HOMEWORK N. 1, MATHEMATICAL LOGIC FOR COMPUTER SCIENCE 2019/2020

ASSIGNMENT: DO 3 EXERCISES CHOSEN FROM GROUP 1 AND 3 EXERCISES CHOSEN FROM GROUP 2.
DEADLINE: 28 MARCH 2020.

1. Group 1

Exercise 1 Let R be a predicate symbol of arity 1. Show that $\exists x(R(x) \to \forall y R(y))$ is logically valid.

Exercise 2 Show that the following sentence is unsatisfiable, where S is any formula with two free variables: $\exists x \forall y(S(x,y) \leftrightarrow \neg S(y,y))$.

Exercise 3 Prove or disprove: for any formulas G and F,

$$F \models G \text{ if and only if } \models (F \to G).$$

Exercise 4 Is the following formula logically valid for any formula F and any term t?

$$\forall x F(x) \to F(t).$$

If not, give an example of a formula F, a structure $\mathfrak{A}$ and an assignment α witnessing this fact.

Exercise 5 Is the following implication true for any choice of formulas? Is it true for sentences?
If

$$\text{If } \models G \text{ then } \models F,$$

then

$$\models (G \to F).$$

Recall that for a formula G, $\models G$ means that for all structures $\mathfrak{A}$, for all assignments α in $\mathfrak{A}$, $\mathfrak{A} \models G[\alpha]$.

Exercise 6 Let F be a formula with no quantifiers, function symbols, or constants. Prove the following two statements.

(1) A closed formula of the form $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m F$ with $m \geq 0$ and $n \geq 1$ is valid if and only if it is true in every non-empty structure with $\leq n$ elements.
(2) A closed formula (a sentence) of the form $\exists y_1 \ldots \exists y_m F$ is valid if and only if it is true in every structure with 1 element.

Can we draw some conclusion about the decidability of the validity of formulas in item (1)?

Exercise 7 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures for the same predicative language with no constant or function symbols. Prove that if f is a bijection from A to B such that, for all atomic formulas G the following holds

$$\mathfrak{A} \models G(x_1,\ldots,x_n)\left[\begin{pmatrix} x_1,\ldots,x_n \\ a_1,\ldots,a_n \end{pmatrix}\right] \text{ if and only if } \mathfrak{B} \models G(x_1,\ldots,x_n)\left[\begin{pmatrix} x_1,\ldots,x_n \\ f(a_1),\ldots,f(a_n) \end{pmatrix}\right],$$

then $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences.

(Hint: Induction of sentences is not a viable option (subformulas of a sentence may not be sentences). So typically one proves a result about formulas. In this case one would prove by induction on formulas the following: For any formula $F(x_1,\ldots,x_n)$, for any $(a_1,\ldots,a_n) \in A^n$, $\mathfrak{A} \models F(x_1,\ldots,x_n)[a_1,\ldots,a_n]$ if and only if $\mathfrak{B} \models F(x_1,\ldots,x_n)[f(a_1),\ldots,f(a_n)]$. The result for sentences then follows.)

Exercise 8 Consider the empty language (only logical symbols, including $=$, but no further relation, function or constant symbols).

Can you write a sentence that is true only in finite models? Can you write a sentence that is true only in infinite models? Can you write a set of sentences X such that all models satisfying X are infinite?

Does any of the answers change if you use the language $\{<\}$ (one binary relation symbol)?

Exercise 9 In the language $\mathcal{L} = \{<\}$ of **DLO**, write a sentence that distinguishes $(\mathbf{N}, <)$ from $(\mathbf{Q}, <)$ i.e., that is true in one structure but not in the other.

Exercise 10 Assume that the validity of a sentence in a fixed finite model can be algorithmically decided (this is indeed the case). Consider the set V of logically valid sentences (in a fixed first-order language) and the set U of all unsatisfiable sentences. Is there a decision algorithm (i.e. a deterministic 0,1 valued procedure) that separates V from U in the following sense: V is a subset of the inputs on which the algorithm A returns 1 and U is a subset of the inputs on which the algorithm A returns 0?

(If your answer is yes, describe (informally) an algorithm that separates the two sets; else if your answer is no give an informal proof.)

