The rainbow

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The rainbow, with its arc spanning the sky, is a glorious sight which never ceases to amaze. Rainbows have always been a source of wonder [1], sometimes as a symbol of the gods, sometimes as an evil omen, and have inspired artists (who often get it wrong) and poets such as the Lakes poet, Wordsworth [2]:

> " My heart leaps up when I behold
> A rainbow in the sky… "

For physicists, part of the wonder of rainbows is the way that they literally illustrate so many aspects of the nature of light: most obviously, breaking up white light into the spectrum of colours. They also combine ray aspects of light, determining the angular size of the primary and secondary rainbows, and the wave nature of light which produces weak "supernumerary" bows, often visible inside the main primary. Some of the properties which we shall discuss in this article are apparent in the rainbow shown figure 1, in which we can just see the weaker secondary, some supernumeraries, and the property that the sky is light inside the rainbow and dark outside.

Rays of light

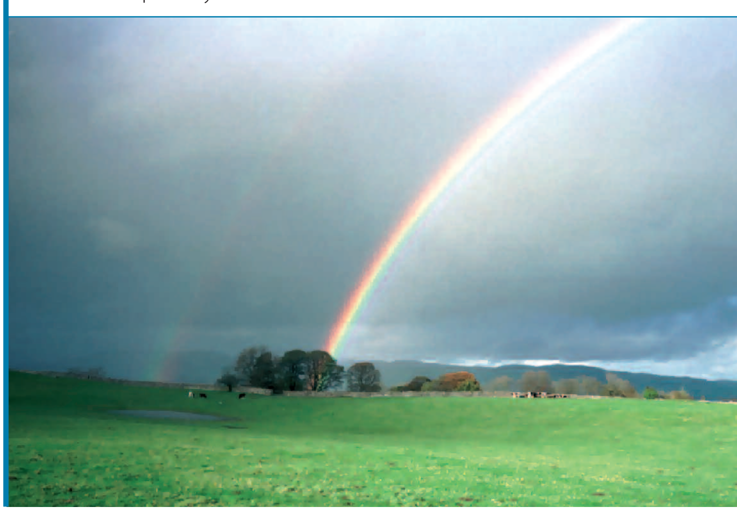
We all know of Newton's experiment with a prism, breaking up a beam of light into the spectrum of colours, and indeed it was Newton who explained the way that the wavelength-dependence of the refractive index of the raindrop produces the colours of the rainbow. But what is startling about the rainbow is the sheer intensity of the spectrum, a result of the way that the raindrop concentrates the different colours in different directions. And to understand this we must go back to Descartes [1], who in 1637 described the paths of rays of light through the raindrop, using the sine law for refraction which we know today as Snell's law. (It is not clear whether Descartes knew of the work of Snell; Newton incorrectly credited de Dominis with the explanation of the rainbow, and was rather casual about Descartes' real explanation [1].)

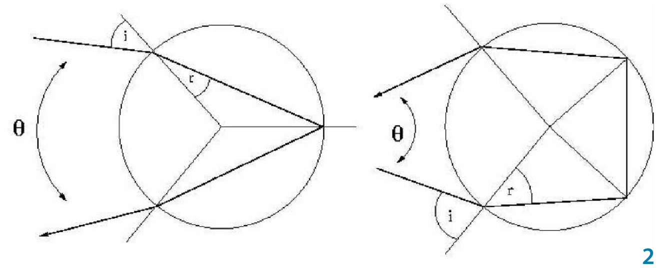
Rays of light are refracted as they enter the drop, and are then reflected inside the drop at the air-water interface – once in the formation of the intense primary rainbow, twice for the much weaker secondary bow – and refracted a second time as they leave the drop (figure 2). Further reflections are possible, but as we shall see, higher order rainbows are rather theoretical. Concentrating on the single reflection, figure 3 shows the paths of different rays incident on the drop: ray 1, incident towards the centre of the drop, is reflected back along its own path, but rays hitting higher up the drop are reflected with increasing angles between the incident and reflected rays. This continues up to a maximum angle of about 42° reached by ray 7, determined by simple geometry from the refractive index of water. Beyond this ray, appropriately called the Descartes ray, the angle decreases. We have a maximum – the rays near the Descartes ray emerge in almost the same direction – and the raindrop scatters most light at an angle of 42° to the incident light. Taking into account the variation of the refractive index n with wavelength – the dispersion – gives a rainbow angle of 42.2° for red light with $n = 1.332$ and 40.6° for violet with $n = 1.343$ (the refractive indices are taken from a very useful web site, [4]). For two reflections (figure 2), we have a minimum angle between the incident and scattered ray giving the secondary bow with an angular size of about 51°. Because the rays of light bend round on themselves, as we see from figure 2, red is on the inside of the secondary rainbow, with violet on the outside.

