INTRODUCTION TO LIE ALGEBRAS – SOLUTION 14

First suppose that (i) the Lie algebra $\mathfrak{g}$ is solvable. Then in the sequence

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supset \mathfrak{g}^{(1)} = \mathfrak{g}' \supset \mathfrak{g}^{(2)} = \mathfrak{g}'' \supset \ldots$$

of derived algebras we have $\mathfrak{g}^{(n)} = \mathfrak{g}^{(n+1)} = \ldots = \{0\}$ for some sufficiently large number n. Thus $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(n)} = \{0\}$, and the factor Lie algebra $\mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}$ is Abelian by Example 13. Since, by the Jacobi Identity, the Lie bracket of two ideals is again an ideal, it follows by induction on i that each $\mathfrak{g}^{(i)}$ is an ideal in $\mathfrak{g}$. We have proved that (i) $\Rightarrow$ (ii).

Now suppose that (ii) there exist ideals $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_n$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = \{0\}$, and for every index $i = 0, \ldots, n-1$ the factor Lie algebra $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is Abelian. Let us prove inductively for $i = n, n-1, \ldots, 0$ that the Lie algebra $\mathfrak{g}_i$ is solvable. For $i = 0$ we will then obtain that $\mathfrak{g}$ is solvable. Note that $\mathfrak{g}_n = \{0\}$. In particular, $\mathfrak{g}_n$ is solvable. Now assume that $\mathfrak{g}_{i+1}$ is solvable for some $i < n$. We know that the factor Lie algebra $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is Abelian. In particular, it is solvable. Then $\mathfrak{g}_i$ is solvable, because both $\mathfrak{g}_{i+1}$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ are solvable.

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 15

Consider the image $\operatorname{ad}\mathfrak{g} \subset \operatorname{Der}(\mathfrak{g})$ of $\mathfrak{g}$. The linear map $X \mapsto \operatorname{ad} X$ is a homomorphism of the Lie algebra $\mathfrak{g}$ onto this image. Therefore if $\mathfrak{g}$ is solvable or nilpotent, then so is $\operatorname{ad}\mathfrak{g}$ by Theorem B.

Now we prove the other implication. The kernel of the map $X \mapsto \operatorname{ad} X : \mathfrak{g} \to \operatorname{ad}\mathfrak{g}$ is exactly the centre $\mathcal{Z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$:

$$\operatorname{ad} X = 0 \quad \Leftrightarrow \quad [X, Y] = 0 \ \text{ for any } \ Y \in \mathfrak{g}\,.$$

So, by a general property of Lie algebra homomorphisms, the Lie algebra $\operatorname{ad}\mathfrak{g}$ is isomorphic to the factor Lie algebra $\mathfrak{g}/\mathcal{Z}(\mathfrak{g})$. Hence $\mathfrak{g}/\mathcal{Z}(\mathfrak{g})$ is solvable or nilpotent if and only if this holds for $\operatorname{ad}\mathfrak{g}$. Furthermore, the centre $\mathcal{Z}(\mathfrak{g})$ is abelian and therefore solvable. So the assertion follows from Theorem C.