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I discuss a model for free-for-all passenger boarding which is employed by some discount air carriers. The model is based on the principles of statistical mechanics, where each seat in the aircraft has an associated energy which reflects the preferences of travelers. As each passenger enters the airplane they select their seats using Boltzmann statistics, proceed to that location, load their luggage, sit down, and the partition function seen by remaining passengers is modified to reflect this fact. I discuss the various model parameters and make qualitative comparisons of this passenger boarding model with those that involve assigned seats. The model can be used to predict the probability that certain seats will be occupied at different times during the boarding process. These results might provide a useful description of this boarding method. The model is a relatively unusual application of undergraduate level physics and describes a situation familiar to many students and faculty. © *2008 American Association of Physics Teachers.*
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I. INTRODUCTION

The study of physics allows students to gain knowledge and acquire skills that can be applied to a host of different fields, many of which are not obviously related to physics. Some examples include the traveling salesman problem, which can be solved via simulated annealing[1,2] and entropy considerations in image reconstruction.[2] Bachmat *et al.*[3] used the principles of general relativity to model airplane passenger boarding. They found that the boarding process can be described by a two-dimensional Lorentzian geometry. The time required for passengers to sit in their assigned seats is related to the number of other passengers that are in their future light cone.

A recent study of airplane boarding[4] applied a minimization algorithm that is commonly employed in physics (Markov chain Monte Carlo) to optimize the passenger boarding process. This application is not the first attempt to optimize passenger boarding; computer simulations have often been employed to study the boarding process.[4–6] These studies require that the passengers have assigned seats. Although most carriers assign seats, not all do. Some discount carriers employ a free-for-all boarding approach where passengers can sit anywhere that they wish. Although it may appear prone to disaster, this boarding strategy gives some of the fastest boarding times in the industry.[7] This fact alone merits understanding the underlying principles of the free-for-all boarding procedure.

A cursory attempt to model free-for-all boarding would be to assign each passenger a random seat within the aircraft. However, such a random assignment ignores subtleties that are fundamental to the success of this boarding method. Of particular importance is the fact that passengers choose the seat where they will sit based on the state of the cabin at the time that they enter. Passengers might prefer to sit in either a window seat or an aisle seat if these seats are available. Or, they may prefer to sit near the front or rear of the airplane. Additionally, their preference might change based on the decisions of the passengers who are in front of them in the line. This capacity to adjust based on the state of the airplane makes the free-for-all boarding process successful.

Statistical mechanics provides an excellent framework with which to study free-for-all airplane boarding. In this article I present a model for free-for-all boarding where each seat inside the cabin is assigned an "energy" which reflects the overall desirability of a particular seat. When passengers enter the airplane, they choose which seat to occupy using Boltzmann statistics. As the airplane fills, the partition function is updated to reflect the evolving state of the aircraft cabin. Thus, the free-for-all boarding process can be studied as a sequence of passenger decisions modeled by statistical mechanics considerations.

In addition to studying the boarding process, we may study the outcome or final seating arrangement of the aircraft for airplanes that are only partially filled (the seating arrangements of full flights are trivial) or the seating arrangement at any time during the boarding process. Instead of modeling the seat selection of each passenger during boarding, we determine the probability that a particular seat will be occupied by a passenger at the time of interest by treating the passengers as a fermion gas.

The discussion proceeds as follows. I discuss in Sec. II the model of the airplane and the statistical properties of the passengers. I then discuss the application of passenger model to the boarding process, including how passengers move within the airplane. I discuss in Sec. III the airplane boarding times given by the model and the effects of changes to the model parameters (Sec. III A). Section III B gives a brief comparison of the results of this model to the results of the assigned-seating model used in Ref. 4. Section IV discusses the final states of a partially filled aircraft using Fermi-Dirac statistics to determine the occupancy of the seats at the end of the boarding process (or at any instant during the boarding process). Finally, I discuss the utility of this model for various classes of people including students, educators, and airline executives.

