PHYS 653 - Homework 3

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PROBLEM 1

Part (a)

The supersymmetry transformations are

$$\delta z^i = \bar{\epsilon_R}\psi_L^i \tag{1}$$

$$\delta\psi_L^i = 2\not\partial z^i\epsilon_R + 2f^i\epsilon_L \tag{2}$$

$$\delta f^i = \bar\epsilon_L\not\partial\psi_L^i \tag{3}$$

$$\delta z_i^* = \bar\epsilon_L\psi_{Ri} \tag{4}$$

$$\delta f_i^* = -\bar\epsilon_R\not\partial\psi_{Ri} \tag{5}$$

$$\delta\bar\psi_{Li} = 2\bar\epsilon_R\not\partial z_i^* + 2\bar\epsilon_L f_i^* \tag{6}$$

$$\delta\bar\psi_R^i = 2\bar\epsilon_L\not\partial z^i + 2f^i\bar\epsilon_R \tag{7}$$

The Lagrangian is

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int} = \left(-\partial_\mu z_i^*\partial^\mu z^i - \frac{1}{2}\bar\psi_{Li}\not\partial\psi_L^i + f_i^* f^i\right) - \left(f^i\partial_i W + c\left(\partial_i\partial_j W\right)\bar{\psi_R^i}\psi_L^j + \text{h.c.}\right) \tag{8}$$

First we vary $\mathcal{L}_{kin}$:

$$\delta\mathcal{L}_{kin} = -\left(\partial_\mu z_i^*\right)\partial^\mu\left(\bar\epsilon_R\psi_L^i\right) - \partial_\mu\left(\bar\epsilon_L\psi_{Ri}\right)\partial^\mu z^i + \left(\bar\epsilon_R\not\partial z_i^* - \bar\epsilon_L f_i^*\right)\not\partial\psi_L^i - \bar\psi_{Li}\not\partial\left(\not\partial z^i\epsilon_R + f^i\epsilon_L\right) + f_i^*\left(\bar\epsilon_L\not\partial\psi_L^i\right) + \left(\bar\epsilon_R\not\partial\psi_{Ri}\right)f^i \tag{9}$$

The second and fifth terms cancel after integration by parts and a Majorana flip:

$$\delta\mathcal{L}_{kin_1} = -\partial_\mu\left(\bar\epsilon_L\psi_{Ri}\right)\partial^\mu z^i - \bar\psi_{Li}\not\partial^2 z^i\epsilon_R = -\bar\epsilon_L\left[\partial_\mu\left(\psi_{Ri}\partial^\mu z^i\right) - \psi_{Ri}\partial^2 z^i\right] - \bar{\psi_{Li}}\partial^2 z^i\epsilon_R \tag{10}$$

$$= \bar{\epsilon_L}\psi_{Ri}\partial^2 z^i - \bar{\psi_{Li}}\epsilon_R\partial^2 z^i = 0 \tag{11}$$

The first and third terms cancel as follows:

$$\delta\mathcal{L}_{kin_2} = -\left(\partial_\mu z_i^*\right)\partial^\mu\left(\bar\epsilon_R\psi_L^i\right) + \bar\epsilon_R\left(\not\partial z_i^*\right)\not\partial\psi_L^i \tag{12}$$

$$= -\bar\epsilon_R\left[\left(\partial_\mu z_i^*\right)\left(\partial^\mu\psi_L^i\right) - \left(\gamma^{\mu\nu} + \eta^{\mu\nu}\right)\left(\partial_\mu z_i^*\right)\left(\partial_\nu\psi_L^i\right)\right] \tag{13}$$

The term with $\gamma^{\mu\nu}$ vanishes by symmetry and the other two terms clearly cancel. The fourth and seventh terms are cancel trivially. The sixth and eighth terms cancel after integration by parts and a Majorana flip:

$$\delta\mathcal{L}_{kin_3} = -\bar{\psi}_{Li}\partial\!\!\!/ f^i\epsilon_L + \bar{\epsilon}_R\left(\partial\!\!\!/\psi^i_R\right)f^i = -\bar{\psi}_{Li}\left(\partial\!\!\!/ f^i\right)\epsilon_L - \bar{\epsilon}_R\left(\partial\!\!\!/ f^i\right)\psi_{Ri} = 0 \tag{14}$$

