PHYS 653 - Homework 2

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PROBLEM 1

Part (a)

The classical Lie algebras are

$$su(n) = \{X \in gl(n) \mid \text{tr}(X) = 0 \text{ and } X^\dagger = X\} \tag{1}$$

$$so(n) = \{X \in gl(n) \mid X^T = -X\} \tag{2}$$

$$Usp(2n) = \left\{X \in gl(2n) \mid \Omega X + X^T\Omega = 0 \text{ and } X^\dagger = -X\right\} \tag{3}$$

where $\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

Part (b)

For $usp(2n)$, the invariant tensors are Ω_{ij}, Ω^{ij} and δ_i^j. First, transform the indices of Ω:

$$\delta\Omega_{ij} = \Omega_{ik}X^k{}_j + \Omega_{kj}X^k{}_i = \Omega_{ik}X^k{}_j + X^k{}_i\Omega_{kj} = \Omega_{ik}X^k{}_j + X_i{}^k\Omega_{kj} = (\Omega X + X^T\Omega)_{ij} = 0 \tag{4}$$

Similarly, we can show that Ω^{ij} is also an invariant tensor. We can also use Ω to raise and lower indices. Varying δ gives

$$\delta(\delta_i^j) = \delta_k^j X^k{}_i + \delta_i{}^k X_k{}^j = X^j{}_i + X_i{}^j = 0 \tag{5}$$

From these two tensors, we can construct other invariant tensors if needed.

For $so(n)$, the invariant tensors are δ_{ij}, and $\epsilon_{i_1 \ldots i_n}$. First, transform δ_{ij}:

$$\delta(\delta_{ij}) = \delta_{ik}X^k{}_j + \delta_{kj}X^k{}_i = X^i{}_j + X^j{}_i = X_i{}^j - X_i{}^j = 0 \tag{6}$$

In a similar manner, we see that $\epsilon_{i_1 \ldots i_n}$ is invariant.

For $su(n)$, the invariant tensors are also δ_{ij} and $\epsilon_{i_1 \ldots i_n}$, and the proofs of their invariance are essentially the same.

Info on Calculating the Dimensions of Representations

We can use Young Tableaux to decompose tensor product representations for $su(n)$, $so(n)$, and $usp(2n)$. The rules for combining diagrams don't change, but the methods for calculating the dimension of a particular diagram depend on the algebra in question. The details on one particular method are featured in this section.

The calculation of the dimension of a representation of one of the given algebras has the same general form for the algebras $USp(2n)$ and $SO(n)$. We build a table from the Young diagram, use it to calculate combinatorial factors, then apply a given dimension formula. For $SU(n)$, we can use the standard factors-over-hooks rule.

For $USp(2n)$ the procedure is as follows:

Begin by filling the first column of the table with integers, starting with N and decreasing to 1:

N_i	C_i	S_i
N		
$N-1$		
$\vdots$		
1		

Next, where the top row of the Young diagram corresponds to N in the column N_i, put the number of columns in the i^{th} row into the column C_i. Finally, $S_i = N_i + C_i$. Then, where $i \neq j$, the following expression gives the dimension for the diagram in question:

$$D = \frac{\prod (S_i + S_j) \prod (S_i - S_j) \prod S_i}{\left((2N-1)!!\right)!} \tag{7}$$

An example for $N = 2$ is shown below:

$$\ydiagram{2,1} \sim$$

N_i	C_i	S_i
2	2	4
1	1	2

$$D = \frac{(4+2) \times (4-2) \times 4 \times 2}{3! \times 1!} = 16 \tag{8}$$

Some useful results for $USp(4)$ are:

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \oplus \bullet \rightarrow \mathbf{4} \otimes \mathbf{4} = \mathbf{10} \oplus \mathbf{5} \oplus \mathbf{1} \tag{9}$$

$$\ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{2,1} \oplus \ydiagram{1} \rightarrow \mathbf{4} \otimes \mathbf{5} = \mathbf{16} \oplus \mathbf{4} \tag{10}$$

$$\ydiagram{1} \otimes \ydiagram{2} = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{1} \rightarrow \mathbf{4} \otimes \mathbf{10} = \mathbf{20} \oplus \mathbf{16} \oplus \mathbf{4} \tag{11}$$

$$\ydiagram{1} \otimes \ydiagram{2,1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2} \oplus \ydiagram{1,1} \rightarrow \mathbf{4} \otimes \mathbf{16} = \mathbf{35} \oplus \mathbf{14} \oplus \mathbf{10} \oplus \mathbf{5} \tag{12}$$

For $SO(2n)$, the details of the procedure change slightly. The column N_i should begin at $N-1$ and decrease to 0.

For $SO(2n+1)$, the column N_i should begin at $N+1/2$ and decrease to 1/2. The dimension formula is the same as in the $SO(2n)$ case.

