Sigma Models and Hyper-Kähler and Quaternionic-Kähler Geometry

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BACKGROUND AND MOTIVATION

Sigma models are integral parts of many aspects of modern research. The worldsheet action in string theory takes the form of a sigma model coupled to two-dimensional gravity, the scalar fields in supergravity are often sigma models, and the close connection to geometry makes them interesting to mathematicians.

This review will briefly discuss some of the geometric aspects of supersymmetric nonlinear sigma models. First, we will discuss the basics of Kähler, hyper-Kähler, and quaternionic Kähler geometry. We will explain the fundamentals of a general sigma model, and give two important geometrical examples: the $D = 6$ sigma model, which incorporates hyper-Kähler geometry, and the coupling of a $D = 4$, $\mathcal{N} = 2$ sigma model to supergravity, where we find quaternionic Kähler geometry.

KÄHLER, HYPER-KÄHLER, AND QUATERNIONIC GEOMETRY

Complex Structures

An "almost complex structure" J is a second rank tensor with real components that satisfies[5]

$$J_i{}^j J_j{}^k = -\delta_i^k \tag{1}$$

Here there is a clear analogy to $i^2 = -1$; this allows multiplication by "i" on the tangent space of a given manifold. However, not all manifolds endowed with an almost complex structure are complex. It can be shown[6] that the necessary and sufficient condition for this to hold is the vanishing of the Nijenhuis tensor:

$$N_{ij}^k = J_i{}^\ell \left(\partial_\ell J_j{}^k - \partial_j J_\ell{}^k\right) - J_j{}^\ell \left(\partial_\ell J_i{}^k - \partial_i J_\ell{}^k\right) = 0 \tag{2}$$

Using complex coordinates, the complex structure can be put in a canonical form:

$$J_i{}^j = i \begin{pmatrix} \delta_i^j & 0 \\ 0 & -\delta_{\bar{i}}^{\bar{j}} \end{pmatrix} \tag{3}$$

Kähler Manifolds

A metric is hermitian with respect to the complex structure if it obeys

$$J_i{}^k J_{jm} g_{km} = g_{ij} \tag{4}$$

On a manifold with such a metric, we can define the fundamental two-form as

$$\omega = J_{ij} dx^i \wedge dx^j \tag{5}$$

Using the canonical form of J, we find that

$$\omega = 2i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \tag{6}$$

Finally, we can define a Kähler manifold as one where ω is closed, that is, it obeys $d\omega = 0$. In terms of complex coordinates, this tells us that

$$\frac{\partial g_{i\bar{j}}}{\partial \phi^k} - \frac{\partial g_{k\bar{j}}}{\partial \phi^i} = 0 \qquad \frac{\partial g_{i\bar{j}}}{\partial \bar{\phi}^k} - \frac{\partial g_{i\bar{k}}}{\partial \bar{\phi}^j} = 0 \tag{7}$$

which further imply that the metric can be written in terms of the Kähler potential K:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K\left(\phi, \bar{\phi}\right) \tag{8}$$

An equivalent condition[2] is that J is covariantly constant; that is, $\nabla J = 0$, where ∇ is the usual Levi-Civita connection.

Holonomy Groups

It is natural to wonder whether the process of generalizing ordinary geometry to complex geometry can be further extended. To answer this question, we first need to briefly discuss the concept of the holonomy group. The holonomy group of a connected n-dimensional Riemannian manifold is the group of transformations generated by parallel transporting vectors around all possible closed curves on the manifold[6].

Given a vector V in the tangent space of a manifold and any closed curve γ, parallel transport of V around γ defines a new vector V'. The elements of the holonomy group are then the linear operators $A^i_j(\gamma)$ that transform V into V'. Tensors transform under elements of the holonomy group as the usual tensor product of vectors, and one can show[7] that a covariantly constant tensor (like our complex structure) commutes with all elements of the holonomy group:

$$A^i_k(\gamma) J_j{}^k - J_k{}^i A^k_j(\gamma) = 0 \tag{9}$$

From the representation theory of the special linear group[4], it can be shown that this constraint puts a strong limit on the ways we can generalize conventional geometry. Objects that commute with elements of the holonomy group must form an associative division algebra over $\mathbb{R}$. The only possibilities are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (the quaternions). In more practical terms, this implies that a manifold can admit up to three almost complex structures, but no more[7]. However, the case of two complex structures is not relevant, since their product will automatically define a third[6].

