PHY 623 - Exercises 5-??

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EXERCISE 6

Splitting the indices as $\{\mathbf{a}\} = \{a, \bar{a}\}$, $\{\mu\} = \{\mu, \bar{\mu}\}$, the vielbein and $R^{\mathbf{a}}{}_{\mathbf{b}}$ can be written as

$$e^{\mathbf{a}}{}_{\mu} = \left(\begin{array}{c|c} e^{a}{}_{\mu} & e^{\bar{a}}{}_{\mu} \\ \hline e^{a}{}_{\bar{\mu}} & e^{\bar{a}}{}_{\bar{\mu}} \end{array}\right) \qquad R^{\mathbf{a}}{}_{\mathbf{b}} = \left(\begin{array}{c|c} R^{a}{}_{b} & R^{\bar{a}}{}_{b} \\ \hline R^{a}{}_{\bar{b}} & R^{\bar{a}}{}_{\bar{b}} \end{array}\right)$$

Then, transforming e using R:

$$\tilde{e}^{\mathbf{b}}{}_{\mu} = R^{\mathbf{b}}{}_{\mathbf{a}} e^{\mathbf{a}}{}_{\mu} \implies \left(\begin{array}{c|c} e^{b}{}_{\mu} & 0 \\ \hline 0 & e^{b}{}_{\bar{\mu}} \end{array}\right) = \left(\begin{array}{c|c} R^{b}{}_{a} & R^{\bar{b}}{}_{a} \\ \hline R^{b}{}_{\bar{a}} & R^{b}{}_{\bar{a}} \end{array}\right) \left(\begin{array}{c|c} e^{a}{}_{\mu} & e^{\bar{a}}{}_{\mu} \\ \hline e^{a}{}_{\bar{\mu}} & e^{\bar{a}}{}_{\bar{\mu}} \end{array}\right)$$

which gives

$$R^{\bar{b}}{}_{a} = \left(R^{b}{}_{a} e^{\bar{a}}{}_{\mu}\right) e_{\bar{a}}{}^{\bar{\mu}} \qquad R^{b}{}_{\bar{a}} = \left(R^{b}{}_{\bar{a}} e^{a}{}_{\mu}\right) e_{a}{}^{\bar{\mu}}$$

EXERCISE 7

On $\mathbb{CP}^2$, we have coordinates $a_1 = \frac{z_2}{z_1}$ and $a_2 = \frac{z_3}{z_1}$ when $z_1 \neq 0$, $b_1 = \frac{z_1}{z_2}$ and $b_2 = \frac{z_3}{z_2}$ when $z_2 \neq 0$, and $c_1 = \frac{z_1}{z_3}$ and $c_2 = \frac{z_2}{z_3}$ when $z_3 \neq 0$. The Kähler potentials are

$$K_A = R^2 \ln(1 + a_1\bar{a}_1 + a_2\bar{a}_2) \qquad K_B = R^2 \ln(1 + b_1\bar{b}_1 + b_2\bar{b}_2) \qquad K_C = R^2 \ln(1 + c_1\bar{c}_1 + c_2\bar{c}_2)$$

The coordinates are related by

$$a_1 = \frac{1}{b_1} \qquad a_2 = \frac{1}{c_1} \qquad b_2 = \frac{1}{c_2}$$

$$\frac{b_1}{c_2} = b_2 \qquad \frac{c_2}{a_1} = c_1 \qquad \frac{a_2}{b_2} = a_1$$

To transform between K_A and K_B:

$$K_A = R^2 \ln\left(1 + \frac{1}{b_1\bar{b}_1} + \frac{1}{c_1\bar{c}_1}\right) = R^2 \ln\left(\frac{1 + b_1\bar{b}_1 + \frac{b_1\bar{b}_1}{c_1\bar{c}_1}}{b_1\bar{b}_1}\right) = R^2 \ln\left(\frac{1 + b_1\bar{b}_1 + b_2\bar{b}_2}{b_1\bar{b}_1}\right) = K_B - R^2 \ln(b_1\bar{b}_1)$$

so $\lambda_{AB} = -R^2 \ln(b_1\bar{b}_1)$. Similarly, we find $\lambda_{BC} = -R^2 \ln(c_2\bar{c}_2)$ and $\lambda_{CA} = -R^2 \ln(a_2\bar{a}_2)$.

EXERCISE 8

For our manifold to be Calabi-Yau, we must have $\det g_{m\bar{n}} = 1$. From the Kähler potential, we get

$$g_{m\bar{n}} = \begin{pmatrix} \frac{\partial}{\partial z^1}\frac{\partial}{\partial \bar{z}^1} & \frac{\partial}{\partial z^1}\frac{\partial}{\partial \bar{z}^2} \\ \frac{\partial}{\partial z^2}\frac{\partial}{\partial \bar{z}^1} & \frac{\partial}{\partial z^2}\frac{\partial}{\partial \bar{z}^2} \end{pmatrix} \sqrt{\frac{(1+z^1z^2)(1+\bar{z}^1\bar{z}^2)}{(1+z^1\bar{z^1})(1+z^2\bar{z}^2)}}$$