II. DESCRIPTION OF MODEL

The airplane model and algorithm has three components: the geometry and characteristics of the airplane, the movements of the passengers inside the airplane, and the decision-making process of the passengers. The first two components are largely details and are less important in terms of the

behavior of the model. The third component is where statistical mechanics plays its major role; the decisions of the typical passenger are modeled here.

A. Decisions, energy, and temperature

Each seat in the airplane is assigned an energy which characterizes how desirable a seat appears to the average passenger. If a seat has a small energy or a large negative energy, it is more likely to be chosen. (I use negative energies here solely for the image of having the passengers "bind" to the seats—adding a constant to the seat energies does not affect the results.) When passengers enter the airplane, they choose a seat with probability $\exp(-\epsilon_i/T)/Z$, where ϵ_i is the energy of seat i, T is the temperature, and

$$Z = \sum_i e^{-\epsilon_i/T} \tag{1}$$

is the partition function; the counter i runs over all accessible (unoccupied) seats. Boltzmann's constant k_B is set to unity so that temperature is measured in units of energy. I use electron volts (eV), but the choice is arbitrary. To choose a seat a random number between 0 and 1 is generated and the Boltzmann factors are summed until the total probability is greater than the random number. The last seat to be added to the sum is the selected seat. (The order of the summation does not matter provided that the ordering scheme does not depend on the value of the random number.) Because the seat decisions are made by individual passengers based on the currently available seats, they are analogous to assigning a single particle a state from all available single particle states; thus, the sum of the Boltzmann factors in Eq. (1) is the appropriate partition function.

The temperature of the passengers is a measure of their apathy in the seat selection process. For very high temperatures ($T \gg |\epsilon|$, where ϵ represents the typical magnitude of the energies of the seats) all of the seats are essentially equal, the energies being a very small fraction of the "thermal" energy of the passengers when they make their seat selection. For very low temperatures the differences between seats appear large and passengers choose the best available seat at all times. Presumably the temperature is determined by human nature and by the environment. Although most passengers are not likely to be at precisely the same temperature because of their different histories, using a representative temperature to characterize the mean behavior should be appropriate, particularly if the boarding process is reproducible across several different flights. Given the consistent performance of free-for-all boarding,[7] this assumption is likely to be justified.

B. Airplane description

Like the study described in Ref. 4, the standard airplane used for this study seats 120 passengers and has six passengers per row with 20 rows. There is no first-class cabin. The energies of the seats are symmetric about the aisle so that a window, middle, or aisle seat for a given row has the same energy as its counterpart across the aisle. In addition, there is a linear increase in the energies from the front to the back of the airplane. This increase initially serves to make the front of the plane more desirable than the back. I investigate changes to this assumption and in other parameters in Sec. III A. Having the energies of the seats vary with higher-order
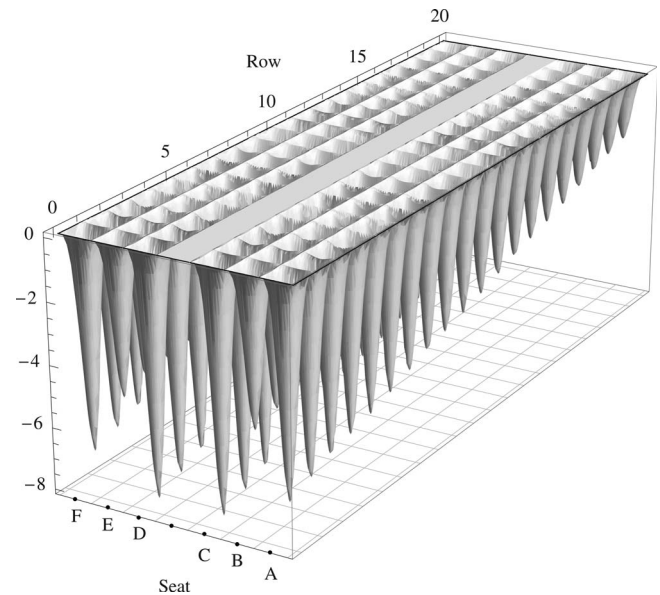

Fig. 1. Plot of the seat energies for the standard airplane model. The depth of the potential well for each seat corresponds to the probability that it will be selected by a passenger; the deeper the well, the more likely it will be chosen. A constant offset can be added to these energies without affecting the results.

polynomials may be a more accurate description, but as with almost all parameters for this model it would need to be compared empirically with passenger data—an exercise that is beyond the scope of this work.

