PHY 622 - Homework 6

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PART (A)

Note: I can't figure out how to typeset the double-plus symbol, so I'm just going to write ++ everywhere instead.

The spinning string action is

$$S = \frac{1}{\pi l^2} \int d\sigma^{++} d\sigma^{=} \left(2\partial_{++} X \cdot \partial_{=} X + \frac{i}{2} \left(\psi^+ \cdot \partial_{=} \psi^+ + \psi^- \cdot \partial_{++} \psi^- \right) + \frac{1}{2} F \cdot F \right)$$

The individual terms transform as

$$(\partial_{++} X \partial_{=} X)' = \Lambda \left[\partial_{++} X\right] \Lambda^{-1} \left[\partial_{=} X\right] = \partial_{++} X \partial_{=} X$$

$$\left(\psi^+ \partial_{=} \psi^+\right)' = \Lambda^{1/2} \psi^+ \Lambda^{-1} \partial_{=} \Lambda^{1/2} \psi^+ = \psi^+ \partial_{=} \psi^+$$

$$\left(\psi^- \partial_{++} \psi^-\right)' = \Lambda^{-1/2} \psi^- \Lambda \partial_{++} \Lambda^{-1/2} \psi^- = \psi^- \partial_{++} \psi^-$$

$$\left(F^\mu F^\nu \eta_{\mu\nu}\right)' = \Lambda^\mu{}_\rho F^\rho \Lambda^\nu{}_\sigma F^\sigma \eta_{\mu\nu} = F^\rho F^\sigma \eta_{\rho\sigma}$$

so the entire action is invariant.

PART (B)

Since X^μ is a scalar field, it transforms as $\left(X^\mu(\sigma')\right)' = X^\mu(\sigma)$. If we Taylor expand the left side, we find (to first order)

$$X^{\mu\prime}(\sigma) + \partial_\nu X^\mu(\sigma)(\sigma' - \sigma)^\nu = X^\mu(\sigma) \implies \delta X^\mu = \lambda \left(\sigma^{=} \partial_{=} - \sigma^{++} \partial_{++}\right) X^\mu$$

From the rule $\psi^\pm(\sigma') = \Lambda^{\pm 1/2} \psi^\pm(\sigma)$ for spinors, Taylor expanding gives

$$\psi^{\pm\prime}(\sigma) + \partial_\mu \psi^\pm(\sigma)(\sigma' - \sigma)^\mu = \left(1 \pm \frac{1}{2}\right) \psi^\pm(\sigma) \implies \delta\psi^\pm = \lambda \left(\pm\frac{1}{2} + \sigma^{=} \partial_{=} - \sigma^{++} \partial_{++}\right) \psi^\pm$$

PART (C)

The variation of the action is

$$\delta S = \frac{1}{\pi l^2} \int \Bigg[2\partial_{++} \left(\lambda\sigma^{=} \partial_{=} X - \lambda\sigma^{++} \partial_{++} X\right) \cdot \partial_{=} X + \partial_{++} X \cdot \partial_{=} \left(\lambda\sigma^{=} \partial_{=} X - \lambda\sigma^{+} \partial_{+} X\right)$$
$$+ \frac{i}{2} \left(\left[\frac{\lambda}{2} + \lambda\sigma^{=} \partial_{=} - \lambda\sigma^{++} \partial_{++}\right] \psi^+ \partial_{=} \psi^+ + \psi^+ \partial_{=} \left(\left[\frac{\lambda}{2} + \lambda\sigma^{=} \partial_{=} - \lambda\sigma^{++} \partial_{++}\right] \psi^+ \right) \right)$$
$$+ \frac{i}{2} \left(\left[-\frac{\lambda}{2} + \lambda\sigma^{=} \partial_{=} - \lambda\sigma^{++} \partial_{++}\right] \psi^- \partial_{++} \psi^- + \psi^- \partial_{++} \left(\left[-\frac{\lambda}{2} + \lambda\sigma^{=} \partial_{=} - \lambda\sigma^{++} \partial_{++}\right] \psi^- \right) \right) \Bigg]$$

If we look at the bosonic terms, we see that we can make them all proportional to $\partial_{++}\partial_{=}X$ if we integrate the first and third terms by parts. If we do this and drop the total derivatives, we get

$$\delta S_{bosonic} = \frac{2\lambda}{\pi l^2} \int \left[\sigma^{++}\partial_{++}X + \sigma^{=}\partial_{=}X - \sigma^{=}\partial_{=}X - \sigma^{++}\partial_{++}X\right] \partial_{++}\partial_{=}X = 0$$

In the fermion terms, we immediately see that the terms with $\left(\partial_{++}\psi^{+}\right)^2$ and $\left(\partial_{=}\psi^{-}\right)^2$ are identically zero. The first line of fermionic terms is then

$$\psi^{+}\partial_{=}\psi^{+} - \sigma^{++}\left(\partial_{=}\psi^{+}\right)\partial_{++}\psi^{+} + \psi^{+}\partial_{=}\left(\sigma^{=}\partial_{=}\psi^{+}\right) - \psi^{+}\partial_{=}\left(\sigma^{++}\partial_{++}\psi^{+}\right)$$

$$= \psi^{+}\partial_{=}\psi^{+} - \sigma^{++}\left(\partial_{++}\psi^{+}\right)\partial_{=}\psi^{+} + \partial_{=}\left(\psi^{+}\sigma^{=}\partial_{=}\psi^{+}\right) - \left(\partial_{=}\psi^{+}\right)^2\sigma^{=} - \psi^{+}\partial_{=}\psi^{+} - \sigma^{++}\psi^{+}\partial_{=}\partial_{++}\psi^{+}$$

$$= -\sigma^{++}\left(\partial_{++}\psi^{+}\right)\partial_{=}\psi^{+} - \partial_{=}\left(\sigma^{++}\psi^{+}\partial_{++}\psi^{+}\right) + \sigma^{++}\left(\partial_{=}\psi^{+}\right)\partial_{++}\psi^{+} = 0$$

The second line vanishes by similar manipulations.

PART (D)

$SO(1,1)$ is abelian, so all commutators are trivially zero.

PART (E)

I'm not sure that the idea of semi-local Lorentz symmetry makes any physical sense.