Now we need to vary the interaction terms:

$$\begin{aligned}\delta\mathcal{L}_{int} =& -\left(\bar{\epsilon}_L\partial\!\!\!/\psi^i_L\right)\partial_i W - \left\{f^i\bar{\epsilon}_R\psi^j_L + 2c\left[\bar{\epsilon}_L\left(\partial\!\!\!/ z^i\right)\psi^j_L + f^i\bar{\epsilon}_R\psi^j_L + \bar{\psi}^i_R\left(\partial\!\!\!/ z^j\right)\epsilon_R + \bar{\psi}^i_R f^j\epsilon_L\right]\right\}\partial_i\partial_j W\\ &- c\bar{\epsilon}_R\psi^k_L\bar{\psi}^i_R\psi^j_L\partial_i\partial_j\partial_k W - \text{h.c.}\end{aligned}$$

The terms with four fermions can be shown to vanish by Fierz rearrangement. The third and fifth terms cancel after a Majorana flip. After flipping the fourth and sixth terms and combining with the second, the variation simplifies to

$$\delta\mathcal{L}_{int} = -\left(\bar{\epsilon}_L\partial\!\!\!/\psi^i_L\right)\partial_i W - (1+4c)f^i\bar{\epsilon}_R\psi^j_L\partial_i\partial_j W + \text{h.c.} \tag{15}$$

The first term of this expression can be made a total derivative in x^μ when ϵ is constant, so it vanishes when integrated. For the rest of the variation to vanish, we find that $c = -1/4$. The conjugate terms vanish or cancel in a similar manner.

To find the conserved current, we make ϵ nonconstant i.e. $\epsilon \to \epsilon(x)$ and collect terms in the variation that will include $\partial_\mu\epsilon(x)$. The kinetic terms give

$$\delta\mathcal{L}_{kin} = -\left(\partial_\mu z^*_i\right)\left(\partial^\mu\bar{\epsilon}_R\right)\psi^i_L - \left(\partial_\mu\bar{\epsilon}_L\right)\psi_{Ri}\left(\partial^\mu z^i\right) + \ldots \tag{16}$$

where $\ldots$ indicates terms that have already been shown to cancel without introducing derivatives of ϵ. Considering the first term of $\delta\mathcal{L}_{int}$, integrating by parts gives

$$\delta\mathcal{L}_{int} = -(\partial_\mu\bar{\epsilon}_L)\gamma^\mu\psi^i_L\partial_i W + \text{h.c.} \tag{17}$$

Combining all terms gives us the conserved current:

$$J^\mu = \left(\partial^\mu z^*_i\right)\psi^i_L + \psi_{Ri}\left(\eta^{\mu\nu}\partial_\nu z^i\right) + \gamma^\mu\psi^i_L\partial_i W + \gamma^\mu\psi_{Ri}\partial^i W^* \tag{18}$$

To see that it is conserved on-shell:

$$\partial_\mu J^\mu = (\partial^2 z^*_i)\psi^i_L + \left(\partial\!\!\!/ z^*_i - f^*_i\right)\partial\!\!\!/\psi^i_L + (\partial^2 z^i)\psi_{Ri} + \left(\partial\!\!\!/ z^i - f^i\right)\partial\!\!\!/\psi_{Ri} \tag{19}$$

Using the field equations:

$$\begin{aligned}\partial_\mu J^\mu =& 2c\partial^i\partial^j W^*\psi_{Rj}\left[\partial\!\!\!/ z^*_i + \partial_i W\right] + 2c\partial_i\partial_j W\psi^j_L\left[\partial\!\!\!/ z^i + \partial^i W^*\right] + \psi^i_L\left[-\partial^i W^*\partial_i\partial_j W + c\partial_i\partial_j\partial_k W\bar{\psi}^j_R\psi^k_L\right]\\ &+ \psi_{Ri}\left[-\partial_i W\partial^i\partial^j W^* - c\partial^i\partial^j\partial^k\bar{\psi}_{Lj}\psi_{Rk}\right] + \partial\!\!\!/ z^*_j\partial^i\partial^j W^*\psi_{Ri} + \partial\!\!\!/ z^j\partial_i\partial_j W\psi^i_L\end{aligned}$$

