MA1214 Extra Exercises 1

(0) Prove some of the results that were stated but not (completely) proved in the lectures. For example:

(i) $\mathrm{range}(g \circ f) = g(\mathrm{range}(f))$.

(ii) If $g \circ f$ and $h \circ g$ are defined, then $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined and equal.

$\vdots$

(1) Consider the sine and cosine function $\sin, \cos : \mathbb{R} \to \mathbb{R}$. Are they injective? What is their range?

(2) Consider the exponential map $\exp : \mathbb{R} \to \mathbb{R}$. Is it injective? What is its range? When you replace the codomain of $\exp$ by its range, does the resulting map have an inverse?

(3) Answer the same questions as in (2) for the complex exponential $\exp : \mathbb{C} \to \mathbb{C}$.
Hint. $\exp(x + iy) = \exp(x)\exp(iy) = \exp(x)\big(\cos(y) + i\sin(y)\big)$.
If you are really ambitious, answer the same questions as in (2) for the complex sine $\sin(z) = 1/2i\big(\exp(iz) - \exp(-iz)\big)$ and cosine $\cos(z) = 1/2\big(\exp(iz) + \exp(-iz)\big)$.

(4) A *congruence* on a semigroup S is a relation $\equiv$ on S such that $x_1 \equiv x_2$ and $y_1 \equiv y_2$ implies $x_1y_1 \equiv x_2y_2$ for all $x_1, x_2, y_1, y_2 \in S$. Now let $\equiv$ be a congruence on a semigroup S. We denote the set of equivalence classes by $S/{\equiv}$.

(i) Show that the map $([x]_\equiv, [y]_\equiv) \mapsto [xy]_\equiv : S/{\equiv} \times S/{\equiv} \to S/{\equiv}$ is well-defined and show that this turns $S/{\equiv}$ into a semigroup. Show that if S is a monoid or a group, then so is $S/\equiv$.

(ii) Consider the monoid $M = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with operation $(k, l)(m, n) = (km, ln)$. Show that the relation $(k, l) \equiv (m, n) \Leftrightarrow \exists_{s,t\in\mathbb{Z}}\big(s, t \neq 0 \wedge (sk, sl) = (tm, tn)\big)$ (or: $kn = lm$) is a congruence on M. Denote $[(k, l)]_\equiv$ by k/l and denote $[(1, 1)]_\equiv$ and $[(0, 1)]_\equiv$ by 1 (one) and 0 (zero). Show that $M/{\equiv}$ is a monoid and that all nonzero elements in $M/{\equiv}$ are invertible. The monoid $M/{\equiv}$ is nothing but the multiplicative monoid of $\mathbb{Q}$, the field of rational numbers.

(iii) The notation is as in (ii). Show that $\mathbb{Z}$ embeds in $M/{\equiv}$ via $k \mapsto k/1$. How would you extend the addition of the integers to $M/{\equiv}$?