The three energies for the seats in a given row are nominally assigned values of −8 eV for aisle seats, −7 eV for window seats, and −5 eV for middle seats; the temperature is fixed at 1 eV. The effect of these energies is that the aisle seats are chosen 70% of the time, the window seats are chosen 26% of the time, and the middle seats are chosen roughly 4% of the time. The energies in each successive row increases by 0.25 eV. The result of this choice is that the probability of the first passenger sitting in the last row of the airplane is roughly 1% of the probability that he/she will sit in the first row. Figure 1 shows the energies of the seats inside the aircraft.

C. Passenger movement

For the boarding process each passenger is assigned an amount of time that they require to load their luggage, a random number between 0 and 100 time units. One time unit corresponds to the time that it takes for a passenger to walk the distance between adjacent rows in the airplane. All passengers walk at the same rate. Other distributions from which the luggage loading times are selected, such as a Gaussian or exponential distribution, were studied in Ref. 4 and are not pursued here. The use of different distributions does not significantly alter the results. The values of the various model parameters are chosen to compare our results with those in Ref. 4.

As the passengers walk down the aisle of the airplane they obey the following rules: (1) A person will not begin to move unless there are two spaces between them and the person in front of them—a space being equal to a row; (2) If a person is moving, then they will occupy any empty space in front of them prior to stopping (thus, the passengers bunch up as they

come to a halt); (3) Passengers require one space either in front of or behind them to load their luggage; and (4) Passengers only load their luggage into the bins above their assigned row. I do not account for the time that it takes for a passenger to slide past someone in the aisle or middle seats to get to the window seat. Such a delay would be straightforward to incorporate, but would not serve any significant purpose here.

When a passenger sits down, the energy of their seat is changed to positive infinity (causing the probability that that particular seat will be chosen again to vanish) and the partition function is recalculated for subsequent passengers. Once a passenger has chosen which seat to occupy, he/she does not change that decision unless the selected seat becomes occupied while he/she is walking down the aisle (for example, by a different passenger who was in front of them in the line). In this event passengers choose another seat based on the partition function that results from using only the seats between their current row to the back of the airplane. Thus, if a person's seat is taken before they reach it, they will select a new seat somewhere in front of them.

The exception to this rule is if an intended seat becomes occupied and there are no other available seats to the rear of the airplane. In this case passengers will reverse their direction and proceed toward the front of the cabin. Unlike the forward moving case, backward moving passengers will sit in the first row that they come to which has an empty seat (again using the partition function for that particular row to select their seat). These final rules only affect a small fraction of the total passengers; the important idea is that each time there is a decision to be made regarding seat selection, the seat is chosen according to the probabilistic rules we have given. Although the passenger movement algorithm given here may not match exactly what occurs in practice, the assumptions applied in this study are reasonable. The overall order that the seats become filled is largely independent of these rules; similar seat selection occurs if the passengers are instantly teleported to their seats (though the airplane fills much faster in such a scenario).

III. RESULTS FOR FULL FLIGHTS

Here I report the results that this model gives for the time required to completely fill an aircraft. I discuss the effects of changes to the various parameters including the mean time required by passengers to load their luggage. Then, I discuss the effects of changing the slope of the linear increase in the seat energies. Finally, I present some of the effects of changing the different energies for the seats within a given row. I do not give a complete exploration of the effects of changing the energies of the individual seats—instead, I focus on a few cases that identify the primary effects of changing these parameters. Also, I do not explore changes to the airplane geometry, though some discussion of that issue can be found.[4]

