PHY 622 - Homework 2

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(Dated: September 11, 2019)

PROBLEM 1

The induced metric is $h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \delta_{\mu\nu}$. The components are

$$h_{\theta\theta} = \partial_\theta X^\mu \partial_\theta X_\mu = \left[\partial_\theta \left(R\cos\phi\sin\theta\right)\right]^2 + \left[\partial_\theta \left(R\sin\phi\sin\theta\right)\right]^2 + \left[\partial_\theta \left(R\cos\theta\right)\right]^2 = R^2$$

$$h_{\theta\phi} = h_{\phi\theta} = \partial_\phi X^\mu \partial_\theta X_\mu = \partial_\theta \left(R\cos\phi\sin\theta\right)\partial_\phi \left(R\cos\phi\sin\theta\right) + \partial_\theta \left(R\sin\phi\sin\theta\right)\partial_\phi \left(R\sin\phi\sin\theta\right) = 0$$

$$h_{\phi\phi} = \partial_\phi X^\mu \partial_\phi X_\mu = R^2\sin^2\theta$$

So as a matrix, h is

$$h_{\alpha\beta} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

with $\sqrt{\det h_{\alpha\beta}} = R^2\sin\theta$. The action is

$$\int d\theta d\phi \sqrt{h} = 2\pi R^2 \int_0^\pi d\theta \sin\theta = 4\pi R^2$$

In stereographic coordinates, the points $X^\mu(x, y, z)$ on S^2 are the points that lie on the line connecting the points $(0, 0, R)$ and $(\xi_1, \xi_2, -R)$ such that $x^2 + y^2 + z^2 = R^2$. The equation of this line is

$$\frac{x}{\xi_1} = \frac{y}{\xi_2} = \frac{R - z}{2R}$$

From these two equations, we can find expressions for x, y, and z. The first equality gives

$$x^2 = \frac{R^2 - z^2}{1 + \frac{\xi_2^2}{\xi_1^2}}$$

and the second gives

$$z = R - \frac{2Rx}{\xi_1}$$

Combining these:

$$x^2 = -\frac{4R^2x^2}{\xi^2} + \frac{4R^2\xi_1 x}{\xi^2}$$

which simplifies to

$$x = \frac{4R^2\xi_1}{4R^2 + \xi^2}$$

In a similar manner, we find that

$$y = \frac{4R^2\xi_2}{4R^2 + \xi^2}$$

To find $z(\xi_1, \xi_2, R)$, we can simply combine the previous two expressions using $x^2 + y^2 + z^2 = R^2$:

$$z^2 = R^2 - \frac{16R^4\xi^2}{\left(4R^2 + \xi^2\right)^2} = \frac{R^2\left(16R^4 + \xi^4 - 8R^2\xi^2\right)}{\left(4R^2 + \xi^2\right)^2}$$

so we see that

$$z = \frac{R\left(\xi^2 - 4R^2\right)}{4R^2 + \xi^2}$$

With these new coordinates, we can now calculate the induced metric. A few useful formulas are:

$$\frac{\partial x^i}{\partial \xi^j} = \frac{4R^2\delta^i_j\left(\xi^2 + 4R^2\right) - 8R^2\xi^i\xi_j}{\left(\xi^2 + 4R^2\right)^2} \qquad \frac{\partial z}{\partial \xi^i} = \frac{16R^3\xi_i}{\left(\xi^2 + 4R^2\right)^2}$$

The determinant of the induced metric is

$$h = \left[\left(\frac{\partial x^1}{\partial \xi^1}\right)^2 + \left(\frac{\partial x^2}{\partial \xi^1}\right)^2 + \left(\frac{\partial x^3}{\partial \xi^1}\right)^2\right]\left[\left(\frac{\partial x^1}{\partial \xi^2}\right)^2 + \left(\frac{\partial x^2}{\partial \xi^2}\right)^2 + \left(\frac{\partial x^3}{\partial \xi^2}\right)^2\right] - \left[\frac{\partial x^1}{\partial \xi^1}\frac{\partial x^1}{\partial \xi^2} + \frac{\partial x^2}{\partial \xi^1}\frac{\partial x^2}{\partial \xi^2} + \frac{\partial x^3}{\partial \xi^1}\frac{\partial x^3}{\partial \xi^2}\right]^2$$

which after some tedious algebra can be shown to be

$$h = \frac{256R^8}{\left(4R^2 + \xi^2\right)^4}$$

The action is

$$S = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{16R^4}{\left(4R^2 + \xi^2\right)^2} = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{16R^4}{\left(4R^2 + \xi_1^2 + \xi_2^2\right)^2} = \int_{-\infty}^{\infty} d\xi_1 \frac{8\pi R^4}{\left(4R^2 + \xi_1^2\right)^{3/2}} = 4\pi R^2$$