In some cases, it is more convenient to use the Weyl dimension formula in its abstract form:

$$D = \prod_{\alpha > 0} \frac{\langle \alpha, \Lambda + \delta \rangle}{\langle \alpha, \delta \rangle} \tag{13}$$

where α are root vectors, Λ is the highest weight, and $\delta = (1, \ldots, 1)$. This requires knowledge of the root system of the algebra. Fortunately, we can obtain the necessary information from the algebra's Dynkin diagram.

Part (c)

The three smallest representations of $SU(n)$ are the singlet, the fundamental representation, which has dimension n, and the smallest totally antisymmetric traceless tensor, which has dimension $\frac{n(n-1)}{2}$.

The three smallest representations of $SO(n)$ are the singlet, the smallest spinor, and the smallest antisymmetric tensor, which correspond to a scalar, spinor, and some p-form gauge field.

The three smallest representations of $USp(2n)$ are the singlet, the fundamental representation (dimension $2n$), and the smallest traceless antisymmetric tensor (dimension $\frac{2n(2n-1)}{2}$)

PROBLEM 2

The automorphism group is $SO(2) \times O(1,1) \times SU(6) \times U(1)$. The algebra is

$$\{Q^i_\alpha, Q^j_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij} \qquad \{Q^i_\alpha, Q^j_\beta\} = 0 \qquad \{Q^i_{\dot\alpha}, Q^j_{\dot\beta}\} = 0 \tag{14}$$

Choosing $P = (E, 0, 0, E)$, we find

$$\{Q^i_\alpha, Q^j_{\dot\beta}\} = \Big(\mathbb{1} + \sigma^3 E\Big)_{\alpha\dot\beta} \delta^{ij} \tag{15}$$

which implies

$$\{Q^i_1, Q^j_{\dot 1}\} = 2E\delta^{ij} \qquad \{Q^i_2, Q^j_{\dot 2}\} = 0 \tag{16}$$

We can create creation/annihilation operators:

$$a_i^\dagger = \frac{1}{\sqrt{2E}} Q^i_1 \qquad a_j = \frac{1}{\sqrt{2E}} Q^j_{\dot 1} \tag{17}$$

which satisfy $\{a_i^\dagger, a_j\} = \delta_{ij}$. Now we can create more states:

$$|\Omega\rangle \quad a_1^\dagger |\Omega\rangle \quad a_1^\dagger a_2^\dagger |\Omega\rangle \quad \ldots \quad a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger |\Omega\rangle \tag{18}$$

Each $a_i^\dagger$ is a $\mathbf{6}$ of $SU(6)$ and $|\Omega\rangle$ is a $\mathbf{1}$. We also assign helicity to each state, starting with the vacuum, which has helicity -2. Each successive operator increases the helicity by 1/2.

We can use Young tableaux to calculate the representations for the states with more creation operators. For the state $a_1^\dagger a_2^\dagger |\Omega\rangle$:

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \rightarrow \mathbf{6} \otimes \mathbf{6} = \mathbf{21} \oplus \mathbf{15} \tag{19}$$

Since we need this state to be antisymmetric in 1 and 2, we can only keep $\begin{array}{c}\square\\[-4pt]\square\end{array}$, so this state is in the $\mathbf{15}$ of $SU(6)$. For the other states with n creation operators, we will find that the only diagram that is allowed is the one with n vertical boxes. The conjugate representation is given by the diagram with $6 - n$ vertical boxes. The results are summarized in the following table. Subscripts on representations denote helicity.

State	Representation
$\|\Omega\rangle$	$\mathbf{1}_{-2}$
$a_1^\dagger \|\Omega\rangle$	$\mathbf{6}_{-3/2}$
$a_1^\dagger a_2^\dagger \|\Omega\rangle$	$\mathbf{15}_{-1}$
$a_1^\dagger a_2^\dagger a_3^\dagger \|\Omega\rangle$	$\mathbf{20}_{-1/2}$
$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger \|\Omega\rangle$	$\overline{\mathbf{15}}_0$
$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger \|\Omega\rangle$	$\overline{\mathbf{6}}_{1/2}$
$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger \|\Omega\rangle$	$\mathbf{1}_1$

To get the complete multiplet, we need to act on each representation with CPT. The end result is the multiplet

$$\Big(\mathbf{1}_{-2} + \mathbf{6}_{-3/2} + \mathbf{15}_{-1} + \mathbf{20}_{-1/2} + \overline{\mathbf{15}}_0 + \overline{\mathbf{6}}_{1/2} + \mathbf{1}_1\Big) + \Big(\mathbf{1}_{-1} + \mathbf{6}_{-1/2} + \mathbf{15}_0 + \mathbf{20}_{1/2} + \overline{\mathbf{15}}_1 + \overline{\mathbf{6}}_{3/2} + \mathbf{1}_2\Big) \tag{20}$$