A. Differing parameter values

The time required for all of the passengers to board the aircraft scales linearly with the time that they require to load their luggage. Figure 2 shows the time required to fill the aircraft for different values of the mean luggage loading times. Recall that the loading times are assigned to the passengers from a uniform distribution. These data are gener-
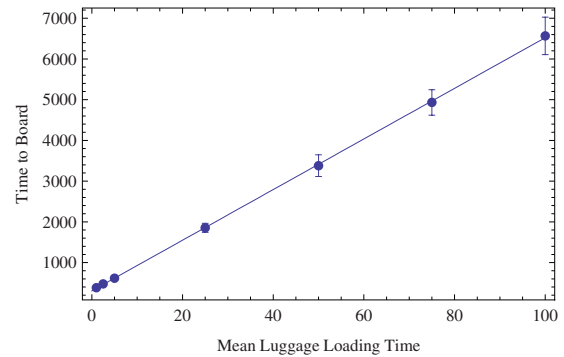

Fig. 2. Plot of the time required to fill the aircraft for different values of the mean luggage loading times. These data are generated from 100 realizations of the boarding process, including a reassignment of the luggage loading times for each passenger. An individual point represents the mean and the bar is the standard deviation of the resulting distribution of boarding times for the sample.

ated from 100 realizations of the boarding process (including a reassignment of the luggage loading times) and the bars represent the standard deviation of the resulting distribution of boarding times.

The best fit line for these data is

$$\text{Boarding time} = 62.2\tau + 306, \qquad (2)$$

where τ is the mean time required for passengers to load their luggage. The only regime where this relation deviates from linear is when τ is much less than the amount of time required by a passenger to walk the length of the airplane. Although this change from linear behavior is very small, it motivates the use of uniform weights for the data when fitting the line instead of inverse variances because there is less spread in the distribution of final boarding times for smaller values of τ, which results in an inappropriate fit for the points with larger τ. Thus, Eq. (2) is a least-squares fit as though the points had equal uncertainties.

Another modification that affects the time to board the airplane is to change the slope of the linear increase in the energies of the seats. The standard model has a slope of +0.25 eV per row. Figure 3 shows the effect of changing this slope while keeping all other parameters fixed. The resulting data are best fitted (using uniform weights) by
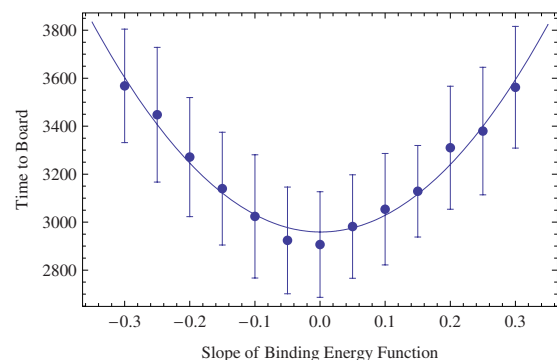

Fig. 3. The time required to board the airplane as a function of the slope of the linear increases in the energies of the different rows of seats. The bars represent the standard deviation of the distributions of the airplane boarding times. The minimum occurs near a slope of zero.

$$\text{Boarding time} = 2960 - 14.7s + 7100s^2, \qquad (3)$$

where s is the slope of the linear increase in the energies of the seats.

We see from Fig. 3 that the airplane boards faster if there is no increase (or nearly so) because, if there is some preference for one portion of the aircraft over another, the passengers will tend to crowd in that area as they prepare to sit. This crowding causes an inefficient use of the aisle where the passengers must stand to load their luggage. When there is no preference for the front or back of the aircraft, passengers will more readily spread themselves out along the length of the aisle and usually more passengers can load their luggage simultaneously. This result is similar to what would happen if the temperature of the passengers were increased; the preference for a particular portion of the cabin would become less important and the passengers would spread out more along the aisle.

Altering the energies along a row of seats (the cross section) also affects the boarding times. For example, if the sign of the energies in a row is changed, such that the middle seats are preferred over the window and aisle seats, then the mean is very near that of the standard model, changing from 3380 time units to 3450. In contrast, if each seat in a row has the same energy, then the airplane boards more slowly—the mean changing to 3800 time units. This increase in boarding time is due to the fact that, coupled with the linear trend, passengers are more likely to sit in the same portion of the aircraft. The effect of having different energies for the different seats in a row is that the airplane will tend to fill in phases with the aisle seats largely filling first (for the standard model), then the window seats, and then the middle seats. This scenario is again a more efficient use of the aisle of the aircraft because passengers stowing their luggage will be distributed along a larger portion of the length of the aircraft. Therefore, the passengers are able to load their luggage simultaneously rather than concentrating in a small region of the cabin and loading one at a time.

