SVM (Support Vector Machine)

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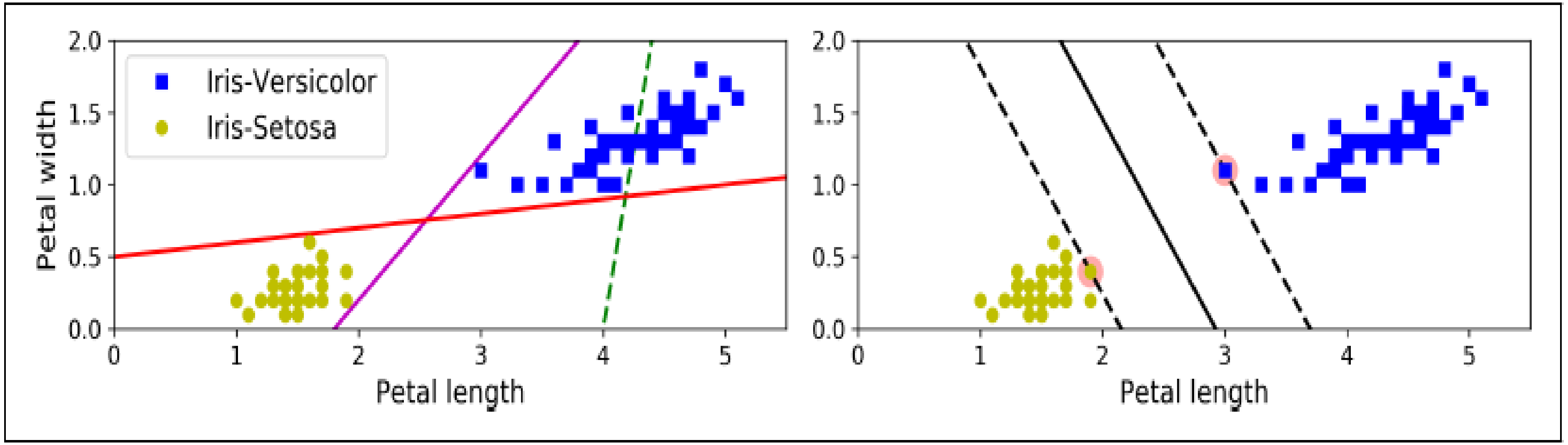
[참고] 본 자료에는 책이나 인터넷, 또는 외부 강의자료에서 인용하여 사용한 그림이나 수식들이 있으니 다른 용도로 사용하거나 외부로 유출을 금해 주시기 바랍니다.

[1] Aurellian Geron, Hands-on Machine Learning with Scikit, Keras, and Tensorflow
[2] https://www.robots.ox.ac.uk/~az/lectures (Prof. Zisserman's lecture slide)

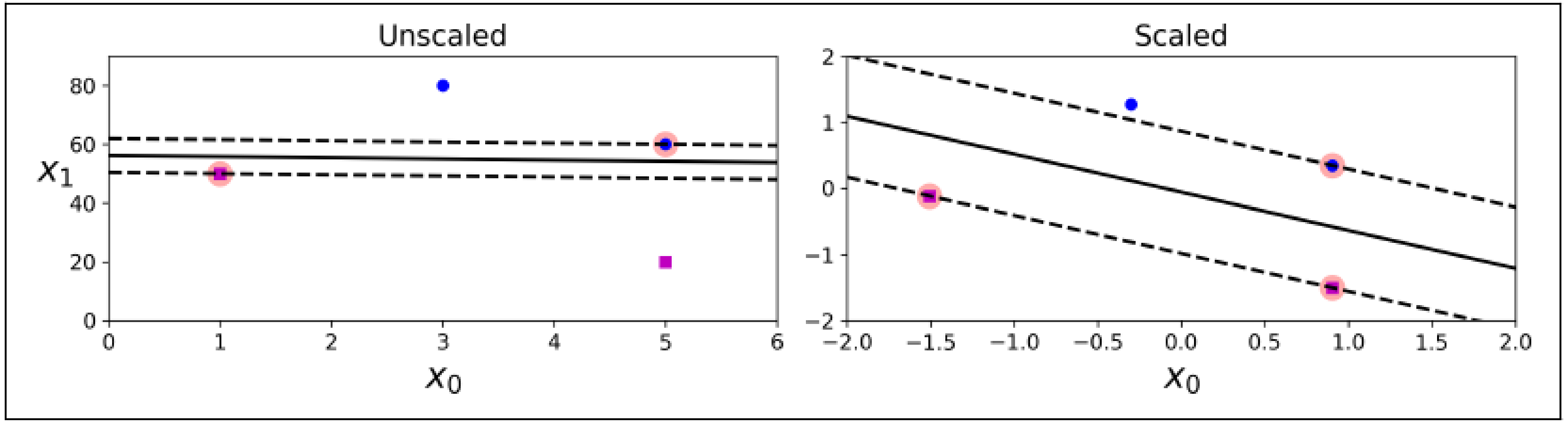
SVM Classifier

- **Linear classification and SVM Classifier**



SVM Classifier

- **Sensitivity to feature scales**

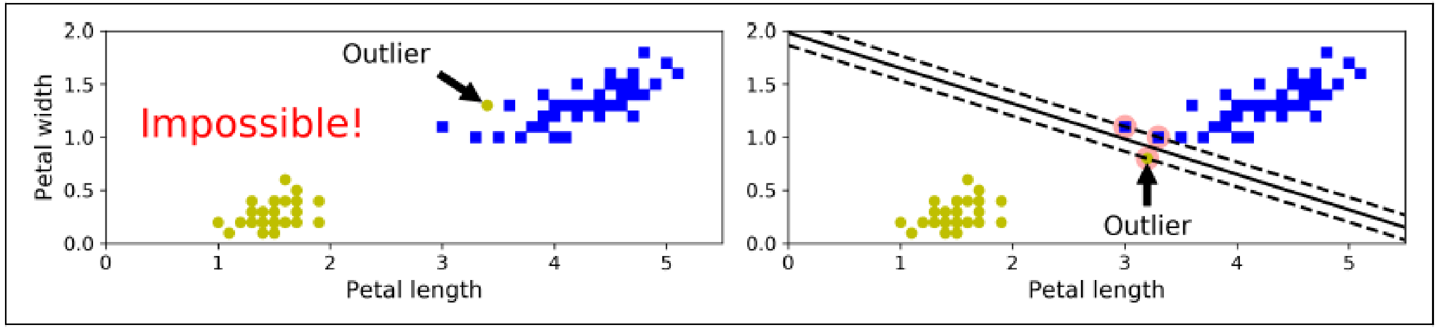


Before Scaling

After Scaling

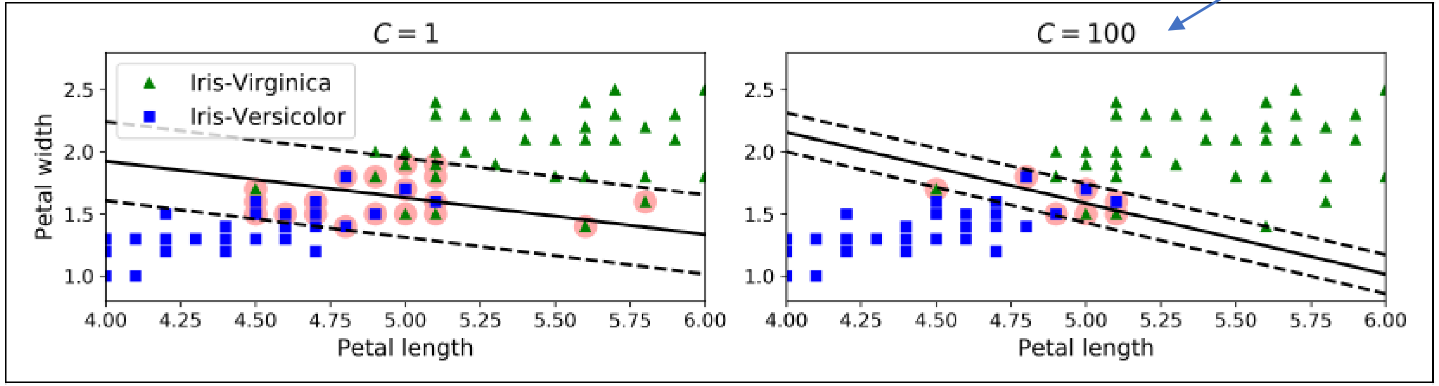
SVM Classifier

- **Hard margin and Soft margin**
  - **Hard margin**



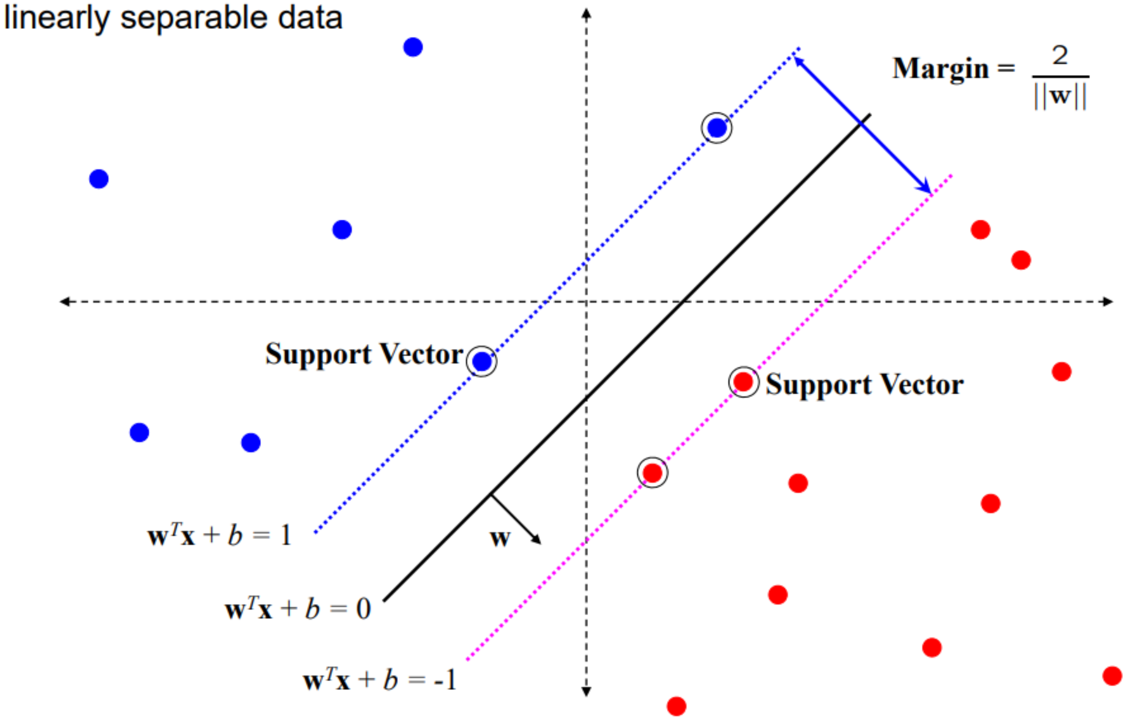
  - **Soft margin**
    - Large margin and fewer margin violations

Overfitting 가능성



SVM Classifier

- **Quadratic optimization problem subject to linear constraints**
  - There is a unique minimum.


linearly separable data
Margin = $\frac{2}{||\mathbf{w}||}$
Support Vector
Support Vector
$\mathbf{w}^T\mathbf{x}+b=1$
$\mathbf{w}$
$\mathbf{w}^T\mathbf{x}+b=0$
$\mathbf{w}^T\mathbf{x}+b=-1$

$$\begin{array}{l}\vec{w}\bullet\vec{x_+}+b\geq\delta\\ \vec{w}\bullet\vec{x_-}+b\leq-\delta\end{array} \xRightarrow{\text{normalize}} \begin{array}{l}\vec{w}\bullet\vec{x_+}+b\geq 1\\ \vec{w}\bullet\vec{x_-}+b\leq -1\end{array}$$

- SVM is formulated as an optimization:

$$\max_{\mathbf{w}} \frac{2}{||\mathbf{w}||} \text{ subject to } \mathbf{w}^\top\mathbf{x}_i+b \begin{array}{l}\geq 1\\ \leq -1\end{array} \quad \begin{array}{l}\text{if } y_i=+1\\ \text{if } y_i=-1\end{array} \quad \text{for } i=1\ldots N$$

Or equivalently

$$\min_{\mathbf{w}} ||\mathbf{w}||^2 \quad \text{subject to } y_i\left(\mathbf{w}^\top\mathbf{x}_i+b\right)\geq 1 \text{ for } i=1\ldots N$$

training

$$h_\theta(x)=\begin{cases}1 & \text{if } \theta^T x >= 0\\ 0 & \text{otherwise}\end{cases}$$

