PHY 623 - Homework 4

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PROBLEM 1

The contraction of X with itself is

$$-\frac{l^2}{4}\left[\ln(z-w)-\ln(\bar{z}-\bar{w})-\ln(z-\bar{w})-\ln(\bar{z}-w)\right]$$

The OPE is

$$\begin{aligned}\partial_z X(z,\bar{z})\partial_z X(z,\bar{z})e^{ikX(w,\bar{w})} &= \Big(\partial_z\left[\ln(z-w)-\ln(z-\bar{w})\right]\Big)^2\left[\frac{(ik)^2}{2}\left(\frac{-l^2}{4}\right)+\frac{(ik)^3}{3!}3\left(\frac{-l^2}{4}\right)X(w,\bar{w})+\dots\right]\\ &+2ik\partial_z\left[\ln(z-w)-\ln(z-\bar{w})\right]\left(1+ikX(w,\bar{w})+\frac{(ik)^2}{2}X(w,\bar{w})X(w,\bar{w})+\dots\right)\end{aligned}$$

$$=\frac{l^2k^2}{8}\left(\frac{1}{z-w}-\frac{1}{z-\bar{w}}\right)^2 e^{ikX(w,\bar{w})}+2ik\left(\frac{1}{z-w}-\frac{1}{z-\bar{w}}\right)e^{ikX(w,\bar{w})}$$

We see that the conformal dimension of e^{ikX} is $\frac{l^2k^2}{8}$.

PROBLEMS 2,3,4

The stress tensor for a general bc system is

$$T=(1-\lambda)(\partial b)c-\lambda b\partial c$$

Now we consider the OPE of $T(z)T(w)$. The terms with two contractions are

$$(1-\lambda)^2\partial_z\left(\frac{\epsilon}{z-w}\right)\partial_w\left(\frac{1}{z-w}\right)-\lambda(1-\lambda)\left[\partial_z\partial_w\left(\frac{\epsilon}{z-w}\right)\frac{1}{z-w}+\frac{\epsilon}{z-w}\partial_z\partial_w\left(\frac{1}{z-w}\right)\right]+\lambda^2\partial_w\left(\frac{\epsilon}{z-w}\right)\partial_z\left(\frac{1}{z-w}\right)$$

$$=\frac{-\epsilon(1-\lambda)^2}{(z-w)^2}\frac{1}{(z-w)^2}+\frac{\epsilon\lambda(1-\lambda)}{(z-w)^4}+\frac{-\epsilon\lambda^2}{(z-w)^4}=\frac{-\epsilon\left(12\lambda^2-12\lambda+2\right)}{2(z-w)^4}$$

The terms with one contraction are

$$\begin{aligned}&(1-\lambda)^2\partial_z\left(\frac{\epsilon}{z-w}\right)c(z)\partial_w b+(1-\lambda)^2\partial_w\left(\frac{1}{z-w}\right)(\partial_z b)c(w)\\ &-\lambda(1-\lambda)\partial_z\partial_w\left(\frac{\epsilon}{z-w}\right)c(z)b(w)-\lambda(1-\lambda)\frac{1}{z-w}(\partial_z b)(\partial_w c)\\ &-\lambda(1-\lambda)\frac{\epsilon}{z-w}(\partial_z c)\partial_w b-\lambda(1-\lambda)b(z)c(w)\partial_z\partial_w\left(\frac{1}{z-w}\right)\\ &+\lambda^2\partial_w\left(\frac{\epsilon}{z-w}\right)(\partial_z c)b(w)+\lambda^2 b(z)(\partial_w c)\partial_z\left(\frac{1}{z-w}\right)\end{aligned}$$

$$= -(1-\lambda)\epsilon\frac{1}{(z-w)^2}c(z)\partial_w b + (1-\lambda)\frac{1}{(z-w)^2}(\partial_z b)c(w) + \lambda\epsilon\frac{1}{(z-w)^2}(\partial_z c)b(w) - \lambda\frac{1}{(z-w)^2}b(z)\partial_w c$$

Taylor expanding gives

$$\frac{1}{(z-w)}\left\{(1-\lambda)^2\left[\partial_w b + (z-w)\partial_w^2 b + \dots\right]c(w) - (1-\lambda)\epsilon\left[c(w) + (z-w)\partial_w c + \dots\right]\partial_w b\right\}$$
$$+\frac{1}{(z-w)^2}\left\{\lambda\epsilon\left[\partial_w c + (z-w)\partial_w^2 c + \dots\right]b(w) - \lambda\left[b(w) + (z-w)\partial_w b\right]\partial_w c\right\}$$

$$= \frac{1}{(z-w)^2}\left[(1-\lambda)(\partial b)c - \epsilon(1-\lambda)c\partial b + \lambda\epsilon(\partial c)b - \lambda b\partial c\right] + \frac{1}{z-w}\left[(1-\lambda)(\partial^2 b)c - \epsilon(1-\lambda)(\partial c)\partial b + \lambda\epsilon(\partial^2 c)b - (\partial b)\partial c\right]$$

$$= \frac{2}{(z-w)^2}\left[(1-\lambda)(\partial b)c - \lambda b\partial c\right] + \frac{1}{z-w}\left[(1-\lambda)\partial\left((\partial b)c\right) - \lambda\partial\left(b\partial c\right)\right]$$

$$= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

All together, the TT OPE is

$$T(z)T(w) = \frac{-\epsilon\left(12\lambda^2 - 12\lambda + 2\right)}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

We can now look at specific b and c. If we take $\psi \equiv \frac{l}{\sqrt{2}}(b+c)$, $\epsilon = 1$, and $\lambda = 1/2$, then we obtain the result for the TT OPE with $T^{(\psi)}(z) = -\frac{1}{2l}\psi(z)\partial_z\psi(z)$. We find a central charge of 1/2 in this case, thanks to the extra factor of 1/2 contributed by our definition of ψ.

For the susy ghosts, we take $b = \beta$, $c = \gamma$, $\lambda = 3/2$ and $\epsilon = -1$, which gives a central charge of 11. For the coordinate ghosts, we choose $\lambda = 2$ and $\epsilon = 1$, which gives a central charge of -26.

We already know that the $T^{(x)}T^{(x)}$ OPE gives a central charge of 1. For a system with D coordinate scalars and the corresponding coordinate ghosts, we find central charge cancellation only in $D = 26$. If we include fermions and susy ghosts as well, then we find that D must satisfy $D + D/2 - 26 + 11 = 0$. The only solution is $D = 10$.