}{n} e^{-int} \cos(n\sigma)\right] = [p^-, x^-] + i\ell \sum_{n\neq 0} \frac{e^{-int}}{n} \cos(n\sigma)[p^-, \alpha_n^-] =$$

PART (D)

The states

$$\alpha_{-3}^i|0,p\rangle \qquad \alpha_{-2}^i\alpha_{-1}^j|0,p\rangle \qquad \alpha_{-1}^i\alpha_{-1}^j\alpha_{-1}^k|0,p\rangle$$

all satisfy $\alpha' M^2 = 3 - a(\text{open}) = 2$. Since $\left[\alpha_{-2}^i, \alpha_{-1}^j\right] = -2\eta^{ij}\delta_{-2-1,0} = 0$, we don't also have to consider $\alpha_{-1}^i\alpha_{-2}^i|0,p\rangle$. We know that i runs from 1 to $D-2$, so for the first set of states, we get $D-2$ states. For the second, we get $(D-2)^2$ states. For the third, we get $\frac{1}{6}D(D-1)(D-2)$ states. The total number of states is

$$\frac{D^3 + 3D^2 - 16D + 12}{6} = \frac{D^3 + 3D^2 - 4D}{6} + 2(D-1)$$

These massive states should transform as irreps of the little group $SO(25)$, but we don't have to really do any group theory in this case, since the number of states happened to come out so nicely and we can already see that we get a massive, traceless, symmetric, rank three tensor and two massive vectors.

For the closed string with $N = \tilde{N} = 2$, we have the following possibilities:

$$\alpha_{-2}^i\tilde{\alpha}_{-2}^j|0,p\rangle \qquad \alpha_{-1}^i\alpha_{-1}^j\tilde{\alpha}_{-1}^k\tilde{\alpha}_{-1}^\ell|0,p\rangle \qquad \tilde{\alpha}_{-2}^i\alpha_{-1}^j\alpha_{-1}^k|0,p\rangle \qquad \alpha_{-2}^i\tilde{\alpha}_{-1}^j\tilde{\alpha}_{-1}^k|0,p\rangle$$

Here, the total number of states is

$$(D-2)^2 + \binom{2+(D-2)-1}{2}^2 + 2(D-2)\binom{2+(D-2)-1}{2} = \frac{\left(D^2-D-2\right)^2}{4} = \left(\frac{D(D-1)}{2} - 1\right)^2$$

With a little effort, we can see that the states here are two copies (one left-moving and one right-moving) of the traceless symmetric tensor representation of $SO(25)$ that we found in the open string at $N = 2$. Schematically:

$$\left(\alpha_{-2}^i\tilde{\alpha}_{-2}^j\right) \oplus \left(\alpha_{-1}^i\alpha_{-1}^j\tilde{\alpha}_{-1}^k\tilde{\alpha}_{-1}^\ell\right) \oplus \left(\tilde{\alpha}_{-2}^i\alpha_{-1}^j\alpha_{-1}^k\right) \oplus \left(\alpha_{-2}^i\tilde{\alpha}_{-1}^j\tilde{\alpha}_{-1}^k\right) = \mathbf{324}_L \otimes \mathbf{324}_R$$