From this geometry, the primary and secondary bows appear as arcs making angles of 42° and 51° around the extension of a line from the sun and passing through the observer's head (figure 4). As a consequence, when the sun is high in the sky the rainbow may appear against the ground. All rainbows have the same angular size, whether they are due to a shower several miles away, or the spray from the garden hose above the lawn. We may ask where we see the rainbow – do we see it at the drops, perhaps? From figure 3 it seems that our eyes focus the rays contributing to the rainbow at infinity, and the rainbow has only a direction rather than a position. There is another point of view (literally), which also suggests that if the rainbow has a location, this is at infinity – if we, as observers, move, the rainbow moves with us. This means that a stereoscopic view of the rainbow, with our two eyes, or the rangefinder of an old-fashioned camera, will place the rainbow at infinity [1].

The ray theory of the rainbow can be neatly represented as polar plots of the scattered intensity in different directions after one, two, or more reflections. To calculate these, we combine Fresnel's formulae for the intensity of reflected and transmitted light [5] with simple ray geometry for different angles of scattering. We obtain the results shown in figure 5 for the scattered intensity for the two polarizations of light. In these figures, we have light coming horizontally from the left-hand side, incident on a raindrop at the centre; the left-hand diagram corresponds to light polarized with its electric field perpendicular to the plane of the diagram (s-polarisation), and on the right the electric field is in the plane (p-polarisation). The large lobe to the right of the drop in both figures represents light passing through the drop, refracted but without any reflection. The primary bow corresponds to the singularity in the scattering after one reflection at ±42° with respect to the incident light, showing up very strongly in s-polarisation, but much weaker in p-polarisation: the rainbow is *strongly polarised.* The figure shows very clearly how light undergoing one reflection

▼ **Fig. 1:** Rainbow above the Lake District fells: the much weaker secondary bow is just visible, with reversed colours. Several supernumerary bows, with alternating green and violet, can be seen inside the brilliant primary. The sky is distinctly darker outside the primary.





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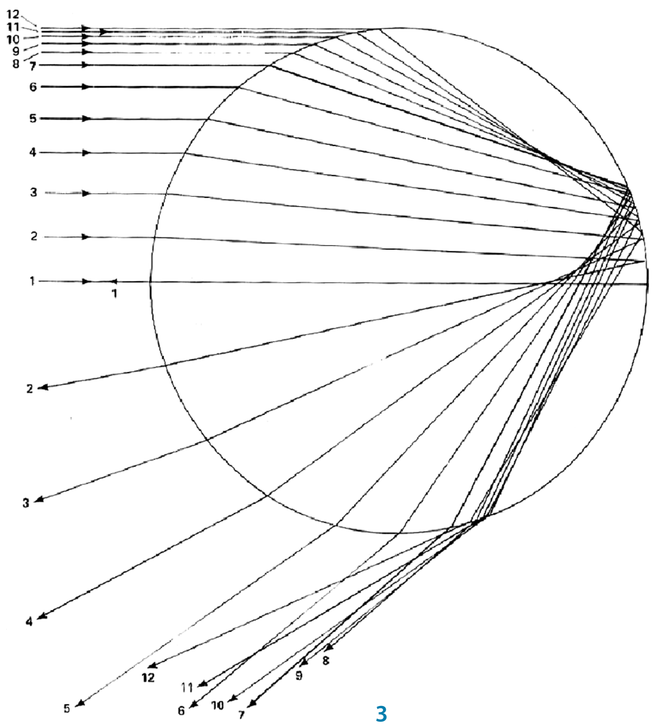