Exercise 11 Let T be a theory (i.e., a set of sentences) in some language $\mathcal{L}$. Let $F(x)$ be a formula in the language $\mathcal{L}$. Let c be a constant symbol not present in the language $\mathcal{L}$. Let $\mathcal{L}'$ be the language $\mathcal{L} \cup \{c\}$. Show that

$$T \models \forall x F(x) \text{ if and only if } T \models F(c).$$

Note that in the left-hand side we are dealing with structures adequate for $\mathcal{L}$ while on the right-hand side we are dealing with structures adequate for $\mathcal{L}'$.

2. Group 2

Definition $\mathfrak{B}$ is a **substructure** of $\mathfrak{A}$ if: $B \subseteq A$; for every constant symbol c, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$, every relation $R^{\mathfrak{B}}$ (resp. function $f^{\mathfrak{B}}$) is the restriction of $R^{\mathfrak{A}}$ (resp. $f^{\mathfrak{B}}$) to B.

Exercise 1 i) If $\mathfrak{B}$ is a substructure of $\mathfrak{A}$, then for any atomic formula $F(x_1, \dots, x_n)$, for all $b_1, \dots, b_n$ in B, $\mathfrak{B} \models F[b_1, \dots, b_n]$ iff $\mathfrak{A} \models F[b_1, \dots, b_n]$. ii) Let T be a set of purely universal sentences (i.e. sentences starting with universal quantifiers followed by an atomic formula). If $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ and $\mathfrak{A} \models T$ then also $\mathfrak{B} \models T$.

Definition $\mathfrak{B}$ substructure of $\mathfrak{A}$ is called **elementary** if for all formulas $F(x_1, \dots, x_n)$ for all $b_1, \dots, b_n$ in B, $\mathfrak{A} \models F[b_1, \dots, b_n]$ iff $\mathfrak{B} \models F[b_1, \dots, b_n]$. That is, $\mathfrak{A}$ and $\mathfrak{B}$ agree on elements of B.

Exercise 2 Show that **DLO** has the Extension Property. Similarly for the Random Graph Theory.

Exercise 3 Show that the structure $\mathcal{Q} = (\mathbb{Q}, +, \times, 0, 1)$ is a substructure of the structure $\mathcal{R} = (\mathbb{R}, +, \times, 0, 1)$ but not an elementary substructure.

Exercise 4 Prove that the structure $\mathfrak{A}$ defined (in Handout 5) by closing a subset $X \subseteq B$ of an arbitrary model of **DLO** under all Skolem functions is an elementary substructure of $\mathfrak{B}$.

(Hint: Show that it satisfies the Existential Closure Property defined in Tarski-Vaught Test).

Exercise 5 Prove the following: Let $\mathfrak{A}$ be a finite structure for a finite relational language $\mathcal{L}$. Then there exists a sentence $S_{\mathfrak{A}}$ in $\mathcal{L}$ such that the following holds: for any finite structure $\mathfrak{B}$ for language $\mathcal{L}$

$$\mathfrak{B} \models S_{\mathfrak{A}} \Leftrightarrow \mathfrak{A} \simeq \mathfrak{B}.$$

In other words, the isomorphism type of a finite structure can be completely described by a single sentence in the language. For simplicity you can consider the language of graphs.

Exercise 6 Prove that the following structures are not isomorphic:

(1) $(\mathbb{N}, +, \times, 0, 1, <)$ and $(\mathbb{Q}, +, \times, 0, 1, <)$
(2) $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$
(3) $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$.

(Hint: in some case you can use the fact that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic then they satisfy the same sentences).

Exercise 7 A theory T has property M if the following holds: For $\mathfrak{A}$ and $\mathfrak{B}$ models of T, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ then $\mathfrak{A}$ is also an elementary substructure of $\mathfrak{B}$. Prove that if a theory admits Quantifier Elimination then it has property M.

Exercise 8 Apply the Quantifier Elimination procedure for the theory DLO to the following sentence E:

$$\exists x \exists y \exists z \forall u (x < y \wedge x < z \wedge z < y \wedge (u = z \vee u < y \vee u = x)).$$

(You can assume that the language contains constants $\top$ and $\perp$ for true and false, respectively.) Decide if $\mathsf{DLO} \models E$ or $\mathsf{DLO} \models \neg E$.