Homework #1

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Question 1

Show that $\exists x(R(x) \longrightarrow \forall y R(y))$ is logically valid.

- $\forall x R(x) \longrightarrow \forall y R(y)$ (Applying Prenex Laws, since the consequent of implication doesn't contain any occurrences of x)
- now the formula is the form : $\alpha \longrightarrow \alpha$ which is trivially true.

Question 3

For all structures $\mathfrak{A}$ and for all assignments α in $\mathfrak{A}$: show that $F \models G \iff \models (F \longrightarrow G)$

(1)By definition of logical consequence, Let Γ be a set (finite or infinite) of formulas and S be a formula. We say that Γ logically implies S if for all adequate structures $\mathfrak{A}$, for all assignment α in A, we have that if then $\mathfrak{A} \models S[\alpha]$. We write $\Gamma \models S[\alpha]$ in this case and say that S is a logical consequence of Γ.

$\Longrightarrow$ Proof:
Let pick Γ as a set with just one element (F),assuming by contradiction that our thesis is false, then we this situation : $\models F \land \neg G$, using the definition of ($\models$) this contradicts our hypothesis (1), since we have an assignment which holds in F and doesn't in G

$\Longleftarrow$ Proof.
Using the definition of semantic relation+definition of valid formula: $\models (F \longrightarrow G)$ states the same thing as (1).

Question 4

For any formula F and for any term t, the following formula is valid?
$\forall x F(x) \longrightarrow F(t)$

Answer The formula is **valid**, we know that according the definition of validity, given a formula F, for all structures Γ and for all assign. α the formula is **satisfied**.
If we assume that is not valid, we must being able to find some structure β which doesn't satisfy the formula, this means that we have a situation where the implication's consequence is false whereas the antecedent must be true.
If we assume that is not valid, we have at least one assignment α s.t satisfies $\forall x F(x)$ and doesn't satisfy $F(t)$, but this contradicts the semantic definition of $\models$
Therefore cannot exists an assignment α which satisfies antecedent and doesn't the consequence!

Question 8

Answer 1: Let Γ a finite structure, the cardinality of its domain is equal to **n**;

$$\underset{i=0..n}{\exists} x_i(\bigwedge_{i=0}^{n-1} x_i \neq x_{i-1} \longrightarrow \forall y(\bigvee_{i=0}^{n} x_i = y))$$

Answer 2: I couldn't find it.

Answer 3: I couldn't find it.

Answer 4: Yes, using the **DLO**'s axiom A4,A5 and putting them in a disjunction. i.e. :

$$\forall x \exists y(x < y) \vee \forall x \exists y(y < x)$$

Question 9

Answer: The existence of a lower bound in $\mathcal{N}$.

$$\exists x \forall y(x < y \vee x = y)$$

Question 10

Answer: Assuming that there's a way to decide the validity's sentence.
We know that a sentence, must be either true or false (whereas a formula with free variables can be both true and false, it depends on assignment).
The algorithm take a sentence and if it's valid, add it into the set **V**, otherwise its negation it's unsatisfiable, then add it in **U**

Group 2

Question 1

If $\mathfrak{B}$ is substr. $\mathfrak{A}$, then for any atomic fml $F(x_1, ...x_n)$ for all $b_1...b_n \in B$, $\mathfrak{B} \models F[b_1...b_n] \iff \mathfrak{A} \models F[b_1...b_n]$

Proof It's easy to see, that $b_1 \dots b_n$ belongs both in A and B, by def of **s.s. of**. $(B \subseteq A$

If $\mathfrak{B}$ is substr. $\mathfrak{A}$, then for some universal sentence T $\mathfrak{A} \models T \implies \mathfrak{B} \models T$.
If $\mathfrak{B} \not\models T$ means that we have some element $b_i \in B$ s.t. the satisfaction relation doesn't holds, but it's impossible by def of substr. $(B \subseteq A)$ and definition of for all quantifier.

Question 3

Answer Q s.s. of R, it's obvius, since we have $\subseteq \mathbb{R}$, the constants are the same and the relations and functions in Q are just a restriction of the relations and functions defined in R.
Now to show that Q isn't **elementary s.s.**, we need to prove that there exist a formula s.t. doesn't hold in Q, let take for instance the existence of $\sqrt{2}$ in $\mathbb{R}$:

$$\exists x(x \times x = (1 + 1))$$

Question 6

- $\forall x \exists y(y < x)$
- $\forall x \exists y(y < x)$
- we know that $\mathbb{R}$ is uncountable and Q is countable, therefore, we can't find any isomorphic function between them.

Question 5

Let $S_{\mathfrak{A}}$ a single sentence that completely describes the domain and the relations that hold between elements in the structure $\mathfrak{A}$.

We could say that : $\mathfrak{A} \models S_{\mathfrak{A}}$ (1)

Proof $\Longleftarrow$ By def $\mathfrak{A} \simeq \mathfrak{B} \iff (\mathfrak{A} \models S \iff \mathfrak{B} \models S)$ for all S sentences.
Using (1), we can also conclude that $\mathfrak{B} \models S_{\mathfrak{A}}$

Proof $\implies$ By contradiction:

- $\mathfrak{B} \models S_{\mathfrak{A}} \implies \mathfrak{A} \not\simeq \mathfrak{B}$
- $\mathfrak{A} \simeq \mathfrak{B} \implies \mathfrak{B} \not\models S_{\mathfrak{A}}$ (cotronominal)
- using (1) we have a sentence S s.t. $\mathfrak{A} \models S$ and $\mathfrak{B} \not\models S$, this contradicts our hypothesis.