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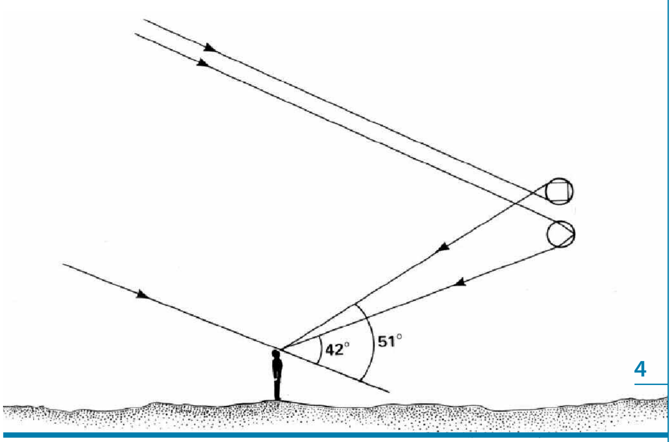

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▲ **Fig. 2:** Rays of sunlight refracted and reflected inside a raindrop: the left-hand figure, with one reflection, shows the paths which lead to the primary, and the right-hand figure, with two reflections, the secondary.

▲ **Fig. 3:** Parallel rays of light incident on the raindrop, with one reflection. Ray 7, the Descartes ray, emerges at the greatest angle, and rays pile up in this direction. (Figure from [3].)

▲ **Fig. 4:** Drops at an angle of 42° to the line from the sun, passing through the observer's head, scatter sunlight to form the primary bow. Those drops at an angle of 51° scatter sunlight to give the secondary. The diagram should be rotated about this line to form the complete bows. (Figure from [3].)

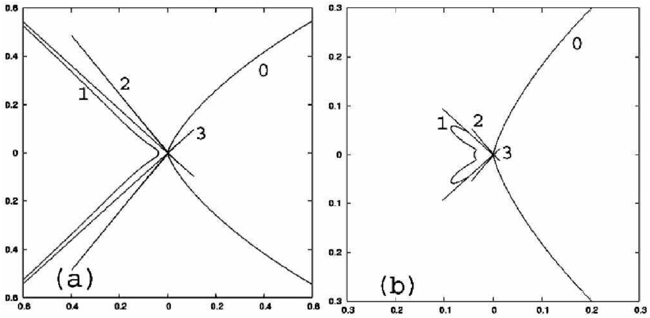
is back-scattered up to this maximum angle, and beyond ±42° no light is scattered. At ±51° there is a singularity for scattering after two reflections – the secondary bow – again strongly polarised. This singularity is the opposite way round from the primary bow singularity, with two reflections scattering some light beyond ±51°, though this does not show up on the scale of the plots.

These singularities in the scattering intensity I as a function of angle ϑ have the form $I \propto |\vartheta - \vartheta_0|^{-1/2}$, where ϑ_0 is the rainbow angle [7]. To derive this we must be clear what we mean by I – as usual in treating scattering, $I(\vartheta)d\vartheta$ is the intensity of light scattered into a small range of angles $d\vartheta$ (this is strictly speaking in two dimensions, and in three dimensions there is an extra factor of $\sin\vartheta$). It follows that together with the various reflection and transmission coefficients, I contains the term $|dy/d\vartheta|$, where y is the "impact parameter" of the ray of light incident on the raindrop, the height above ray 1 in figure 3. As $\vartheta(y)$ is parabolic at the maximum corresponding to the Descartes ray, $|dy/d\vartheta|$ varies like $|\vartheta - \vartheta_0|^{-1/2}$, hence the singularity in the scattering. Except for the primary rainbow in s-polarisation, the singularity is too narrow to be apparent in figure 5. We shall see shortly what happens to this singularity when the wave nature of light is considered.

The light scattered up to ±42° produces a bright sky inside the primary, the brightness increasing as the rainbow is approached. This can be seen quite clearly in figure 1. As no light is scattered for one or two reflections between ±42° and ±51°, the sky appears dark in this range of angles – this phenomenon goes under the name Alexander's band, perhaps reminiscent of some dance band from the 1920's, but in fact named for the Greek philosopher Alexander of Aphrodisias (the names get more and more unlikely) [1]. Are there higher order rainbows, corresponding to three, four and more reflections? From figure 5 we see that three reflections produce weak singularities at ±42° in the forward direction (for n = 1.332), and though it is not clear, four reflections produce still weaker singularities at ±43°, almost on top of the three reflections peak. In principle these correspond to higher order rainbows *around* the sun, but the fact that the singularities are extremely weak and lie inside the large forward scattering lobe means that they must be practically invisible. Bernoulli thought that the sharp-eyed lynx or eagle might discern these higher order bows [1] – alas, it seems very unlikely [3]. Quite frequently we have been told about "rainbows" visible around the sun – these are invariably ice crystal halos, and on such occasions we refer our friends to Greenler's "Rainbows, Halos, and Glories" [3]. Very higher order bows can in fact be measured in laser experiments [6].

