PHY 623 - Homework 2

M. Ross Tagaras
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PROBLEM 1

Expanding the sum in ${b_a}^{\alpha}{\delta_\alpha}^a = 0$ gives ${b_+}^+ + {b_-}^- = 0$. Then, $0 = {b_a}^{\alpha}\delta_{\alpha b} = {b_b}^{\alpha}\delta_{\alpha a}$ gives either ${b_+}^+$ or ${b_-}^- = 0$, depending on the choice of a and b.

Now we use $\rho^0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\rho^1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to find $\rho_{++} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho_{=} = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$. Expanding the given expression for β:

$$0 = {\beta_B}^{\alpha}\left(\rho_\alpha\right)^B{}_A = {\beta_+}^{++}\left(\rho_{++}\right)^+{}_A + {\beta_+}^{=}\left(\rho_{=}\right)^+{}_A + {\beta_-}^{++}\left(\rho_{++}\right)^-{}_A + {\beta_-}^{=}\left(\rho_{=}\right)^-{}_A$$

For $A = +$, this becomes $-{\beta_-}^{=} = 0$ and for $A = -$, we find ${\beta_+}^{++} = 0$.

PROBLEM 2

From the definition of the vielbein, we find

$$\delta_W h_{\alpha\beta} = \lambda_W h_{\alpha\beta} = 2\left(\delta_W {e_\alpha}^m\right)\eta_{mn}{e_\beta}^n \implies \delta_W {e_\alpha}^m = \frac{1}{2}\lambda_W {e_\alpha}^m$$

We also have

$$\delta_W e = -\frac{1}{2}e h_{\alpha\beta}\delta_W h^{\alpha\beta} = \frac{e\lambda_W}{2}h_{\alpha\beta}h^{\alpha\beta} = e\lambda_W$$

The variation of the scalar kinetic term is

$$\delta_W \mathcal{L}_{scalar} = -\frac{Te\lambda_W}{2}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X + \frac{Te\lambda_W}{2}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X - Teh^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta\left(\delta_W X\right) = -Teh^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta\left(\delta_W X\right)$$

The variation of the Dirac term is

$$\delta_W \mathcal{L}_{Dirac} = -\frac{Te}{2}\bar{\psi}\rho^\alpha\partial_\alpha\left(\delta_W\psi\right) - \frac{Te\lambda_W}{8}\bar{\psi}\rho^\alpha\partial_\alpha\psi$$

The variation of the four fermion term is

$$\frac{Te\lambda_W}{16}\bar{\psi}\psi\bar{\chi}_\alpha\rho^m\rho^n\chi_\beta {e_m}^\beta {e_n}^\alpha + \frac{Te\lambda_W}{16}\left(-\frac{\lambda_W}{2}\right)\bar{\psi}\psi\bar{\chi}_\alpha\rho^m\rho^n\chi_\beta {e_m}^\beta {e_n}^\alpha + \frac{Te\lambda_W}{8}\bar{\psi}\psi\bar{\chi}_\alpha\rho^m\rho^n\left(\delta_W\chi_\beta\right){e_m}^\beta {e_n}^\alpha$$
$$+ \frac{Te\lambda_W}{8}\bar{\psi}\psi\bar{\chi}_\alpha\rho^m\rho^n\chi_\beta\left(-\frac{\lambda_W}{2}\right){e_m}^\beta {e_n}^\alpha$$

$$= -\frac{Te\lambda_W}{32}\bar{\psi}\psi\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta + \frac{Te\lambda_W}{8}\bar{\psi}\psi\bar{\chi}_\alpha\rho^\beta\rho^\alpha\left(\delta_W\chi_\beta\right)$$

From these three expressions, we find the potential transformations

$$\delta_W X = 0 \qquad \delta_W\psi = -\frac{\lambda_W}{4}\psi \qquad \delta_W\chi_\alpha = \frac{\lambda_W}{4}\chi_\alpha$$

The last term in the full classical Lagrangian is invariant under these transformations:

$$\left(\frac{Te\lambda_W}{2}+\frac{Te\lambda_W}{8}-\frac{Te\lambda_W}{2}-\frac{Te\lambda_W}{8}+0\right)\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\partial_\beta X=0$$

There is also the possibility of a term with $\partial\lambda_W$ in $\delta_W\mathcal{L}_{Dirac}$, but $\bar{\psi}\rho^\alpha\psi\partial_\alpha\lambda_W=0$ after a Majorana flip, so our current transformation is sufficient.

