PHY 623 - Exercises 1-4

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EXERCISE 1

We can rewrite equation (2.39) as

$$\frac{1}{2}\left(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a\right) dx^\mu \wedge dx^\nu + \frac{1}{2}\left(\Omega_\mu{}^a{}_b e_\nu{}^b - \Omega_\nu{}^a{}_b e_\mu{}^b\right) dx^\mu \wedge dx^\nu = \frac{1}{2} T^\rho{}_{\mu\nu}\, e_\rho{}^a dx^\mu \wedge dx^\nu$$

which gives

$$\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \Omega_\mu{}^a{}_b e_\nu{}^b - \Omega_\nu{}^a{}_b e_\mu{}^b = T^\rho{}_{\mu\nu}\, e_\rho{}^a$$

Now, if we antisymmetrize equation (2.40), we find

$$0 = \nabla_\mu e_\nu{}^a - \nabla_\nu e_\mu{}^a = \left[\partial_\mu e_\nu{}^a + \Omega_\mu{}^a{}_b e_\nu{}^b - \Gamma^\rho{}_{\mu\nu} e_\rho{}^a\right] - \left[\partial_\nu e_\mu{}^a + \Omega_\nu{}^a{}_b e_\mu{}^b - \Gamma^\rho{}_{\nu\mu} e_\rho{}^a\right]$$

which gives

$$\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \Omega_\mu{}^a{}_b e_\nu{}^b - \Omega_\nu{}^a{}_b e_\mu{}^b = \left(\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}\right) e_\rho{}^a = T^\rho{}_{\mu\nu}\, e_\rho{}^a$$

so the two expressions are consistent.

EXERCISE 2

Using the explicit expression for $\nabla_\mu e_\rho{}^a$, we can find the relation between the two expressions for R:

$$0 = \left[\nabla_\mu, \nabla_\nu\right] e_\rho{}^a = \nabla_\mu \left[\partial_\nu e_\rho{}^a + \Omega_\nu{}^a{}_b e_\rho{}^b - \Gamma^\sigma{}_{\nu\rho} e_\sigma{}^a\right] - (\mu \leftrightarrow \nu)$$

$$= \partial_\nu \left[\partial_\mu e_\rho{}^a + \Omega_\mu{}^a{}_b e_\rho{}^b - \Gamma^\sigma{}_{\mu\rho} e_\sigma{}^a\right] + \Omega_\nu{}^a{}_b \left[\partial_\mu e_\rho{}^b + \Omega_\mu{}^b{}_c e_\rho{}^c - \Gamma^\sigma{}_{\mu\rho} e_\sigma{}^b\right] - \Gamma^\sigma{}_{\nu\rho} \left[\partial_\mu e_\sigma{}^a + \Omega_\mu{}^a{}_b e_\sigma{}^b - \Gamma^\tau{}_{\mu\sigma} e_\tau{}^a\right] - (\mu \leftrightarrow \nu)$$

$$= \left[\partial_\mu \Gamma^\tau{}_{\nu\rho} - \partial_\nu \Gamma^\tau{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\rho}\Gamma^\tau{}_{\mu\sigma} - \Gamma^\sigma{}_{\mu\rho}\Gamma^\tau{}_{\nu\sigma}\right] e_\tau{}^a - \left[\partial_\mu \Omega_\nu{}^a{}_b - \partial_\nu \Omega_\mu{}^a{}_b + \Omega_\mu{}^a{}_c \Omega_\nu{}^c{}_b - \Omega_\nu{}^a{}_c \Omega_\mu{}^c{}_b\right] e_\rho{}^b$$

The first term is the expression (2.41) and the second term is the expanded form of (2.42). We can see that $R_{\mu\nu}{}^\rho{}_\sigma e^a{}_\rho e^\sigma{}_b = R_{\mu\nu}{}^a{}_b$.

EXERCISE 3

First, act on a test function with the commutator of the derivatives:

$$\left[\partial_{+\mu}, \partial_{+\nu}\right] f = \frac{1}{4}\left[\partial_\mu, \partial_\nu\right] f + \frac{i}{4}\left[\partial_\mu, J^\sigma{}_\nu \partial_\sigma\right] f + \frac{i}{4}\left[J^\rho{}_\mu \partial_\rho, \partial_\nu\right] f - \frac{1}{4}\left[J^\rho{}_\mu \partial_\rho, J^\sigma{}_\nu \partial_\sigma\right] f$$

$$= \frac{i}{4}\left[\left(\partial_\mu J^\sigma{}_\nu\right) \partial_\sigma f - \left(\partial_\nu J^\sigma{}_\mu\right) \partial_\sigma f\right] + \frac{1}{4}\left[\left(\partial_\sigma J^\rho{}_\mu\right) J^\sigma{}_\nu \partial_\rho f - \left(\partial_\rho J^\sigma{}_\nu\right) J^\rho{}_\mu \partial_\sigma f\right]$$

