INTRODUCTION TO LIE ALGEBRAS – SOLUTION 4

In each of the cases, we have to show that any element of $\mathfrak{g}$ can be obtained as a linear combination of commutators $[X, Y] = XY - YX$ where $X, Y \in \mathfrak{g}$. In each case, we will choose a basis in $\mathfrak{g}$ and show that every basis element equals a certain commutator in $\mathfrak{g}$, up to a scalar multiple.

(i) We will give the proof for arbitrary $n \geqslant 2$. Consider the matrix units $E_{ij} \in \mathfrak{gl}_n\mathbb{C}$ where $i, j = 1, \ldots, n$. The $n \times n$ matrix E_{ij} has only one non-zero entry, in the row i and the column j. The matrices E_{ij} where $i \neq j$, and the matrices $E_{ii} - E_{i+1,i+1}$ where $i < n$, form a basis in the special linear Lie algebra $\mathfrak{sl}_n\mathbb{C}$. For $i < n$ we get

$$[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}\,.$$

For $i \neq j$ we have

$$[E_{ii} - E_{jj}, E_{ij}] = 2E_{ij}\,.$$

Note that here we took the commutators of $n \times n$ matrices, each of which has zero trace and therefore belongs to $\mathfrak{sl}_n\mathbb{C}$.

(ii) We will give the proof for arbitrary $n \geqslant 3$. Again consider the matrix units E_{ij} where $i, j = 1, \ldots, n$. The matrices $E_{ij} - E_{ji}$ where $i < j$, constitute a basis in the orthogonal Lie algebra $\mathfrak{so}_n\mathbb{C}$. Since $n \geqslant 3$, for any $i \neq j$ we can find an index $k \in \{1, \ldots, n\}$ such that $k \neq i, j$. Then

$$[E_{ik} - E_{ki}, E_{kj} - E_{jk}] = E_{ij} - E_{ji}\,.$$

(iii) We will give the proof for arbitrary $n \geqslant 1$. The symplectic Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ is a certain subalgebra in the general linear Lie algebra $\mathfrak{gl}_{2n}\mathbb{C}$. We will use the matrix units from $\mathfrak{gl}_{2n}\mathbb{C}$. We can denote them by E_{ij} where $i, j = 1, \ldots, 2n$. But it will be more convenient to let the indices i, j still range over $1, \ldots, n$, and to write

$$E_{ij}\ ,\ E_{i,j+n}\ ,\ E_{i+n,j}\ ,\ E_{i+n,j+n}\,.$$

Choose the basis in $\mathfrak{sp}_{2n}\mathbb{C}$ consisting of the elements

$$E_{ij} - E_{j+n,i+n}\ ,\ E_{i,j+n} + E_{j,i+n}\ ,\ E_{i+n,j} + E_{j+n,i}\,.$$

For any $i, j = 1, \ldots, n$ we have the equalities

$$\begin{aligned}
[E_{ij} - E_{j+n,i+n}\,, E_{j,j+n}] &= E_{i,j+n} + E_{j,i+n}\,,\\
[E_{ij} - E_{j+n,i+n}\,, E_{i+n,i}] &= -(E_{i+n,j} + E_{j+n,i})\,,\\
[E_{i,j+n} + E_{j,i+n}\,, E_{j+n,j}] &= (1 + \delta_{ij})\,(E_{ij} - E_{j+n,i+n})\,.
\end{aligned}$$

Here we took commutators of $2n \times 2n$ matrices, each of which belongs to $\mathfrak{sp}_{2n}\mathbb{C}$.

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 5

For any two matrices $X, Y \in \mathfrak{gl}_n\mathbb{C}$ their commutator $[X, Y] = XY - YX$ has the zero trace. Therefore the derived algebra $(\mathfrak{gl}_n\mathbb{C})'$ is contained in $\mathfrak{sl}_n\mathbb{C}$. To prove the equality $(\mathfrak{gl}_n\mathbb{C})' = \mathfrak{sl}_n\mathbb{C}$, we have to show that any matrix from $\mathfrak{sl}_n\mathbb{C}$ (that is, of trace zero) can be obtained as a linear combination of the commutators in $\mathfrak{gl}_n\mathbb{C}$. When $n = 1$ we have $\mathfrak{sl}_n\mathbb{C} = \{0\}$, so there is nothing to show. We will assume that $n \geqslant 2$. Let us use the basis of the Lie algebra $\mathfrak{gl}_n\mathbb{C}$ consisting of the matrix units E_{ij} where $i, j = 1, \ldots, n$. The $n \times n$ matrix E_{ij} has only one non-zero entry, in the row i and the column j. The commutator of any two such matrices is

$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}\,.$$

The matrices E_{ij} where $i \neq j$, and the matrices $E_{ii} - E_{i+1,i+1}$ where $i < n$, form a basis in $\mathfrak{sl}_n\mathbb{C}$. It suffices to present only these basis matrices as commutators. But for $i \neq j$ we have

$$[E_{ij}, E_{jj}] = E_{ij}\,.$$

For $i < n$ we get

$$[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}\,.$$