Homework 2

Due on: Monday, February 17

Problem 1

From integrating out the BRST-auxiliary fields and the quantum vielbein and quantum gravitino, the Weyl ghost, the Lorentz ghosts, and the super-Weyl ghosts, and setting the background fields to their super-flat-space values, $e_{\alpha,back}^{a} = \delta_a{}^a$ and $\chi_{\alpha,back}^{A} = 0$, we found the following field equations for the Weyl ghost c_W, the Lorentz ghosts c^{ab}, and the super-Weyl ghosts γ_{cs}^{A}

$$b_a{}^\alpha \delta_\alpha{}^a = 0\,; \quad b_a{}^\alpha \eta_{\alpha b} - b_b{}^\alpha \eta_{\alpha a} = 0\,; \quad (\beta^\alpha \rho_\alpha)_A = 0\,. \tag{1.1}$$

Show that in $+-$ notation this implies

$$b_{++}{}^{++} = 0\,, \quad b_{=}{}^{=} = 0\,, \quad \beta_{+}{}^{++} = 0\,, \quad \beta_{-}{}^{=} = 0\,. \tag{1.2}$$

The quantum action for the spinning string now takes on the following form

$$\mathcal{L} = T\left[2\partial_{++}X\cdot\partial_{=}X + \tfrac{i}{2}\psi_+\cdot\partial_{=}\psi_+ + \tfrac{i}{2}\psi_-\cdot\partial_{++}\psi_-\right] \\ + \tfrac{i}{\pi}\left[b_{++++}\partial_{=}c^{++} + \beta_{+++}\partial_{=}\gamma^{+} + (+\leftrightarrow -)\right]. \tag{1.3}$$

Problem 2

The classical action (the $d = 1+1$ WZ model coupled to external worldsheet supergravity) has 4 bosonic and 4 fermionic local (gauge) symmetries: Einstein, Weyl, local Lorentz, local susy and local super-Weyl (= local conformal susy) invariance. We focus in this problem on Weyl invariance. We normalize Weyl rescalings by $\delta_W h_{\alpha\beta} = \lambda_W h_{\alpha\beta}$. Requiring invariance of the classical action under Weyl rescaling, derive the following transformation rules under Weyl transformations

$$\delta_W e_\alpha{}^a = \tfrac{1}{2}\lambda_W e_\alpha{}^a\,; \quad \delta_W \psi^A = -\tfrac{1}{4}\lambda_W \psi^A\,; \quad \delta_W \chi_\alpha{}^A = +\tfrac{1}{4}\lambda_W \chi_\alpha{}^A\,; \quad \delta_W X^\mu = 0\,. \tag{2.1}$$

Did we use any of these transformation laws in the construction of the BRST-invariant quantum action? If so, which one(s)?

Problem 3

Before integrating out any fields, and keeping arbitrary background fields, define the gravitational stress tensor by

$$\frac{\delta}{\delta e_{\alpha,back}^{\ \ a}} S = T_a{}^\alpha \tag{3.1}$$

and the supercurrent by

$$\frac{\delta}{\delta \chi_{\alpha,back}^{\ \ A}} S = i J_A{}^\alpha \tag{3.2}$$

Derive the following field equations

$$\frac{\delta}{\delta e_{\alpha,qu}^{\ \ a}} S = d_a{}^\alpha + T_a{}^\alpha = 0 \tag{3.3}$$

$$\frac{\delta}{\delta \chi_{\alpha,qu}^{\ \ A}} S = \Delta_A{}^\alpha + i J_A{}^\alpha = 0\,. \tag{3.4}$$

(Recall that Δ is imaginary, so with this definition J is real.) Now set $e_{\alpha,back}^{\ \ a} = \delta_\alpha{}^a$ and $\chi_{\alpha,back}^{\ \ A} = 0$, and derive the following field equations

$$\frac{\delta}{\delta b_a{}^a} S = c_W + \partial_\alpha c^\alpha = 0 \quad (\text{with } b_a{}^a = b_a{}^\alpha \delta_\alpha{}^a) \tag{3.5}$$

$$\frac{\delta}{\delta b^{[ab]}} S = c_{[ab]} - \tfrac{1}{2}(\partial_a c_b - \partial_b c_a) = 0 \quad (\text{with } b^{[ab]} = \tfrac{1}{2}(b^{ab} - b^{ba})) \tag{3.6}$$

$$\frac{\delta}{\delta \beta_A} S = \gamma^A_{cs} + \tfrac{1}{2}(\partial\!\!\!/\gamma)^A = 0 \quad (\text{with } \beta_A = \beta_B{}^\alpha (\rho_\alpha)^B{}_A)\,. \tag{3.7}$$

Problem 4

Using all field equations (so on-shell), derive the expressions for the gravitational stress tensor and the supercurrent, assuming a flat super-background ($e_{\alpha,back}^{\ \ a} = \delta_a{}^a$ and $\chi_{\alpha,back}^{\ \ A} = 0$).

$$T^{(X\psi)}_{+++\!+} = T\left(\partial_{+\!+} X \cdot \partial_{+\!+} X + \tfrac{i}{4}\psi_+ \partial_{+\!+}\psi_+\right) \tag{4.1}$$

$$T^{(bc)}_{+++\!+} = \tfrac{i}{2\pi}\left[(\partial_{+\!+} b_{+++\!+})c^{+\!+} + 2 b_{+++\!+}\partial_{+\!+} c^{+\!+}\right] \tag{4.2}$$

$$T^{(\beta\gamma)}_{+++\!+} = \tfrac{i}{2\pi}\left[\tfrac{1}{2}(\partial_{+\!+}\beta_{+++})\gamma^+ + \tfrac{3}{2}\beta_{+++}\partial_{+\!+}\gamma^+\right] \tag{4.3}$$

Hint: Use $\bar\psi = \psi^T i\rho^0$ with $\rho^0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\rho^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\psi^+ = \psi_-$ but $\psi^- = -\psi_+$.

$$J^{(X\psi)}_{+++} = \tfrac{1}{\pi l^2}\psi_+ \cdot \partial_{+\!+} X \tag{4.4}$$

$$J^{(\text{ghosts})}_{+++} = -\tfrac{1}{\pi} b_{+++\!+}\gamma^+ - \tfrac{i}{2\pi}\left[(\partial_{+\!+}\beta_{+++})c^{+\!+} + \tfrac{3}{2}\beta_{+++}\partial_{+\!+} c^{+\!+}\right] \tag{4.5}$$

Check that all parts of T and J are real.