PHYS 653 - Homework 4

M. Ross Tagaras
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PROBLEM 1

The sigma model action is

$$I = \int dtd^2\theta \ \frac{1}{2}\epsilon^{AB} D_A\Phi^i D_B\Phi^j g_{ij}(\Phi)$$

with the superfield defined as

$$\Phi^i = \phi^i(t) + \theta^A\psi^i_A + \frac{i}{2}\epsilon^{AB}\theta_A\theta_B F^i$$

The superfield derivatives are

$$D_A\Phi^i = (\partial_A + i\theta_A\partial_t)\left(\phi^i + \theta^B\psi^i_B - \frac{i}{2}\theta^2 F^i\right)$$

$$= \psi^i_A + i\theta_A F^i + i\theta_A\partial_t\phi^i + i\theta_A\theta^B\partial_t\psi^i_B$$

$$D_C D_A\Phi^i = i\epsilon_{CA}F^i + i\epsilon_{CA}\partial_t\phi^i + i\epsilon_{CA}\theta^B\partial_t\psi^i_B - i\delta^B_C\theta_A\partial_t\psi^i_B + i\theta_C\partial_t\psi^i_A - \theta_C\theta_A\partial_t F^i - \theta_C\theta_A\partial^2\phi^i$$

$$D^D D_A\Phi^i = \epsilon^{DC} D_C D_A\Phi^i$$

$$D^C D_C D_A\Phi^i = i\partial_t\psi^i_A + \theta_A\partial_t F^i + 2\theta_A\theta^B\partial^2\psi^i_B - \theta^2\partial^2\psi^i_A$$

$$D_A g_{ij} = \frac{\partial g_{ij}}{\partial\Phi^k} D_A\Phi^k$$

$$D^2 g_{ij} = \frac{\partial^2 g_{ij}}{\partial\Phi^m\partial\Phi^k} D^A\Phi^m D_A\Phi^k + \frac{\partial g_{ij}}{\partial\Phi^k} D^2\Phi^k$$

Evaluating these at $\theta = 0$, we find

$$D_A\Phi^i\Big|_{\theta=0} = \psi^i_A \qquad D_C D_A\Phi^i\Big|_{\theta=0} = i\epsilon_{CA}\left(F^i + \partial_t\phi^i\right) \qquad D^2 D_A\Phi^i\Big|_{\theta=0} = i\partial_t\psi^i_A \qquad D_A g_{ij}\Big|_{\theta=0} = \frac{\partial g_{ij}}{\partial x^k}\psi^k_A$$

$$D^2 g_{ij}\Big|_{\theta=0} = \frac{\partial^2 g_{ij}}{\partial\Phi^m\partial\Phi^k} D^C\Phi^m D_C\Phi^k\Big|_{\theta=0} + \frac{\partial g_{ij}}{\partial\Phi^k} D^C D_C\Phi^k\Big|_{\theta=0} = \frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^k}\psi^{Cm}\psi^k_C + 2i\frac{\partial g_{ij}}{\partial\phi^k}\left(F^k + \partial_t\phi^k\right)$$

We can write the action as

$$I = -\frac{1}{4}\int dt\ \epsilon^{AB} D^2 \left[D_A\Phi^i D_B \Phi^j g_{ij}\right]\Big|_{\theta=0}$$

Distributing the derivatives gives (with some indices suppressed)

$$\begin{aligned}-\frac{1}{4}\int dt\ \epsilon^{AB} \Big[&D^2 D_A\Phi D_B\Phi g + D_C D_A \Phi D^C D_B \Phi g - D_C D_A\Phi D_B \Phi D^C g \\ &- D^C D_A \Phi D_C D_B \Phi g + D_A\Phi D^2 D_B\Phi g + D_A \Phi D_C D_B \Phi D^C g \\ &+ D^C D_A\Phi D_B \Phi D_C g - D_A\Phi D^C D_B \Phi D_C g + D_A\Phi D_B\Phi D^2 g\Big]\Big|_{\theta=0}\end{aligned}$$