PROBLEM 3

The algebra is

$$\{Q^i_\alpha, Q^j_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij} \qquad \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \qquad \{Q^i_{\dot\alpha}, Q^j_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} Z^{ij} \tag{21}$$

The automorphism group is $SO(3) \times USp(4) \times O(1,1)$.
With $P = (m, 0, 0, 0)$ we find that

$$\{Q^i_\alpha, Q^j_{\dot\beta}\} = m\delta_{\alpha\dot\beta}\delta^{ij} \tag{22}$$

Splitting our index i (which ranges from 1 to 4) into indices a and A (which each range from 1 to 2), the algebra becomes

$$\{Q^{aA}_\alpha, Q^{bB}_{\dot\beta}\} = m\delta_{\alpha\dot\beta}\delta^{ab}\delta^{AB} \qquad \{Q^i_\alpha, Q^j_\beta\} = m\epsilon_{\alpha\beta}\epsilon^{AB}\delta^{ab} \qquad \{Q^i_{\dot\alpha}, Q^j_{\dot\beta}\} = m\epsilon_{\dot\alpha\dot\beta}\epsilon^{AB}\delta^{ab} \tag{23}$$

The only anticommutators that survive are

$$\{Q^{aA}_1, Q^{bB}_{\dot 1}\} = m\delta^{ab}\delta^{AB} \qquad \{Q^{aA}_2, Q^{bB}_{\dot 2}\} = m\delta^{ab}\delta^{AB} \qquad \{Q^{aA}_1, Q^{bB}_2\} = m\delta^{ab}\epsilon^{AB} \qquad \{Q^{aA}_{\dot 1}, Q^{bB}_{\dot 2}\} = m\delta^{ab}\epsilon^{AB} \tag{24}$$

Now define

$$S^{aA} := \frac{1}{\sqrt{2m}}\Big(Q^{aA}_1 + \epsilon^{AC} Q^{aC}_{\dot 2}\Big) \qquad R^{aA} := \frac{1}{\sqrt{2m}}\Big(Q^{aA}_1 - \epsilon^{AC} Q^{aC}_{\dot 2}\Big) \tag{25}$$

We can then calculate the anticommutators of these objects with their conjugates to find

$$\{S^{aA}, S^\dagger_{bB}\} = m\delta^{ab}\Big(\delta^{AB} + \epsilon^{BC}\epsilon^A{}_C + \epsilon^{AC}\epsilon_C{}^B + \epsilon^{AC}\epsilon^{BC}\Big) = \delta^{ab}\delta^{AB} \tag{26}$$

$$\{R^{aA}, R^\dagger_{bB}\} = m\delta^{ab}\Big(\delta^{AB} - \epsilon^{BC}\epsilon^A{}_C - \epsilon^{AC}\epsilon_C{}^B + \epsilon^{AC}\epsilon^{BC}\Big) = 0 \tag{27}$$

This tells us that $S^\dagger_{aA}$ can be used as a creation operator, and that states created by R will have zero norm, so they can be ignored. We can also recombine our indices here, so the operator that will actually be used is $S^\dagger_i$, where i again ranges from 1 to 4.

This gives the following possible states with $SO(2)$ helicities:

Helicity	-1	-1/2	0	1/2	1
Multiplicity	1	4	6	4	1

We begin at helicity -1 to avoid massive gravitons/gravitinos. We also need the dimensions of the representations of $USp(2N)$ carried by each state. We can calculate these by the methods detailed in Problem 1:

State	Representation of $USp(4)$
$\|\Omega\rangle$	$\mathbf{1}$
$S_1^\dagger \|\Omega\rangle$	$\mathbf{4}$
$S_1^\dagger S_2^\dagger \|\Omega\rangle$	$\mathbf{5}$
$S_1^\dagger S_2^\dagger S_3^\dagger \|\Omega\rangle$	$\mathbf{4}$
$S_1^\dagger S_2^\dagger S_3^\dagger S_4^\dagger \|\Omega\rangle$	$\mathbf{1}$

Combining our $SO(2)$ information and $USp(4)$ information, we find

Helicity	-1	-1/2	0	1/2	1
Multiplicity	1	4	5+1	4	1

So we see that we can combine our states into representations of the full automorphism group $SO(3) \times USp(4)$ as $(3,1) + (2,4) + (1,5)$. These correspond to a vector, 4 spinors, and 5 scalars respectively.

PROBLEM 4

Part (a)

We start with eight real supercharges in the $(2,8)$ representation of $SO(3)\times USp(8)$. If we complexify and decompose under the subgroup $SO(2) \times USp(8)$, the supercharges split as $(2,8) = 8_{1/2} + 8_{-1/2}$.