B. Comparison with other boarding methods

If all of the seats in the aircraft have the same energy (or if the high temperature limit is taken), the boarding is nearly equivalent to boarding passengers with assigned seats in a random order. The mean of an ensemble of realizations of this case is 2900 ± 220 time units, which is very close to the mean of 2846 time units reported in Ref. 4 for this case. Most of the difference can be accounted for by passengers who must reverse their direction and return to the front of the aircraft to find an empty seat, which would not occur with the assigned seat/random order model.

The overall boarding times that are obtained with the free-for-all model are roughly the same as the best boarding strategies with assigned seats[4] such as boarding window seats first, then middle seats, and then aisle seats. Although a direct quantitative comparison between these models is not warranted without proper calibration, the assumptions that are employed here give results that are in good qualitative agreement with those in Ref. 4 and in industry practice. Thus, the model described here can be used to qualitatively compare the benefits of different boarding procedures including free-for-all boarding and methods that employ assigned seats.

IV. PARTIALLY FILLED AIRCRAFT

If the flight is only partially filled, the model can predict the likely seating arrangements of the passengers after the boarding process has completed. It also can be used to predict the distribution of seated passengers at various times during the boarding of a full flight.

In this case it is more straightforward to consider the passengers as a Fermi gas (which is true for all passengers except adults carrying infants in their lap) and use the Fermi-Dirac distribution to determine the probability that a particular seat is occupied at a particular time. The occupancy of a seat n with energy ϵ (that is, the expected number of passengers in a seat with energy ϵ) is given by

$$n = \frac{1}{e^{(\epsilon - \mu)/T} + 1}, \qquad (4)$$

where μ is the chemical potential of the system. The chemical potential characterizes the energy of the seat that is likely to be occupied by the next passenger to board the airplane.

The chemical potential depends on both the number of passengers in the aircraft and the temperature. At zero temperature the first unoccupied seat gives the chemical potential and is equal to the Fermi energy. As the temperature increases, the chemical potential may be found by recognizing that the number of passengers is fixed and solving

$$N = \sum_i \frac{1}{e^{(\epsilon_i - \mu)/T} + 1} \qquad (5)$$

for μ at the desired temperature. The result of this calculation is not included here but the calculation is suitable for an undergraduate level computational problem. Figure 4 shows how the seating distribution changes from a degenerate arrangement to a random arrangement as the temperature increases.

Figure 5 shows the results of 100 realizations of the passenger boarding algorithm at various temperatures and compares the results to the corresponding prediction of the Fermi-Dirac occupancy distribution. Here, there are only 60 passengers, the mean boarding time is 15 time units (to lessen the computational expense), and the temperature is given by the stated values; otherwise the input parameters are those of the standard model described earlier. Of particular interest is that the Boltzmann-type decisions of the individual passengers coupled with the fact that passengers do not move once seated recovers the Fermi-Dirac distribution, the largest discrepancy being for very low temperatures where both distributions exhibit mathematical pathologies (that is, division by zero) and where the discrete nature of the allowed seat energies becomes more important (the chemical potential is generally not equal to one of the allowed energy values so the distribution crosses a value of $1/2$ between the energy levels).