Generally, an n-dimensional Riemannian manifold has holonomy group $O(n)$, but in the presence of additional complex structures, we find that the holonomy group is instead some subgroup of $O(n)$. Manifolds with a holonomy group that leaves a single complex structure invariant are $2n$-dimensional Kähler manifolds and have holonomy group $G \subseteq U(n) \subseteq O(2n)$. If instead three complex structures are left invariant, the manifold is $4n$-dimensional with holonomy group $G \subseteq Sp(n) \subseteq O(4n)$. These are called hyper-Kähler manifolds, and the complex structures obey the following relation[3]:

$$J^{(n)}{}_i{}^j J^{(m)}{}_j{}^k = -\delta^{nm}\delta_i^k + \varepsilon^{nm\ell} J^{(\ell)}{}_i{}^k \tag{10}$$

where n, m are simply labels that run from 1 to 3 and $i, j = 1, \ldots, 4n$. This is easily recognized as the usual quaternion (or $SU(2)$) algebra.

Quaternionic Kähler vs. Hyper-Kähler Manifolds

A quaternionic Kähler manifold is a $4n$-dimensional manifold whose holonomy group is a subgroup of $Sp(n) \times Sp(1)/\mathbb{Z}_2$[3]. Like hyper-Kähler manifolds, these manifolds admit three covariantly constant complex structures that obey the quaternion algebra. Since the holonomy group is a product of $Sp(n)$ and $Sp(1)$, the Riemann curvature must be the sum of the $Sp(n)$ and $Sp(1)$ curvatures, and furthermore, it can be shown[3] that

$$R = \lambda R_{\mathbb{H}P(n)} + R_0 \tag{11}$$

where $R_{\mathbb{H}P(n)}$ is the curvature of the quaternionic protective space, λ is a constant, and R_0 is the Ricci-flat part of the $Sp(n)$ curvature. In this scenario, R_0 functions as the curvature for a manifold with a holonomy group contained in $Sp(n)$ - exactly the criteria for a hyper-Kähler manifold. Thus, we see that quaternionic Kähler geometry is a generalization of hyper-Kähler geometry. When the scalar curvature vanishes - that is, when $\lambda = 0$ - we recover hyper-Kähler geometry.

BASICS OF SIGMA MODELS

Nonlinear sigma models are the field theories of scalar fields interpreted as harmonic maps from spacetime into a Riemannian manifold. When these models are made supersymmetric, strong constraints are placed on the geometry of the manifold. The general sigma model Lagrangian takes the form[1]

$$\mathcal{L} = \frac{1}{2} g_{AB} \partial_\mu \phi^A \partial^\mu \phi^B + \text{supersymmetrization} \tag{12}$$

where g_{AB} is the metric for a manifold M with coordinates given by the scalar fields. The structure of this manifold is determined by the spacetime dimension and the amount of supersymmetry desired. Theorems by Alvarez-Gaume and Freedman[7] and Zumino[9] specify the geometry of the manifold for a given dimension and amount of supersymmetry. This is summarized in the following table[2]:

$D =$	6 4 2	Geometry
$\mathcal{N} =$	1 2 4	Hyper-Kähler
$\mathcal{N} =$	1 2	Kähler
$\mathcal{N} =$	1	Riemannian

$D = 6$ SIGMA MODELS

The $D = 6$ sigma model was first constructed by Sierra and Townsend[1]. We will see that the scalar fields in this model define coordinates on a hyper-Kähler manifold, as summarized above. To define the model, we begin with the algebra of superfield derivatives:

$$\{D^a_\alpha, D^b_\beta\} = i\varepsilon^{ab}\partial_{\alpha\beta} \tag{13}$$

where $\partial_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\partial_\mu$. The fields of interest are the superfield ϕ^{ia} and the Majorana-Weyl spinor ψ^i_α, which can be defined by $\psi^i_\alpha = D^a_\alpha \phi^{ib}\varepsilon_{ab}|_{\theta=0}$. The action of this model is given by

$$S = \int d^6x \, \left(\frac{1}{2} g_{ia\ jb} \partial^{\alpha\beta}\phi^{ia}\partial_{\alpha\beta}\phi^{jb} + \frac{i}{2}\varepsilon_{ij}\psi^i_\alpha \mathcal{D}^{\alpha\beta}\psi^j_\beta + \frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta} R_{imnp}\psi^i_\alpha\psi^m_\beta\psi^n_\gamma\psi^p_\delta \right) \tag{14}$$