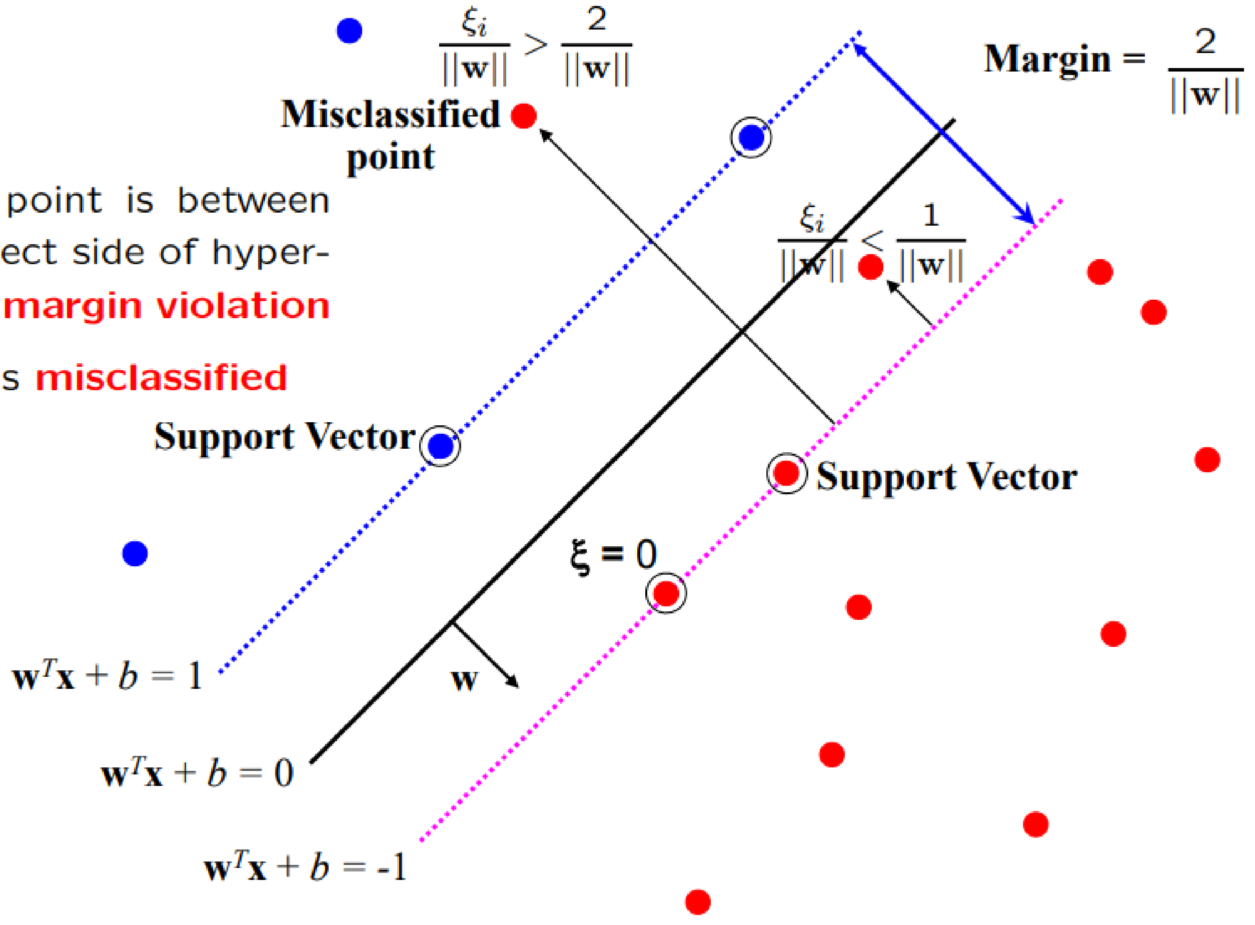
testing(inferencing)

SVM Classifier

- **Introduce Slack variables**
  - amount of error (hinge loss) from the correct side of hyperplane

$$\xi_i \geq 0$$

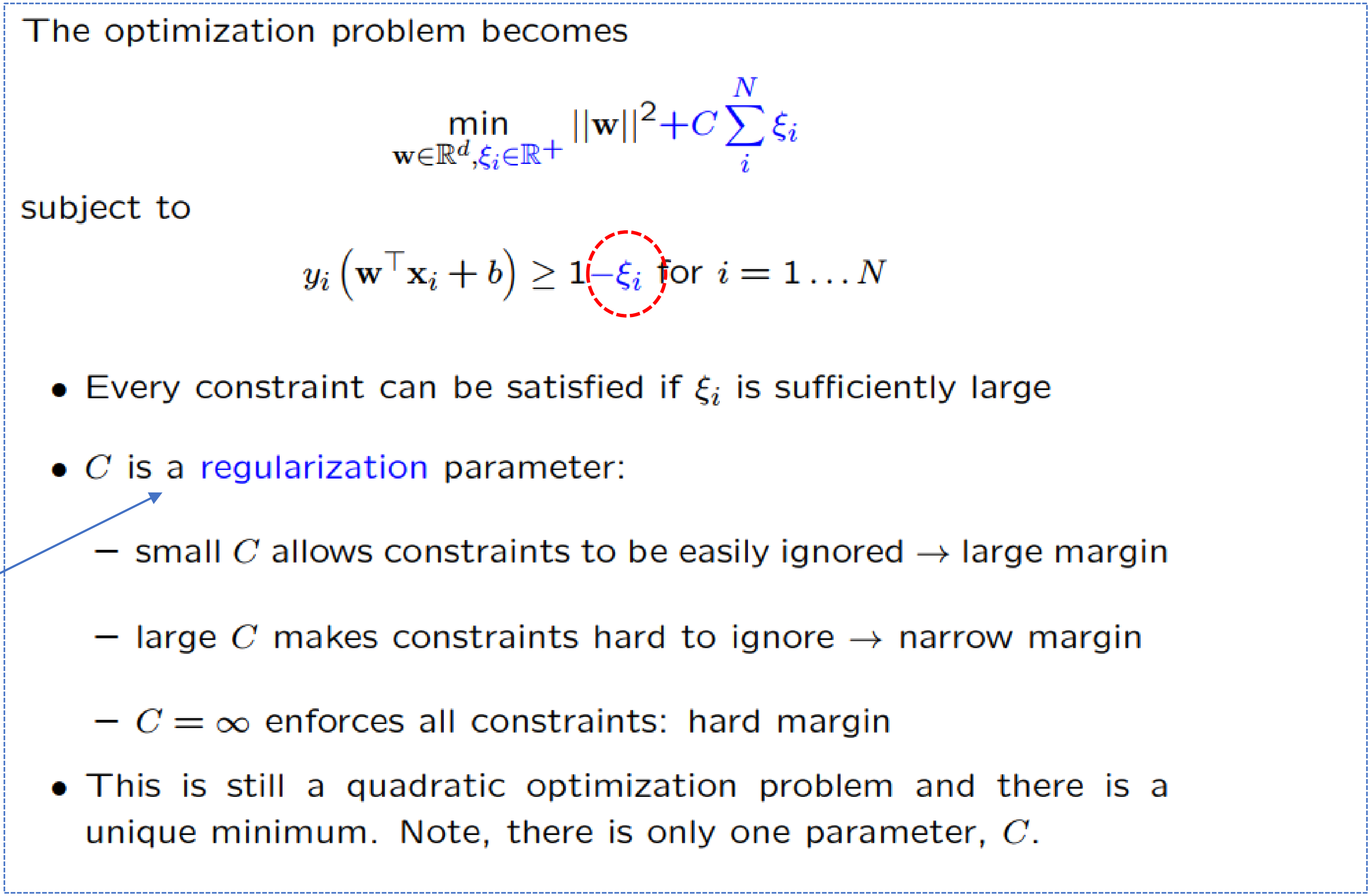
- for $0 < \xi \leq 1$ point is between margin and correct side of hyperplane. This is a **margin violation**
- for $\xi > 1$ point is **misclassified**



$\frac{\xi_i}{||\mathbf{w}||} > \frac{2}{||\mathbf{w}||}$
Misclassified point
Margin = $\frac{2}{||\mathbf{w}||}$
$\frac{\xi_i}{||\mathbf{w}||} < \frac{1}{||\mathbf{w}||}$
Support Vector
Support Vector
ξ = 0
$\mathbf{w}$
$\mathbf{w}^T\mathbf{x} + b = 1$
$\mathbf{w}^T\mathbf{x} + b = 0$
$\mathbf{w}^T\mathbf{x} + b = -1$

SVM Classifier

- **Soft margin**



The optimization problem becomes

$$\min_{\mathbf{w}\in\mathbb{R}^d,\xi_i\in\mathbb{R}^+} ||\mathbf{w}||^2 + C\sum_{i}^{N} \xi_i$$

subject to

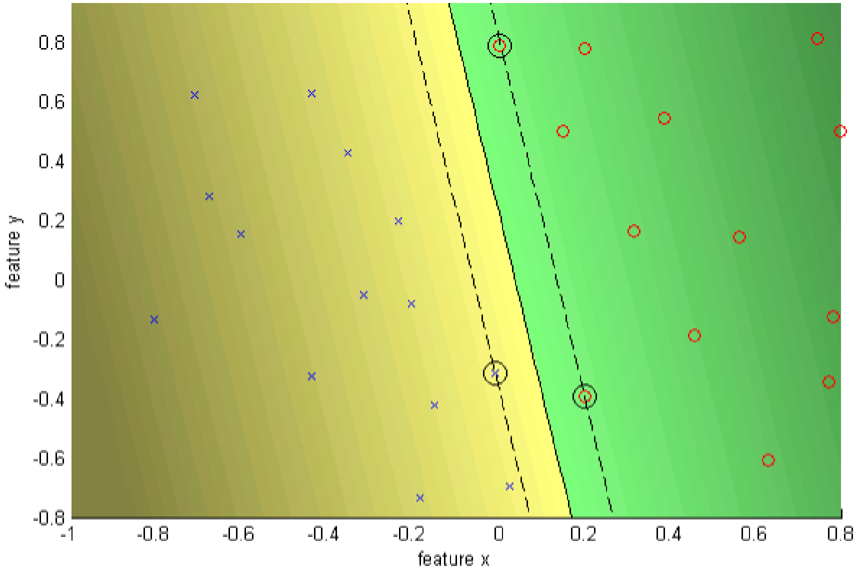
$$y_i\left(\mathbf{w}^\top \mathbf{x}_i + b\right) \geq 1 - \xi_i \text{ for } i = 1 \dots N$$

- Every constraint can be satisfied if ξ_i is sufficiently large
- C is a regularization parameter:
  - small C allows constraints to be easily ignored → large margin
  - large C makes constraints hard to ignore → narrow margin
  - $C = \infty$ enforces all constraints: hard margin
- This is still a quadratic optimization problem and there is a unique minimum. Note, there is only one parameter, C.

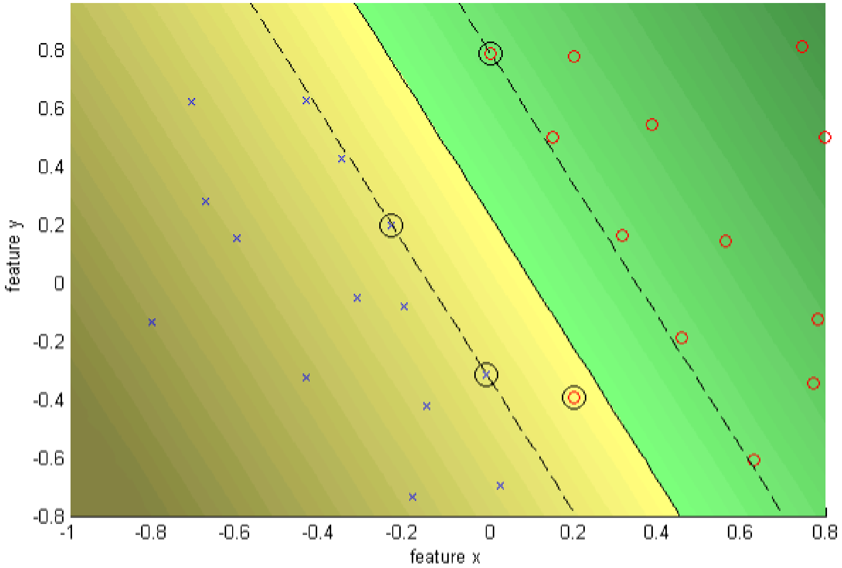
Or,
cost parameter (or
penalty parameter)

SVM Classifier

- **Hard margin and Soft margin**



C = Infinity, hard margin



C = 10, Soft margin

SVM Optimization

- Constrained optimization problem

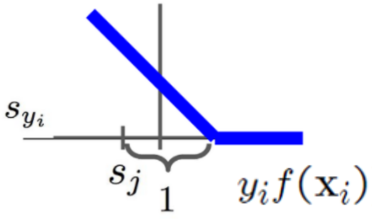
$$\min_{\mathbf{w}\in\mathbb{R}^d, \xi_i \in \mathbb{R}^+} ||\mathbf{w}||^2 + C\sum_i^N \xi_i \text{ subject to } y_i\left(\mathbf{w}^\top \mathbf{x}_i + b\right) \geq 1 - \xi_i \text{ for } i = 1 \dots N$$

$$y_i\left(\mathbf{w}^\top \mathbf{x}_i + b\right) \geq 1 - \xi_i$$

$$\xi_i \geq 0$$

$$\Rightarrow$$

$$\xi_i = \max\left(0, 1 - y_i f(\mathbf{x}_i)\right)$$

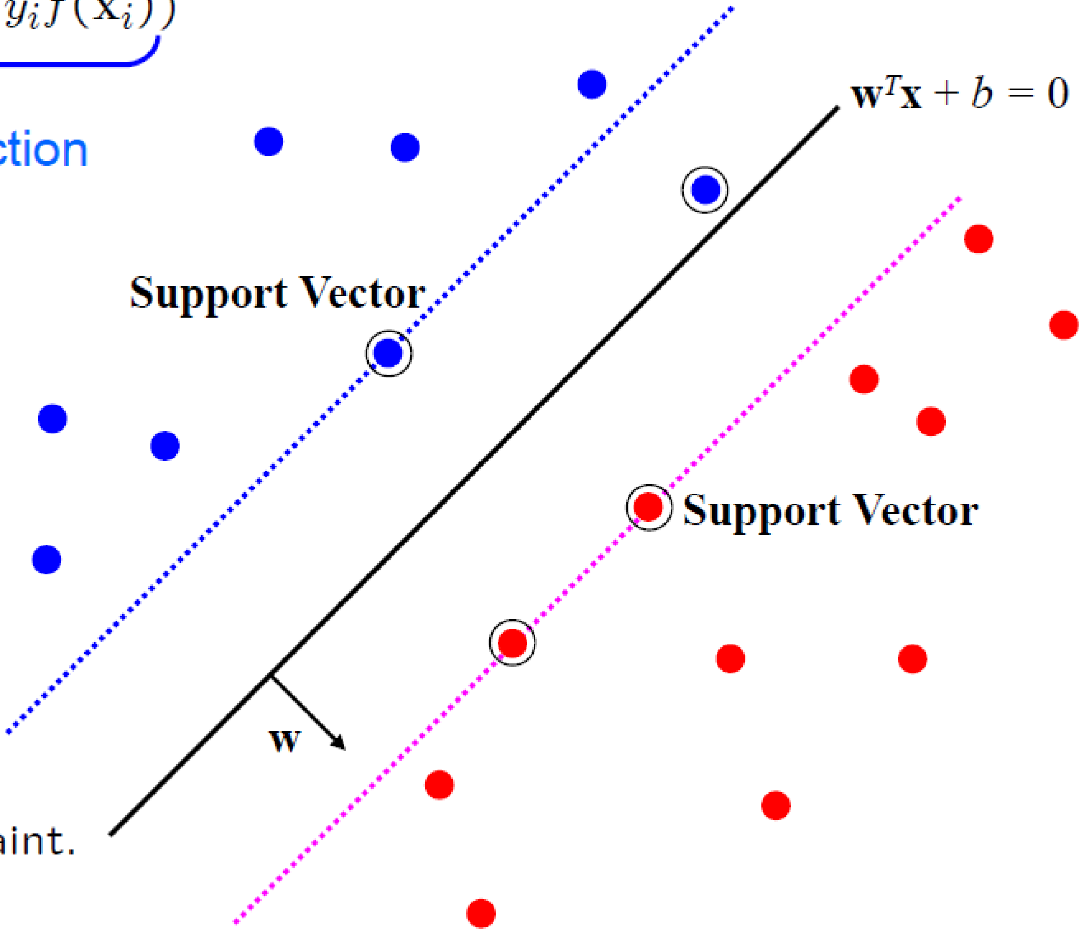


- The learning problem is now equivalent to the unconstrained optimization problem over w

$$\min_{\mathbf{w}\in\mathbb{R}^d} \underbrace{||\mathbf{w}||^2}_{\text{regularization}} + C\sum_i^N \underbrace{\max\left(0, 1 - y_i f(\mathbf{x}_i)\right)}_{\text{loss function}}$$