which vanishes. We can add terms to make the current gamma-traceless. Write the modified current as

$$\tilde{J}^\mu = J^\mu + \gamma^{\mu\nu}\partial_\nu T \tag{20}$$

which clearly satisfies $\partial_\mu\tilde{J}^\mu = 0$. Gamma-tracelessness implies

$$\gamma_\mu\tilde{J}^\mu = 0 \implies \gamma_\mu J^\mu + \gamma_\mu\gamma^{\mu\nu}\partial_\nu T = 0 \implies \gamma_\mu J^\mu = (1-D)\partial\!\!\!/ T \tag{21}$$

Using the explicit form of the superpotential, the field equation for ψ_L^i becomes

$$\not{\partial}\psi_L^i = -c\lambda^{ijk} z_k^* \psi_{Rj} \tag{22}$$

The current is then

$$J^\mu = (\partial^\mu z_i^*)\psi_L^i + \gamma^\mu \psi_{Ri}\lambda^{ijk} z_j^* z_k^* + \text{c.c.} = (\partial^\mu z_i^*)\psi_L^i - \frac{1}{c}\gamma^\mu z_i^* \not{\partial}\psi_L^i + \text{c.c.} \tag{23}$$

Multiplying by γ_μ and substituting $c = -1/4$:

$$\gamma_\mu J^\mu = (\not{\partial} z_i^*)\psi_L^i + z_i^* \not{\partial}\psi_L^i + \text{c.c.} = \not{\partial}\left(z_i^*\psi_L^i + z^i \psi_{Ri}\right) \tag{24}$$

This gives us the desired improvement terms:

$$T = -\frac{1}{3}\left(z_i^*\psi_L^i + z^i\psi_{Ri}\right) \tag{25}$$

PROBLEM 2

Part (a)

$$[\delta_1, \delta_2]A_\mu^a = \bar{\epsilon}_2\gamma_\mu\left(c\gamma^{\nu\rho}F_{\nu\rho}^a\epsilon_1\right) - (1 \leftrightarrow 2) \tag{26}$$

$$= c\bar{\epsilon}_2\left(\gamma_\mu{}^{\nu\rho} + 2\gamma^\rho\eta_\mu^\nu\right)\epsilon_1 F_{\nu\rho}^a - (1 \leftrightarrow 2) \tag{27}$$

After a Majorana flip, the terms with $\gamma^{\mu\nu\rho}$ cancel and the others combine. We are left with

$$[\delta_1, \delta_2]A_\mu^a = 4c\bar{\epsilon}_2\gamma^\mu\epsilon_1 F_{\nu\mu}^a = 4c\bar{\epsilon}_1\gamma^\mu\epsilon_2 F_{\mu\nu}^a \tag{28}$$

Defining $\xi^\mu = \bar{\epsilon}_1\gamma^\mu\epsilon_2$, and with $c = -1/8$ (though we could really pick any constant we want, as long as we redefine the fields correctly), the final result is

$$[\delta_1, \delta_2]A_\nu^a = -\frac{1}{2}\xi^\mu F_{\mu\nu}^a \tag{29}$$

If we fix one of the derivative terms in $F_{\mu\nu}$ to be zero by gauge freedom, then we see that this resembles the standard translation.

Now we transform λ^a:

$$[\delta_1, \delta_2]\lambda^a = c\gamma^{\mu\nu}\left(\delta_1 F_{\mu\nu}^a\right)\epsilon_2 - (1 \leftrightarrow 2) \tag{30}$$

$$= c\gamma^{\mu\nu}\left[D_\mu(\bar{\epsilon_1}\gamma_\nu\gamma^a) - D_\nu(\bar{\epsilon_1}\gamma_\mu\gamma^a)\right]\epsilon_2 - (1 \leftrightarrow 2) \tag{31}$$

$$= c\gamma^{\mu\nu}\bar{\epsilon}_1\left(\gamma_\nu D_\mu\lambda^a - \gamma_\mu D_\nu\lambda_a\right)\epsilon_2 - (1 \leftrightarrow 2) \tag{32}$$

