PHY 623 - Homework 3

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PROBLEM 1

Scalar fields do not transform under local Lorentz transformations, spinors transform as $\delta_{LL}\psi = \frac{1}{4}\lambda^{mn}\rho_{mn}\psi$, and vectors (with flat indices) transform as $\delta_{LL}v^m = \lambda^m{}_n v^n$. First, we will check $\mathcal{L}_{cl}$:

$$\frac{1}{T}\mathcal{L}_{cl} = -\frac{e}{2}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X - \frac{e}{4}\bar{\psi}\rho^\alpha\partial_\alpha\psi + \frac{e}{2}F\cdot F + \frac{e}{2}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\cdot\partial_\beta X + \frac{e}{16}(\bar{\psi}\cdot\psi)\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta$$

It is obvious that the scalar kinetic term and the auxiliary term are invariant, so we only need to check the remaining terms.

$$\delta_{LL}\left(\bar{\psi}\rho^\alpha\partial_\alpha\psi\right) = \frac{1}{4}\bar{\psi}\rho^\alpha\rho_{mn}\psi\partial_\alpha\lambda^{mn} = -\frac{1}{4}\bar{\psi}\rho^\alpha\rho_{mn}\psi\partial_\alpha\lambda^{mn} = 0$$

$$\delta_{LL}\left(\frac{e}{2}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\cdot\partial_\beta X\right) = \bar{\chi}_\alpha\rho^\beta\left(-\frac{1}{4}\lambda^{mn}\rho_{mn}\right)\rho^\alpha\psi\cdot\partial_\beta X + \bar{\chi}_\alpha\rho^\beta\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\right)\rho^\alpha\psi\cdot\partial_\beta X = 0$$

$$\delta_{LL}\left(\bar{\psi}\psi\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta\right) = \left[\bar{\psi}\left(-\frac{1}{4}\lambda^{mn}\rho_{mn}\right)\psi + \bar{\psi}\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\psi\right)\right]\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta$$
$$+\bar{\psi}\psi\left[\bar{\chi}_\alpha\left(-\frac{1}{4}\lambda^{mn}\rho_{mn}\psi\right)\rho^\alpha\chi_\beta + \bar{\chi}_\alpha\rho^\beta\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\psi\right)\rho^\alpha\chi_\beta\right] = 0$$

so the classical Lagrangian is invariant. The gauge fixing Lagrangian transforms as

$$\lambda_a{}^b d_b{}^\alpha e_\alpha{}^a{}_{(qu)} + d_a{}^\alpha\lambda^a{}_b e_\alpha{}^b{}_{(qu)} + \bar{\Delta}_B{}^\alpha\left(-\frac{1}{4}\lambda^{mn}\left(\rho_{mn}\right)^B{}_A\right)\chi_\alpha{}^A{}_{(qu)} + \bar{\Delta}_A{}^\alpha\left(\frac{1}{4}\lambda^{mn}\left(\rho_{mn}\right)^A{}_B\right)\chi_\alpha{}^B{}_{(qu)}$$

$$= \lambda_\alpha{}^b d_b{}^\alpha + \lambda^a{}_\alpha d_a{}^\alpha + 0 = 0$$

The bc terms transform as

$$\left[\lambda_a{}^b b_b{}^\alpha c^\gamma\partial_\gamma e_\alpha{}^a + b_a{}^\alpha c^\gamma\partial_\gamma\left(\lambda^a{}_b e_\alpha{}^b\right)\right] + \left[\lambda_a{}^b b_b{}^\alpha\left(\partial_\alpha c^\gamma\right)e_\gamma{}^a + b_a{}^\alpha\left(\partial_\alpha c^\gamma\right)\lambda^a{}_b e_\gamma{}^b\right] + \left[\lambda^a{}_c b_c{}^\alpha c^a{}_b e_\alpha{}^b + b_a{}^\alpha\lambda^a{}_c c^c{}_b e_\alpha{}^b\right]$$