Loss Function

$$\min_{\mathbf{w}\in\mathbb{R}^d} ||\mathbf{w}||^2 + C\sum_{i}^{N} \underbrace{\max\left(0, 1 - y_i f(\mathbf{x}_i)\right)}_{\text{loss function}}$$

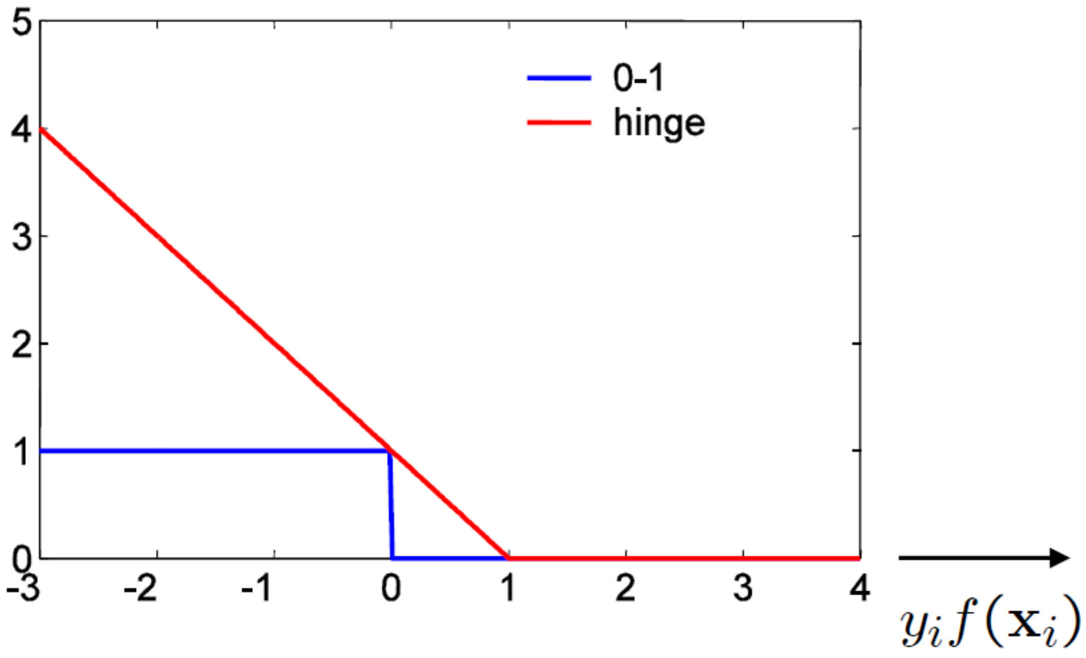


Points are in three categories:

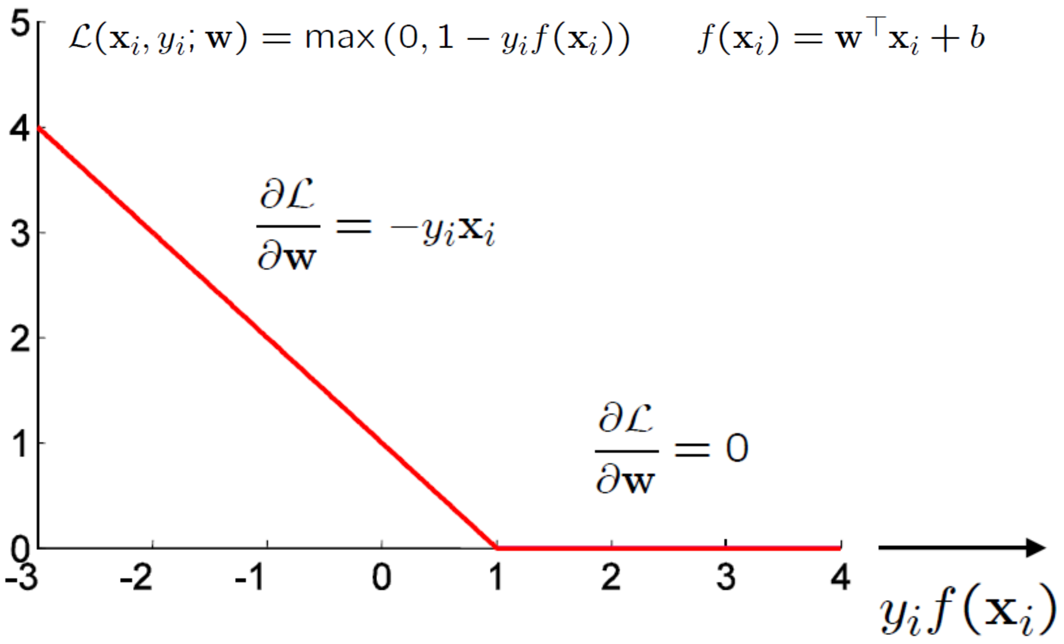
1. $y_i f(x_i) > 1$
   Point is outside margin.
   No contribution to loss
2. $y_i f(x_i) = 1$
   Point is on margin.
   No contribution to loss.
   As in hard margin case.
3. $y_i f(x_i) < 1$
   Point violates margin constraint.
   Contributes to loss

Hinge Loss

- **Hinge loss**



- **Sub-gradient for Hinge loss**



$$\mathcal{L}(\mathbf{x}_i, y_i; \mathbf{w}) = \max(0, 1 - y_i f(\mathbf{x}_i)) \qquad f(\mathbf{x}_i) = \mathbf{w}^\top \mathbf{x}_i + b$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = -y_i \mathbf{x}_i$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = 0$$

- SVM uses “hinge” loss $\max(0, 1 - y_i f(\mathbf{x}_i))$
- an approximation to the 0-1 loss

Sub-gradient descent algorithm for SVM

$$\mathcal{C}(\mathbf{w}) = \frac{1}{N}\sum_{i}^{N}\left(\frac{\lambda}{2}||\mathbf{w}||^2 + \mathcal{L}(\mathbf{x}_i, y_i; \mathbf{w})\right)$$

The iterative update is

$$\begin{aligned}\mathbf{w}_{t+1} &\leftarrow \mathbf{w}_t - \eta\nabla_{\mathbf{w}_t}\mathcal{C}(\mathbf{w}_t)\\ &\leftarrow \mathbf{w}_t - \eta\frac{1}{N}\sum_{i}^{N}(\lambda\mathbf{w}_t + \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{x}_i, y_i; \mathbf{w}_t))\end{aligned}$$

where η is the learning rate.

Then each iteration t involves cycling through the training data with the updates:

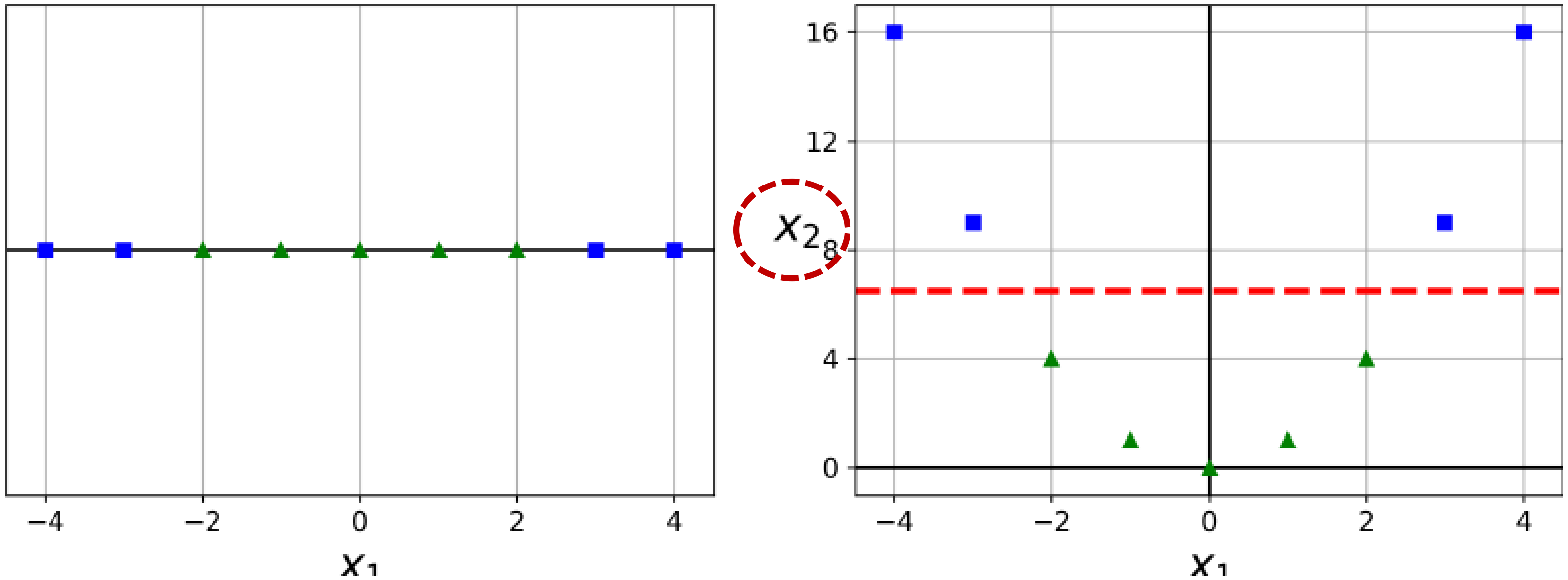
$$\boxed{\begin{aligned}\mathbf{w}_{t+1} &\leftarrow \mathbf{w}_t - \eta(\lambda\mathbf{w}_t - y_i\mathbf{x}_i) && \text{if } y_i f(\mathbf{x}_i) < 1\\ &\leftarrow \mathbf{w}_t - \eta\lambda\mathbf{w}_t && \text{otherwise}\end{aligned}}$$

Why not solve SVM with Gradient Descent

- **Why not solve SVM with Gradient Descent instead of Quadratic Programming?**
  - Gradient descent is a first-order method, which means it utilizes minimal information about the problem (only gradients) and thus converges slowly and might suffer from convergence issues. – SGDClassifier(loss='hinge') in sklearn
  - Taking into account more information about the problem, such as its quadratic nature and linear constraints, can yield faster and more robust algorithms. There are also specific quadratic optimization algorithms developed for SVM. (https://en.wikipedia.org/wiki/Sequential_minimal_optimization)

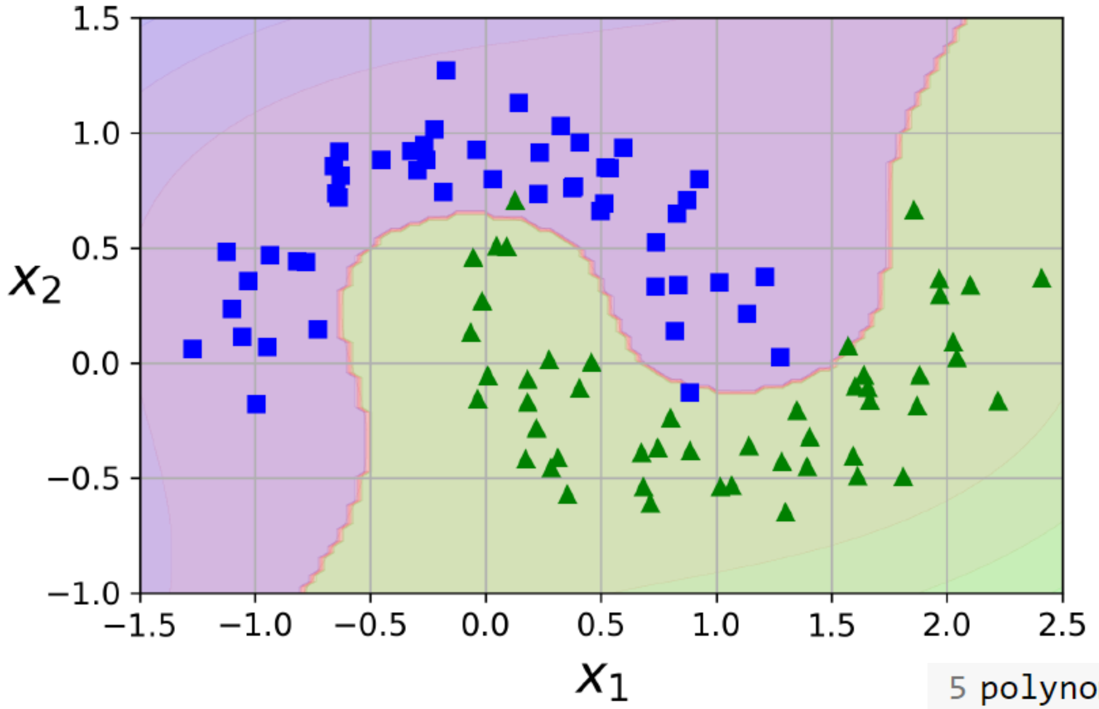
Nonlinear SVM Classifier

- **Adding features to make a dataset linearly separable**
  - Add a second feature $x_2 = (x_1)^2$