Substituting the superfield derivatives:

$$I = \frac{1}{4}\int dt\ \left[2i\partial_t\psi^{Ai}\psi^j_A g_{ij} + 4\left(F^i+\partial_t\phi^i\right)\left(F^j+\partial_t\phi^j\right)g_{ij} + 4i\left(F^i+\partial_t\phi^i\right)\psi^{Aj}\frac{\partial g_{ij}}{\partial\phi^k}\psi^k_A + \psi^{Ai}\psi^j_A\psi^{Bm}\psi^k_B\frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^k}\right]$$

The field equation for F is

$$F_i = -i\psi^{jA}\psi^k_A\frac{\partial g_{ij}}{\partial\phi^k} - 2\partial_t\phi_i$$

Substituting this into the action gives

$$I = \int dt\ \left[\frac{i}{2}\psi^{iA}\left(\partial_t\psi_{iA} + \partial_t\phi^k\frac{\partial g_{ik}}{\partial\phi^j}\psi^j_A\right) + \partial_t\phi^i\partial_t\phi_i + \frac{1}{4}\psi^{iA}\psi^j_A\psi^{mB}\psi^n_B\frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^n}\right]$$

Now consider $\psi^{iA}\psi^n_A\Gamma^j{}_{mn}$. From the definition of the Christoffel symbol:

$$\psi^{iA}\psi^n_A\Gamma^j{}_{mn} = \frac{1}{2}\psi^{iA}\psi^n_A\left(\frac{\partial g_{im}}{\partial\phi^n} + \frac{\partial g_{in}}{\partial\phi^m} - \frac{\partial g_{mn}}{\partial\phi^i}\right)$$

The second term vanishes by antisymmetry and the first and third combine after index relabeling, so we find that

$$\psi^{iA}\psi^n_A\Gamma^j{}_{mn} = \psi^{iA}\psi^n_A\frac{\partial g_{im}}{\partial\phi^n}$$

which allows us to replace the first two terms in the new action with the covariant derivative D. The Riemann tensor can be written as

$$R_{imnj} = \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^n} + \frac{\partial^2 g_{mn}}{\partial\phi^i\partial\phi^j} - \frac{\partial^2 g_{in}}{\partial\phi^m\partial\phi^j} - \frac{\partial^2 g_{mj}}{\partial\phi^i\partial\phi^n}\right) + g_{k\ell}\left(\Gamma^k{}_{mn}\Gamma^\ell{}_{ij} - \Gamma^k{}_{mj}\Gamma^\ell{}_{in}\right)$$

$\psi^{iA}\psi^j_A\psi^{mB}\psi^n_B$ is symmetric under $(i\leftrightarrow n)$ and $(j\leftrightarrow m)$ and antisymmetric under $(i\leftrightarrow j)$ and $(m\leftrightarrow n)$, so by symmetry, we can write

$$\psi^{iA}\psi^j_A\psi^{mB}\psi^n_B R_{imnj} = \psi^{iA}\psi^j_A\psi^{mB}\psi^n_B\left[\frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^n} + g_{k\ell}\Gamma^k{}_{mn}\Gamma^\ell{}_{ij}\right]$$

From the definition of the Christoffel symbol:

$$\psi^{iA}\psi^j_A\psi^{mB}\psi^n_B R_{imnj} = \psi^{iA}\psi^j_A\psi^{mB}\psi^n_B \left[\frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^n} + \frac{1}{4}\left(\partial_n g_{\ell m} + \partial_m g_{\ell n} - \partial_\ell g_{mn}\right) g^{\ell k}\left(\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}\right)\right]$$

$$= \psi^{iA}\psi^j_A\psi^{mB}\psi^n_B \frac{\partial^2 g_{ij}}{\partial\phi^m\partial\phi^n}$$

All together, the action can be rewritten in its final form:

$$I = \int dt \left[\frac{i}{2}\psi^{iA} D_t \psi^j_A g_{ij} + \partial_t\phi^i\partial_t\phi_i + \frac{1}{4}\psi^{iA}\psi^j_A\psi^{mB}\psi^n_B R_{imnj}\right]$$

The supersymmetry transformations are

$$\delta\phi^i = \delta\Phi^i\Big|_{\theta=0} = \varepsilon^A D_A \Phi^i\Big|_{\theta=0} = \varepsilon^A \psi^i_A$$

$$\delta\psi^i_A = \delta D_A\Phi^i\Big|_{\theta=0} = \varepsilon^B D_B D_A \Phi^i\Big|_{\theta=0} = \varepsilon^B i\epsilon_{BA}\left(F^i + \partial_t\phi^i\right) = i\partial_t\phi^i\varepsilon_A - \psi^{jB}\psi^k_B\Gamma^i{}_{jk}\varepsilon_A$$