We can move spinor bilinears around, which makes Fierz rearrangement easier:

$$= c\gamma^{\mu\nu}\epsilon_2\bar{\epsilon}_1\left(\gamma_\nu D_\mu\lambda^a - \gamma_\mu D_\nu\lambda_a\right) - (1\leftrightarrow 2) \tag{33}$$

$$= -c\sum_A \gamma^{\mu\nu}\Gamma_A\gamma_\nu(D_\mu\lambda^a)\bar{\epsilon}_1\Gamma^A\epsilon_2 - (1\leftrightarrow 2) \tag{34}$$

In $D = 3$, bilinears with rank-0 and rank-3 gamma matrices are symmetric and bilinears with rank-1 and rank-2 matrices are antisymmetric, so we can kill terms in the sum using Majorana flips:

$$= -2c\left[\gamma^{\mu\nu}\gamma_\rho\gamma_\nu D_\mu\lambda^a\bar{\epsilon}_1\gamma^\rho\epsilon_2 + \gamma^{\mu\nu}\gamma_{\rho\sigma}\gamma_\nu D_\mu\lambda^a\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2\right] \tag{35}$$

After some matrix algebra, we find that this simplifies to

$$4c\left(-\bar{\epsilon_1}\gamma^\mu\epsilon_2 + \gamma^\mu{}_{\rho\sigma}\bar{\epsilon}_1\gamma^{\rho\sigma}\epsilon_2\right)D_\mu\lambda^a \tag{36}$$

The second term can be made to resemble the first with duality relations:

$$4c\left(-\bar{\epsilon_1}\gamma^\mu\epsilon_2 + \varepsilon^\mu{}_{\rho\sigma}\varepsilon^{\nu\rho\sigma}\bar{\epsilon}_1\gamma_\nu\epsilon_2\right)D_\mu\lambda^a \tag{37}$$

$$= 4c\left(-\bar{\epsilon_1}\gamma^\mu\epsilon_2 + 2\delta^{\mu\nu}\bar{\epsilon}_1\gamma_\nu\epsilon_2\right)D_\mu\lambda^a = 4c\left(\bar{\epsilon_1}\gamma^\mu\epsilon_2\right)D_\mu\lambda^a \tag{38}$$

which gives the final result

$$[\delta_1,\delta_2]\lambda^a = -\frac{1}{2}\xi^\mu D_\mu\lambda^a \tag{39}$$

Part (b)

The field equations are

$$\varepsilon^{a\nu} \equiv D_\mu F^{\mu\nu a} = -\frac{1}{2}g f_{bc}{}^a\bar{\lambda}^b\gamma_\nu\lambda^c \qquad \varepsilon^a \equiv \not{D}\lambda^a = 0 \tag{40}$$

Varying $\varepsilon^{\nu a}$:

$$\delta\left(D_\mu F^{\mu\nu a}\right) = \partial_\mu\left[2D^{[\mu}\delta A^{\nu]a}\right] + g f_{bc}{}^a\bar{\epsilon}\gamma_\mu\lambda^b F^{\mu\nu c} + g f_{bc}{}^a A^b_\mu\left[2D^{[\mu}\delta A^{\nu]c}\right] \tag{41}$$

The first and third terms combine to give

$$\delta\left(D_\mu F^{\mu\nu a}\right) = D_\mu\left[2D^{[\mu}\delta A^{\nu]a}\right] + g f_{bc}{}^a\bar{\epsilon}\gamma_\mu\lambda^b F^{\mu\nu c} \tag{42}$$

$$= D_\mu\left[D^\mu\left(\bar{\epsilon}\gamma^\nu\lambda^a\right) - D^\nu\left(\bar{\epsilon}\gamma^\mu\lambda^a\right)\right] - g f_{bc}{}^a\bar{\epsilon}\gamma_\mu F^{\mu\nu b}\lambda^c \tag{43}$$

$$= \bar{\epsilon}\gamma^\nu D^2\lambda^a - \left(D^\nu D^\mu - [D^\nu, D^\mu]\right)\left(\bar{\epsilon}\gamma_\mu\lambda^a\right) + \bar{\epsilon}\gamma_\mu\left[D^\mu, D^\nu\right]\lambda^a \tag{44}$$

$$= \bar{\epsilon}\gamma^\nu D^2\lambda^a - \bar{\epsilon}D^\nu\not{D}\lambda^a \tag{45}$$