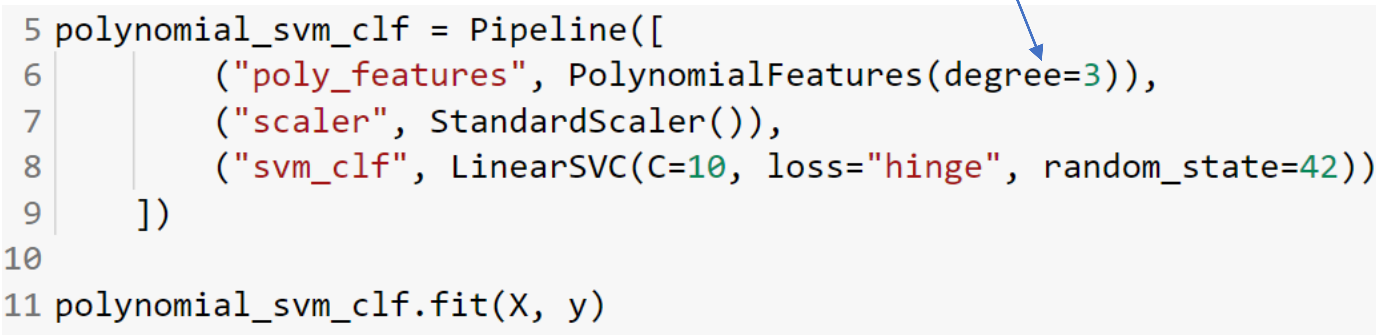


Nonlinear SVM Classifier

- **Linear SVM classifier using polynomial features**



10 features: c, x1, x2, x1^2, x2^2, x1x2, x1^3, x2^3, x1^{2}x2, x1x2^2



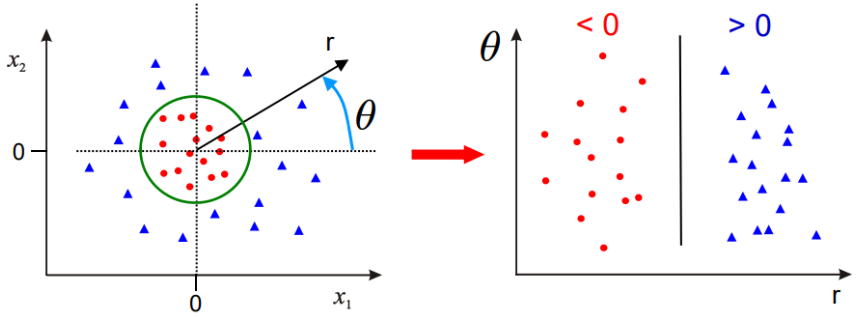
```
polynomial_svm_clf = Pipeline([
        ("poly_features", PolynomialFeatures(degree=3)),
        ("scaler", StandardScaler()),
        ("svm_clf", LinearSVC(C=10, loss="hinge", random_state=42))
    ])

polynomial_svm_clf.fit(X, y)
```

Nonlinear SVM Classifier

- **General (linearly) non-separable data**

Use polar coordinates

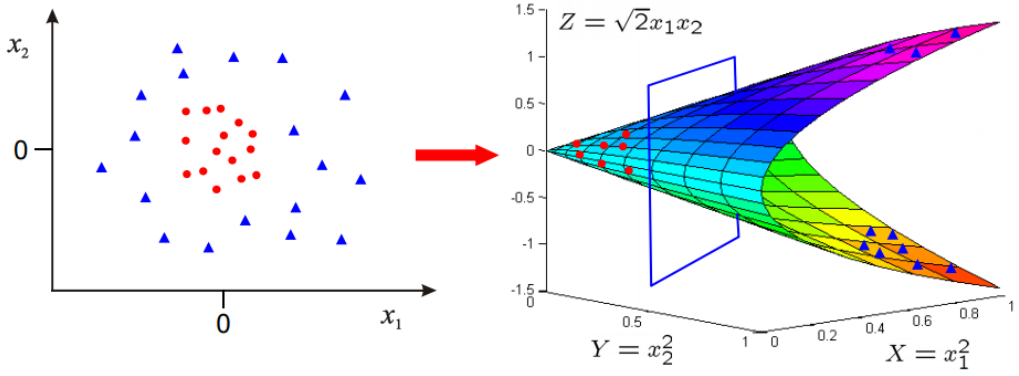


- Data is linearly separable in polar coordinates
- Acts non-linearly in original space

$$\Phi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} r \\ \theta \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Map data to higher dimension

$$\Phi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}x_1 x_2 \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



Feature space

Non-linear Classification - SVM Kernels

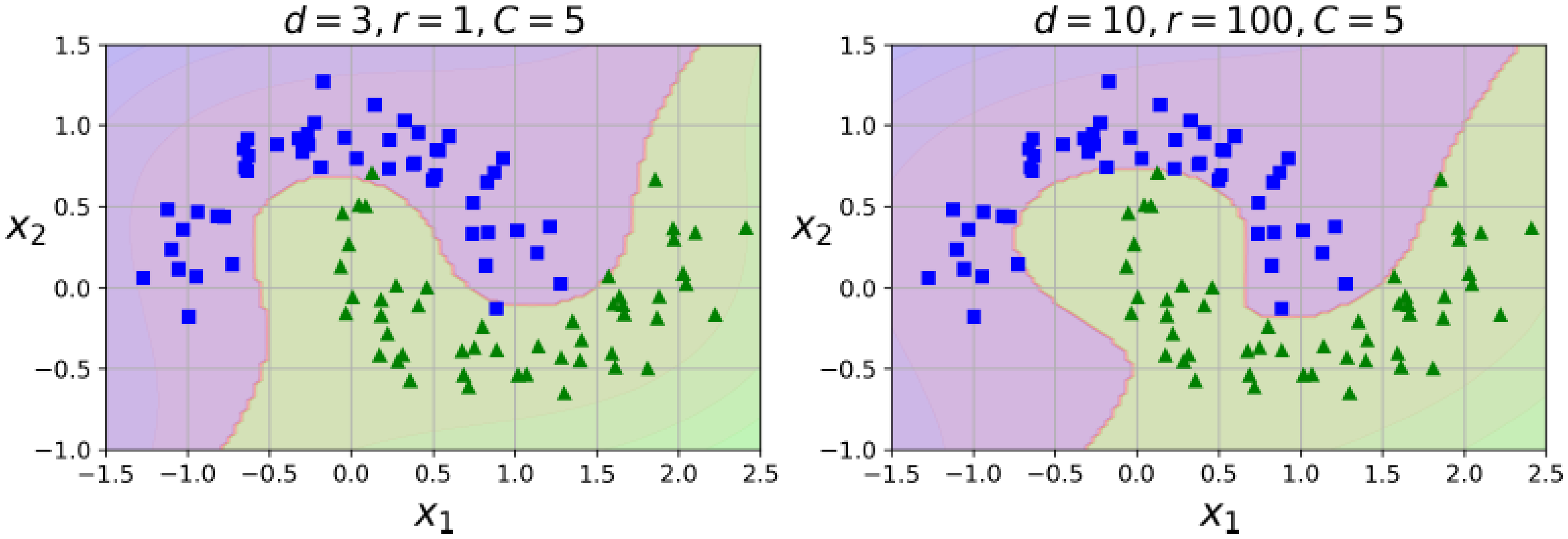
- **SVM classifiers with a polynomial kernel**

$$\mathbf{K}(a,b) = (a \times b + r)^d$$

- a, b: 서로 다른 데이터

- r: polynomial의 coefficient를 결정

- d: polynomial의 차수

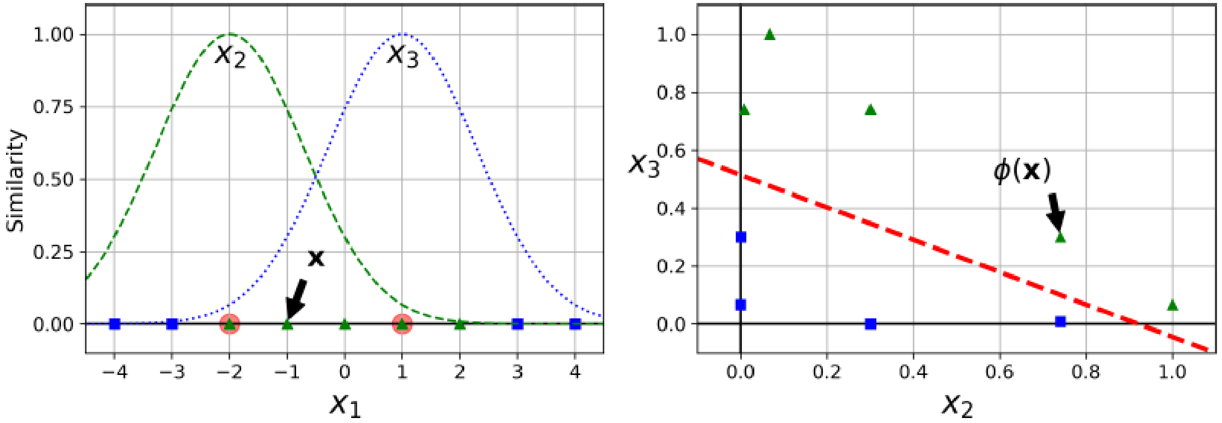


SVM Kernels

- **SVM with Similarity features**
  - Use a **similarity function** that measures how much each instance resembles a particular *landmark*.
  - define the similarity function: **Gaussian Radial Basis Function (*RBF*)**

$$\phi_{\gamma}(\mathbf{x}, \ell) = \exp\left(-\gamma \| \mathbf{x} - \ell \|^2\right)$$

original feature:
x_1 = -1

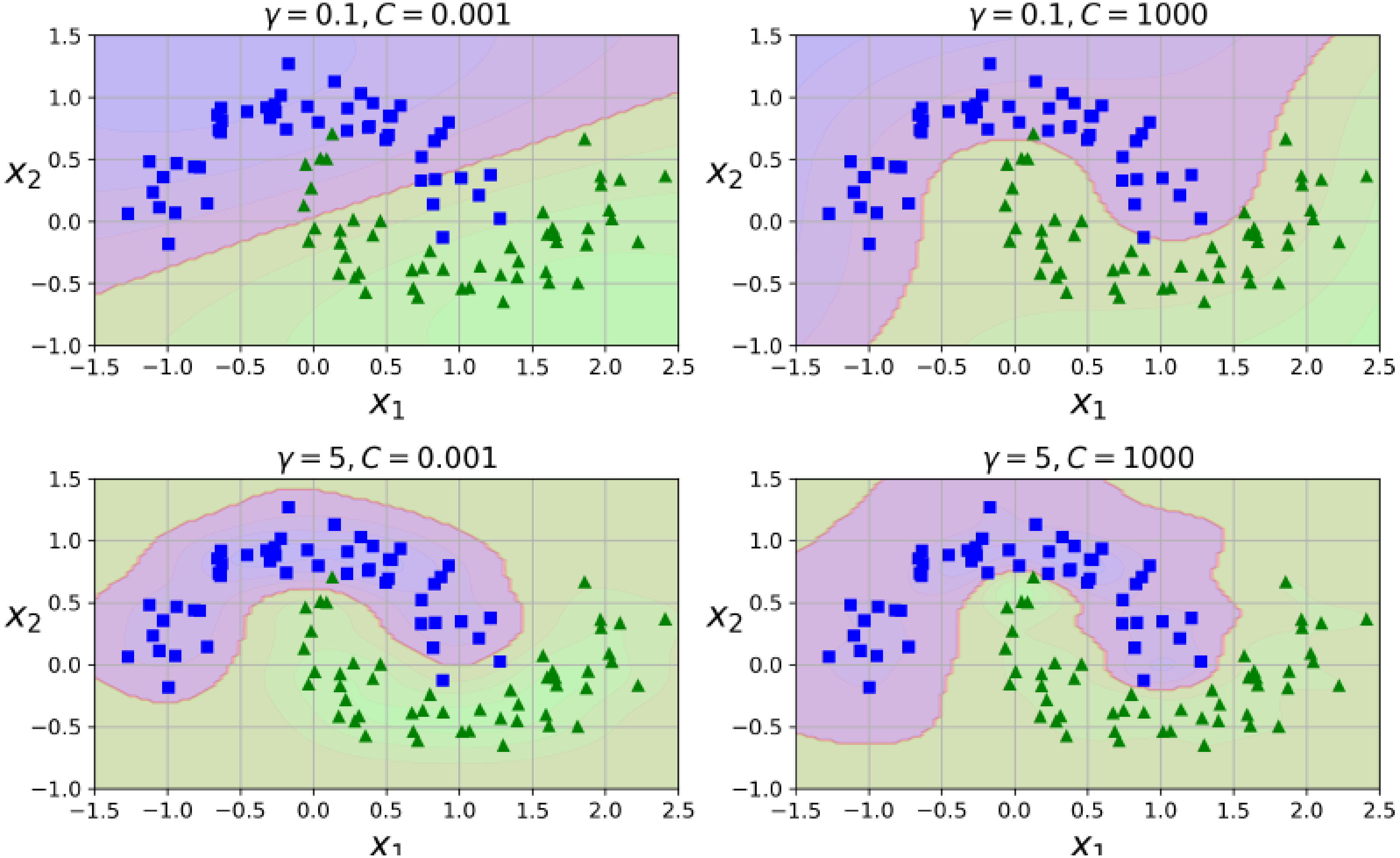


Take two landmarks at x_1=-2, x_1=1
then, new features are:
x_2 = exp (–0.3 × 1^2) ≈ 0.74
x_3 = exp (–0.3 × 2^2) ≈ 0.30
now, linearly separable.