Close to the singularity, the scattering in the primary bow is 96% *s*-polarised, and in the secondary 90%. This polarisation results from the fact that the angle at which the light is reflected inside the drops is close to the Brewster angle, at which the reflection coefficient for *p*-polarised light is zero [5]. Taking the refractive index for water for green light as n = 1.335, the Brewster angle is 37°, and the angle of reflection of the Descartes ray for the primary bow is 40°. At this angle the ratio of the *p* to *s* reflectivities is 0.03. For the secondary bow the angle of reflection is 45°, giving a ratio of the reflectivities of 0.26 at each of the two reflections. The *s*-polarisation of the rainbow corresponds to the electric field vector being tangential to the bow, and consequently it is interesting to view the bow through Polaroid. The segment of the rainbow which is tangential to the plane of polarisation of the Polaroid appears relatively brighter compared with the background sky, quite a striking effect [8].

The scattering intensity for single reflection in *p*-polarisation displays a curious angular variation within the primary rainbow singularity (figure 5). This is a consequence of the fact for a range



▲ **Fig. 5:** Polar diagram of scattering of light incident horizontally from the left on a raindrop at the centre of the figures.
(a) *s*-polarisation (electric field perpendicular to plane of figure);
(b) *p*-polarisation (electric field in plane). The numbers indicate the number of reflections. The refractive index is taken as $n = 1.332$ (corresponding to red light).

of angles inside the rainbow angle, there are two rays which contribute to the intensity at each $\vartheta(15° < \vartheta < 42°)$, each with its variation in reflectivity as the Brewster angle is approached. We doubt whether this scattering has been seen, as these lobes are very weak compared with the *s*-polarisation rainbow. We should note here that a scattering angle of 15°, the minimum angle for which two rays contribute, corresponds to the incident ray which grazes the raindrop, the only ray for which total internal reflection occurs.

Wave-fronts and waves

Rainbows produced by raindrops about 1mm in diameter or smaller often show several extra bands of colour, typically alternating green and violet [8], inside the primary (figure 6) – these are the supernumerary bows, produced by interference of the light waves [3]. The two rays which leave the drop for a range of directions inside the rainbow angle have different path-lengths, and interfere with one another. To obtain quantitative results, we first construct the geometrical wave-front, a surface perpendicular to the classical rays, on which the phase of the waves is constant. In the Huygens-Fresnel semi-classical approach, each point on the wave-front is considered as a source of spherical waves, which interfere with each other – this is not a full solution of the wave equations, but is a good approximation when the wavelength is small compared with the dimensions of the object scattering the light [5].

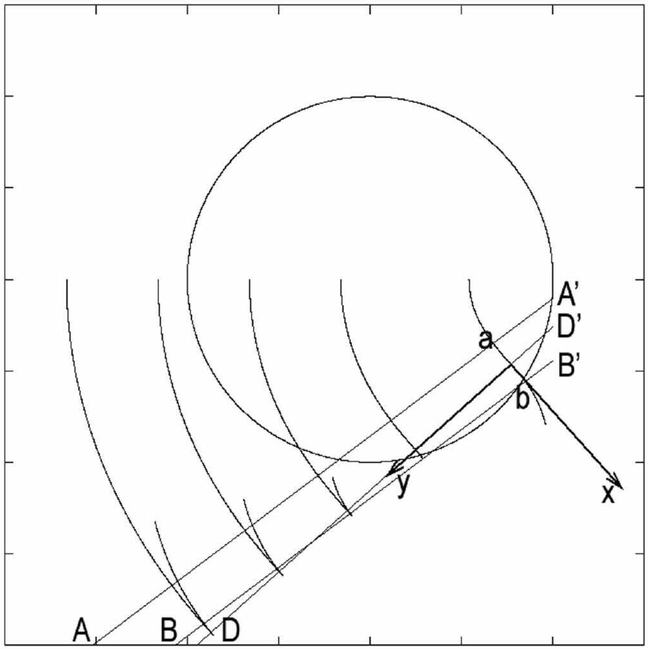
Geometrical wave-fronts corresponding to light leaving the raindrop are shown in figure 7, the different curves corresponding to different phases of the waves, or different path-lengths the waves travel. The fronts which intersect the drop are, in fact, *virtual*, formed by extending the actual wave-fronts backwards through air rather than through the drop. They correspond to the rays leaving the drop extended backwards as straight lines. What we immediately notice are the cusps in the wave-fronts, which lie on the Descartes ray (except close to the drop), and trace out a caustic. (The bright patterns on the surface of the breakfast cup of tea reflected from the kitchen spotlight are a familiar example of caustics.) It is the interference between the wave-fronts on either side of the cusps which give rise to the supernumeraries.