PROBLEM 2

Computing the initial variation of the action, we find that

$$\delta I = \int dt \left(\epsilon\partial_t\psi^i A_i + \partial_t\phi^i\partial_j A_i\epsilon\psi^j\right) = \int dt\, \epsilon\left(\partial_t\psi^i A_i + \partial_t\phi^i\partial_t A_i\psi^j\right) = \int dt\, \epsilon\left(\partial_t\left(\psi^i A_i\right) - \psi^i\partial_j A_i\partial_t\phi^j + \partial_t\phi^i\partial_j A_i\psi^j\right)$$

Now we need to add extra terms to the action that cancel this variation. The term $\psi^i\psi^j\partial_i A_j$ has the variation

$$\delta\left(\psi^i\psi^j\partial_i A_j\right) = \partial_t\phi^i\epsilon\psi^j\partial_i A_j + \psi^i\epsilon\partial_t\phi^j\partial_i A_j + \psi^i\psi^j\partial_i\left(\partial_k A_j\epsilon\psi^k\right)$$

The last term vanishes by antisymmetry and by $\partial_i\psi^k = 0$, resulting in

$$\delta\left(\psi^i\psi^j\partial_i A_j\right) = \epsilon\left(\partial_t\phi^i\psi^j\partial_i A_j - \psi^i\partial_t\phi^j\partial_i A_j\right)$$

which cancels exactly with the initial variation. The supersymmetric action is

$$\tilde{I} = \int dt \left(\partial_t\phi^i A_i + \psi^i\psi^j\partial_i A_j\right)$$

To see that the transformations form a closed algebra, we calculate:

$$[\delta_1, \delta_2]\phi^i = \epsilon_2\partial_t\phi^i\epsilon_1 - (1 \leftrightarrow 2) = 2\epsilon_2\epsilon_1\partial_t\phi^i$$

$$[\delta_1, \delta_2]\psi^i = \epsilon_1\partial_t\psi^i\epsilon_2 - (1 \leftrightarrow 2) = 2\epsilon_2\epsilon_1\partial_t\psi^i$$

Now we consider the new transformations

$$\delta\phi^i = \beta^r V^{ir}(\phi) \qquad\qquad \delta\psi^i = \beta^r \partial_j V^{ir} \psi^j$$

With these, we find

$$\delta\tilde{I} = \int dt \left(\beta^r \partial_j V^{ir} \partial_t \phi^j A_i + \partial_t \phi^i \partial_j A_i \beta^r V^{jr} + \beta^r \partial_k V^{ir} \psi^k \psi^j \partial_i A_j + \psi^i \beta^r \partial_k V^{jr} \psi^k \partial_i A_j + \psi^i \psi^j \partial_i \left(\partial_k A_j \beta^r V^{kr} \right) \right)$$

$$= \int dt\ \beta^r \left[\partial_j V^{ir} \partial_t \phi^j A_i + \partial_t \phi^i \partial_j A_i V^{jr} + \partial_k V^{ir} \psi^k \psi^j F_{ij} + \psi^i \psi^j \partial_i \left(\partial_k A_j V^{kr} \right) \right]$$

$$= \int dt\ \beta^r \left[\partial_t \phi^i \left(\partial_i \left(V^{jr} A_j \right) + V^{jr} F_{ji} \right) + \psi^k \psi^j \left(\partial_k V^{ir} F_{ij} + \partial_k \left(\partial_i A_j V^{ir} \right) \right) \right]$$

$$= \int dt\ \beta^r \left[\partial_t \phi^i \left(\partial_i \left(V^{jr} A_j \right) + V^{jr} F_{ji} \right) + \psi^k \psi^j \left(\partial_k V^{ir} F_{ij} + \partial_k \left(F_{ij} V^{ir} \right) \right) \right]$$

For the variation to equal zero, we see that

$$V^{jr} F_{ij} = \partial_i \left(V^{jr} A_j \right)$$