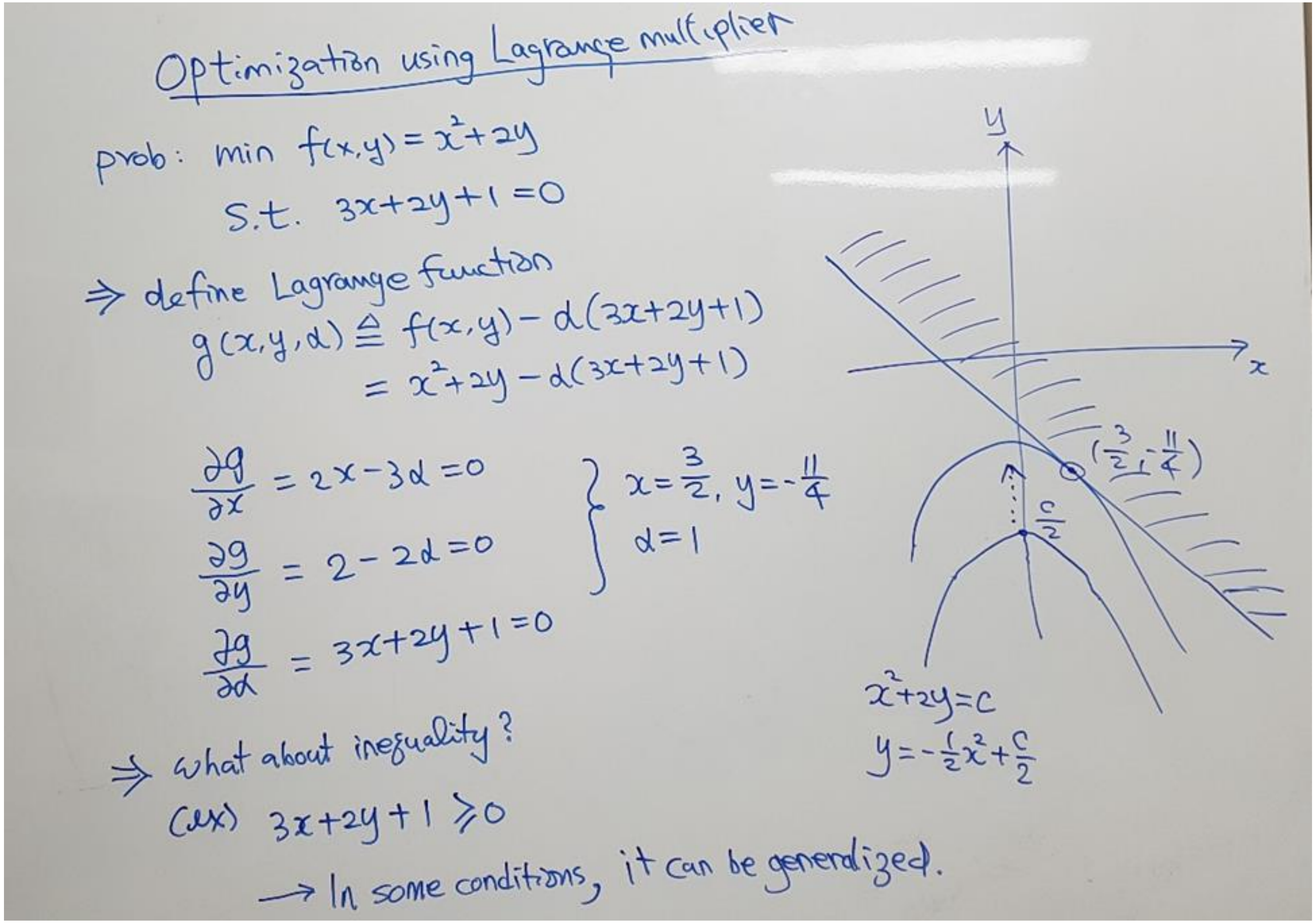
- Create landmarks at the location of each instance, then (assuming drop of the original features)
  m instances, *n* features -> *m* instances, *m* features (large features !)

SVM Kernels

- Gaussian RBF Kernel



Optimization – Lagrangian multiplier



Optimization using Lagrange multiplier

prob: min $f(x,y) = x^2+2y$

s.t. $3x+2y+1=0$

⇒ define Lagrange function

$$g(x,y,\alpha) \triangleq f(x,y) - \alpha(3x+2y+1)$$

$$= x^2+2y-\alpha(3x+2y+1)$$

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= 2x-3\alpha=0 \\ \frac{\partial g}{\partial y} &= 2-2\alpha=0 \\ \frac{\partial g}{\partial \alpha} &= 3x+2y+1=0 \end{aligned}\right\} \begin{aligned} & x=\frac{3}{2},\ y=-\frac{11}{4} \\ & \alpha=1 \end{aligned}$$

⇒ what about inequality?

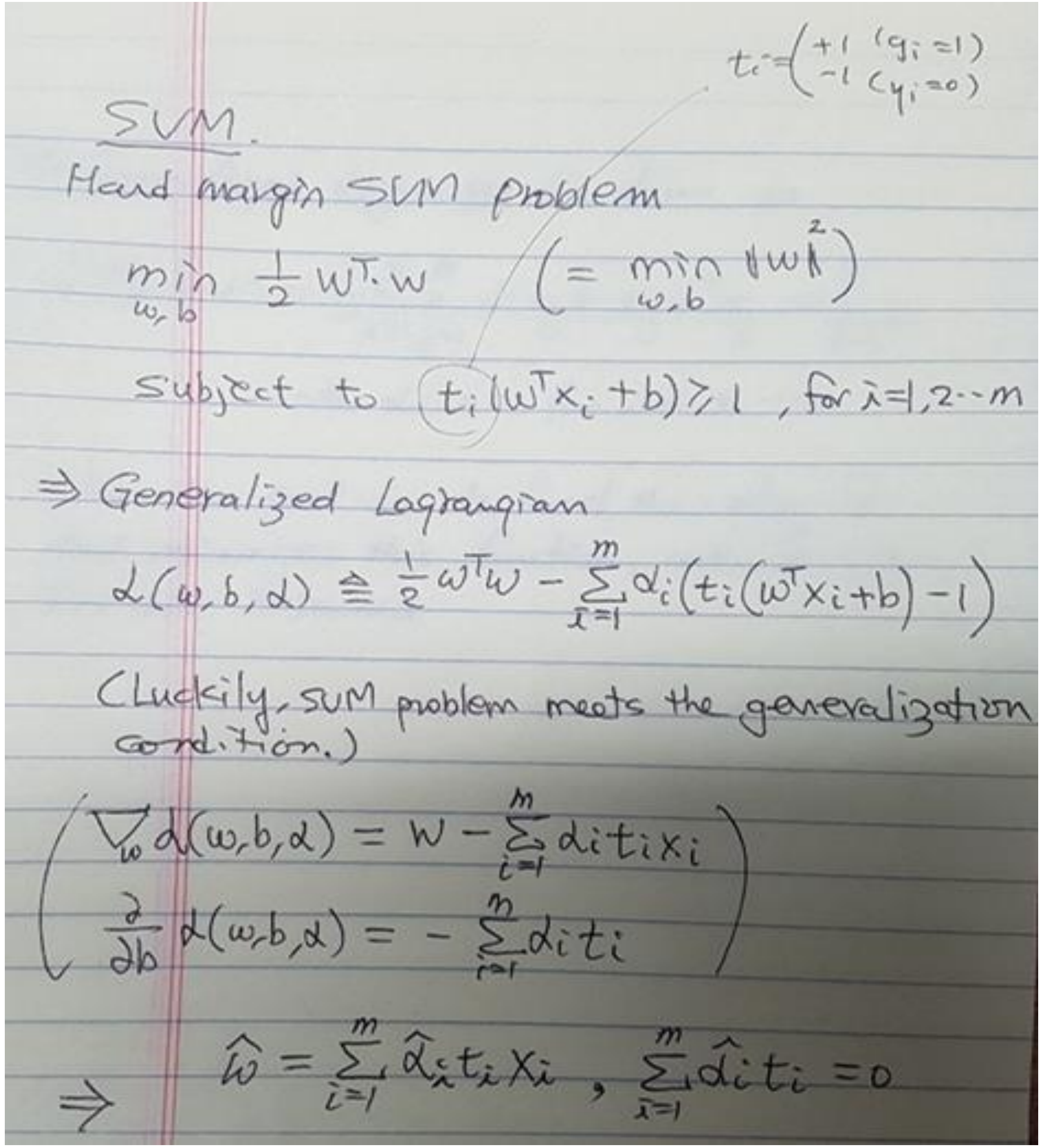
(ex) $3x+2y+1 \geqslant 0$

→ In some conditions, it can be generalized.

$x^2+2y=c$

$y=-\frac{1}{2}x^2+\frac{c}{2}$

Optimization – SVM **Hard** margin



$t_i = \begin{cases} +1 & (y_i=1) \\ -1 & (y_i=0) \end{cases}$

SVM.

Hard margin SVM problem

$$\min_{w,b} \frac{1}{2} w^T \cdot w \quad \left(= \min_{w,b} \|w\|^2\right)$$

subject to $t_i(w^T x_i + b) \geq 1$, for $i=1,2\cdots m$

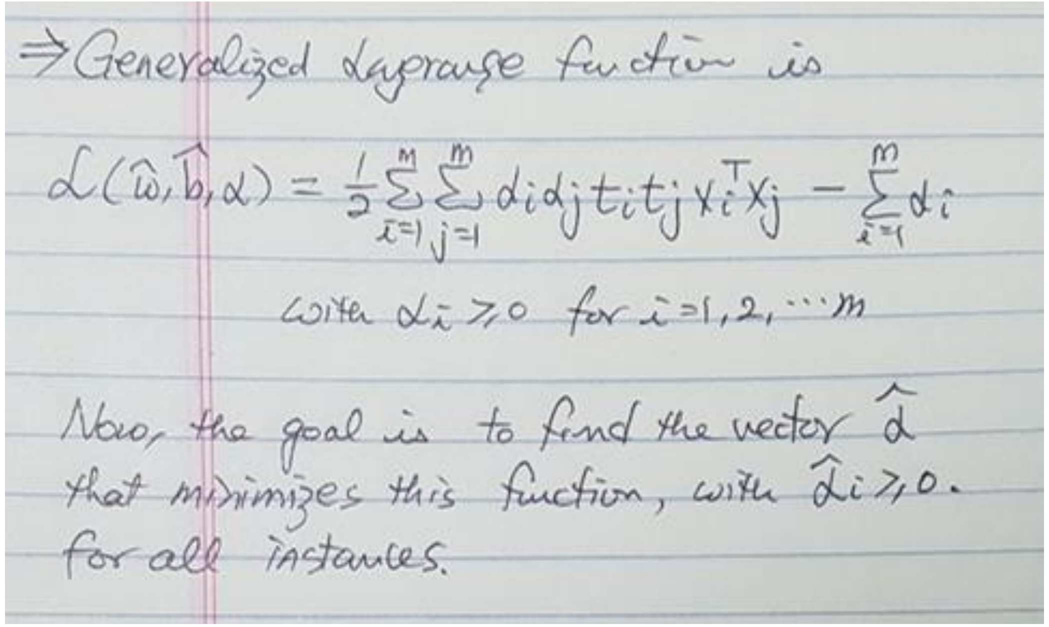
⇒ Generalized Lagrangian

$$\mathcal{L}(w,b,\alpha) \triangleq \frac{1}{2} w^T w - \sum_{i=1}^{m} \alpha_i \left(t_i (w^T x_i + b) - 1\right)$$

(Luckily, SVM problem meets the generalization condition.)

$$\left(\begin{aligned} \nabla_w \mathcal{L}(w,b,\alpha) &= w - \sum_{i=1}^{m} \alpha_i t_i x_i \\ \frac{\partial}{\partial b} \mathcal{L}(w,b,\alpha) &= - \sum_{i=1}^{m} \alpha_i t_i \end{aligned} \right)$$

$$\Rightarrow \quad \hat{w} = \sum_{i=1}^{m} \hat{\alpha}_i t_i x_i \, , \quad \sum_{i=1}^{m} \hat{\alpha}_i t_i = 0$$



⇒ Generalized Lagrange function is

$$\mathcal{L}(\hat{w}, \hat{b}, \alpha) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j t_i t_j x_i^T x_j - \sum_{i=1}^{m} \alpha_i$$

with $\alpha_i \geq 0$ for $i=1,2,\cdots m$

Now, the goal is to find the vector $\hat{\alpha}$ that minimizes this fuction, with $\hat{\alpha}_i \geq 0$. for all instances.

Optimization – SVM Soft margin

$$\begin{aligned} \min_{\boldsymbol{w},b,\boldsymbol{\xi}} \quad & \frac{1}{2}\|\boldsymbol{w}\|^2 + C\sum_{n=1}^{N}\xi_n \\ \text{subject to} \quad & y_n(\langle \boldsymbol{w}, \boldsymbol{x}_n\rangle + b) \geqslant 1 - \xi_n \\ & \xi_n \geqslant 0 \end{aligned}$$

$$\min_{\boldsymbol{w},b} \quad \underbrace{\frac{1}{2}\|\boldsymbol{w}\|^2}_{\text{regularizer}} + \underbrace{C\sum_{n=1}^{N}\max\{0, 1 - y_n(\langle \boldsymbol{w}, \boldsymbol{x}_n\rangle + b)\}}_{\text{error term}}$$

Lagrangian

$$\begin{aligned} \mathfrak{L}(\boldsymbol{w}, b, \xi, \alpha, \gamma) = & \frac{1}{2}\|\boldsymbol{w}\|^2 + C\sum_{n=1}^{N}\xi_n \qquad (12.34) \\ & \underbrace{-\sum_{n=1}^{N}\alpha_n(y_n(\langle \boldsymbol{w}, \boldsymbol{x}_n\rangle + b) - 1 + \xi_n)}_{\text{constraint (12.26b)}} \underbrace{-\sum_{n=1}^{N}\gamma_n\xi_n}_{\text{constraint (12.26c)}} \,. \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathfrak{L}}{\partial \boldsymbol{w}} &= \boldsymbol{w}^\top - \sum_{n=1}^{N}\alpha_n y_n \boldsymbol{x}_n{}^\top \,, \\ \frac{\partial \mathfrak{L}}{\partial b} &= \sum_{n=1}^{N}\alpha_n y_n \,, \\ \frac{\partial \mathfrak{L}}{\partial \xi_n} &= C - \alpha_n - \gamma_n \,. \end{aligned}$$

Optimization – SVM Soft margin

$$\mathfrak{D}(\xi,\alpha,\gamma) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j \alpha_i \alpha_j \left\langle \boldsymbol{x}_i, \boldsymbol{x}_j \right\rangle + \sum_{i=1}^{N} \alpha_i + \sum_{i=1}^{N} (C - \alpha_i - \gamma_i)\xi_i \,. \tag{12.40}$$

Dual SVM

$$\begin{aligned} \min_{\boldsymbol{\alpha}} \quad & \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j \alpha_i \alpha_j \left\langle \boldsymbol{x}_i, \boldsymbol{x}_j \right\rangle - \sum_{i=1}^{N} \alpha_i \\ \text{subject to} \quad & \sum_{i=1}^{N} y_i \alpha_i = 0 \\ 0 \leqslant \alpha_i \leqslant C \quad & \text{for all} \quad i = 1, \ldots, N \,. \end{aligned} \tag{12.41}$$

Primal and Dual formulations (Summary)

N is number of training points, and d is dimension of feature vector $\mathbf{x}$.