The derivatives are

$$\frac{\partial^2 K}{\partial z^1 \partial \bar{z}^1} = \frac{z^1(-2\bar{z}^1z^2\bar{z}^2 + \bar{z}^1 - 3z^2) - 3\bar{z}^1\bar{z}^2 + z^2\bar{z}^2 - 2}{4(z^1\bar{z}^1+1)^3(z^2\bar{z}^2+1)\sqrt{\frac{(z^1z^2+1)(\bar{z}^1\bar{z}^2+1)}{(z^1\bar{z}^1+1)(z^2\bar{z}^2+1)}}}$$

$$\frac{\partial^2 K}{\partial z^1 \partial \bar{z}^2} = -\frac{(\bar{z}^1 - z^2)^2}{4(z1\bar{z}^1+1)^2(z^2\bar{z}^2+1)^2\sqrt{\frac{(z^1z^2+1)(\bar{z}^1\bar{z}^2+1)}{(z^1\bar{z}^1+1)(z^2\bar{z}^2+1)}}}$$

$$\frac{\partial^2 K}{\partial z^2 \partial \bar{z}^1} = -\frac{(z^1 - \bar{z}^2)^2}{4(z^1\bar{z}^1+1)^2(z^2\bar{z}^2+1)^2\sqrt{\frac{(z^1z^2+1)(\bar{z}^1\bar{z}^2+1)}{(z^1\bar{z}^1+1)(z^2\bar{z}^2+1)}}}$$

$$\frac{\partial^2 K}{\partial z^2 \partial \bar{z}^2} = \frac{z^1(-2\bar{z}^1z^2\bar{z}^2 + \bar{z}^1 - 3z^2) - 3\bar{z}^1\bar{z}^2 + z^2\bar{z}^2 - 2}{4(z^1\bar{z}^1+1)(z^2\bar{z}^2+1)^3\sqrt{\frac{(z^1z^2+1)(\bar{z}^1\bar{z}^2+1)}{(z^1\bar{z}^1+1)(z^2\bar{z}^2+1)}}}$$

Which gives

$$\det(g_{m\bar{n}}) = \frac{z^1\bar{z}^1(z^2\bar{z}^2-1) + 2z^1z^2 + 2\bar{z}^1\bar{z}^2 - z^2\bar{z}^2 + 1}{4(z^1\bar{z}^1+1)^3(z^2\bar{z}^2+1)^3}$$

and it isn't possible to make this 1. I'm clearly doing something wrong...

EXERCISE 9

Using $\omega_k = \frac{1}{k!}\omega_{m_1\ldots m_k} dz^{m_1}\wedge\cdots\wedge dz^{m_k}$, we have

$$\omega_k \wedge \bar{\omega}_k = \frac{1}{(k!)^2}\omega_{m_1\ldots m_k}\bar{\omega}_{\bar{n}_1\ldots\bar{n}_k} dz^{m_1}\wedge\cdots\wedge dz^{m_k}\wedge d\bar{z}^{\bar{n}_1}\wedge\cdots\wedge d\bar{z}^{\bar{n}_k}$$

$$= \frac{(-1)^{\frac{k(k-1)}{2}}}{(k!)^2}\omega_{m_1\ldots m_k}\bar{\omega}_{\bar{n}_1\ldots\bar{n}_k}\left[dz^{m_1}\wedge d\bar{z}^{\bar{n}_1}\right]\wedge\cdots\wedge\left[dz^{m_k}\wedge d\bar{z}^{\bar{n}_k}\right]$$

Now, define $\omega_{m_1\ldots m_k} = iN\bar{\epsilon}^*\gamma_{m_1\ldots m_k}\epsilon$, where N is an undetermined normalization constant. This gives

$$\omega_k\wedge\bar{\omega}_k = \frac{(-1)^{\frac{k(k-1)}{2}+1}N^2}{(k!)^2}\left[\langle 1\ldots k|\gamma_{m_1}\ldots\gamma_{m_k}|0\rangle + asym.\right]\left[\langle 1\ldots k|\gamma_{\bar{n}_1}\ldots\gamma_{\bar{n}_k}|0\rangle + asym.\right]\left[dz^{m_1}\wedge d\bar{z}^{\bar{n}_1}\right]\wedge\cdots\wedge\left[dz^{m_k}\wedge d\bar{z}^{\bar{n}_k}\right]$$

$$= \frac{(-1)^{\frac{k(k-1)}{2}+1}N^2}{2^k(k!)^2}\left(\langle 1,\ldots,k|m_1\ldots m_k\rangle + asym.\right)\left(\langle \bar{n}_1\ldots\bar{n}_k|1\ldots k\rangle + asym.\right)\left[dz^{m_1}\wedge d\bar{z}^{\bar{n}_1}\right]\wedge\cdots\wedge\left[dz^{m_k}\wedge d\bar{z}^{\bar{n}_k}\right]$$

$$= \frac{(-1)^{\frac{k(k-1)}{2}+1}N^2}{2^k(k!)^2}\left(\delta_{1,m_1}\ldots\delta_{k,m_k} + asym.\right)\left(\delta_{1,\bar{n}_1}\ldots\delta_{k,\bar{n}_k} + asym.\right)\left[dz^{m_1}\wedge d\bar{z}^{\bar{n}_1}\right]\wedge\cdots\wedge\left[dz^{m_k}\wedge d\bar{z}^{\bar{n}_k}\right]$$

$k!$ of the terms in the expansion of the product will have the same sign, and the others will vanish by the antisymmetry of the wedge product. This leaves us with

$$\omega_k \wedge \bar{\omega}_k = \frac{(-1)^{\frac{k(k-1)}{2}+1} N^2}{2^k k!} \left(\delta_{m_1,\bar{n}_1} \dots \delta_{m_k,\bar{n}_k}\right) \left[dz^{m_1} \wedge d\bar{z}^{\bar{n}_1}\right] \wedge \dots \wedge \left[dz^{m_k} \wedge d\bar{z}^{\bar{n}_k}\right]$$

$$= \frac{(-1)^{\frac{k(k-1)}{2}+1} N^2}{2^k} \frac{\omega^k}{k!}$$

so we should take

$$N = (-1)^{\frac{-k^2+k-2}{4}} 2^{k/2}$$