INTRODUCTION TO LIE ALGEBRAS – SOLUTION 6

Let us use the matrix units E_{ij}. The $n \times n$ matrix E_{ij} has only one non-zero entry, in the row i and column j. The commutator of any two such matrices is

$$[E_{ij}, E_{pq}] = E_{ij}E_{pq} - E_{pq}E_{ij} = \delta_{pj}E_{iq} - \delta_{iq}E_{pj} \,.$$

The matrices E_{ij} where $j - i \geqslant k$, form a basis in $\mathfrak{g}_k$. Now suppose that $E_{ij} \in \mathfrak{g}_k$ and $E_{pq} \in \mathfrak{g}_l$. This means that $j - i \geqslant k$ and $q - p \geqslant l$. Taking the sum of these two inequalities, we get

$$j - i + q - p \geqslant k + l \,.$$

In particular, we get $q - i \geqslant k + l$ when $p = j$, and $j - p \geqslant k + l$ when $q = i$. Our formula for the commutator of two matrix units now shows that $[E_{ij}, E_{pq}] \in \mathfrak{g}_{k+l}$. Hence

$$[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}. \quad (*)$$

Furthermore, for any indices $q, i \in \{1, \dots, n\}$ such that $0 > q - i \geqslant k + l$, we have

$$[E_{i,i+k}, E_{i+k,q}] = E_{iq} \,.$$

Here $E_{i,i+k} \in \mathfrak{g}_k$ and $E_{i+k,q} \in \mathfrak{g}_l$, since $q - i - k \geqslant l$. Therefore

$$[\mathfrak{g}_k, \mathfrak{g}_l] \supset \mathfrak{g}_{k+l} \,. \quad (**)$$

The proof of (*) was valid for all integers k and l, so $[\mathfrak{t}, \mathfrak{g}_k] \subseteq [\mathfrak{g}_0, \mathfrak{g}_k] \subseteq \mathfrak{g}_k$. Since $[E_{ii}, E_{ij}] = E_{ij}$ for all i, j with $1 \leqslant i < j \leqslant n$, we get that $[\mathfrak{t}, \mathfrak{g}_k] = \mathfrak{g}_k$ for all $k \geqslant 1$. Thus $[\mathfrak{g}_0, \mathfrak{g}_k] = [\mathfrak{t}, \mathfrak{g}_k] + [\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_k$ for all $k \geqslant 1$. Note that this also follows from the fact that the proof of (**) was valid for all $k, l \geqslant 0$ such that $k > 0$ or $l > 0$. Furthermore, $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{t}, \mathfrak{g}_1] + [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_1$. It follows that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{gl}_n\mathbb{F}$ and that each $\mathfrak{g}_k$ is an ideal of $\mathfrak{g}_0$.

Let us use the induction on $n = 1, 2, \dots$. In the case $n = 1$ the equality to be proved is

$$D[X,Y] = [D(X),Y] + [X,D(Y)]\,.$$

This is exactly the Leibniz rule, which holds by the definition of a derivation. We will write

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

as usual. Let us now assume that the equality

$$D^n[X,Y] \,=\, \sum_{k=0}^{n} \binom{n}{k} \cdot [\,D^k X, D^{n-k} Y\,]$$

holds for a given $n \geqslant 1$ and all $X, Y \in \mathfrak{g}$. Using the definition of D^{n+1}, we then get

$$D^{n+1}[X,Y] = D\big(D^n[X,Y]\big) \,=\, \sum_{k=0}^{n} \binom{n}{k} \cdot D[\,D^k X, D^{n-k} Y\,] \,=$$

$$\sum_{k=0}^{n} \binom{n}{k} \cdot [\,D^{k+1} X, D^{n-k} Y\,] + \sum_{k=0}^{n} \binom{n}{k} \cdot [\,D^k X, D^{n-k+1} Y\,] \,=$$

$$\sum_{k=1}^{n+1} \binom{n}{k-1} \cdot [\,D^k X, D^{n+1-k} Y\,] + \sum_{k=0}^{n} \binom{n}{k} \cdot [\,D^k X, D^{n+1-k} Y\,] \,=$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \cdot [\,D^k X, D^{n+1-k} Y\,]\,,$$

which makes the induction step. Here the last equality holds because

$$\binom{n}{0} = 1 = \binom{n+1}{0}\;,\;\binom{n}{n} = 1 = \binom{n+1}{n+1}$$

and

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

for any $k = 1, \dots, n$.

Recall that the eigenspace V_λ of the eigenvalue λ of a linear operator X on V is defined by

$$V_\lambda := \{v \in V \mid X(v) = \lambda v\}.$$

Lemma. *Suppose X is a linear operator on a finite dimensional vector space V and let $\lambda_1, \dots, \lambda_r$ be disctinct eigenvalues of X. Then the sum $\sum_{i=1}^r V_{\lambda_i}$ of the eigenspaces V_{λ_i} is direct.*

Proof. We prove this by induction on r. If $r = 1$, there is nothing to prove. Assume the assertion holds for r. We prove it for $r+1$. For each $i = 1, \dots, r+1$ let $v_i \in V_{\lambda_i}$ and assume that $v_1 + \cdots + v_{r+1} = 0$ (*). Applying X to (*) we get $\lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} = 0$ (**). Multiplying (*) by λ_{r+1} and subtracting it from (**) we obtain $(\lambda_1 - \lambda_{r+1})v_1 + \cdots + (\lambda_r - \lambda_{r+1})v_r = 0$. Now the induction hypothesis gives us $(\lambda_i - \lambda_{r+1})v_i = 0$ for all $i \in \{1, \dots, r\}$. Since $\lambda_i \neq \lambda_{r+1}$ for all $i \in \{1, \dots, r\}$, we get $v_i = 0$ for all $i \in \{1, \dots, r\}$. Now (*) gives us $v_{r+1} = 0$. $\square$

Now we continue with the exercise. For $i = 1, \dots, n$ pick $v_i \in V_{a_i}$ nonzero. Then the v_i are linearly independent by the above lemma and since we have n of them, they must form a basis of V. So X is semisimple. Note also that each V_{a_i} must be one dimensional.

Now assume X is semisimple with eigenvalues $a_1, \dots, a_n$. Let $(v_1, \dots, v_n)$ be a basis of V such that v_i is an X-eigenvector with eigenvalue a_i. Let us identify the Lie algebra $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n\mathbb{F}$ by using this basis. Then consider the matrix units $E_{ij} \in \mathfrak{gl}_n\mathbb{F}$. The operator X is then identified with the diagonal matrix $A = a_1E_{11} + \ldots + a_nE_{nn}$. The elements E_{ij} form a basis of eigenvectors of the operator $\operatorname{ad} A$ in the vector space $\mathfrak{gl}_n\mathbb{F}$:

$$\big(\operatorname{ad} A\big)(E_{ij}) = [a_1E_{11} + \ldots + a_nE_{nn}, E_{ij}] = (a_i - a_j)E_{ij}\,; \quad i,j = 1, \dots, n\,.$$

So $\operatorname{ad} X$ is semisimple with eigenvlues $a_i - a_j$, $i,j = 1, \dots, n$.