The next step is to use one of the wave-fronts as a source of waves, to find the intensity as a function of scattering angle. This calculation was first performed by Airy, described in a classic paper published in 1838 [9], following Young's realization in 1804 that interference causes the supernumeraries [10]. A wave-front which we may use is the right-hand one in figure 7 and we use the axes shown on the diagram, with the y-axis in the direction of the Descartes ray. Then the Fresnel formula tells us that the amplitude of the diffracted ray at a large distance from the drop, at angle ϑ to the Descartes ray, is proportional to the integral along the wave-front, $\int \psi \exp[i2\pi(x\sin\vartheta + y(x)\cos\vartheta)/\lambda]\sqrt{1+(dy/dx)^2}\,dx$. Here, λ is the wavelength of the light, and ψ is the amplitude of the electromagnetic field over the wave-front, whose equation is $y(x)$ [5]. The exponential gives the phase of the contribution over the wave-front, and the square root gives the length of the element of the wave-front; positive ϑ corresponds to scattering outside the Descartes ray. Why do we choose this particular wave-front for the Fresnel integral? We want to avoid the singularity of the later wave-fronts with cusps, and the wave-front where the cusp just finishes has a large amplitude right at the end, a point through which many classical rays pass. Moreover, this virtual wave-front has a simple analytic form in the region which mainly contributes to the integral, $y \approx \alpha x^3$ – a result which Airy used to evaluate the integral in terms of his famous function.

In evaluating the diffraction integral, Airy assumed that the amplitude ψ was constant over the wave-front; neglecting the square root for the length of wave-front, the diffraction amplitude for small angle ϑ is given by the Airy integral $\Psi(\vartheta) \propto \int_0^\infty \cos[2\pi(x\vartheta + \alpha x^3)/\lambda]dx$. This may be expressed in terms of the Airy function Ai, $\Psi(\vartheta) \propto \mathrm{Ai}(\vartheta/y)$, with $y = (3\alpha[\frac{\lambda}{2\pi}]^2)^{1/3}$ – Airy evaluated his function by hand, but it is now immediately available in computer packages. The cubic coefficient α depends on the refractive index, and for $n = 1.335$ it is given by $\alpha = 1.62/R^2$, where R is the radius of the raindrop [11]. The resulting diffraction intensity is shown in figure 8, for light of wavelength $\lambda = 500$ nm scattered by a drop of radius 0.5 mm, plotted as a function of angle from the Descartes ray. We see that the singularity in ray theory is replaced by a finite peak, the primary bow in diffraction theory, with a maximum at about 1/4° inside the Descartes ray. The subsequent peaks constitute the supernumerary bows.

The integrand in the Airy integral oscillates very rapidly as x varies over the wave-front, when angle ϑ is negative, except where the wave-front is perpendicular to the direction in which the amplitude is evaluated. For a range of angles inside the Descartes ray there are two points at which the wave-front is perpendicular, corresponding to two classical rays travelling in this direction. Rays AA' and BB' in figure 7 are two such rays, travelling at $\vartheta \approx -5°$ from the Descartes ray, and at the points of intersection with the wave-front a' and b' the front is perpendicular. Around these two points,

► **Fig. 6:** Rainbow above Penyghent, North Yorkshire, with several supernumerary bows inside the primary. The secondary bow is barely visible.

