# Homework 2

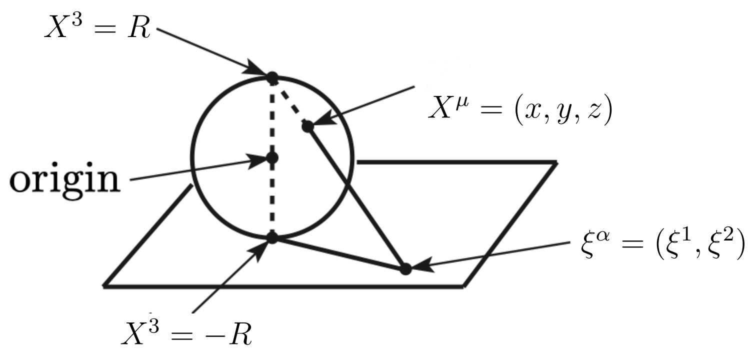
Due on: Wednesday, September 11

## Problem 1

In order to illuminate the concept of an induced metric, we consider a familiar model: the two-sphere $S^2$ with radius $R$ and coordinates $\sigma^1 = \theta$ and $\sigma^2 = \varphi$ embedded in 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates $X^\mu(\sigma^1, \sigma^2) = (x(\sigma^1, \sigma^2), y(\sigma^1, \sigma^2), z(\sigma^1, \sigma^2))$. The metric in $\mathbb{R}^3$ space is $\delta_{\mu\nu}$. Using $x = R\cos\varphi\sin\theta$, $y = R\sin\varphi\sin\theta$, $z = R\cos\theta$, construct the induced metric $h^{\text{ind}}_{\alpha\beta}(\sigma^\gamma)$ on the "worldsheet" $S^2$. What is the value of the action

$$S = \int d\theta\, d\varphi \sqrt{\det(h^{\text{ind}}_{\alpha\beta})} \quad ? \tag{1.1}$$

Consider next stereographic coordinates $\xi^1$ and $\xi^2$ on $S^2$ where




First show that $X^\mu(\xi^1, \xi^2)$ are given by

$$x^k = \xi^k \frac{4R^2}{\xi^2 + 4R^2}; \quad z = \frac{R(\xi^2 - 4R^2)}{\xi^2 + 4R^2} \qquad (k = 1, 2)\ . \tag{1.2}$$

Then evaluate again the action $S$ in this choice of coordinates, and check that the value of $S$ is the same.

**Comment:** you should find that $h^{\text{ind}}_{\alpha\beta}$ is proportional to $\delta_{\alpha\beta}$. Such metrics are called conformal; they preserve the angles between two curves on $S^2$ and the two corresponding curves in the plane.

## Problem 2

In order to check the two minus signs in the Nambu-Goto Lagrangian,

$$L_{NG} = -T \int_0^{\pi \text{ or } 2\pi} d\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}\ , \tag{2.1}$$

first consider a worldsheet with $X^0 = \ell t$, $X^1 = \ell\sigma$, $X^j = 0$ for $j \neq 1$. The dimension of $X^0$ and $X^j$ is that of a length, and we prefer to take $\sigma$ and $t$ as dimensionless, so we need the constant $\ell$ with the dimensions of a length. (Then dimensionless $t$ and $\sigma$ will appear in the exponents in the mode expansions.) Show that one needs a minus sign inside the square root.

Next consider a worldsheet with $X^0 = \ell t$, $X^1 = f(\sigma)$. Draw a picture of this worldsheet and the coordinate grid of constant $\sigma$ and constant $t$ lines. Show that in order that $L_{NG} = T^{\text{kin}} - V$ with positive $T^{\text{kin}}$, one needs the overall minus sign in $L_{NG}$.

**Comment:** Some textbooks begin with coordinates $t$ and $\sigma$ with the dimensions of time and length, respectively. Later they switch to dimensionless parameters $t'$ and $\sigma'$. We prefer to use only one set of (dimensionless) $\sigma$ and $t$.

## Problem 3

In order to identify the constant $T\ell$ in the string action as the tension of the string, we want to show that the potential energy $V$ of an open static stretched relativistic string of length $a$ is equal to $V = T\ell a$.

Choose the static gauge $X^0 = \ell t$ and

$$\begin{aligned} &X^1(t,\sigma) = f(\sigma); \quad X^2 = \cdots = X^d = 0. \\ &f(\sigma = 0) = 0; \qquad f(\sigma = \pi) = a \end{aligned} \tag{3.1}$$

The function $f(\sigma)$ has the dimension of a length and is strictly increasing and smooth on the interval $0 \leq \sigma \leq \pi$ but further arbitrary. Einstein invariance (reparametrization invariance) should give an answer for $V$ that is independent of the choice of $f(\sigma)$.

(a) Evaluate the Nambu-Goto action (not the Lagrangian) for this string. Show that is of the form $S_{NG} = \int K\, dt$ where $K$ is a constant, and explain why $K = -V$.

(b) Show that $K = -T\ell a$.

(c) Show that this string satisfies the equations of motion which follow from the Nambu-Goto action.