HOMEWORK N. 4, MATHEMATICAL LOGIC FOR COMPUTER SCIENCE 2019/2020

DEADLINE: MAY 29 2020.

Choose two exercises from each group

1. Group 1

Exercise 1 Prove that ω-consistency implies consistency. Prove that the reverse implication is false.

(Hint: consider the theory $\mathsf{PA} + \neg Con(\mathsf{PA})$ or $\mathsf{MA} + \neg Con(\mathsf{MA})$)

Exercise 2 Consider Gödel's unprovable sentence G for some theory $T \supseteq \mathbf{MA}$, satisfying:

$$T \vdash G \leftrightarrow \forall x \neg Proof_T(x, \overline{code(G)}).$$

Prove that if T is consistent then $T + \{\neg G\}$ is consistent but not ω-consistent.

Exercise 3 Apply the fix-point theorem to obtain sentences in the language of arithmetic that express the following:

(1) I am decidable in MA (i.e. either provable or disprovable).
(2) I am undecidable in MA.
(3) I am not refutable in MA (i.e. I am consistent with MA).
(4) I am provable in MA.

For each of the above say as much as possible of the following questions: Is the sentence provable, refutable or undecidable? Is the sentence true in the standard model?

Exercise 4 A theory T is called 1-consistent if the following holds: For every formula $R(x)$ of type Δ_0 (i.e., with bounded quantifiers only), if for all $n \in \mathbf{N}$ we have $T \vdash R(\overline{n})$, then $T \nvdash \exists x \neg R(x)$. Let T be a theory in the language of arithmetic such that for every sentence E of type Σ_1, if $\mathbf{N} \models E$ then $T \vdash E$. Prove that, for every sentence A, $T \cup \{A\}$ is 1-consistent if and only if for every sentence E of type Π_1 true in $\mathbf{N}$, $T \cup \{A, E\}$ is consistent. (A sentence of type Π_1 is of the form $\forall x_1 \dots \forall x_k H$ where H contains only bounded quantifiers. Note that the negation of a Σ_1 formula is Π_1, and viceversa).

Exercise 5 Recall the Rosser sentence (see handout notes).

$$E := (\forall y)(F(\overline{m}, y) \to (\exists z)(z \leq y \wedge H(\overline{m}, z))),$$

where m is the code of the formula $(\forall y)(F(x, y) \to (\exists z)(z \leq y \wedge H(x, z)))$ F represents the relations R and H represents the relation S introduced in the handouts. Show that if $T \supseteq \mathsf{MA}$ is consistent then $T \nvdash E$ and $T \nvdash \neg E$. One can prove that if T is consistent then $\neg E$ is not provable.

Exercise 6 Modify our proof that $\{n : \forall x \neg T(n, n, x)\}$ in in $\Pi_1 - \Sigma_1$ and $\{n : \exists x T(n, n, x)\}$ is in $\Sigma_1 - \Pi_1$ to prove that there is a set in $\Sigma_2 - \Pi_2$ and that there is a set in $\Pi_2 - \Sigma_2$ (don't use what we proved in class about Fin and Tot!).

(Hint: You can assume the following Normal Form Theorem for relations: for each $k \geq 1$ there is a computable predicate $T(i, a_1, \dots, a_k, x)$ such that if $R(n_1, \dots, n_k, x)$ is a computable relation there is a number e such that for all $n_1, \dots, n_k$ in $\mathbf{N}$: $\exists x R(n_1, \dots, n_k, x)$ if and only if $\exists x T(e, n_1, \dots, n_k, x)$.)

Exercise 7 Deduce from Tarski's Theorem (on the non-representability of the theorem of a theory within a theory - proved in class) the following version of Godel's Theorem: If T is a consistent decidable set of sentence extending MA then T is incomplete.

(Hint: use the Representation Theorem for MA as well).

2. Group 2

Exercise 1 Prove that $\Box\bot \leftrightarrow \Box\neg\Box\bot$ is a theorem of **GL**. You can either write a formal proof in **GL** or show that it holds at every point x of any frame $(X, \prec)$ with respect to any model over the frame.

Exercise 2 Prove the following: A frame $(X, \prec)$ is such that the accessibility relation $\prec$ is reflexive if and only if the formula $\Box E \to E$ is valid in $(X, \prec)$.

Exercise 3 Prove that the formula $\Box E \leftrightarrow \neg\Diamond\neg E$ is true in all frames under all assignments.

Exercise 4 Is the following formula valid (i.e. true in all frames for all assignments): $(\Box E \to \Box F) \to \Box(E \to F)$? Prove or disprove.

Exercise 5 Is the following formula valid (i.e. true in all frames for all assignments): $\Box E \to \Diamond E$? If your answer is negative formulate a condition on a frame $(X, \prec)$ sufficient to satisfy the formula.

Exercise 6 Describe a frame $(X, \prec)$ and an assignment on it that falsifies the formula $(\Diamond E \wedge \Diamond F) \to \Diamond(E \wedge F)$.

Exercise 7 Describe a frame $(X, \prec)$ and an assignment on it that falsifies the formula $(\Box\Diamond E \wedge \Box\Diamond F) \to \Box\Diamond(E \wedge F)$.