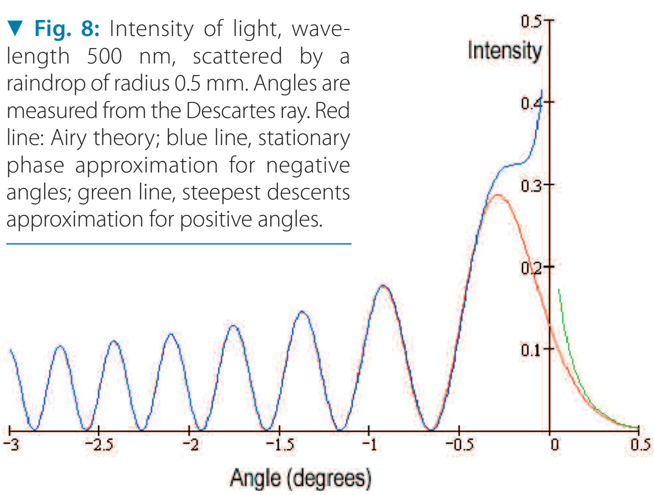


◀ **Fig. 7:** Wave-fronts of rays leaving the raindrop, $n = 1.335$, with rays incident on the drop as in figure 3. The wave-fronts intersecting the drop are "virtual", extended backwards through air. DD' is the Descartes ray leaving the drop, and AA', BB' are the two rays leaving the drop at a scattering angle inside the rainbow angle, which can interfere. All these rays are extended backwards through air. Local axes x and y are for the Fresnel integration over the wave-front.

the phase in the cosine integrand varies very little, and this dominates the integral. The method of stationary phase shows how the integral may be determined in terms of the contributions from these two regions [12], and the result is that the amplitude is given approximately by $\Psi(\vartheta) \approx \cos[\pi/4 - \frac{2}{3}(|\vartheta|/y)^{3/2}]/[\sqrt{\pi}(|\vartheta|/y)^{1/4}]$, ϑ negative. The corresponding intensity is shown by the dark blue line in figure 8, and we see that apart from a very small range of angles as we approach the Descartes ray, this approximation is amazingly accurate. The method of stationary phase gives the interference pattern from the two rays travelling in direction ϑ, producing supernumeraries. Something surprising, which does not come out of a straightforward interference picture, is the phase shift of $\pi/4$ found in the stationary phase result given above, shifting the first maximum and the supernumeraries. Such phase shifts, and the general study of wave forms near ray caustics and singularities, are a very active topic of current research, and there are many papers by Berry and co-workers in this area [10].

For positive angles, the method of steepest descents [12] may be used to obtain an approximation to the Airy integral (this method is the same as stationary phase if we go into the complex plane), giving $\Psi(\vartheta) \approx \exp[-\frac{2}{3}(\vartheta/y)^{3/2}]/[2\sqrt{\pi}(|\vartheta|/y)^{1/4}]$, ϑ positive. This gives the intensity shown by the green line in figure 8, again remarkably accurate beyond $\vartheta \approx +\frac{1}{4}°$. The Airy function crops up again and again in physics, in particular as the solution of the Schrödinger equation in a linear potential, and these approximate expressions for the Airy function (actually the first terms in an asymptotic expansion) are important in the mathematical analysis of this equation [12].

▼ **Fig. 8:** Intensity of light, wavelength 500 nm, scattered by a raindrop of radius 0.5 mm. Angles are measured from the Descartes ray. Red line: Airy theory; blue line, stationary phase approximation for negative angles; green line, steepest descents approximation for positive angles.