PROBLEM 3

The gauge-fixed ghost action is

$$\mathcal{L}_{gh}^{fix}=\left[b_\gamma{}^\alpha\partial_\alpha c^\gamma+b_a{}^\alpha c^a{}_\alpha+\frac{1}{2}b_a{}^a c\right]+\bar{\beta}_A{}^\alpha\left[\left(D_\alpha(\hat{\omega})\gamma\right)^A+\left(\rho_\alpha i\gamma_{sc}\right)^A\right]$$

$$=\left[b_\gamma{}^\alpha\partial_\alpha c^\gamma+b_a{}^\alpha c^a{}_\alpha+b_a{}^a c\right]+\left[i\beta_B\gamma_{sc}^B+(\beta^T)^\alpha{}_A i\left(\rho_\beta\right)^A{}_B\left(\rho^\beta\right)^B{}_A\rho^0\partial_\alpha\gamma^C+\dots\right]$$

This directly gives us

$$0=\frac{\delta\mathcal{L}_{gh}^{fix}}{\delta b_r{}^s}=\frac{1}{2\pi}\left[\delta_\gamma^r\delta_s^\alpha\partial_\alpha c^\gamma+\delta_a^r\delta_s^\alpha c^a{}_\alpha+\frac{1}{2}\delta_{ar}\delta^{as}c\right]=\partial_s c^r+c^r{}_s+\frac{1}{2}\delta_r^s c\implies\partial_a c^a+c=0$$

To find the next field equation, we can write the fields in the first part of the action in terms of their symmetric and antisymmetric parts:

$$\frac{1}{2}\left[b^{[ab]}+b^{(ab)}\right](\partial_b c_a+c_{ab})+\frac{1}{4}\left[b^{[aa]}+b^{(aa)}\right]c=\frac{1}{2}b^{[ab]}\left(\partial_{[b}c_{a]}+c_{[ab]}\right)+\frac{1}{2}b^{(ab)}\partial_b c_a+\frac{1}{4}b^{(aa)}c$$

Taking the derivative with respect to the antisymmetric part of b^{ab} then gives

$$0=c_{[ab]}-\partial_{[a}c_{b]}$$

Finally, the last field equation is

$$0=\frac{\delta\mathcal{L}_{gh}^{fix}}{\delta\beta_A}=i\gamma_S^A+(\not{\partial}\gamma)^A$$

PROBLEM 4

The scalar part of the Lagrangian is

$$\mathcal{L}_{scalar}=-\frac{T}{2}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X=-\frac{T}{2}e^\alpha{}_m\eta^{mn}e^\beta{}_n\partial_\alpha X\cdot\partial_\beta X$$

Its contribution to the stress tensor is

$$\frac{\delta\mathcal{L}_{scalar}}{\delta e^\gamma{}_r{}^{(q)}}=-Te\delta_\gamma^\alpha\delta_m^r\eta^{mn}e^\beta{}_n\partial_\alpha X\cdot\partial_\beta X=-Te\partial_\gamma X\cdot\partial^r X\implies T_{++}^{(X)}=-Te\partial_+X\cdot\partial_+X$$

The Dirac term is

$$\mathcal{L}_{Dirac} = -\frac{T}{4}\psi^T i\rho^0 e^{\alpha}{}_m \rho^m \partial_\alpha \psi$$

Its contribution to the stress tensor is

$$T^{(\psi)}_{++} = \frac{\delta \mathcal{L}_{Dirac}}{\delta e_{++}{}^{(q)}} = -\frac{Ti}{4}\begin{pmatrix}\psi^+ & \psi^-\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\partial_+\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} = -\frac{Ti}{4}\psi^-\partial_+\psi^- = -\frac{Ti}{4}\psi_+\partial_+\psi_+$$