where R_{imnp} is the Riemann tensor for the manifold formed by the scalar fields. The covariant derivative of ψ comes from the superfield constraint

$$e^{i(c}_{jb} D^{a)}_\alpha \phi^{jb} = 0 \tag{15}$$

and is defined as

$$\mathcal{D}_{a\alpha} = (D_{a\alpha}\delta^i_k + \omega_{jb}{}^i{}_k e^{jb}_{la}\psi^l_\alpha)\psi^k_\beta \tag{16}$$

where ω is a complicated function of the vielbein that is related to the spin connection. Differentiating (16), we arrive at the field equation for ψ:

$$i\partial^{\alpha\beta}\psi_\beta^i + i\omega_{jb}{}^i{}_m\partial^{\alpha\beta}\phi^{jb}\psi_\beta^m + R^i{}_{mnp}\psi_\beta^m\psi_\gamma^n\psi_\delta^p\varepsilon^{\alpha\beta\gamma\delta} = 0 \tag{17}$$

It can be observed[1] that this is only derivable from an action if ω and R satisfy

$$\omega_{jb}{}^i{}_i = 0 \qquad\qquad R^i{}_{inp} = 0 \tag{18}$$

After a technical calculation involving the relationships between ω and the spin connection $\hat{\omega}$, these lead to the constraint

$$\hat{\omega}_{mc}{}^a{}_b = 0 \tag{19}$$

Given this constraint, it can be shown that the only nonzero component of the Riemann tensor is $R_{(imnp)}$. By similar reasoning, it can be shown that the full tensor $\hat{R}_{iajb} = 0$, which is a necessary but not sufficient condition for our manifold to be hyper-Kähler. We can completely show that the manifold is hyper-Kähler by constructing three complex structures that obey the required quaternion algebra relation. These are

$$F^A{}_{kc}{}^{ld} = \left(-i\sigma^A\right)_a{}^b e^{ia}_{kc}e^{ld}_{ib} \tag{20}$$

where $A = 1, 2, 3$ labels the individual matrices. Note that these can be shown to be covariantly constant.

COUPLING $\mathcal{N} = 2$ SIGMA MODELS TO SUPERGRAVITY IN $D = 4$

Witten and Bagger[8] were the first to discover the presence of quaternionic Kähler geometry in $\mathcal{N} = 2$ supergravity. Since it is known that the scalar fields of sigma models with global $\mathcal{N} = 2$ supersymmetry are coordinates on a hyper-Kähler manifold[7], we should consider a theory with $4n$ scalar fields and $2n$ Majorana spinors. From dimensional arguments, the transformation law for the scalar fields should be of the form

$$\delta\phi^i = \gamma^i_{AZ}\left(\bar{\epsilon}^A_R\chi^Z_L + \bar{\epsilon}^A_L\chi^Z_R\right) \tag{21}$$

where γ^i_{AZ} are undetermined functions of the scalar fields. From geometric arguments, we find that γ^i_{AZ} are covariantly constant generalizations of the ordinary gamma matrices, that A is an $Sp(1)$ index, and that Z is an $Sp(n)$ index. Since γ^i_{AZ} are covariantly constant, $[\nabla_i, \nabla_j]\gamma^k_{AZ} = 0$, which implies that

$$R_{ijkl}\gamma^l_{AY}\gamma^k_{BZ} = \varepsilon_{AB}R_{ijYZ} + \varepsilon_{YZ}R_{ijAB} \tag{22}$$

where R_{ijAB} is the $Sp(1)$ Riemann curvature and R_{ijYZ} is the $Sp(n)$ Riemann curvature.

