PHY 623 - Homework 1

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PROBLEM 1

Defining $p(c^+) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t c^+)} = -ikb_{++}$ and substituting the given oscillators into the canonical relation $\{p(c^+(\sigma,t)), c^+(\sigma',t)\} = -i\hbar\delta(\sigma-\sigma')$ we find

$$-\frac{i\hbar}{2\pi}\sum_q e^{-iq(\sigma-\sigma')} = ik\left\{\sum_n b_n e^{-in(t+\sigma)}, \sum_m c_m e^{-im(t+\sigma')}\right\}$$

$$= ik\sum_n\sum_m e^{-it(n+m)}e^{-in\sigma-im\sigma'}\{b_n, c_m\} = i\hbar k\sum_n e^{-in(\sigma-\sigma')} \implies k = -\frac{1}{2\pi}$$

If we take $p(\gamma^+) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \gamma^+)} = -i\kappa\beta_{+,++}$ and use the commutation relation $[p(\gamma^+(\sigma,t)), \gamma^+(\sigma',t)] = -i\hbar\delta(\sigma-\sigma')$ with $[\beta_n, \gamma_m] = -i\hbar\delta_{m+n,0}$, then by the same steps, we find $\kappa = -\frac{1}{2\pi}$.

PROBLEM 2

Expanding the covariant derivative gives

$$\bar{\psi}\rho^\mu\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{mn}\rho_{mn}\right)\psi$$

We can rewrite the second term as

$$\frac{1}{4}\omega_\mu{}^{mn}e^\mu{}_r\bar{\psi}\rho^r\rho_{mn}\psi = \frac{1}{4}\omega_\mu{}^{mn}e^\mu{}_r\bar{\psi}\left(\rho^r{}_{mn} + 2\rho_{[n}\eta^r_{m]}\right)\psi$$

In $d=2$, the rank-3 gamma matrix is identically zero, and $\bar{\epsilon}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\epsilon$ when ϵ, χ are Majorana spinors, so the entire expression vanishes.

PROBLEM 3

The classical part of the Lagrangian is

$$\mathcal{L}_{cl} = Te\left[-\frac{1}{2}\partial_\alpha X\cdot\partial_\beta X h^{\alpha\beta} - \frac{1}{4}\bar{\psi}\rho^\alpha\partial_\alpha\psi + \frac{1}{2}F\cdot F + \frac{1}{2}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\cdot\partial_\beta X + \frac{1}{16}\bar{\psi}\cdot\psi\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta\right]$$

The gauge fixing term is

$$\mathcal{L}_{GF} = d_a{}^\alpha e_\alpha{}^a{}_{(q)} + \Delta_A{}^\alpha\chi_\alpha{}^A{}_{(q)}$$

Integrating over F in the part integral trivially sets $F=0$. Taking flat space as the background for the metric and integrating over d gives factors of $\delta(e_\alpha{}^a{}_{(q)})$. Integrating over e sets the quantum part of the vielbein to zero, and using lightcone coordinates, the first term becomes $2T\partial_+X\cdot\partial_-X$. Using

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The second term becomes

$$\frac{Ti}{4}\begin{pmatrix} \psi^+ & \psi^- \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\partial_\sigma\right]\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = -\frac{Ti}{8}\left(\psi^+\partial_+\psi^+ + \psi^-\partial_-\psi^-\right) = \frac{Ti}{2}\left(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\right)$$

By gauge choice, the gravitino is equal to the background gravitino only, which we take to vanish, so the remaining terms vanish as well.

Define $p(\psi_+) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_+)} = \frac{iT}{4}\psi_+$. Since $p(\psi) \sim \psi$ we will need to include an extra factor of 1/2 in the commutation relation. We have

$$\left\{p\left(\psi_+^\mu(\sigma,t)\right), \psi_+^\nu\left(\sigma',t\right)\right\} = -\frac{i\hbar}{2}\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\frac{iT}{4}\left\{\psi_+^\mu(\sigma,t), \psi_+^\nu\left(\sigma',t\right)\right\} = -\frac{i\hbar}{2}\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$\left\{\psi_+^\mu(\sigma,t), \psi_+^\nu\left(\sigma',t\right)\right\} = 2\hbar\pi\ell^2\eta^{\mu\nu}\delta(\sigma-\sigma')$$

We find a similar result for ψ_-.