$$= b_a{}^b c^\gamma\partial_\gamma\lambda^a{}_b = 0$$

The $\beta\gamma$ terms transform as

$$\bar{\beta}_A{}^\alpha\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\right)c^\gamma\partial_\gamma\chi_\alpha{}^A - \bar{\beta}_A{}^\alpha c^\gamma\partial_\gamma\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\chi_\alpha{}^A\right) + \bar{\beta}_A{}^\alpha\left(-\frac{1}{4}\lambda^{mn}\rho_{mn}\right)\left(D_\alpha(\hat{\omega})\gamma\right)^A + \bar{\beta}_A{}^\alpha\left(\frac{1}{4}\lambda^{mn}\rho_{mn}\right)\left(D_\alpha(\hat{\omega})\gamma\right)^A + \ldots$$

$$= -\frac{1}{4}\bar{\beta}_A{}^\alpha c^\gamma\rho_{mn}\chi_\alpha{}^A\partial_\gamma\lambda^{mn}$$

PROBLEM 2

$$\delta_{conf}\psi = (\delta_E + \delta_{LL} + \delta_W)\,\psi = \xi^\alpha\partial_\alpha\psi + \frac{1}{4}\lambda^{mn}\rho_{mn}\psi - \frac{1}{4}\lambda_W\psi$$

with $\lambda^{mn} = \frac{1}{2}\left(\partial^m\xi^n - \partial^n\xi^m\right)$ and $\lambda_W = -\partial_\alpha\xi^\alpha$. Using $\rho_{++} = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\rho_{=} = \left(\begin{smallmatrix}0&0\\-1&0\end{smallmatrix}\right)$, this becomes

$$\delta_{conf}\begin{pmatrix}\psi_-\\-\psi_+\end{pmatrix} = \xi^\alpha\partial_\alpha\begin{pmatrix}\psi_-\\-\psi_+\end{pmatrix} + \frac{1}{4}\left(\partial_{++}\xi^{++} - \partial_{=}\xi^{=}\right)\begin{pmatrix}\psi_-\\0\end{pmatrix} + \frac{1}{4}\left(\partial_{=}\xi^{=} - \partial_{++}\xi^{++}\right)\begin{pmatrix}0\\-\psi_+\end{pmatrix} + \frac{1}{4}\left(\partial_\alpha\xi^\alpha\right)\begin{pmatrix}\psi_-\\-\psi_+\end{pmatrix}$$

Or, in terms of the individual components:

$$\delta_{conf}\psi_+ = \xi^{++}\partial_{++}\psi_+ + \xi^{=}\partial_{=}\psi_+ + \frac{1}{2}\left(\partial_{=}\xi^{=}\right)\psi_+$$

$$\delta_{conf}\psi_- = \xi^{++}\partial_{++}\psi_{=} + \xi^{=}\partial_{=}\psi_{=} + \frac{1}{2}\left(\partial_{++}\xi^{++}\right)\psi_{=}$$

The first two terms are the standard transport terms in the infinitesimal transformation rule for a tensor, and the third term is like the infinitesimal transformation of an index, but half as fast as usual. This suggests that instead of using

$$\psi_+(x') = \frac{\partial x}{\partial x'}\psi_+(x)$$

we should use instead

$$\psi_+(x') = \sqrt{\frac{\partial x}{\partial x'}}\psi_+(x)$$

For the transformation from the worldsheet to the complex z-plane, we get

$$\psi_+(t+\sigma) = \sqrt{\frac{\partial z}{\partial(t+\sigma)}}\psi_z(z)$$

After Wick rotating, this is

$$\sqrt{\frac{\partial e^{\tau+i\sigma}}{\partial(-i\tau+\sigma)}}\psi_z(z) = \sqrt{iz}\psi_z(z) \equiv \sqrt{z}\psi(z)$$

We have previously worked out the contribution of ψ to the stress tensor:

$$T^{(\psi)}_{++\ ++} = \frac{iT}{4}\psi_+\partial_{++}\psi_+$$

Substituting this into the expression for C gives

$$C = i\int_0^{2\pi} d\sigma\xi^{++}T_{++\ ++} = \frac{2\pi i}{2\pi i}\frac{i^2T}{4}\oint\frac{dz}{iz}\frac{d\sigma^{++}}{dz}\frac{dz}{d\sigma^{++}}\xi^z\sqrt{z}^2\psi\partial_z\psi = \frac{1}{2\pi i}\oint dz\ \xi^z\left(-\frac{\pi T}{2}\psi\partial_z\psi\right)$$

so we see that $T(z) = -\frac{\pi T}{2}\psi\partial_z\psi$.