Varying ε^a:

$$\delta\left(\gamma^\mu D_\mu \lambda^a\right) = \gamma^\mu \left[\partial_\mu\left(c\gamma^{\nu\rho}F^a_{\nu\rho}\epsilon\right) - gf_{bc}{}^a\bar{\epsilon}\gamma_\mu\lambda^b\lambda^c - gf_{bc}{}^a A^b_\mu c\gamma^{\nu\rho}F^c_{\nu\rho}\epsilon\right] \tag{46}$$

$$= c\gamma^\mu\gamma^{\nu\rho}\epsilon D_\mu F_{\nu\rho} - gf_{bc}{}^a\gamma^\mu\bar{\epsilon}\gamma_\mu\lambda^b\lambda^c \tag{47}$$

$$= c\left(\gamma^{\mu\nu\rho} + 2\gamma^{[\rho}\eta^{\nu]\mu}\right)\epsilon D_\mu F_{\nu\rho} - gf_{bc}{}^a\gamma^\mu\bar{\epsilon}\gamma_\mu\lambda^b\lambda^c \tag{48}$$

The first term vanishes by the Bianchi identity, and the second simplifies:

$$= 2c\gamma^\mu\epsilon D^\nu F^a_{\nu\mu} - gf_{bc}{}^a\gamma^\mu\bar{\epsilon}\gamma_\mu\lambda^b\lambda^c \tag{49}$$

The identity $\gamma^\mu\lambda_{[1}\bar{\lambda}_2\gamma_\mu\lambda_{3]} = 0$ gives us

$$-gf_{bc}{}^a\left(\gamma^\mu\lambda^c\bar{\epsilon}\gamma_\mu\lambda^b + \gamma^\mu\lambda^b\bar{\lambda}^c\gamma_\mu\epsilon + \gamma^\mu\epsilon\bar{\lambda}^b\gamma_\mu\lambda^c\right) = 0 \tag{50}$$

After relabeling $b \leftrightarrow c$ on the second term and using the antisymmetry of $f_{bc}{}^a$, we find that

$$-gf_{bc}{}^a\gamma^\mu\lambda^c\bar{\epsilon}\gamma_\mu\lambda^b = \frac{1}{2}gf_{bc}{}^a\gamma^\mu\epsilon\bar{\lambda}^b\gamma_\mu\lambda^c \tag{51}$$

Notice that this closely resembles the field equation for $\varepsilon^{a\nu}$. We can combine the two remaining terms, after which we find

$$\delta\varepsilon^a = \delta\left(\gamma^\mu D_\mu\lambda^a\right) = (2c-1)\,\gamma^\mu\varepsilon^a_\mu\epsilon \tag{52}$$

PROBLEM 3

The $D = 6$ $(1,0)$ hypermultiplet consists of a 4-component symplectic Majorana-Weyl spinor ψ^A and four scalars ϕ^{iA}. This spinor (and the transformation parameter) are chirally projected. The transformation laws are

$$\delta\psi^A = a\partial\!\!\!/\phi^{iA}\epsilon^j\varepsilon_{ji} \qquad\qquad \delta\phi^{iA} = b\bar{\epsilon}^i\psi^A \tag{53}$$

where a and b have not been determined. We can compute the action of the transformations:

$$[\delta_1,\delta_2]\phi^{iA} = ab\bar{\epsilon}^i_2\partial\!\!\!/\phi^{jA}\epsilon^k_1\varepsilon_{kj} - (1\leftrightarrow 2) = ab\bar{\epsilon}^i_2\gamma^\mu\epsilon^j_1\partial_\mu\phi^A_j - (1\leftrightarrow 2) \tag{54}$$

$$= ab\left(\bar{\epsilon}^i_2\gamma^\mu\epsilon^j_1 - \bar{\epsilon}^i_1\gamma^\mu\epsilon^j_2\right)\partial_\mu\phi^A_j = -ab\left(\bar{\epsilon}^i_1\gamma^\mu\epsilon^j_2 - \bar{\epsilon}^j_1\gamma^\mu\epsilon^i_2\right)\partial_\mu\phi^A_j \tag{55}$$