Primal problem: for $\mathbf{w} \in \mathbb{R}^d$

$$\min_{\mathbf{w}\in\mathbb{R}^d} ||\mathbf{w}||^2 + C\sum_{i}^{N} \max\left(0, 1 - y_i f(\mathbf{x}_i)\right)$$

Dual problem: for $\boldsymbol{\alpha} \in \mathbb{R}^N$ (stated without proof):

$$\max_{\alpha_i \geq 0} \sum_i \alpha_i - \frac{1}{2}\sum_{jk} \alpha_j \alpha_k y_j y_k (\mathbf{x}_j{}^\top \mathbf{x}_k) \text{ subject to } 0 \leq \alpha_i \leq C \text{ for } \forall i, \text{ and} \sum_i \alpha_i y_i = 0$$

- Need to learn d parameters for primal, and N for dual
- If $N << d$ then more efficient to solve for α than $\mathbf{w}$
- Dual form only involves $(\mathbf{x}_j{}^\top \mathbf{x}_k)$.

Primal and Dual in transformed Feature space

Primal Classifier in transformed feature space

Classifier, with $\mathbf{w} \in \mathbb{R}^D$:

$$f(\mathbf{x}) = \mathbf{w}^\top \Phi(\mathbf{x}) + b$$

Learning, for $\mathbf{w} \in \mathbb{R}^D$

$$\min_{\mathbf{w} \in \mathbb{R}^D} ||\mathbf{w}||^2 + C \sum_i^N \max(0, 1 - y_i f(\mathbf{x}_i))$$

- Simply map $\mathbf{x}$ to $\Phi(\mathbf{x})$ where data is separable
- Solve for $\mathbf{w}$ in high dimensional space $\mathbb{R}^D$
- If $D >> d$ then there are many more parameters to learn for $\mathbf{w}$. Can this be avoided?

Dual Classifier in transformed feature space

Classifier:

$$f(\mathbf{x}) = \sum_i^N \underline{\alpha_i y_i \mathbf{x}_i{}^\top} \mathbf{x} + b \quad \longleftarrow \text{ wx + b}$$

$$\rightarrow f(\mathbf{x}) = \sum_i^N \alpha_i y_i \Phi(\mathbf{x}_i)^\top \Phi(\mathbf{x}) + b$$

Learning:

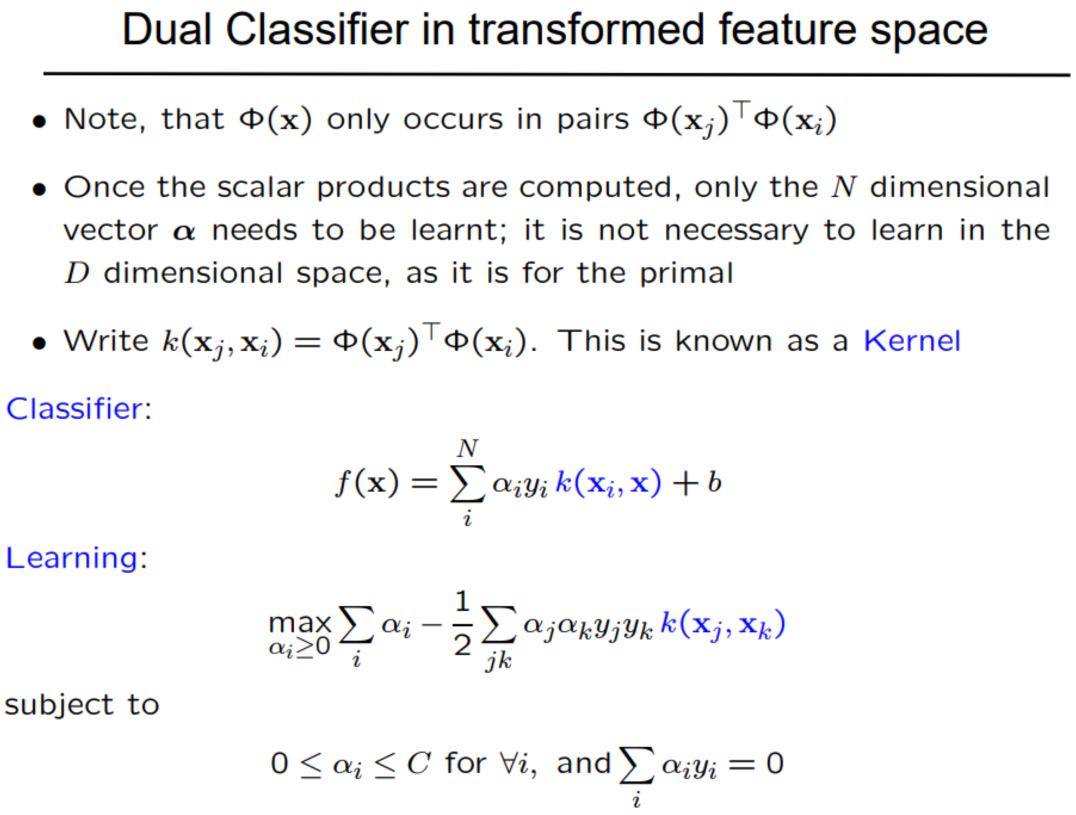
$$\max_{\alpha_i \geq 0} \sum_i \alpha_i - \frac{1}{2} \sum_{jk} \alpha_j \alpha_k y_j y_k \mathbf{x}_j{}^\top \mathbf{x}_k$$

$$\rightarrow \max_{\alpha_i \geq 0} \sum_i \alpha_i - \frac{1}{2} \sum_{jk} \alpha_j \alpha_k y_j y_k \Phi(\mathbf{x}_j)^\top \Phi(\mathbf{x}_k)$$

subject to

$$0 \leq \alpha_i \leq C \text{ for } \forall i, \text{ and } \sum_i \alpha_i y_i = 0$$

SVM Kernels



Dual Classifier in transformed feature space

- Note, that $\Phi(\mathbf{x})$ only occurs in pairs $\Phi(\mathbf{x}_j)^\top \Phi(\mathbf{x}_i)$
- Once the scalar products are computed, only the N dimensional vector $\boldsymbol{\alpha}$ needs to be learnt; it is not necessary to learn in the D dimensional space, as it is for the primal
- Write $k(\mathbf{x}_j, \mathbf{x}_i) = \Phi(\mathbf{x}_j)^\top \Phi(\mathbf{x}_i)$. This is known as a Kernel

Classifier:

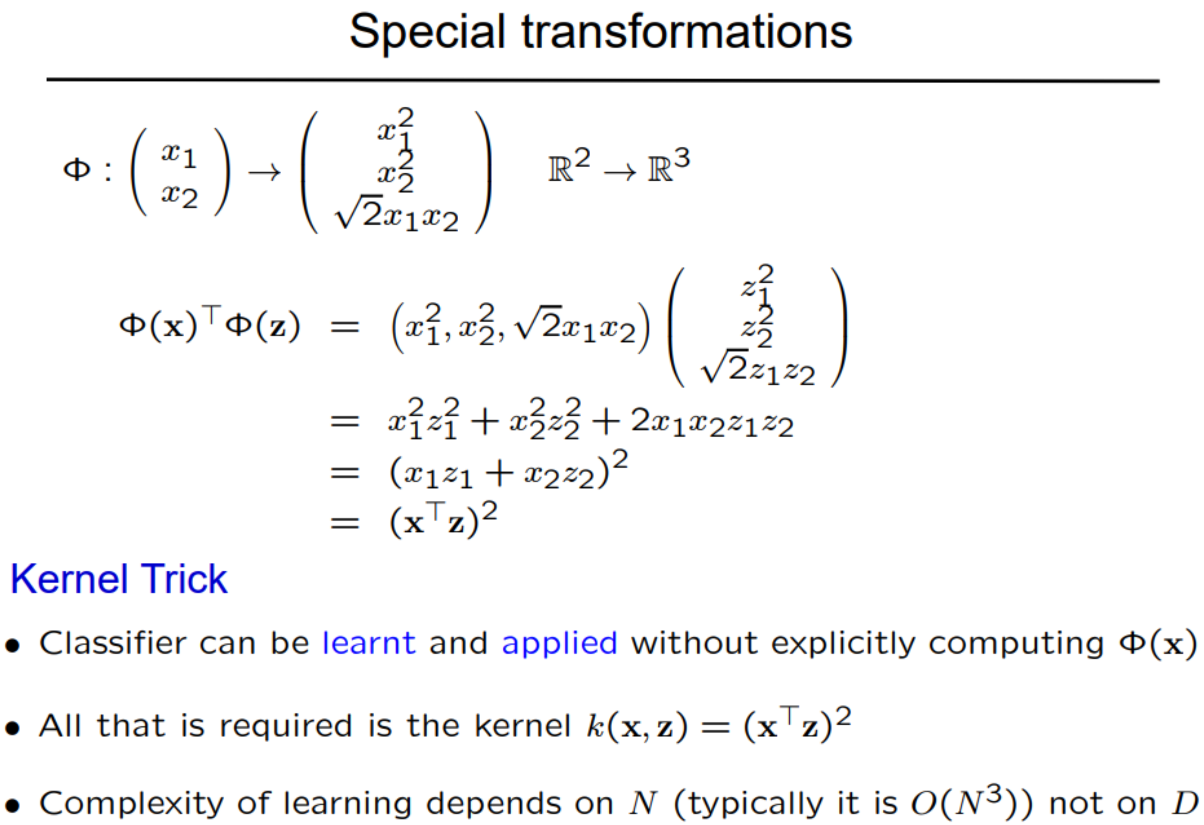
$$f(\mathbf{x}) = \sum_i^N \alpha_i y_i \, k(\mathbf{x}_i, \mathbf{x}) + b$$

Learning:

$$\max_{\alpha_i \geq 0} \sum_i \alpha_i - \frac{1}{2} \sum_{jk} \alpha_j \alpha_k y_j y_k \, k(\mathbf{x}_j, \mathbf{x}_k)$$

subject to

$$0 \leq \alpha_i \leq C \text{ for } \forall i, \text{ and } \sum_i \alpha_i y_i = 0$$



Special transformations

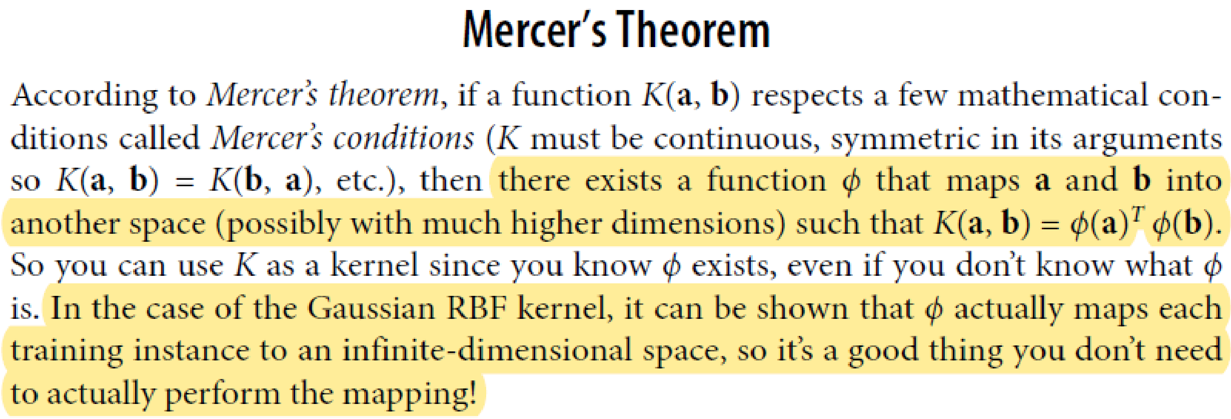
$$\Phi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2} x_1 x_2 \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\begin{aligned} \Phi(\mathbf{x})^\top \Phi(\mathbf{z}) &= \left(x_1^2, x_2^2, \sqrt{2} x_1 x_2\right) \begin{pmatrix} z_1^2 \\ z_2^2 \\ \sqrt{2} z_1 z_2 \end{pmatrix} \\ &= x_1^2 z_1^2 + x_2^2 z_2^2 + 2 x_1 x_2 z_1 z_2 \\ &= (x_1 z_1 + x_2 z_2)^2 \\ &= (\mathbf{x}^\top \mathbf{z})^2 \end{aligned}$$

Kernel Trick

- Classifier can be learnt and applied without explicitly computing $\Phi(\mathbf{x})$
- All that is required is the kernel $k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^2$
- Complexity of learning depends on N (typically it is $O(N^3)$) not on D