The bc contribution to the stress tensor is

$$T^{(bc)}_{++} = \frac{i}{2\pi}\left[\partial_\alpha\left(b_{++}c^\alpha\right) - b_+{}^\alpha\partial_\alpha c_+ - b_{\alpha+}c^\alpha{}_+ - \frac{1}{2}b_{++}c\right]$$

$$= \frac{i}{2\pi}\left[\partial_+ b_{++}c^+ + \partial_- b_{++}c^- + 2b_{++}\partial_- c_+ - \frac{1}{2}b_{++}\partial^+ c_+ - \frac{1}{2}b_{-+}\partial^- c_+ + \frac{1}{2}b_{++}\partial_+ c^+ + \frac{1}{2}b_{-+}\partial_+ c^- + \frac{1}{2}b_{++}\partial_+ c^+ + \frac{1}{2}b_{++}\partial_- c^-\right]$$

$$= \frac{i}{2\pi}\left[\partial_+ b_{++}c^+ + 2b_{++}\partial_+ c^+\right]$$

The contribution to the supercurrent from X and ψ is given by

$$J^{(X\psi)}_{++} = \frac{\delta}{\delta\chi^{(q)}_{++,+}}\left[\frac{T}{2}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\partial_\beta X\right] = \frac{T}{2}\left(\rho^+\rho_+\psi\right)_+\partial_+ X + \frac{T}{2}\left(\rho^-\rho_+\psi\right)_+\partial_- X$$

$$= \frac{T}{2}\left[\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}\right]_+\partial_+ X + \frac{T}{2}\left[\begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}\right]_+\partial_- X = -\frac{T}{2}2\psi^-\partial_+ X = T\psi_+\cdot\partial_+ X$$

The ghost contribution to the supercurrent is

$$J^{(bc\beta\gamma)}_{++} = -\frac{1}{2\pi}b_{a+}\left(i\bar{\gamma}\rho^a\right)_+ + \frac{1}{2\pi}\left[\partial_\alpha\left(\beta_{++}c^\alpha\right) - \beta^\alpha{}_+\partial_\alpha c_+ - \frac{1}{4}\beta_{++}c + \frac{1}{4}\left(\rho_a\rho_b\beta_+\right)_+ c^{ab}\right]$$

$$= -\frac{1}{2\pi}b_{++}\left(i\bar{\gamma}\rho^+\right)_+ - \frac{1}{2\pi}b_{-+}\left(i\bar{\gamma}\rho^-\right)_+ + \frac{1}{2\pi}\left[\partial_+\beta_{++}c^+ + \partial_-\beta_{++}c^- + 2\beta_{++}\partial_- c_+ + \frac{1}{4}\beta_{++}\partial_+ c^+ + \frac{1}{4}\beta_{++}\partial_- c^- \right.$$
$$\left. -\frac{1}{4}\beta_{++}\left(\partial^+ c^- - \partial^- c^+\right)\right]$$

$$= \frac{1}{2\pi}b_{++}\left[\begin{pmatrix}\gamma^+ & \gamma^-\end{pmatrix}\begin{pmatrix}-2 & 0\\ 0 & 0\end{pmatrix}\right]_+ + \frac{1}{2\pi}b_{-+}\left[\begin{pmatrix}\gamma^+ & \gamma^-\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\right]_+ + \frac{1}{2\pi}\left[\partial_+\beta_{++}c^+ + \frac{3}{2}\beta_{++}\partial_+ c^+\right]$$

$$= \frac{1}{\pi}b_{++}\gamma^+ + \frac{1}{2\pi}\left[\partial_+\beta_{++}c^+ + \frac{3}{2}\beta_{++}\partial_+ c^+\right]$$

X, ψ, c, and β are real; γ and b are imaginary. It is easy to see that each term in the stress tensor/supercurrent is real.