HOMEWORK N. 3, MATHEMATICAL LOGIC FOR COMPUTER SCIENCE 2019/2020

DEADLINE: MAY 15 2020.

Choose two exercises from each group!

1. Compactness

Exercise 1 Let T be a theory that axiomatizes a property P of structures. Prove that if P is also finitely axiomatizable then P is finitely axiomatizable by a subset of T.

Exercise 2
Let T_1 and T_2 be two theories in a language $\mathcal{L}$. Suppose that for each structure $\mathfrak{A}$ adequate for $\mathcal{L}$, $\mathfrak{A} \models T_1$ if and only if $\mathfrak{A} \not\models T_2$. Then T_1 and T_2 are finitely axiomatizable.
(Hint: Reason by way of contradiction and use the Compactness Theorem to obtain a model of an unsatisfiable theory).

Exercise 3
A coloring of a of a set $X \subseteq \mathbf{N}$ in $c \in \mathbf{N}$ colors is a function $f : X \to \{1, \ldots, c\}$.
Consider the following theorem.

Theorem For any $c \in \mathbf{N}$ there is an $n \in \mathbf{N}$ such thatfor every coloring of the set $\{1, \ldots, n\}$ with c colors, there exists $a, b, a + b \in \{1, \ldots, n\}$ such that $c(a) = c(b) = c(a + b)$ (Note that a and b are not necessarily distinct).

Its natural infinite version is the following:

Theorem For any $c \in \mathbf{N}$, for any coloring of $\mathbf{N}$ with c many colors there exist infinitely many triples $\{a, b, a + b\} \subseteq \mathbf{N}$ such that $c(a) = c(b) = c(a + b)$.

Prove the finite version of the Theorem assuming that the infinite version is true.

(Hint: use Compactness as done for Ramsey's theorem in class. Start by assuming by way of contradiction that for some $c \in \mathbf{N}$ for all $n \in \mathbf{N}$ there is a coloring $f : \{1, \ldots, n\} \to \{1, \ldots, c\}$ such that for no $a, b, a + b \in \{1, \ldots, n\}$ we have $c(a) = c(b) = c(a + b)$.)

Exercise 4 Show that the property of being a non-bipartite graph is not finitely axiomatizable.

(Hint: use a particular property of bipartite graphs concerning cycles and a standard Compactness argument).

Exercise 5 Let $<$ be a strict total order on a set X (that is, $<$ is anti-symmetric, irreflexive and total). We call such a relation *nice* if it admits no infinite decreasing sequences. Prove by a Compactness argument that the notion of being a nice relation is not axiomatizable.

Exercise 6 Prove the following: If a property P and its complement are axiomatizable then P is finitely axiomatizable.

Exercise 7 Let $p_0, p_1, p_2, \ldots$ be the list of all prime numbers in increasing order. Show that for any subset $S \subseteq \mathbf{N}$ there is a model of arithmetic (i.e. a model of all sentences true in the standard model) that contains an element c such that c is divisible by p_i for all and only the p_is such that $i \in S$.

(Hint: use an extra constant and Compactness)

Exercise 8 Let T be a theory that has some finite models and some infinite models. Let E be a sentence such that is $\mathfrak{A} \models T$ and $\mathfrak{A}$ is infinite then $\mathfrak{A} \models E$. Show that there is a bound $b \in \mathbf{N}$ such that if $\mathfrak{A} \models T$ and $\mathfrak{A}$ is of cardinality $\geq b$ then $\mathfrak{A} \models E$.

(Hint: Compactness and some of its corollaries).

2. Ultrafilters and Ultraproducts

Exercise 1 Let F be a filter on $\mathbf{N}$. Prove that the two following conditions on F are equivalent:

(1) for all $S, S' \subseteq \mathbf{N}$, if $S \cup S' \in F$ then $S \in F$ o $S' \in F$.
(2) for all $S \subseteq \mathbf{N}$ either $S \in F$ or $\mathbf{N} \setminus S$ belongs to F.

Exercise 2 Show that the following condition on a family $\mathcal{F}$ of subsets of $\mathbf{N}$ is equivalent to being an ultrafilter: if $A_1 \cup \cdots \cup A_n \in \mathcal{F}$ then for some $i \in \mathbf{N}$: $A_i \in F$.

Exercise 3 If $\mathcal{U}$ is an ultrafilter on $\mathbf{N}$ such that for no $n \in \mathbf{N}$ we have $\mathcal{U} = \{X \subseteq \mathbf{N} : n \in X\}$ then $\mathcal{U}$ contains the filter of cofinite sets.

Exercise 4 A filter $\mathcal{F}$ on $\mathbf{N}$ is a *free filter* is $\bigcap_{X \in \mathcal{F}} X = \varnothing$. Show that $\mathcal{F}$ is not free if and only if there is an $n \in \mathbf{N}$ such that $\mathcal{F} = \{X \subseteq \mathbf{N} : n \in X\}$.

Exercise 5 Show if an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ contains a finite set then there is some $n \in \mathbf{N}$ such that $\mathcal{U} = \{X \subseteq \mathbf{N} : n \in X\}$.

Exercise 6 Prove the following: A class of structures is axiomatizable only if it is closed under elementary equivalence (i.e. if a structure is a member of the class then any structure that is elementarily equivalent to it is also a member) and contains every ultraproduct of its members. (NB: the reverse implication also holds).

Exercise 7 Prove using an ultraproduct construction the following fact proved in class: If a theory T has arbitrarily large finite models then it has an infinite model.

(Hint: use an ultrafilter extending the filter of cofinite sets).

Exercise 8 Consider the language of graphs. For each $n \in \mathbf{N}$ let C_n be the cycle of length n. C_n is an adequate structure for the language of graphs. Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbf{N}$ and consider the ultraproduct $\Pi_{n \in \mathbf{N}} C_n / \mathcal{U}$. How does this structure look like in graph-theoretic terms?

Exercise 9 Show usiing ultraproducts that 3-colorability (of graphs) is axiomatizable but not finitely axiomatizable.

(Hint: The hardest part is showing non finite axiomatizability. Find 3-colorable graphs G_i such that one of their ultraproducts is 3 colorable).