SOLUTIONS PROBLEM SHEET 1

Exercise 1.

(i)The answer is yes: every two lines from the plane can be sent one to another through a euclidian transformation. We have to study the following two cases:

Case 1: Let l_1 and l_2 two lines in the plane intersecting at the angle θ at the point p. Making a change of coordinates we can assume that p is the origin. We define the rotation of angle θ:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Then $R(\theta)(l_1) = l_2$.

Case 2: Let l_1 and l_2 two parallel lines in the plane. Then there exists a translation sending l_1 to l_2.

(ii) The answer is not necessarily : every pairs of two intersecting lines from the plane can be sent one to another trough a euclidian transformation if and only if they form the same angle.

Let l_1, l_2 be two lines in the plane intersecting at the angle θ_1 at the point p_1. Let l_3, l_4 be two lines in the plane intersecting at the angle θ_2 at the point p_2. Then there exists a unique euclidian transformation T such that $T(p_1) = p_2$ and $T(l_1) = l_3$. Now the proof is straightforward.

(iii) The answer is yes.

Exercise 2.

(i) $C : xy + x - 1 = 0$. Using the transformation $(X, Y) = (x, y + 1)$ it follows that $C : XY = 1$ (hyperbola).

(ii) $C : x^2 - 2xy + y^2 + x - 1 = 0$. Then $C : (x - y)^2 + x - 1 = 0$. Using the transformation $(X, Y) = (-x + 1, y - x)$ it follows that $C : X = Y^2$ (parabola).

(iii) $C : x^2 - 2xy + y^2 - 1 = 0$. Then $C : (x - y)^2 = 1$. Using the transformation $(X, Y) = (x, y - x)$ it follows that $C : 1 = Y^2$ (two lines).

(iv) $C : x^2 + y^2 - y - x - 1 = 0$. Then $C : (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + \frac{1}{2} = 0$. Using the transformation $(X, Y) = (x - \frac{1}{2}, y - \frac{1}{2})$ it follows that $C : Y^2 + X^2 = -\frac{1}{2}$ (empty set).

Exercise 3.

(i) In the coordinates $(x, y) = \left(\frac{X}{Z}, \frac{Y}{Z}\right)$ the line $\widetilde{L}$ is defined as follows

$$\widetilde{L} : \frac{Y}{Z} = a\frac{X}{Z} + b, \quad \widetilde{L} : Y = aX + bZ. \tag{0.1}$$

(ii) Using the second equation from (0.1) we obtain the equation of the induced line on B

$$\widetilde{L} : \frac{Y}{X} = a + b\frac{Z}{X}, \quad \widetilde{L} : y' = a + bz'. \tag{0.2}$$

(iii) The ,,infinity point" for $\widetilde{L}$ is $[(X, aX, 0)]$, for $X \neq 0$. This is obtained taking $Z = 0$ in the second equation from (0.1).

(iv) Let $[(X, aX, 0)] \in B$, for $X \neq 0$. Then $[(1, a, 0)] \in B$ and this is true $\forall a, b$.