To construct the sigma model Lagrangian, we first use the fact that hyper-Kähler manifolds are Ricci flat and the cyclic identity $R_{ijkl} + R_{ikjl} + R_{iljk} = 0$ to find

$$R_{ijYZ} = \gamma^{AW}_i\gamma^X_{jA}\Omega_{YZXW} \tag{23}$$

where Ω_{YZXW} is a totally symmetric tensor. The $\mathcal{N} = 2$ sigma model Lagrangian is then

$$\mathcal{L}_\sigma = -g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2}\bar{\chi}_Z\gamma^\mu D_\mu\chi^Z + \frac{1}{16}\Omega_{XYZW}\left(\bar{\chi}^X_L\chi^Y_L\right)\left(\bar{\chi}^Z_L\gamma^\mu\chi^W_L\right) \tag{24}$$

where $\chi_{L,R}$ are left and right projected spinors. The covariant derivative is

$$D_\mu\chi^Z = \partial_\mu\chi^Z + \Gamma_i{}^Z{}_Y\partial_\mu\phi^i\chi^Y \tag{25}$$

This can be shown to be supersymmetric under the following transformations:

$$\delta\phi^i = \gamma^i_{AZ}\left(\bar{\epsilon}^A_R\chi^Z_L + \bar{\epsilon}^A_L\chi^Z_R\right) \tag{26}$$

$$\delta\chi^Z_L = 2\partial_\mu\phi^i\gamma_i^{AZ}\gamma^\mu\epsilon_{RA} - \Gamma_i{}^Z{}_Y\delta\phi^i\chi^Y_L \tag{27}$$

To gauge these transformations, we use the Noether procedure. The first step is to add the $\mathcal{N} = 2$ supergravity Lagrangian to $\mathcal{L}_\sigma$:

$$\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_{SG}$$

$$\begin{aligned}\mathcal{L}_{SG} = &-\frac{1}{2\kappa^2}eR - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu A}\gamma_5\gamma_\nu D_\rho\psi^A_\sigma \\ &- \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} + \frac{\sqrt{2}}{4}e\kappa\bar{\psi}_{\mu A}\left(F^{\mu\nu} + \frac{1}{2}e^{-1}\tilde{F}^{\mu\nu}\gamma_5\right)\psi^A_\nu \\ &- \frac{1}{8}e\kappa^2\bar{\psi}_{\mu A}\left(\bar{\psi}^\mu_B\psi^{\nu B} + \frac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\rho B}\psi^B_\sigma\gamma_5\right)\psi^A_\nu\end{aligned} \tag{28}$$

We can add couplings via the Noether current:

$$\begin{aligned}\mathcal{L}_N &= -\frac{1}{2}\bar{\psi}_{\mu A}J^{\mu A} \\ &= e\kappa\gamma_{iAZ}\left(\bar{\chi}^Z_R\gamma^\mu\gamma^\nu\psi^A_{\mu L} + \bar{\chi}^Z_L\gamma^\mu\gamma^\nu\psi^A_{\mu R}\right)\end{aligned} \tag{29}$$

This ensures that $\mathcal{L}$ is invariant under local supersymmetry transformations to order κ. Most of the cancellations after transforming the Lagrangian are straightforward, but the $\bar{\epsilon}\psi$ terms require that

$$R_{ijAB} = \kappa^2\left(\gamma_{iAZ}\gamma_{jB}{}^Z - \gamma_{jAZ}\gamma_{iB}{}^Z\right) \tag{30}$$

This is, in general, nonzero, so when considering local supersymmetry, we find the requirement that the $Sp(1)$ connection must not vanish. This gives a sigma model manifold with a holonomy group that is a subgroup of $Sp(n) \times Sp(1)$, which is exactly the previously stated definition of a quaternionic Kähler manifold. Furthermore, it can be shown that the scalar curvature is fixed in terms of κ:

$$R = -8\kappa^2(n^2 + 2n) \tag{31}$$

Therefore, we see that our manifold must have negative scalar curvature. Note that quaternionic manifolds with negative scalar curvature are typically non-compact.

CONCLUSIONS

We have discussed hyper-Kähler and quaternionic Kähler geometry and the connection to sigma models. In these theories, scalar fields act as coordinates on Riemannian manifolds. The structure of such manifolds is dictated largely by requirements imposed by supersymmetry. We gave the examples of the $D = 6$ sigma model and the coupling of $\mathcal{N} = 2$ supergravity to a sigma model, which prominently feature hyper-Kähler and quaternionic Kähler geometries respectively.

However, these models have an extremely rich structure, and there are many aspects that were not covered here. Such topics include gauging isometries of the target manifold, finiteness and renormalizability, the role of sigma models in gauge/gravity dualities, their appearance in string theory, and the ways the mathematics community has utilized supersymmetric sigma models in classifying and creating new manifolds in special geometries.

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