A short side calculation that exploits the antisymmetry of the $USp(2)$ metric gives

$$[\delta_1,\delta_2]\phi^{iA} = -ab\bar{\epsilon}^m_1\gamma^\mu\epsilon^n_2\varepsilon_{mn}\partial_\mu\phi^{iA} = -ab\bar{\epsilon}^j_1\gamma^\mu\epsilon_{2j}\partial_\mu\phi^{iA} \tag{56}$$

which is the standard transformation when $ab = \frac{1}{2}$. For the fermion:

$$[\delta_1,\delta_2]\psi^A = ab\gamma^\mu\bar{\epsilon}^i_1\partial_\mu\psi^A\epsilon^j_2\varepsilon_{ji} - (1\leftrightarrow 2) \tag{57}$$

A Fierz identity gives

$$[\delta_1,\delta_2]\psi^A = -\frac{ab\varepsilon_{ji}}{8}\gamma^\mu \sum_B \left[\Gamma_B\partial_\mu\psi^A\bar{\epsilon}_1^i\Gamma^B\epsilon_2^j + \Gamma^B\partial_\mu\psi^A\bar{\epsilon}_2^j\Gamma^B\epsilon_1^i\right] \tag{58}$$

$$= -\frac{ab\varepsilon_{ji}}{8}\gamma^\mu \sum_B \Gamma_B\partial_\mu\psi^A\bar{\epsilon}_1^i\Gamma^B\epsilon_2^j(1+t_B) \tag{59}$$

Since the spinors are left handed, we can insert left projectors into the bilinear freely:

$$\bar{\epsilon}_1\Gamma^B\epsilon_2 = \overline{P_L\epsilon_1}\gamma^B P_L\epsilon_2 = \bar{\epsilon}_1 P_R\Gamma^B P_L\epsilon_2 \tag{60}$$

Since γ_* commutes with gamma matrices of even rank leading to a factor of P_RP_L in the bilinear, these terms vanish in the sum. Terms that are antisymmetric after a Majorana flip also vanish. Therefore, we have

$$[\delta_1,\delta_2]\psi^A = -\frac{ab\varepsilon_{ji}}{4}\left[\gamma^\mu\gamma_\nu\bar{\epsilon}_1^i\gamma^\nu\epsilon_2^j + \gamma^\mu\gamma_{\nu\rho\sigma\tau\lambda}\bar{\epsilon}_1^i\gamma^{\nu\rho\sigma\tau\lambda}\epsilon_2^j\right]\partial_\mu\psi^A \tag{61}$$

Combining the gammas:

$$[\delta_1,\delta_2]\psi^A = -\frac{ab\varepsilon_{ji}}{4}\left[\left(\gamma^\mu{}_\nu + \eta^\mu_\nu\right)\bar{\epsilon}_1^i\gamma^\nu\epsilon_2^j + \left(\gamma^\mu{}_{\nu\rho\sigma\tau\lambda} + 5\gamma_{[\rho\sigma\tau\lambda}\eta^\mu_{\nu]}\right)\bar{\epsilon}_1^i\gamma^{\nu\rho\sigma\tau\lambda}\epsilon_2^j\right]\partial_\mu\psi^A \tag{62}$$

Using the definition of γ_* in terms of the rank-6 gamma matrix, a duality relation, and the fact that the spinors are chirally projected, the third and fourth terms can be rewritten:

$$[\delta_1,\delta_2]\psi^A = -\frac{ab\varepsilon_{ji}}{4}\left[\left(\gamma^\mu{}_\nu + \eta^\mu_\nu\right)\bar{\epsilon}_1^i\gamma^\nu\epsilon_2^j + \frac{1}{5!}\varepsilon^{\lambda\tau\sigma\rho\nu\alpha}\left(\varepsilon^\mu{}_{\nu\rho\sigma\tau\lambda} + 5\gamma_{[\rho\sigma\tau\lambda}\eta^\mu_{\nu]}\right)\bar{\epsilon}_1^i\gamma_\alpha\epsilon_2^j\right]\partial_\mu\psi^A \tag{63}$$

