Sheet 7

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(1) reflexive: $e \cdot x = x$, so $x \sim x$ for all $x \in S$

3 symmetric: If $x \sim y$, then $x = g \cdot y$ for some $g \in G$. So $y = g^{-1} \cdot x$ and $y \sim x$.

transitive: If $x \sim y \sim z$, then $x = g \cdot y$, $y = h \cdot z$ for certain $g, h \in G$.
So $x = g \cdot (h \cdot z) = (gh) \cdot z$ and $x \sim z$.

We have $O_x = \{g \cdot x \mid g \in G\} = \{y \in S \mid \exists_{g \in G}\, y = g \cdot x\} = [x]_\sim$.
So the orbits of G on S are the equivalence classes of $\sim$.

(2) (i) Clearly $\mathrm{id} \in \mathrm{Aut}(G)$. If $\varphi, \psi \in \mathrm{Aut}(G)$, then we have for $x, y \in G$
2 $(\varphi \circ \psi)(xy) = \varphi(\psi(x)\psi(y)) = (\varphi\circ\psi)(x)(\varphi\circ\psi)(y)$, so $\varphi\circ\psi \in \mathrm{Aut}(G)$. If $\varphi \in \mathrm{Aut}(G)$ and $x, y \in G$, then $xy = \varphi(\varphi^{-1}(x))\varphi(\varphi^{-1}(y)) = \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$, so $\varphi^{-1}(xy) = \varphi^{-1}(x)\varphi^{-1}(y)$ and $\varphi^{-1} \in \mathrm{Aut}(G)$.

2 (ii) $\mathrm{int}(g)(h_1 h_2) = g h_1 h_2 g^{-1} = g h_1 g^{-1} g h_2 g^{-1} = \mathrm{int}(g)(h_1)\, \mathrm{int}(g)(h_2)$. So $\mathrm{int}(g)$ is a homomorphism. Clearly $\mathrm{int}(g)$ is invertible with inverse $\mathrm{int}(g^{-1})$, so $\mathrm{int}(g) \in \mathrm{Aut}(G)$.

2 (iii) $\mathrm{int}(g_1 g_2)(h) = g_1 g_2 h (g_1 g_2)^{-1} = g_1 g_2 h g_2^{-1} g_1^{-1} = (\mathrm{int}(g_1) \circ \mathrm{int}(g_2))(h)$.
So $\mathrm{int}(g_1 g_2) = \mathrm{int}(g_1) \circ \mathrm{int}(g_2)$ and $\mathrm{int}: G \to \mathrm{Aut}(G)$ is a homomorphism.

3 (i) If $(a_1 \ldots a_m) \in S_n$ is a cycle and $\pi \in S_n$, then $\pi (a_1 \ldots a_m)\pi^{-1} = (\pi(a_1) \ldots \pi(a_m))$ which
3 is a cycle of the same length. If $(a_1 \ldots a_m), (b_1 \ldots b_m) \in S_n$ are cycles of the same length, then we can make a bijection from $\{1, \ldots, n\}$ to itself by picking a bijection $\pi : \{1, \ldots, n\} \setminus \{a_1, \ldots, a_m\} \to \{1, \ldots, n\} \setminus \{b_1, \ldots, b_m\}$ and extending it to $\{1 \ldots n\}$ by the rule $\pi(a_i) = b_i$, $i = 1, \ldots, m$; and for this π we will have $\pi(a_1 \ldots a_m)\pi^{-1} = (\pi(a_1) \ldots \pi(a_m)) = (b_1 \ldots b_m)$.

(ii) If $\tau = \sigma_1 \ldots \sigma_r$ is the disjoint cycle form of τ, then $\pi\tau\pi^{-1} = \pi\sigma_1\pi^{-1} \cdots \pi\sigma_r\pi^{-1}$
3 is the disjoint cycle form of $\pi\tau\pi^{-1}$! So τ and $\pi\tau\pi^{-1}$ have for each cycle length the same number of cycles of that length in their disjoint cycle form.
Now assume $\tau = \sigma_1 \ldots \sigma_r$ and $\tau' = \sigma_1' \ldots \sigma_r'$ are the disjoint cycle forms of $\tau, \tau' \in S_n$ such that σ_i and σ_i' have the same length. Let S_i and S_i' be the sets of numbers that occur in σ_i and σ_i'. Now we pick an arbitrary bijection $\pi : \{1 \ldots n\} \setminus \bigcup_{i=1}^{r} S_i \to \{1 \ldots n\} \setminus \bigcup_{i=1}^{r} S_i'$ and extend it to a bijection $\pi : \{1 \ldots n\} \to \{1 \ldots n\}$ by mapping S_i onto S_i' in such a way that $\pi\sigma_i\pi^{-1} = \sigma_i'$; this can be done as in part (i). Then we have $\pi\tau\pi^{-1} = \tau'$.