Common SVM Kernels

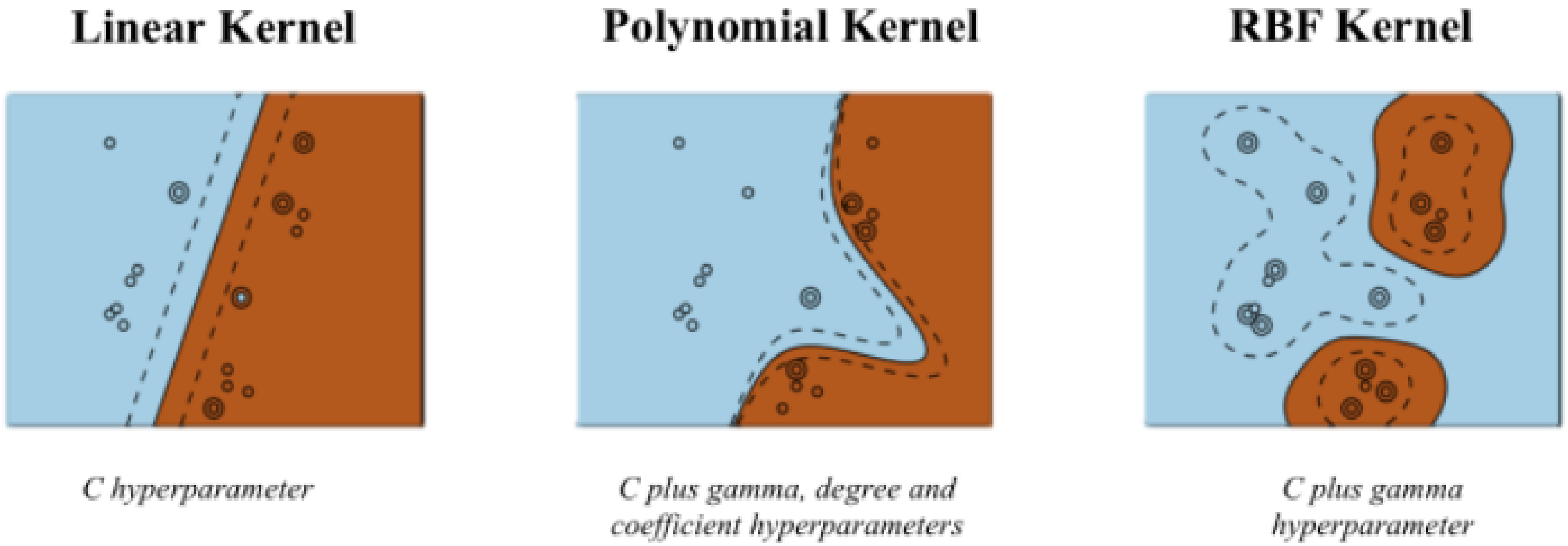


Mercer's Theorem

According to *Mercer's theorem*, if a function $K(\mathbf{a}, \mathbf{b})$ respects a few mathematical conditions called *Mercer's conditions* (K must be continuous, symmetric in its arguments so $K(\mathbf{a}, \mathbf{b}) = K(\mathbf{b}, \mathbf{a})$, etc.), then there exists a function ϕ that maps $\mathbf{a}$ and $\mathbf{b}$ into another space (possibly with much higher dimensions) such that $K(\mathbf{a}, \mathbf{b}) = \phi(\mathbf{a})^T \phi(\mathbf{b})$. So you can use K as a kernel since you know ϕ exists, even if you don't know what ϕ is. In the case of the Gaussian RBF kernel, it can be shown that ϕ actually maps each training instance to an infinite-dimensional space, so it's a good thing you don't need to actually perform the mapping!

$$\begin{aligned}
\text{Linear:} \quad & K(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T\mathbf{b} \\
\text{Polynomial:} \quad & K(\mathbf{a}, \mathbf{b}) = \left(\gamma\mathbf{a}^T\mathbf{b} + r\right)^d \\
\text{Gaussian RBF:} \quad & K(\mathbf{a}, \mathbf{b}) = \exp\left(-\gamma\| \mathbf{a} - \mathbf{b} \|^2\right) \\
\text{Sigmoid:} \quad & K(\mathbf{a}, \mathbf{b}) = \tanh\left(\gamma\mathbf{a}^T\mathbf{b} + r\right)
\end{aligned}$$

Common SVM Kernels



Linear Kernel

C hyperparameter

Polynomial Kernel

C plus gamma, degree and coefficient hyperparameters

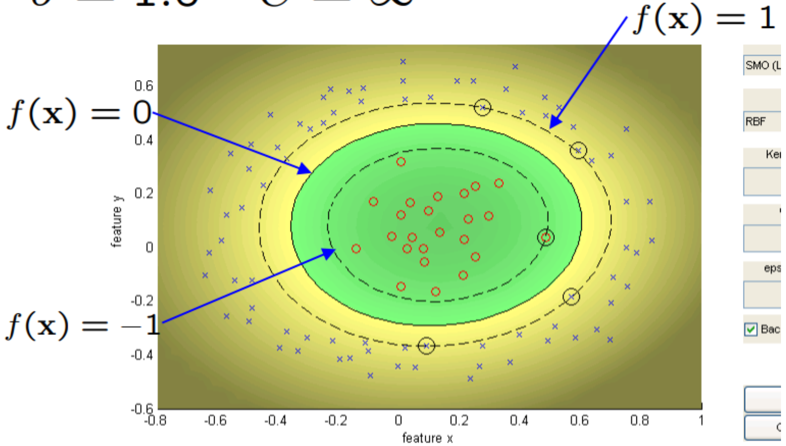
RBF Kernel

C plus gamma hyperparameter

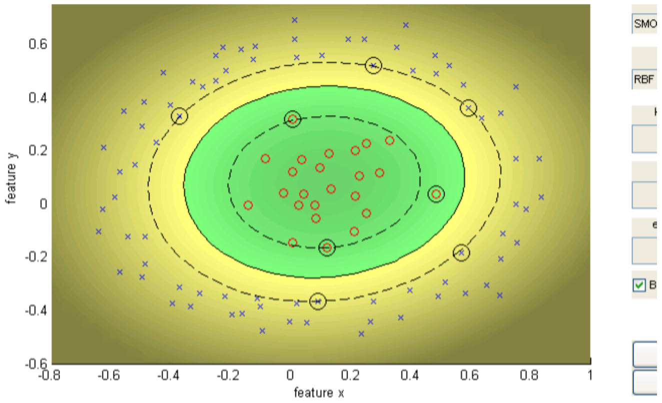
RBF Kernel SVM Example

$$f(\mathbf{x}) = \sum_{i}^{N} \alpha_i y_i \boxed{\exp\left(-||\mathbf{x}-\mathbf{x}_i||^2/2\sigma^2\right)} + b$$

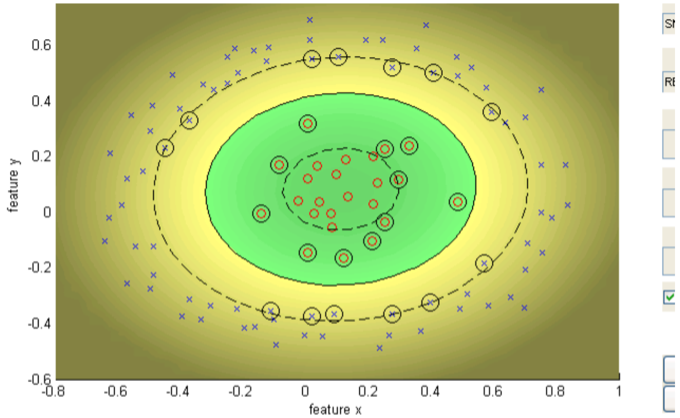
$\sigma = 1.0 \quad C = \infty$



$\sigma = 1.0 \quad C = 100$

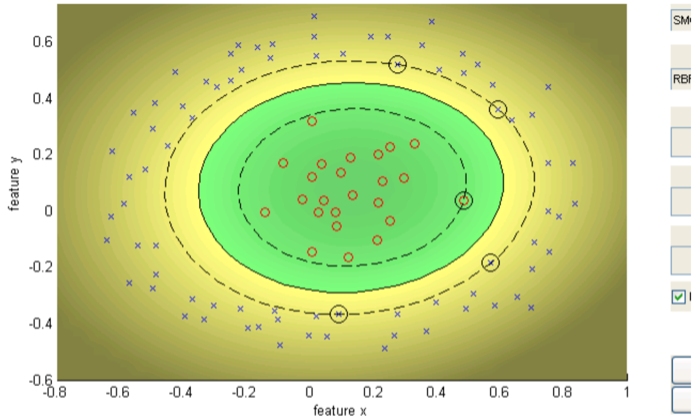


$\sigma = 1.0 \quad C = 10$

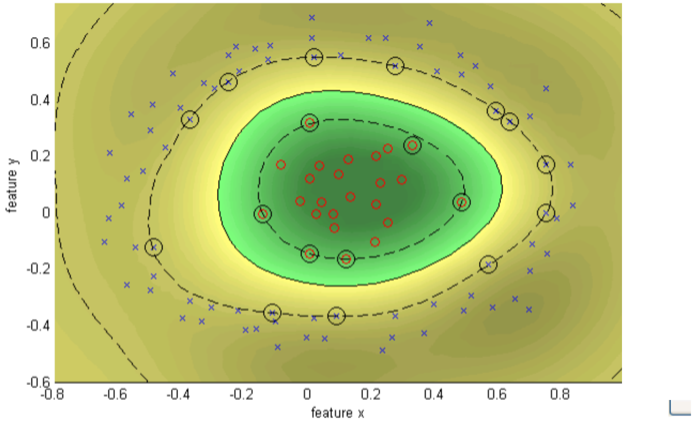


Decrease C -> wider (soft) margin

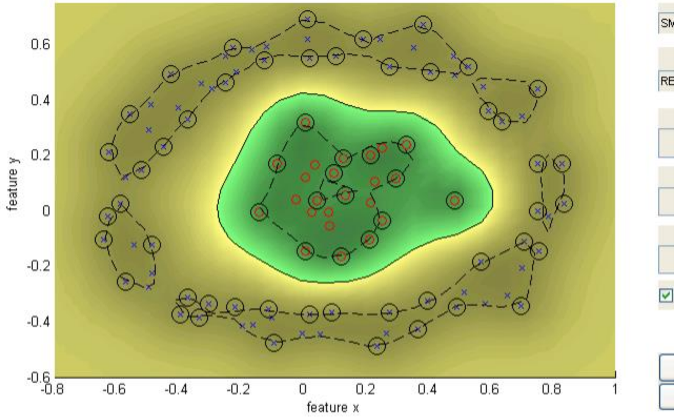
$\sigma = 1.0 \quad C = \infty$



$\sigma = 0.25 \quad C = \infty$



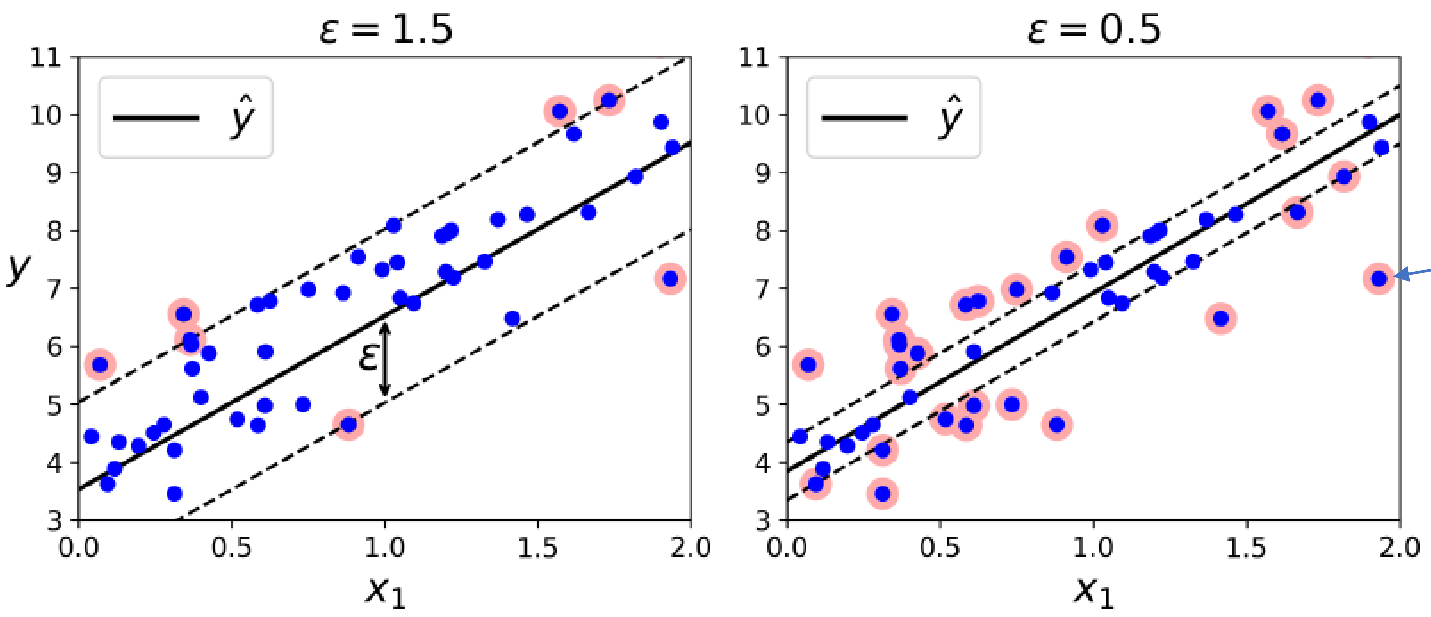
$\sigma = 0.1 \quad C = \infty$



Decrease sigma (increase gamma) -> move towards nearest neighbor classifier

SVM Regression

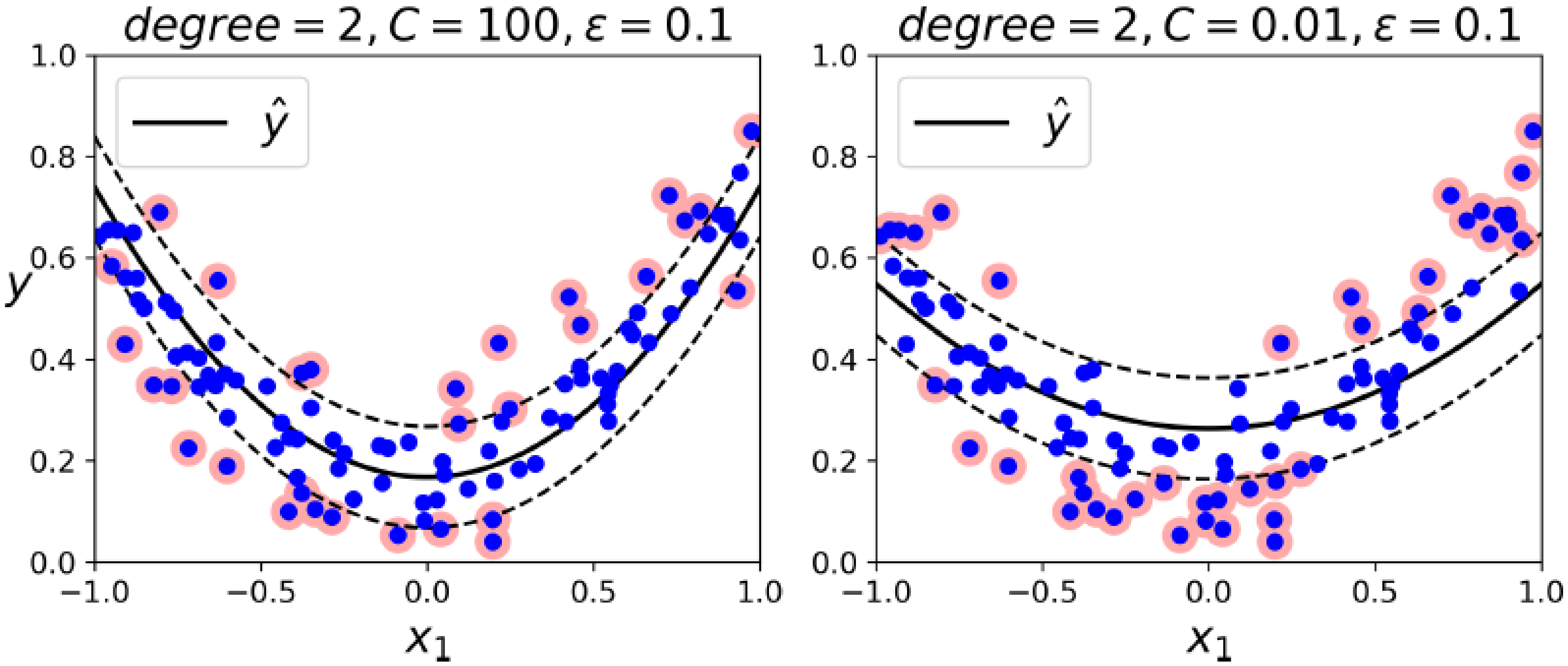
- **It also supports linear and nonlinear regression.**
- **Reverse the object:**
  - instead of trying to fit the largest possible street between two classes while limiting margin violations,
  - SVM Regression tries to **fit as many instances as possible *on* the street** while limiting margin violations (i.e., instances *off* the street).
  - The width of the street is controlled by a hyper-parameter ϵ.