PROBLEM 2

We define the induced metric as

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Its determinant is then

$$\det h_{\alpha\beta} = h_{00}h_{11} - h_{01}h_{10}$$

$$= \ell^4\left\{\left[-(\partial_0 t)^2 + (\partial_0\sigma)^2\right]\left[-(\partial_1 t)^2 + (\partial_1\sigma)^2\right] - \left[-(\partial_0 t)(\partial_1 t) + (\partial_0\sigma)(\partial_1\sigma)\right]\left[-(\partial_1 t)(\partial_0 t) + (\partial_1\sigma)(\partial_0\sigma)\right]\right\} = -\ell^4$$

Therefore, we see that the minus sign under the square root is needed to ensure that $\sqrt{-h}$ is real.

PROBLEM 3

The action is

$$S = -T\int dt\int_0^\pi d\sigma\sqrt{-h} = -T\int dt\int_0^\pi d\sigma\sqrt{\left[\partial_t\left(\ell t\right)\right]^2\left[\partial_\sigma\left(f\left(\sigma\right)\right)\right]^2} = -T\ell\int dt\int_0^\pi d\sigma\ \partial_\sigma f(\sigma)$$

$$= -T\ell\int dt\left[f(\pi)-f(0)\right] = \int dt\left[-T\ell a\right]$$

Since our string has no dynamics by choice of gauge, there should be no kinetic energy, and therefore the remaining term in the Lagrangian is $-V$.

To obtain the equations of motion for the Nambu-Goto action, it is easier to use the form

$$\mathcal{L} = -T\sqrt{\left(\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial\sigma}\right)^2 - \frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial t}\frac{\partial X^\nu}{\partial\sigma}\frac{\partial X_\nu}{\partial\sigma}}$$

The equations of motion are then

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} = \partial_t\left(\frac{\left(\frac{\partial X^\nu}{\partial t}\frac{\partial X_\nu}{\partial\sigma}\right)\frac{\partial X^\mu}{\partial\sigma} - \frac{\partial X^\nu}{\partial\sigma}\frac{\partial X_\nu}{\partial\sigma}\frac{\partial X^\mu}{\partial t}}{\sqrt{\left(\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial\sigma}\right)^2 - \frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial t}\frac{\partial X^\nu}{\partial\sigma}\frac{\partial X_\nu}{\partial\sigma}}}\right) + \partial_\sigma\left(\frac{\left(\frac{\partial X^\nu}{\partial t}\frac{\partial X_\nu}{\partial\sigma}\right)\frac{\partial X^\mu}{\partial t} - \frac{\partial X^\nu}{\partial t}\frac{\partial X_\nu}{\partial t}\frac{\partial X^\mu}{\partial\sigma}}{\sqrt{\left(\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial\sigma}\right)^2 - \frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial t}\frac{\partial X^\nu}{\partial\sigma}\frac{\partial X_\nu}{\partial\sigma}}}\right)$$

In the static gauge, we find that

$$\frac{\partial X^\nu}{\partial t}\frac{\partial X_\nu}{\partial\sigma} = 0 \qquad \frac{\partial X^\nu}{\partial\sigma}\frac{\partial X_\nu}{\partial\sigma} = \left(\partial_\sigma f\right)^2 \qquad \frac{\partial X^\nu}{\partial t}\frac{\partial X_\nu}{\partial t} = -\ell^2$$

so the equations of motion become

$$0 = \partial_t\left(\frac{-\left(\partial_\sigma f\right)^2\frac{\partial X^\mu}{\partial t}}{\ell}\right) + \partial_\sigma\left(\frac{\ell\frac{\partial X^\mu}{\partial\sigma}}{\partial_\sigma f}\right)$$

These are trivially satisfied for $\mu = 2,\ldots,d-1$. For $\mu = 0$, we find

$$0 = -\partial_t\left(\partial_\sigma f\right)^2 + \partial_\sigma(0) = 0$$

For $\mu = 1$, we find

$$0 = \partial_t(0) + \partial_\sigma\left(\frac{\ell\partial_\sigma f}{\partial_\sigma f}\right) = 0$$