HOMEWORK N. 2, MATHEMATICAL LOGIC FOR COMPUTER SCIENCE 20199/2020

ASSIGNMENT: DO 3 EXERCISES.
DEADLINE: 19 APRIL 2020.

Exercise 1 Consider the following two structures for the language of graphs. $\mathfrak{G}_1 = (V_1, E_1)$:

$$V_1 = \{1, 2, 3, 4\}; \;\; E_1 = \{(1, 2), (2, 3), (3, 4), (4, 1)\}\}$$

and $\mathfrak{G}_2 = (V_2, E_2)$

$$V_2 = \{a_1, a_2, a_3, a_4, a_5, a_6\},$$
$$E_2 = \{(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5), (a_5, a_6), (a_5, a_2), (a_1, a_4), (a_6, a_3), (a_6, a_1)\}$$

Show that the Duplicator doesn't win the 3-moves game on $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Exhibit two different sentences of quantifier-rank 3 that distinguish between the two structures.

Exercise 2 Consider the property of being an Eulerian graph. This is equivalent to the fact that every node has even degree. Use the following structure to prove that the Eulerian (boolean) query is not first-order definable in the language of graphs:

$\mathfrak{G}_n$: The set of vertices is $\{a_1, \dots, a_n, v, w\}$. The only edges are as follows: each a_i is connected by an edge to v and to w. Show that for every $m \leq n$ the Duplicator wins the m-moves game on A_n and A_{n+1}.

Exercise 3 Express the following queries with first-order sentences in the language of graphs:

(1) G is a complete graph
(2) G has two isolated nodes
(3) G has degree at least 2
(4) G contains a path of length 4.

Exercise 4 Consider the following two structures for the language of linear orders

$$L_1 = \{\{1, 2, 3, 4, 5, 6, 7\}, <\}; \;\; L_2 = \{\{1, 2, 3, 4, 5, 6, 7, 8\}, <\}$$

where in both cases $<$ is the natural order on the naturals. Show that the Duplicator wins the 3-moves game on L_1 and L_2. Does he also win the 4 moves game?

Exercise 5 Consider the following two (structures in the language of) graphs: $\mathfrak{G}_1$ is a cycle of length $4n$; $\mathfrak{G}_2$ consists of two disjoint cycles of length $2n$. Use these structures to prove that the Connectivity query is not-expressible in the first-order language of graphs, using games.

Exercise 6 Consider the following two graphs: $\mathfrak{G}_1$ is a line of length $4n$; $\mathfrak{G}_2$ consists of a line of length 2n and a cycle of length 2n (the two components are disjoint). Use these structures to prove that the Acyclicity query is not-expressible in the first-order language of graphs, using games.

Exercise 7 Express the Non-Connectivity query in Existential Second Order Logic.

Exercise 8 Express the k-Clique query in Existential Second Order Logic. The k-Clique query is the set of pairs (G, k) where G is a graph, k is a number and G has a clique of size k as a subgraph. Your sentence should be in the language containing the binary relation symbol E (edge relation) and a constant c_k (for the number k). You can assume without loss of generality that the domain of a finite model of size n is the initial segment $\{0, 1, \dots, n-1\}$. You can also use a one-ary function symbol f, which might be helpful to deal with identifying a numbering of the vertices of the graph such that the first k vertices form a clique.