Since we have a massless representation with no central charge, we know that one of the supercharges will function as a creation operator when acting on some vacuum, and the other as an annihilation operator. Define $S_i^\dagger \sim 8_{1/2}$ as the set of 8 creation operators.

We then get states with the following helicities and multiplicities:

Helicity	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2
Multiplicity	1	8	28	56	70	56	28	8	1

Using the same techniques as in the previous problem, we find that the $USp(8)$ representations of interest are

$$\ydiagram{1} = \mathbf{8} \qquad \ydiagram{1,1} = \mathbf{27} \qquad \ydiagram{1,1,1} = \mathbf{48} \qquad \ydiagram{1,1,1,1} = \mathbf{42} \tag{28}$$

So the states can be arranged as

Helicity	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2
Multiplicity	1	8	27+1	48+8	42+27+1	48+8	27+1	8	1

and we can see that when we recombine into the original group $SO(3) \times USp(8)$ we have the multiplet $(5,1) + (4,8) + (3,27) + (2,48) + (1,42)$. This corresponds to a graviton, 8 gravitinos, 27 vectors, 48 spinors, and 42 scalars.

Part (b)

Now we let the vacuum carry the $(3,1)$ representation of $SO(3) \times USp(8)$. We need to calculate $(3,1) \times \big[(5,1)+(4,8)+(3,27)+(2,48)+(1,42)\big]$. The $USp(8)$ part is a $\mathbf{1}$, so we really only need to worry about how the $SO(3)$ parts combine. Since $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, we can use the $SU(2)$ Young Tableaux to find

$$(3,1) \times \big[(5,1)+(4,8)+(3,27)+(2,48)+(1,42)\big] = (3\times 5,1)+(3\times 4,8)+(3\times 3,27)+(3\times 2,48)+(3,42)$$

$$= (3,1)+(5,1)+(7,1)+(6,8)+(4,8)+(2,8)+(5,27)+(3,27)+(1,27)+(4,48)+(2,48)+(3,42) \qquad (29)$$

PROBLEM 5

Part (a)

In the $(1,1)$ Poincaré superalgebra in $D=10$, $Q_{1/2} \sim \mathbf{8}_+ + \mathbf{8}_-$. Since the multiplet should have supercharges with opposite chirality, we can write

$$(\mathbf{8}_v + \mathbf{8}_+) \otimes (\mathbf{8}_v + \mathbf{8}_-)\,|1\rangle = (\mathbf{8}_v \times \mathbf{8}_v) + (\mathbf{8}_v \times \mathbf{8}_-) + (\mathbf{8}_v \times \mathbf{8}_+) + (\mathbf{8}_+ \times \mathbf{8}_-) \qquad (30)$$

Then, using the rules

$$\mathbf{8}_i \otimes \mathbf{8}_i = \mathbf{1} + \mathbf{28}_v + \mathbf{35}_i \qquad \mathbf{8}_i \otimes \mathbf{8}_j = \mathbf{8}_k + \mathbf{56}_k \qquad (31)$$

where i,j,k are cyclic, we find that the multiplet is

$$(\mathbf{1} + \mathbf{28} + \mathbf{35}_v) + (\mathbf{8}_+ + \mathbf{56}_+) + (\mathbf{8}_- + \mathbf{56}_-) + (\mathbf{8}_v + \mathbf{56}_v) \qquad (32)$$

which corresponds to a scalar, a two-form, a graviton, a left-handed spinor, a left-handed gravitino, a right-handed spinor, a right-handed gravitino, a one-form, and a three-form.

Part (b)

For the $(2,0)$ superalgebra, the automorphism group is $SO(8) \times SO(2)$, so $Q_{1/2}$ decomposes as $Q_{1/2} \sim (\mathbf{8})_1 + (\mathbf{8})_{-1}$ where ± 1 denotes the $SO(2)$ weight. This time, we want the supercharges to have the same chirality.

This tells us that the multiplet is given by

$$(\mathbf{8}_v + \mathbf{8}_+) \otimes (\mathbf{8}_v + \mathbf{8}_+)\,|1\rangle = (\mathbf{8}_v \times \mathbf{8}_v) + (\mathbf{8}_v \times \mathbf{8}_+) + (\mathbf{8}_v \times \mathbf{8}_+) + (\mathbf{8}_+ \times \mathbf{8}_+) \qquad (33)$$

$$= (\mathbf{1} + \mathbf{28} + \mathbf{35}_v) + (\mathbf{8}_- + \mathbf{56}_-) + (\mathbf{8}_- + \mathbf{56}_-) + (\mathbf{1} + \mathbf{28} + \mathbf{35}_+) \qquad (34)$$

which corresponds to two scalars, two two-forms, a graviton, a four-form, two spinors with the same chirality, and two gravitinos with the same chirality.