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sabato 9 maggio 2020 09:11

COMPACTNESS

2) Let T_1 and T_2 theories.
for each structure $\mathcal{A}$: $\mathcal{A} \models T_1 \iff \mathcal{A} \not\models T_2 \implies T_1, T_2$ are finitely axiomatizable
① HYPOTHESIS

PROOF: $\implies$ (BY WAY OF CONTRADICTION)
Assume the contrary T_1 is NOT finitely axiomatizable, T_1 is not equivalent to any of its subtheories T_1'
Then for every subtheories T_1' there's some some structure $\mathcal{B}$ s.t. $\mathcal{B} \not\models T_1$ and $\mathcal{B} \models T_1'$
By ① $\mathcal{B} \models T_2$, so $\mathcal{B} \models (T_1' \cup T_2)$, now let $P = T_1' \cup T_2$, every finite subtheories of P is contained in $T_1' \cup T_2$ for some $T_1' \subset T_1$.
By COMPACTNESS, P has model $\mathcal{C} \implies \mathcal{C} \models T_1$ and $\mathcal{C} \models T_2$ (CONTRADICTION)

5) Prove by a Compactness argument that the notion of being a nice relation is NOT axiomatizable
PROOF:
Suppose we've T s.t. $\mathcal{A} \models T \iff \mathcal{A}$ admits nice
Let a sentence $A_m := \exists x_1 \dots \exists x_m \left(\bigwedge_{i \neq j} x_i \neq x_j \wedge (x_m < x_{m-1} < \dots x_1) \right)$ informally it states: "there exist a finite decreasing sequence of length m"
Let $T' = T \cup \{A_m \mid m \in \mathbb{N}\}$

$$\mathcal{A} \models T' \begin{cases} \mathcal{A} \models T, \ \mathcal{A} \text{ has no infinite sequence} \\ \text{for all } m \in \mathbb{N} \ \mathcal{A} \models A_m, \ \mathcal{A} \text{ has decreasing sequence of length } m \implies \mathcal{A} \text{ has an infinite decreasing sequence} \end{cases}$$

CLAIM: T' is SAT
By COMPACTNESS, for every subtheory $X \subseteq T'$ is SAT
/X contain some sentences of T union some $A_{m_1} \dots A_{m_q}$
We can conclude that the model of X is a structure with finite decreasing sequence of length $m = \max\{m_1 \dots m_q\}$

But this bring us to state that T' is SAT, WHICH IS AN ABSURD.

ULTRAFILTERS

3) Let $\mathcal{U}$ ultrafilter s.t. for no $m \in \mathbb{N}$ $\mathcal{U} = \{X \subseteq \mathbb{N} : m \in X\}$ ($\mathcal{U}$ is not a principal ultrafilter)
Let $Cof(\mathbb{N}) = \{A^c \subseteq \mathbb{N} : A \text{ is finite}\}$

$$\mathcal{U} \implies Cof(\mathbb{N}) \subset \mathcal{U}$$

PROOF $\implies$
Let $y \in \mathbb{N}$, since $\mathcal{U}$ is not a principal ultrafilter, the set $\{y\}$, $\{y\}^c \in \mathcal{U}$ and $\{y\} \notin \mathcal{U}$ (MAXIMALITY)
Thus for all $y \in X$. Now we can define a subset H of $\mathbb{N}$

$$H^c = \bigcap_{y \in \mathbb{N}} \{y\}^c$$

Every filter is closed under finite intersection, then $H^c \in \mathcal{U}$
$Cof(\mathbb{N}) = \{H^c \subseteq \mathbb{N} : H \text{ is finite}\}$

then $Cof(\mathbb{N}) \subset \mathcal{U}$

5) $\mathcal{U}$ ultrafilter contains a finite set not empty (since $\mathcal{U}$ is an ultrafilter) ① HYP
$\implies$ for some $m \in \mathbb{N}$ s.t. $\mathcal{U} = \{X \subseteq \mathbb{N} : m \in X\}$
PROOF:

By ① pick $A = \bigcap \mathcal{U}$ since $\mathcal{U}$ it's closed under finite intersection, A is a finite set and it's min of $\mathcal{U}$, ②
1) A must not be equal $\emptyset$
2) We want show that A is a singleton and is equal to $\{m\}$
By way of contradiction: A isn't a singleton $\implies A = B \cup C$
Since A is the minimum of $\mathcal{U}$ (see ②) neither B nor $C \in \mathcal{U}$ ③
Now using MAXIMALITY $B^c \in \mathcal{U}$, using the closed intersection
$A \cap B^c = C \in \mathcal{U}$, by ③ is a CONTRADICTION!

then $A = \{m\}$, $\mathcal{U} = \{X \subseteq X : \{m\} \subseteq X\} = \{X \subseteq X : m \in X\}$