Although the chemical potential is often introduced in statistical mechanics classes by allowing the environment to exchange particles with the system (the grand canonical ensemble), particle exchange is not a necessary condition for its use. A discussion of the relation between the chemical potential, temperature, energy, and number of particles in a system can be found in Ref. 8. Also, when particles are exchanged with the environment it is not necessary that the system and its environment be physically distinct (see, for example, Ref. 9). For the system here, the chemical potential is im-
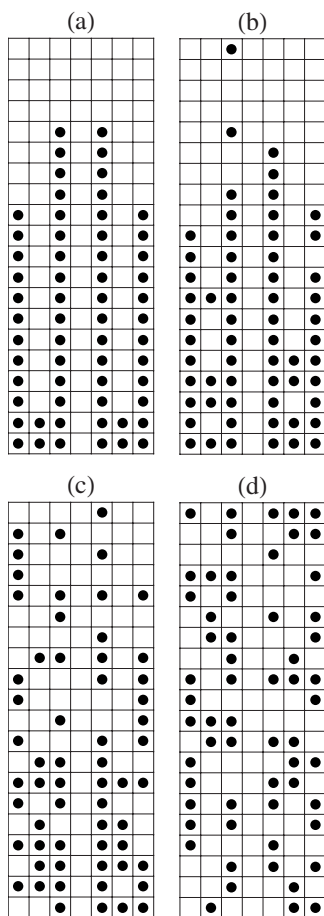
Fig. 4. Final seating arrangement for a half-full airplane for different values of the temperature. (a) Temperature$=|\bar{\epsilon}|/1000\simeq 0$; (b) Temperature $=|\bar{\epsilon}|/10$; (c) Temperature$=|\bar{\epsilon}|$; and (d) Temperature$=10|\bar{\epsilon}|$, where $\bar{\epsilon}$ is the mean energy of the seats in the standard airplane model. The dots represent seated passengers.
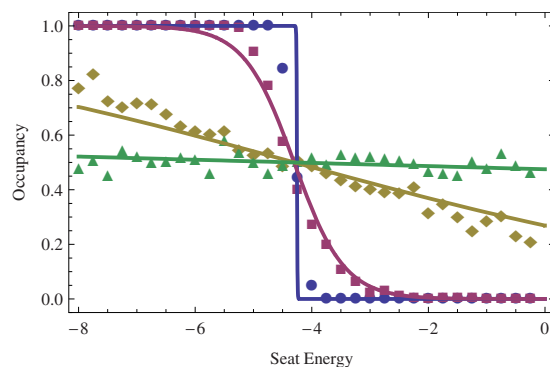

Fig. 5. Seat occupancies (the fraction of the time that a seat with a given energy is occupied) as a function of energy for 100 realizations of the boarding process with various values for the passenger temperatures. The temperature values are those of Fig. 4: $|\bar{\epsilon}|/1000\simeq 0$ (circles), $|\bar{\epsilon}|/10$ (squares), $|\bar{\epsilon}|$ (diamonds), and $10|\bar{\epsilon}|$ (triangles). The curves are predictions using the Fermi-Dirac distribution.

plicitly defined by the temperature and the number of particles. It characterizes the expected energy that the system would gain (or lose) if an additional passenger were added (or removed) regardless of whether or not it is physically possible to do so.

V. CONCLUSIONS

I have shown that the decisions made by the passengers in the free-for-all airplane boarding process may be effectively modeled using the principles of statistical mechanics. That is, the desirability of a particular seat can be expressed as an energy, and the probability of passenger choosing a particular seat can be estimated with Boltzmann statistics. We can also use Fermi-Dirac statistics to identify the probability that a particular seat will be occupied at any time during the boarding process.

Airplane boarding is a somewhat unusual application of statistical mechanics, but it might be a valuable example for an undergraduate course because it is novel and has been experienced by many students and faculty. Discussing such nonstandard applications of physical theory will likely prove valuable to students because most of them will ultimately find employment in industrial settings where "outside-the-box" thinking will be beneficial.

Some additional modifications to this model which would affect the boarding process, such as how the energies of a particular portion of the airplane change as a passenger sits in a given seat, how crowding in the aisle affects the local energies (or the temperature of the waiting passengers), and the increased time required by someone sitting in a window seat compared to an aisle seat might improve the model's accuracy. However, these changes would only serve to complicate the point of this paper that statistical physics is an appropriate framework for modeling free-for-all passenger boarding.

Another modification would be to allow different passengers to enter the aircraft at different temperatures (indicating different arrival histories at the gate). The effect of each passenger being at a different temperature upon entering the airplane would be to change their individual decisions regarding their seat selection. Because this model does not allow passengers to change seats, the seat selection process for most passengers is fixed by the available seats at the time that they enter the airplane (the exception is when two passengers select the same seat). Although some passengers may not care where they sit, others would. It would be interesting to identifying where the uniform temperature approximation might fail. Also, students could use this model to calculate, for example, the heat capacity of a partially filled airplane or identify the temperature of the passengers based on the distribution of the occupied seats.