Using $J_\mu{}^\rho J_\rho{}^\nu = -\delta^\nu_\mu$, we can rewrite the first two terms:

$$\frac{1}{4}\left[J^\sigma{}_\nu \partial_\sigma J^\rho{}_\mu - J^\sigma{}_\mu \partial_\sigma J^\rho{}_\nu\right]\partial_\rho f + \frac{i}{4}\left[J^\rho{}_\sigma \partial_\nu J^\sigma{}_\mu - J^\rho{}_\sigma \partial_\mu J^\sigma{}_\nu\right] J^\lambda{}_\rho \partial_\lambda f$$

Adding and subtracting identical terms and rearranging gives

$$\begin{aligned}&\frac{1}{8}\left[J^\sigma{}_\nu \partial_\sigma J^\rho{}_\mu - J^\sigma{}_\mu \partial_\sigma J^\rho{}_\nu + J^\rho{}_\sigma \partial_\mu J^\sigma{}_\nu - J^\rho{}_\sigma \partial_\nu J^\sigma{}_\mu\right]\left(\partial_\rho f + iJ^\lambda{}_\rho \partial_\lambda f\right)\\ +&\frac{1}{8}\left[J^\sigma{}_\nu \partial_\sigma J^\rho{}_\mu - J^\sigma{}_\mu \partial_\sigma J^\rho{}_\nu - J^\rho{}_\sigma \partial_\mu J^\sigma{}_\nu + J^\rho{}_\sigma \partial_\nu J^\sigma{}_\mu\right]\left(\partial_\rho f - iJ^\lambda{}_\rho \partial_\lambda f\right)\end{aligned}$$

so we see that

$$C_{\mu\nu}{}^\rho = \frac{1}{4}\left[J^\sigma{}_\nu \partial_\sigma J^\rho{}_\mu - J^\sigma{}_\mu \partial_\sigma J^\rho{}_\nu + J^\rho{}_\sigma \partial_\mu J^\sigma{}_\nu - J^\rho{}_\sigma \partial_\nu J^\sigma{}_\mu\right]$$

$$N_{\mu\nu}{}^\rho = \frac{1}{4}\left[J^\sigma{}_\nu \partial_\sigma J^\rho{}_\mu - J^\sigma{}_\mu \partial_\sigma J^\rho{}_\nu - J^\rho{}_\sigma \partial_\mu J^\sigma{}_\nu + J^\rho{}_\sigma \partial_\nu J^\sigma{}_\mu\right]$$

The usual transformations give

$$N_{\mu\nu}{}^\rho \to \frac{\partial x^\sigma}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^\nu} J^\alpha{}_\beta \frac{\partial x^\gamma}{\partial x^\sigma}\frac{\partial}{\partial x^\gamma}\left(\frac{\partial x^\rho}{\partial x^\delta}\frac{\partial x^\epsilon}{\partial x^\mu} J^\delta{}_\epsilon\right) + \dots$$

The terms that potentially stop N and C transforming as tensors are the ones containing second derivatives:

$$J^\alpha{}_\beta J^\delta{}_\epsilon\left[\frac{\partial^2 x^\rho}{\partial x^\alpha \partial x^\delta}\frac{\partial x^\beta}{\partial x^\nu}\frac{\partial x^\epsilon}{\partial x^\mu} + \frac{\partial^2 x^\epsilon}{\partial x^\alpha \partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^\delta} \pm \frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\delta}\frac{\partial x^\rho}{\partial x^\alpha}\frac{\partial x^\epsilon}{\partial x^\nu} \pm \frac{\partial^2 x^\epsilon}{\partial x^\mu \partial x^\nu}\frac{\partial x^\beta}{\partial x^\delta}\frac{\partial x^\rho}{\partial x^\alpha}\right] - (\mu - \leftrightarrow \nu)$$

Here, $+$ corresponds to the transformation of C and $-$ to the transformation of N. It is easy to see that the first and fourth terms cancel after taking $(\mu \leftrightarrow \nu)$ and commuting partial derivatives, but the second and third terms cannot vanish on their own and only cancel each other in the transformation of N. Therefore, N transforms as a tensor but C does not.

