MA1214 Sheet 5

(1) The *centre* of a group G is the set of elements which commute with everything:

$$\{x \in G : \ (\forall y \in G) \ xy = yx\}$$

Call this set C.

(i) Prove that C is a normal subgroup of G.

(ii) Show that if $\pi \in S_n$ commutes with a cycle $\sigma = (a_1 \dots a_k)$, $k \geq 2$, $a_i \neq a_j$ for $i \neq j$, then π maps $\{a_1, \dots, a_k\}$ into itself.
Hint. Use the identity $\pi(a_1 \dots a_k)\pi^{-1} = (\pi(a_1) \dots \pi(a_k))$. For a more direct proof: When π and σ commute, then $\pi^{-1}(a_1 \dots a_k)\pi = (a_1 \dots a_k)$. Now let $i \in \{1, \dots, k\}$ and assume, for a contradiction, that $\pi(a_i) \notin \{a_1, \dots, a_k\}$.

(iii) Deduce from (ii) that if $n \geq 3$, then the centre of S_n is trivial (i.e., $= \{1\}$).
Hint. You only need (ii) for transpositions (i.e. $k = 2$). Use the equality $\{i, j\} \cap \{i, j'\} = \{i\}$ for $i, j, j' \in \{1, \dots, n\}$ distinct.

(2) Let G and G' be groups, let N be a normal subgroup of G and let $f : G \to G'$ be a surjective homomorphism. Show that $f(N)$ is a normal subgroup of G'.

(3) Let G be a group and let H and K be normal subgroups of G such that $H \cap K = \{1\}$.

(i) Show that the elements of H commute with those of K. *Hint*. Let $x \in H$ and $y \in K$ and consider the element $xyx^{-1}y^{-1} = x(yx^{-1}y^{-1}) = (xyx^{-1})y^{-1}$.

(ii) Now assume in addition that H and K generate G, i.e. every element of G can be written as a product of an element of H and an element of K. Show that $G \cong H \times K$ and that $G/H \cong K$.