In addition to passenger boarding, statistical mechanics might be the proper framework for modeling passengers leaving the airplane. Here, the passengers, who are essentially frozen in a crystalline structure, would sublimate and escape into the environment, the terminal. There is a "latent heat" associated with the transition that is characterized by the time required for passengers to retrieve their luggage.

An appropriate model for free-for-all boarding is valuable for the airline industry because the insights gained can be used to make decisions regarding a company's boarding policies and practices or where they should store the blankets and pillows (presumably above seats that are not likely to be occupied). Although sufficient detail is important, the basic principles that can be extracted from the model discussed here, such as the effect of the slope in the linear increase or the effect of more uniform energies in a particular row, can

be informative As seen in Fig. 3, the fastest boarding times occur when the boarding process is nearly random. This fact indicates that raising the temperature (or temperament) of the passengers would increase the speed of the boarding process. Although changing the passenger temperature might lie outside the control of an airline, some influence might exist to influence passengers to view all seats, or at least all the rows, equivalently. This possibility could further improve this already successful boarding strategy.

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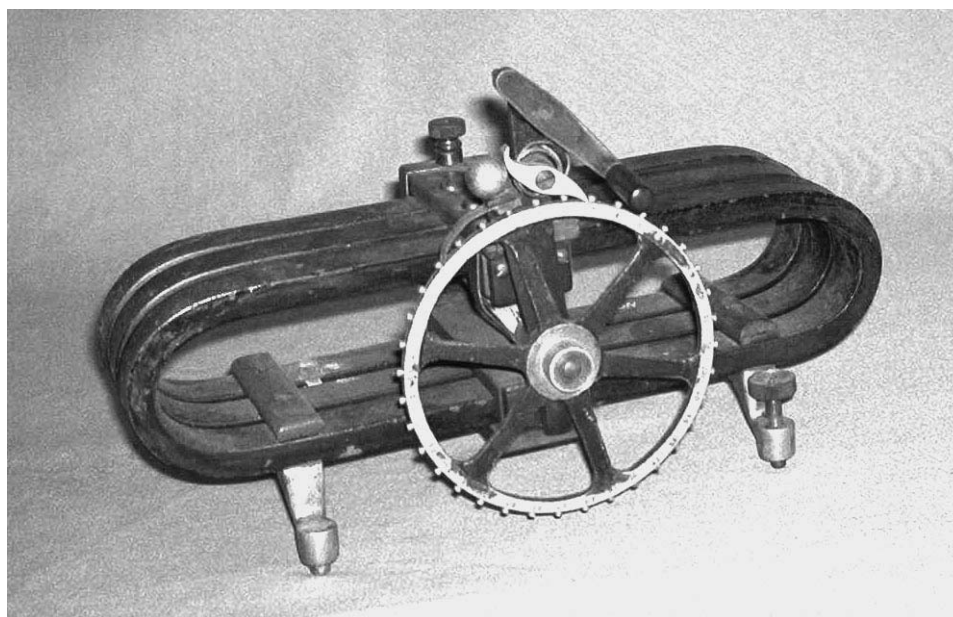
Dynamo Model. The flat, multi-turn coil between the magnet poles rotates 10° every time the ratchet is lifted. The resulting induced EMF is fed to a galvanometer, acting as a direct-reading voltmeter. Plotting the EMF as a function of the angular position of the coil gives a sinusoidal curve. This apparatus was made by the firm of Wm. Gaertner of Chicago, and is discussed in detail in Robert Andrews Millikan and John Mills, *A Short University Course in Electricity, Sound and Light* (Ginn and Company, Boston, 1908), pg. 134. It is very likely that Millikan designed the apparatus for the basic course in physics that he designed for Chicago ca. 1900 that was based primarily on experiment. The instrument is in the Greenslade Collection. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)