After some matrix algebra, we find that the third and fourth terms simplify:

$$[\delta_1,\delta_2]\psi^A = -\frac{ab\varepsilon_{ji}}{4}\left[\left(\gamma^\mu{}_\nu + \eta^\mu_\nu\right)\bar{\epsilon}_1^i\gamma^\nu\epsilon_2^j + \left(\eta^{\alpha\mu} + \gamma^{\mu\alpha}\right)\bar{\epsilon}_1^i\gamma_\alpha\epsilon_2^j\right]\partial_\mu\psi^A = -\frac{ab}{2}\gamma^\mu\gamma^\nu\partial_\mu\phi^A\bar{\epsilon}_1^i\gamma_\nu\epsilon_{2i} \tag{64}$$

Reversing the order of the gamma matrices gives

$$-ab\bar{\epsilon}_1^i\gamma^\mu\epsilon_{2i}\partial_\mu\phi^A + \frac{ab}{2}\gamma^\nu\not{\partial}\phi^A\bar{\epsilon}_1^i\gamma_\nu\epsilon_{2i} \tag{65}$$

The second term vanishes on-shell by the field equation $\not{\partial}\phi^A = 0$, which gives the result

$$[\delta_1,\delta_2]\psi^A = -ab\bar{\epsilon}_1^i\gamma^\mu\epsilon_{2i}\partial_\mu\psi^A \tag{66}$$

which matches the result for the gauge field.

PROBLEM 4

When we reduce from $D = 10$ to $D = 3$, the symmetry group decomposes as $SO(9,1) \to SO(2,1) \times SO(7)$. The gauge field decomposes as

$$\hat{A}^{\hat{\mu}} \to \left(A_\mu, \phi_i\right) \qquad \mu = 0,1,2 \quad i = 1,\ldots,7 \tag{67}$$

And the spinor (where α is a spinor index) decomposes as

$$\hat{\psi}^{\alpha} \to \psi^{aA} \tag{68}$$

The index a carries a 2-dimensional representation of $SO(2,1)$ and A carries an 8-dimensional representation of $SO(7)$. In $D = 10$, we have a Majorana-Weyl spinor with 16 independent components, which will become several 2-dimensional spinors in $D = 3$. The ten-dimensional action is

$$S = \int d^{10}x \;\; \mathrm{tr}\left(-\frac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{\bar{\psi}}\hat{\Gamma}^{\hat{\mu}}\hat{D}_{\hat{\mu}}\hat{\psi}\right) \tag{69}$$

First decompose the gauge field:

$$\hat{F}_{\hat{\mu}\hat{\nu}} \quad = \partial_{\hat{\mu}}\hat{A}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}_{\hat{\mu}} + [\hat{A}_{\hat{\mu}}, \hat{A}_{\hat{\nu}}] \to \begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\left[A_\mu, A_\nu\right] \\ F_{\mu i} = 2\partial_\mu \phi^i - 2\partial_i A_\mu + 2g\left[A_\mu, \phi_i\right] = 2D_\mu \phi_i \\ F_{ij} = g\left[\phi_i, \phi_j\right] \end{cases} \tag{70}$$

Then the field strength term becomes

$$-\frac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}\hat{F}^{\hat{\mu}\hat{\nu}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\phi_i D^\mu\phi^i - \frac{g^2}{4}\left[\phi_i, \phi_j\right]\left[\phi^i, \phi^j\right] \tag{71}$$

For gamma matrices, we use

$$\hat{\Gamma} = \begin{cases} \sigma_1 \otimes \gamma_\mu \otimes \mathbb{1} \;\; , \;\; \mu = 0, 1, 2 \\ i\sigma_2 \otimes \mathbb{1} \otimes \gamma_i \;\; , \;\; i = 1, \ldots, 7 \end{cases} \tag{72}$$

