PHY 623 - Exercise IA2.3

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If we multiply by a test function $f(\theta)$, then

$$D_k = e^{-\frac{1}{2}\theta^i\bar{\theta}_i}\frac{\partial}{\partial\theta^k}e^{\frac{1}{2}\theta^j\bar{\theta}_j}f(\theta^k) = \left(1-\frac{1}{2}\theta^i\bar{\theta}_i\right)\frac{\partial}{\partial\theta^k}\left[\left(1+\frac{1}{2}\theta^j\bar{\theta}_j\right)f(\theta^k)\right] = \left(1-\frac{1}{2}\theta^i\bar{\theta}_i\right)\left[\frac{1}{2}\delta^j_k\bar{\theta}_j f(\theta^k) + \left(1+\frac{1}{2}\theta^j\bar{\theta}_j\right)\frac{\partial f(\theta^k)}{\partial\theta^k}\right]$$

$$= \frac{1}{2}\bar{\theta}^k f(\theta^k) + \frac{\partial f(\theta^k)}{\partial\theta^k} \implies e^{-\frac{1}{2}\theta^i\bar{\theta}_i}\frac{\partial}{\partial\theta^k}e^{\frac{1}{2}\theta^j\bar{\theta}_j} = \frac{\partial}{\partial\theta^k} + \frac{1}{2}\bar{\theta}_k$$

and we find a similar result for $\bar{D}^k$.

Now take f and $\bar{g}$ to be arbitrary functions of $\theta, \bar{\theta}$. Then, using the conditions $\bar{D}f = D\bar{g} = 0$, we find that

$$\left(\partial_{\bar{\theta}} + \frac{1}{2}\theta\right)\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = c - d\theta + \frac{a}{2}\theta + \frac{c}{2}\theta\bar{\theta} = 0 \implies c = 0,\ d = a/2$$

so we have $f = a + b\theta + \frac{a}{2}\theta\bar{\theta}$. By a similar calculation, $g = a + b\bar{\theta} - \frac{a}{2}\theta\bar{\theta}$. Generalizing to more than one θ, it is easy to see that $f = \prod_i \left(a_i + b_i\theta_i + \frac{a_i}{2}\theta^i\bar{\theta}_i\right)$ obeys the condition $D_i f = 0$ (and similarly for g).

We can also construct f by acting on a general function of θ with $e^{\frac{1}{2}\theta\bar{\theta}}$:

$$e^{\frac{1}{2}\theta\bar{\theta}}\,(a + b\theta) = (1 + \frac{1}{2}\theta\bar{\theta})\,(a + b\theta) = a + b\theta + \frac{1}{2}\theta\bar{\theta} = f$$

and this easily generalizes to cases with more θ's. We also find a similar result for $\bar{g}$:

$$e^{-\frac{1}{2}\theta\bar{\theta}}(a + b\bar{\theta}) = \left(1 - \frac{1}{2}\theta\bar{\theta}\right)\left(a + b\bar{\theta}\right) = a + b\bar{\theta} - \frac{1}{2}\theta\bar{\theta} = \bar{g}$$

If we now take Df, we find

$$Df = \left(\partial_\theta + \frac{1}{2}\bar{\theta}\right)\left(a + b\theta + \frac{a}{2}\theta\bar{\theta}\right) = b + \frac{a}{2}\bar{\theta} + \frac{a}{2}\bar{\theta} + \frac{b}{2}\bar{\theta}\theta = b + a\bar{\theta} - \frac{b}{2}\theta\bar{\theta}$$

This matches the expression for $\bar{g}$ with the order of the coefficients reversed. Since we can move $D(\theta_i)$ through $f(\theta_j)$ if $i \neq j$, this generalizes to the case with more than one θ. We also have that

$$\bar{D}\bar{g} = \left(\partial_{\bar{\theta}} + \frac{1}{2}\theta\right)\left(a + b\bar{\theta} - \frac{a}{2}\theta\bar{\theta}\right) = b + a\theta + \frac{b}{2}\theta\bar{\theta}$$

Finally, let f be a function of θ_i, where $i = 1, \ldots, N$. It can be written as a sum of all products of the θ_i's where none repeat e.g. $f(\theta_1, \theta_2) = \alpha^f_1 + \alpha^f_2\theta_1 + \alpha^f_3\theta_2 + \alpha^f_4\theta_1\theta_2$. When we apply the operator $\mathcal{D}_i = (\partial_{\theta_i} + \theta_i)$ to f, the derivative will kill the 2^{N-1} terms without θ_i, and multiplying by θ_i will kill the 2^{N-1} terms with a θ_i, leaving us with $2^{N-1} + 2^{N-1} = 2^N$ terms, which is the same number we started with.

When we act on f with $\mathcal{D}_i$, the overall effect is to reshuffle the coefficients, reducing the index on terms with θ_i and increasing it on terms without θ_i. For example,

$$(\partial_{\theta_1} + \theta_1)\,(\alpha_1 + \alpha_2\theta_1 + \alpha_3\theta_2 + \alpha_4\theta_1\theta_2) = \alpha_2 + \alpha_1\theta_1 + \alpha_4\theta_2 + \alpha_3\theta_1\theta_2$$

As we continue this process, the order of the α_i's completely reverses. If $g = \alpha^g_1 + \alpha_2\theta_1 + \ldots$, then the relationship between the coefficients of f and g is $\alpha^g_1 = \alpha^f_N$, $\alpha^g_2 = \alpha^f_{N-1}$, etc.