EXERCISE 4

We start with

$$T_{\mu\nu\rho} = g_{\rho\sigma} T^\sigma{}_{\mu\nu} = g_{\rho\sigma}\left(\Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu}\right)$$

Using the expression for the general connection and the definition of $\nabla^{(0)}$, we get

$$\begin{aligned}T_{\mu\nu\rho} = g_{\rho\sigma}\Bigg[&\tilde\Gamma^\sigma{}_{\mu\nu} - \frac{1}{2}J^\sigma{}_\tau\left(\partial_\mu J^\tau{}_\nu + \tilde\Gamma^\tau{}_{\mu\alpha}J^\alpha{}_\nu - \tilde\Gamma^\alpha{}_{\mu\nu}J^\tau{}_\alpha\right) + aJ^\sigma{}_\nu\left(\partial_\tau J^\tau{}_\mu + \tilde\Gamma^\tau{}_{\tau\alpha}J^\alpha{}_\mu - \tilde\Gamma^\alpha{}_{\tau\mu}J^\tau{}_\alpha\right)\\ &+ bJ^\sigma{}_\tau\left(\partial_\nu J^\tau{}_\mu + \tilde\Gamma^\tau{}_{\nu\alpha}J^\alpha{}_\mu - \tilde\Gamma^\alpha{}_{\nu\mu}J^\tau{}_\alpha\right) + bJ^\tau{}_\nu\left(\partial_\tau J^\sigma{}_\mu + \tilde\Gamma^\sigma{}_{\mu\alpha}J^\alpha{}_\tau - \tilde\Gamma^\alpha{}_{\tau\mu}J^\sigma{}_\alpha\right)\Bigg] - (\mu \leftrightarrow \nu)\end{aligned}$$

where $\tilde\Gamma$ is the metric connection. Now, if we take $b = -1/4$, the $J\partial J$ terms from the first term in parentheses and the ones proportional to b can be written as

$$\frac{1}{4}J^{\tau}{}_{\nu}\partial_{\mu}J^{\sigma}{}_{\tau} - \frac{1}{4}J^{\tau}{}_{\mu}\partial_{\nu}J^{\sigma}{}_{\tau} - \frac{1}{4}J^{\tau}{}_{\nu}\partial_{\tau}J^{\sigma}{}_{\mu} + \frac{1}{4}J^{\tau}{}_{\mu}\partial_{\tau}J^{\sigma}{}_{\nu} = N^{\sigma}{}_{\mu\nu} = 0$$

After removing these terms and relabeling some indices, we are left with

$$T_{\mu\nu\rho} = g_{\rho\sigma}\left[\tilde{\Gamma}^{\sigma}{}_{\mu\nu} + aJ^{\sigma}{}_{\nu}\partial_{\tau}J^{\tau}{}_{\mu} - \frac{1}{2}\tilde{\Gamma}^{\tau}{}_{\mu\alpha}J^{\sigma}{}_{\tau}J^{\alpha}{}_{\nu} + a\tilde{\Gamma}^{\tau}{}_{\tau\alpha}J^{\sigma}{}_{\nu}J^{\alpha}{}_{\mu} - a\tilde{\Gamma}^{\tau}{}_{\alpha\mu}J^{\sigma}{}_{\nu}J^{\alpha}{}_{\tau} - \frac{1}{4}\tilde{\Gamma}^{\tau}{}_{\nu\alpha}J^{\sigma}{}_{\tau}J^{\alpha}{}_{\mu} + \frac{1}{4}\tilde{\Gamma}^{\tau}{}_{\alpha\mu}J^{\alpha}{}_{\nu}J^{\sigma}{}_{\tau}\right] - (\mu \leftrightarrow \nu)$$

$$= g_{\rho\sigma}a\left[J^{\sigma}{}_{\nu}\partial_{\tau}J^{\tau}{}_{\mu} - J^{\sigma}{}_{\mu}\partial_{\tau}J^{\tau}{}_{\nu} + \tilde{\Gamma}^{\tau}{}_{\tau\alpha}J^{\sigma}{}_{\nu}J^{\alpha}{}_{\mu} - \tilde{\Gamma}^{\tau}{}_{\tau\alpha}J^{\sigma}{}_{\mu}J^{\alpha}{}_{\nu} + \tilde{\Gamma}^{\tau}{}_{\alpha\nu}J^{\sigma}{}_{\mu}J^{\alpha}{}_{\tau} - \tilde{\Gamma}^{\tau}{}_{\alpha\mu}J^{\sigma}{}_{\nu}J^{\alpha}{}_{\tau}\right]$$