Now we consider the spinor kinetic term. We can write the spinor ψ as an $SO(7)$ doublet:

$$\psi = \begin{pmatrix} \psi_1^A \\ \psi_2^A \end{pmatrix} \tag{73}$$

When $\hat{\mu} = \mu$, we have (suppressing gauge indices)

$$-\frac{1}{2}\bar{\psi}\Gamma^\mu D_\mu \psi = -\frac{1}{2}\bar{\psi}^{aA}\sigma_1\gamma^\mu D_\mu \psi^{aA} = -\frac{1}{2}\bar{\psi}_1^A\gamma^\mu D_\mu \psi_1^A - \frac{1}{2}\bar{\psi}_2^A\gamma^\mu D_\mu \psi_2^A \tag{74}$$

When $\hat{\mu} = i$, we have

$$-\frac{1}{2}\bar{\psi}\Gamma^i D_i \psi = -\frac{1}{2}\bar{\psi}_1^A\gamma^i D_i \psi_1^A + \frac{1}{2}\bar{\psi}_2^A\gamma^i D_i \psi_2^A = \frac{g}{2}\bar{\psi}_1^A\gamma^i\left[\phi_i, \psi_1^A\right] - \frac{g}{2}\bar{\psi}_2^A\gamma^i\left[\phi_i, \psi_2^A\right] \tag{75}$$

The final action is

$$\begin{aligned} S = \int d^3x \;\; \mathrm{tr}\Bigg(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\phi_i D^\mu\phi^i - \frac{g^2}{4}\left[\phi_i, \phi_j\right]\left[\phi^i, \phi^j\right] \\ &-\frac{1}{2}\bar{\psi}_1^A\gamma^\mu D_\mu \psi_1^A - \frac{1}{2}\bar{\psi}_2^A\gamma^\mu D_\mu \psi_2^A \frac{g}{2}\bar{\psi}_1^A\gamma^i\left[\phi_i, \psi_1^A\right] - \frac{g}{2}\bar{\psi}_2^A\gamma^i\left[\phi_i, \psi_2^A\right]\Bigg) \end{aligned}$$

The transformation laws in $D = 10$ are (up to a constant)

$$\delta\hat{\psi} = \hat{\Gamma}^{\hat{\mu}\hat{\nu}}F_{\hat{\mu}\hat{\nu}}\hat{\epsilon} \qquad \delta\hat{A}_{\hat{\mu}} = \hat{\bar{\epsilon}}\hat{\Gamma}_{\hat{\mu}}\hat{\psi} \tag{76}$$

First, we need to decompose the second rank gamma matrix:

$$\Gamma^{\hat{\mu}\hat{\nu}} = \begin{cases} I_2 \otimes \gamma^{\mu\nu} \otimes I_2, & \hat{\mu} = \mu, \hat{\nu} = \nu \\ -\sigma_3 \otimes \gamma^{\mu} \otimes \gamma^{i}, & \hat{\mu} = \mu, \hat{\nu} = i \\ -I_2 \otimes I_2 \otimes \gamma^{ij}, & \hat{\mu} = i, \hat{\nu} = j \end{cases} \tag{77}$$

The transformation laws are

$$\delta A_\mu = \hat{\bar{\epsilon}}\sigma_1 \gamma_\mu \hat{\psi} = \bar{\epsilon}_1 \gamma_\mu \psi_1 + \bar{\epsilon}_2 \gamma_\mu \psi_2 \tag{78}$$

$$\delta \phi_i = \hat{\bar{\epsilon}} i \sigma_2 \gamma_i \hat{\psi} = \bar{\epsilon}_1 \gamma_i \psi_1 - \bar{\epsilon}_2 \gamma_i \psi_2 \tag{79}$$

$$\delta \psi_1 = \left(\gamma^{\mu\nu} F_{\mu\nu} - 2\gamma^\mu \gamma^i D_\mu \phi_i - g\gamma^{ij} [\phi_i, \phi_j] \right) \epsilon_1 \tag{80}$$

$$\delta \psi_2 = \left(\gamma^{\mu\nu} F_{\mu\nu} + 2\gamma^\mu \gamma^i D_\mu \phi_i - g\gamma^{ij} [\phi_i, \phi_j] \right) \epsilon_2 \tag{81}$$

where $\epsilon_{1,2}$ are the left and right projections of the $10D$ transformation parameter.