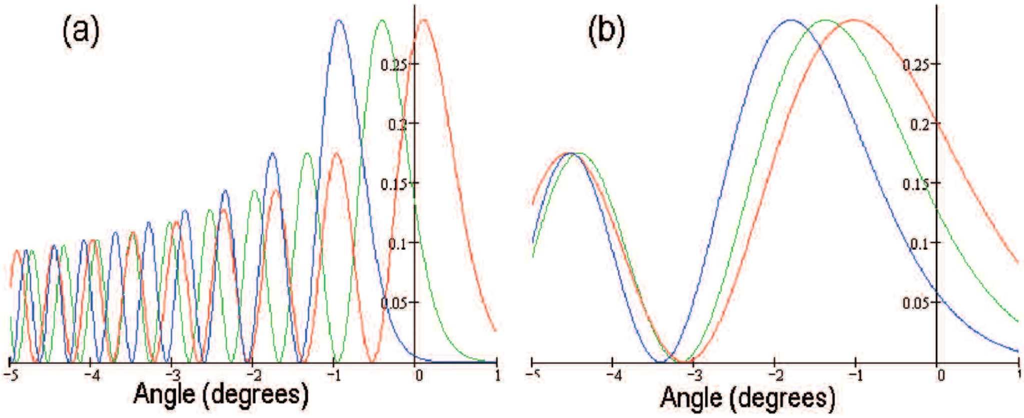


From supernumeraries to fog-bows

It is remarkable that interference fringes show up as supernumeraries in the rainbow, when we consider that the light waves are being scattered by raindrops 1000 times larger – we are used to interference effects in scattering over length scales comparable with the wave-length of light. Supernumeraries are even more remarkable when we consider that this effect of light waves appears in a large-scale phenomenon, traversing the sky! The appearance of these supernumeraries depends on the size of the raindrops scattering the light, and we can explore this using the theory described above.

The intensity of light scattered by the raindrop at varying angles in the primary bow is shown in figures 9a and 9b, for raindrops of radius 0.3 mm and 0.05 mm respectively. On each figure, the three curves correspond to red, green and blue light. For raindrops of radius 0.3 mm (figure 9a) we see that red and green give good strong principal peaks, with the first supernumerary of red overlapping with the first peak of blue – these constitute the primary bow, with the overlapping red and blue enhancing the violet. Inside the primary we see supernumeraries, which are initially alternating green and red + blue. It is a little dangerous to go from this figure directly to the actual appearance of the rainbow; for this we should consider the scattering of the whole spectrum of visible light, and then use the trichromatic nature of colour vision to work out the appearance of the rainbow [1]. But figure 9a, taken at face value, is consistent with the supernumeraries described by Minnaert in his classic book [8], as alternating "violet-pink" and green (the violet-pink comes from the superposition of the red and blue peaks). We can make out several supernumeraries just inside the primary bow in figure 6, and at least on the original photograph these are seen to be alternating violet and green. In nature, only a few supernumeraries are ever visible – for one thing, raindrops are unlikely to have a uniform size, and this varies the phase and wavelength of the oscillations. Moreover, the sun has an angular diameter of about ½°, again smearing out the supernumeraries. The supernumeraries will be less apparent with bigger drops, for which they are more closely spaced, and hence more likely to be lost.

With very small drops of radius 0.05mm, mist rather than rain, the principal peaks broaden and overlap completely (figure 9b). The colours of the primary bow are completely smeared out – we observe in this case a white rainbow, or fog-bow, the figure suggesting that the supernumeraries should be quite strong. Some photographs in the literature [1] and on the web do show supernumeraries with fog-bows, but the only time that one of us observed a complete white rainbow, only the principal bow was visible, and this rather faint (figure 10). As there is no mist or fog (the visibility of the snow-covered Helvellyn ridge is excellent), this must be a cloud-bow, formed by droplets in the clouds.



◀ **Fig. 9:** Intensity of light scattered by raindrops of (a) radius 0.3 mm and (b) 0.05 mm. The different curves correspond to red, green and blue light, and angles are measured from the Descartes ray for green light.

Beyond the rainbow

The theory of the rainbow which we have described explains everything in terms of classical rays of light, even Airy theory boiling down to interference between two rays of light travelling in the same direction. However, these theories are not the whole story of light scattering by water droplets, and they cannot begin to explain another phenomenon involving light scattering from mist – the glory, and the Brocken spectre [3,8]. The full scattering theory of light by a dielectric sphere – Mie theory – is needed to understand the glory, and computer programs are available on the web to explore this [4]. The beauty of the physics of rainbows is that so much can be understood in terms of rays and simple wave theory: rainbows open our eyes to some of the fundamental properties of light. ■

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Owen Davies and **Jeff Wannell** both obtained the MPhys Degree in Cardiff in 2001, and this article is based on their final year project dissertation, supervised by **John Inglesfield**. Jeff has since become a banker. Owen obtained a PhD in Cardiff in 2005, on the theory of DNA, and now works in the City of London.

John Inglesfield received his PhD in 1970 from Cambridge, and is a professor of physics in Cardiff. His research interests are the theory of surface electronic structure, and most recently DNA and photonics.

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▼ **Fig. 10:** Cloud-bow on a winter's day above the Grasmere fells, Lake District. As an aid to the eye (the bow is faint), it appears as an almost complete semicircle, spanning the double photograph.