Introduction to Supersymmetry and Supergravity

Homework 1

February 24, 2019
Due: March 8

1. Consider the following Lagrangian for the interacting Wess-Zumino model

$$\mathcal{L} = -\partial_\mu z_i^\star \partial^\mu z^i - \frac{1}{2}\overline{\psi_L}_i \gamma^\mu \partial_\mu \psi_L^i + f_i^\star f^i - \left(f^i \partial_i W + c_1 \partial_i \partial_j W \, \overline{\psi_R}^i \, \psi_L^j + \text{h.c.} \right)$$

where $W(z)$ is a function of the complex scalars $z^i (i = 1, ..., n)$ and c_1 is a constant to be determined.

(a) Fix c_1 by supersymmetry and, up to quartic fermion terms, verify the invariance of the action $I = \int d^4x \mathcal{L}$ under off-shell supersymmetry transformations

$$\begin{aligned} \delta z^i &= \overline{\epsilon_R} \, \psi_L^i \ , \\ \delta \psi_L^i &= 2\gamma^\mu \partial_\mu z^i \epsilon_R + 2 f^i \epsilon_L \ , \\ \delta f^i &= \overline{\epsilon_L} \gamma^\mu \partial_\mu \psi_L^i \ . \end{aligned} \tag{0.1}$$

(b) Compute the Noether supercurrent J_α^μ associated with the above action and check that it is conserved on-shell. Next, taking $W = \lambda_{ijk} z^i z^j z^k$ where λ_{ijk} are some real constants, find improvement term(s) such that the current is γ-traceless, i.e. $\gamma^\mu J_\mu = 0$.

2. The supersymmetry transformation rules for $N = 1$ super Yang-Mills multiplet in $2 + 1$ dimensions are given by (spinors are Majorana)

$$\begin{aligned} \delta A_\mu &= \bar{\epsilon} \gamma_\mu \lambda \ , \\ \delta \lambda &= c \gamma^{\mu\nu} F_{\mu\nu} \epsilon \end{aligned}$$

(a) Establish the closure of the supersymmetry algebra, and determine the constant c, as well as the field equations that may be necessary for the closure.

(b) Work out the supersymmetry transformations of the field equations

$$\mathcal{E}^{\mu a} \equiv D_\sigma F^{\sigma \mu a} \ , \qquad \mathcal{E}^a \equiv \gamma^\mu D_\mu \lambda^a \ .$$

3. Consider the $N = (1,0), D = 6$ hypermultiplet consisting of the fields (ψ^A, ϕ^{iA}), where $i, A = 1, 2$ are $USp(2)$ doublet indices, ψ^A is symplectic Majorana-Weyl spinor and the scalars are pseudo-real in the sense that $(\phi^{iA})^\star = \epsilon_{ij}\epsilon_{AB}\phi^{jB}$. Construct the on-shell supersymmetry transformations for this multiplet and verify that the supersymmetry transformations close on-shell on both fields. Use the raising and lowering convention convention for $USp(2)$ indices as follows: $\psi^A = \epsilon^{AB}\,\psi_B$ and $\psi_A = \psi^B\,\epsilon_{BA}$ where $\epsilon_{AB}\epsilon^{AC} = \delta^C_B$.

4. Starting from the super Yang-Mills theory in $D = 10$, perform ordinary dimensional reduction down to $D = 3$. Obtain the action and supersymmetry transformation rules of the resulting system. You may use, if you wish, the following representation for the 32×32 Dirac Γ-matrices of ten dimensions:

$$\begin{aligned} \Gamma^\mu &= \sigma_1 \times \gamma^\mu \times 1 \\ \Gamma^i &= i\sigma_2 \times 1 \times \gamma^i \ , \end{aligned}$$

where γ^μ $(\mu = 0, 1, 2)$ are the 2×2 Dirac γ-matrices in $2 + 1$ dimensions, and γ^i $(i = 1, ..., 7)$ are the 8×8 Dirac γ-matrices in euclidean seven dimensions.