margin violation

SVM Regression

- **SVM regression using 2nd degree polynomial kernel**



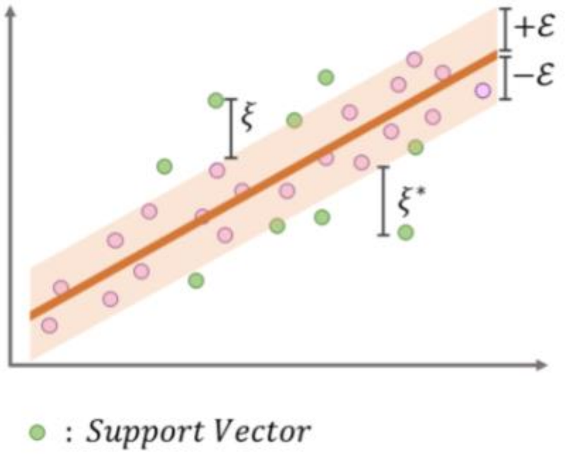
SVM Regression

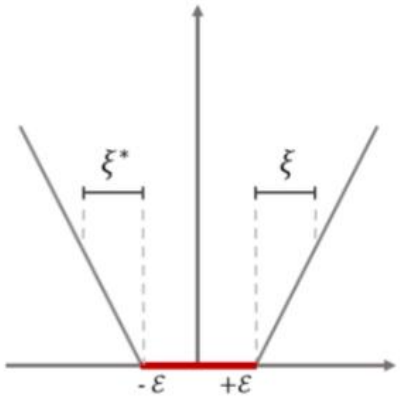
- **Loss function: ε-insensitive loss:**
  - Ignores errors that are within ε distance by treating them as zero
  - Measured based on the distance between observed value y and the ε boundary

$$L_\varepsilon = \begin{cases} 0 & \text{if } |y - f(x)| \le \varepsilon \\ |y - f(x)| - \varepsilon & \text{otherwise} \end{cases}$$

$$L_{SVR} = \min \overbrace{\frac{1}{2}\|w\|^2}^{\text{Robustness}} + \underbrace{C\sum_{i=1}^{n}(\xi_i + \xi_i^*)}_{\text{loss funciton}}$$

$$\begin{aligned} s.t. \quad & (w^T x_i + b) - y_i \le \epsilon + \xi_i \\ & y_i - (w^T x_i + b) \le \epsilon + \xi_i^* \\ & \xi_i, \xi_i^* \ge 0 \end{aligned}$$

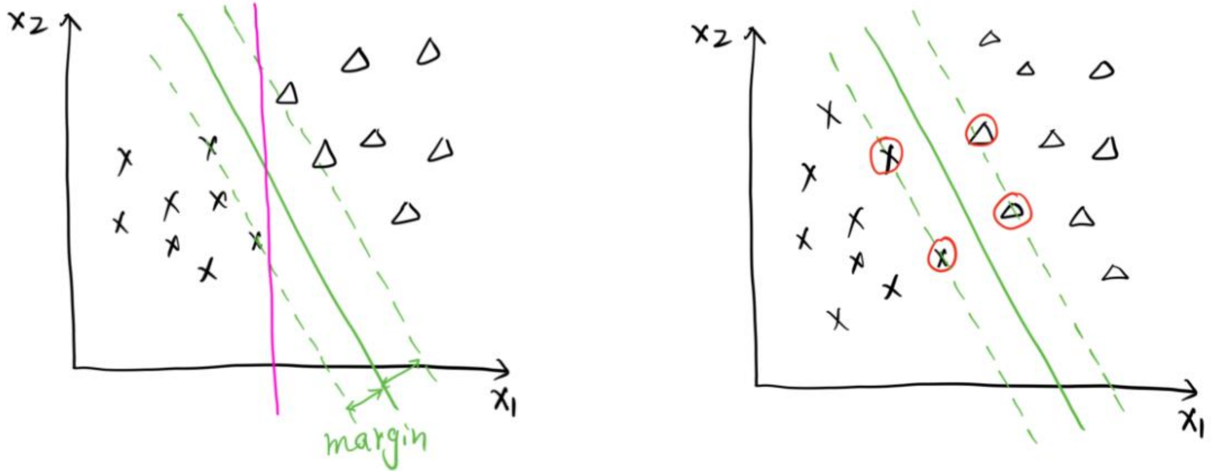


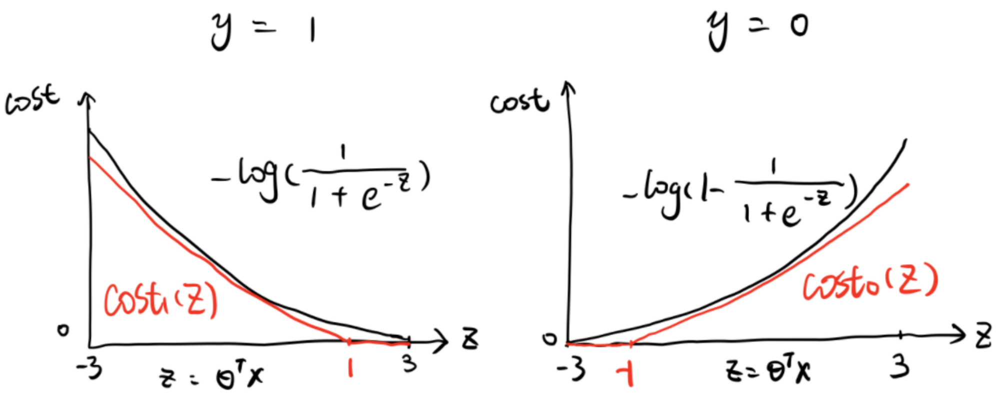


ε − insensitive loss

SVM Summary

- **Linear SVM**
    - Concept
    - support vectors
    - Loss function
    - Hypothesis





$$h_\theta(x) = \begin{cases} 1 & \text{if } \theta^T x >= 0 \\ 0 & \text{otherwise} \end{cases}$$

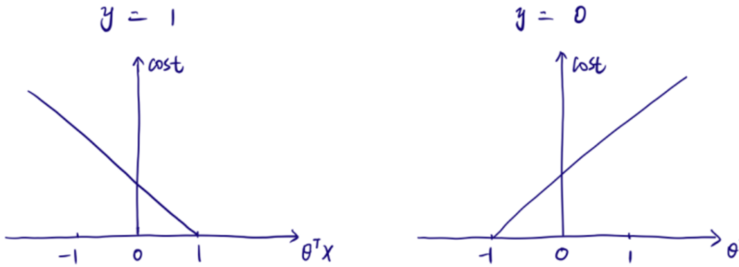
https://towardsdatascience.com/optimization-loss-function-under-the-hood-part-iii-5dff33fa015d

SVM Summary

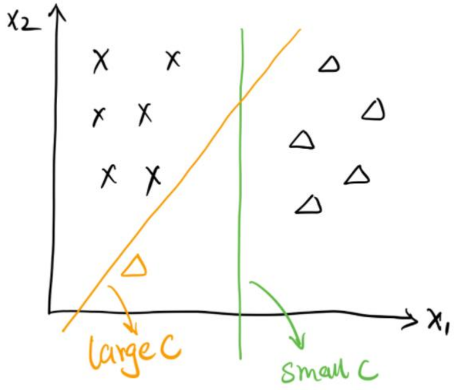
- **SVM cost function**

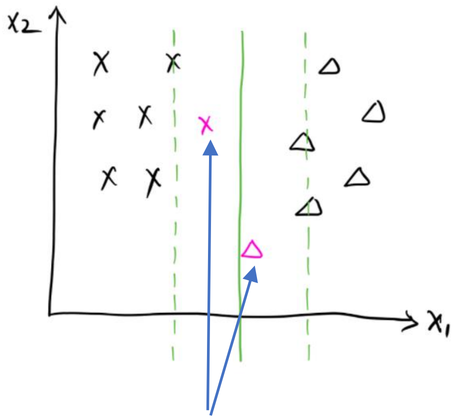
$$J(\theta) = C[\sum_{i=1}^{m} y^{(i)} Cost_1(\theta^T(x^{(i)}) + (1 - y^{(i)})Cost_0(\theta^T(x^{(i)})] + \frac{1}{2}\sum_{j=1}^{n}\theta_j^2$$

$$m = number\ of\ samples, \quad n = number\ of\ features$$



$$Cost(h_\theta(x), y) = \begin{cases} max(0, 1 - \theta^T x) & \text{if } y = 1 \\ max(0, 1 + \theta^T x) & \text{if } y = 0 \end{cases}$$





Violations when C is small

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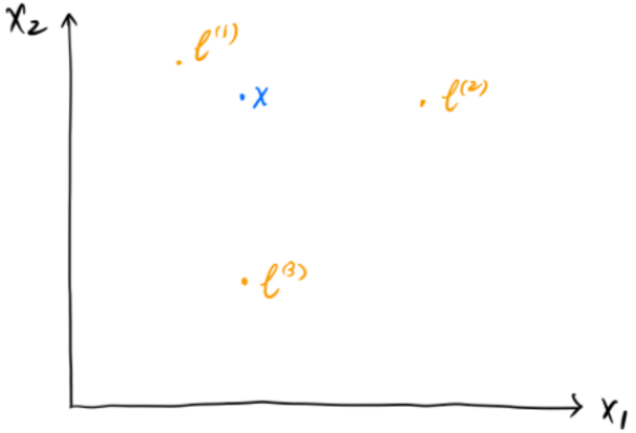
SVM Summary

- **Non-linear (rbf) SVM**
  - Hypothesis and cost function are almost the same (x -> f)
  - Define landmarks to see how close x is to them (similarity) – kernel function
  - Gaussian kernel (RBF: radial basis function) – basically the same (use γ to represent $1/2\sigma^2$)
  - Now we have new features (f1,f2,f3) instead of x1 and x2.
  - Prediction: $\theta^T f = \theta 0 + \theta 1 f1 + \theta 2 f2 + \theta 3 f3$



$$f1 = \mathrm{Similarity}(x, l^{(1)}) \text{ or } k(x, l^{(1)})$$

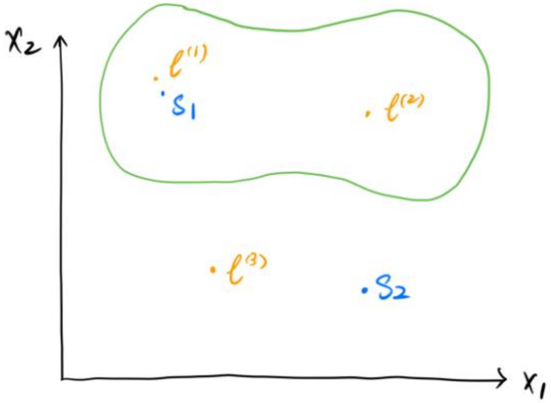
$$f2 = \mathrm{Similarity}(x, l^{(2)}) \text{ or } k(x, l^{(2)})$$

$$f3 = \mathrm{Similarity}(x, l^{(3)}) \text{ or } k(x, l^{(3)})$$

$$f_1 = Similarity(x, l^{(1)}) = exp(\frac{||x - l^{(1)}||^2}{2\sigma^2})$$

$$f_2 = Similarity(x, l^{(2)}) = exp(\frac{||x - l^{(2)}||^2}{2\sigma^2})$$

$$f_3 = Similarity(x, l^{(3)}) = exp(\frac{||x - l^{(3)}||^2}{2\sigma^2})$$



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SVM Summary

- **Non-linear (rbf) SVM**

$$Given\ (x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots\ldots (x^{(m)}, y^{(m)})$$

$$Choose\ l^{(1)} = x^{(1)}, l^{(2)} = x^{(2)}, \ldots\ldots l^{(m)} = x^{(m)}$$

$$m\ =\ number\ of\ samples$$

$$Given\ the\ i^{th}\ sample\ x^{(i)}\ :$$

$$f_1^{(i)} = k(x^{(i)}, l^{(1)})$$

$$f_2^{(i)} = k(x^{(i)}, l^{(2)})$$

$$\ldots\ldots$$

$$f_i^{(i)} = k(x^{(i)}, l^{(i)})$$

$$\ldots\ldots$$

$$f_m^{(i)} = k(x^{(i)}, l^{(m)})$$

$$where\ x^{(i)} = l^{(i)},\ f_i^{(i)} = 1$$

$$Hypothesis :\quad h_\theta(x) = \begin{cases} 1 & \text{if } \theta^T f >= 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\theta^T f = \theta_0 f_0 + \theta_1 f_1 + \ldots + \theta_m f_m$$

$$Regularized\ Cost\ Function\ :$$

$$J(\theta) = C[\sum_{i=1}^{m} [y^{(i)} Cost_1(\theta^T(f^{(i)}) + (1 - y^{(i)}) Cost_0(\theta^T(f^{(i)})] + \frac{1}{2} \sum_{j=1}^{m} \theta_j^2$$

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