FURTHER NOTES FOR THE COURSE LIE ALGEBRAS

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PLEASE NOTE: All material discussed here is NON-EXAMINABLE.

Lie groups and algebraic groups

The notion of a Lie algebra can be justified by the much more natural/obvious notion of a Lie group. The notion of a Lie algebra has the advantage that it is more elementary and easier to use in representation theory.

Throughout k is a field. When dealing with Lie groups $k = \mathbb{R}$ or $k = \mathbb{C}$, and in case $k = \mathbb{C}$ "smooth" should be read as "analytic". When dealing with algebraic groups k is supposed to be algebraically closed.

Definition. *A* **Lie group** *is a group over k which is also a smooth manifold over k such that multiplication and inversion are morphisms of manifolds.*

Definition. *An* **algebraic group** *is a group which is also an algebraic variety over k such that multiplication and inversion are morphisms of algebraic varieties.*

I describe the definition of the Lie algebra of a Lie group or algebraic group. Recall that for a smooth manifold M the space of smooth vector fields can be identified with the space of derivations of the algebra of smooth functions on M:

$$\mathrm{Vect}^\infty(M) \cong \mathrm{Der}(C^\infty(M)).$$

If $v = (v_p)_{p\in M}$ is a smooth vector field, then $f \mapsto \big(p \mapsto (d_pf)(v_p)\big)$ is the corresponding derivation of $C^\infty(M)$. Here $d_pf : T_p(M) \to k$ denotes the differential of f at p. As we have seen in the lectures, $\mathrm{Der}(C^\infty(M))$ is a Lie algebra, so we can carry this structure over to $\mathrm{Vect}^\infty(M)$. Similarly, the tangent space of M at p can be identified with the point derivations of $C^\infty(M)$ at p.

$$T_p(M) \cong \mathrm{Der}_p(C^\infty(M)).$$

If v is a tangent vector at p, then $f \mapsto (d_pf)(v)$ is the corresponding point derivation of $C^\infty(M)$ at p. Recall that a point derivation of $C^\infty(M)$ at p is a linear map $D : C^\infty(M) \to k$ such that $D(fg) = D(f)g(p) + f(p)D(g)$ for all $f, g \in C^\infty(M)$.

Now let G be a Lie group and denote for $g \in G$ the left multiplication $h \mapsto gh : G \to G$ by λ_g. Denote the corresponding automorphism $f \mapsto f \circ \lambda_g$ of $C^\infty(M)$ by $\overline{\lambda}_g$.

Definition. *A smooth vector field $v = (v_g)_{g\in G}$ on G is called* **left invariant** *if $v_{gh} = (d_h\lambda_g)(v_h)$ for all $g, h \in G$.*

Actually it is enough that

$$v_g = (d_e\lambda_g)(v_e) \text{ for all } g \in G. \tag{$*$}$$

Now it is clear that we can construct an isomorphism between $T_e(G)$ and the space of left invariant smooth vector fields on G: If $v \in T_e(G)$, then we put $v_e = v$ and we define v_g by (*). On the other hand, a smooth vector field is left invariant if and only if the corresponding derivation of $C^\infty(G)$ commutes with all the $\overline{\lambda}_g$, $g \in G$. Since those derivations form obviously a Lie subalgebra of $\mathrm{Der}(C^\infty(G))$, the left invariant vector fields form a Lie subalgebra of $\mathrm{Vect}^\infty(G)$.

Definition. *The* **Lie algebra** $\mathrm{Lie}(G)$ **of** G *is* $T_e(G)$ *endowed with the structure of a Lie algebra which comes from the isomorphism with the space of left invariant vector fields.*

With a few small modifications (e.g. one has to replace $C^\infty(G)$ by the algebra of regular functions $k[G]$) the above procedure can also be used to define the Lie algebra of an algebraic group G. In this case there is another construction of the Lie algebra which uses Hopf algebras, see [6].

Poisson brackets

For $f, g \in k[x_1, \dots, x_n, y_1, \dots, y_n]$ put

$$\{f, g\} := \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} \Big) .$$

Then $\{\ ,\ \}$ is a Lie bracket. Moreover, the left Lie multiplications $\{f, -\}$ are derivations of the commutative algebra $k[x_1, \dots, x_n, y_1, \dots, y_n]$. Such an algebra (endowed with both multiplications) is called a **Poisson algebra**. One can do the same for smooth functions in $2n$ variables.

More generally, if M is a symplectic manifold, i.e. a smooth manifold with a nondegenerate smooth symplectic form ω which is closed ($d\omega = 0$), we can define a Poisson bracket by

$$\{f, g\} := \omega((df)^\sharp, (dg)^\sharp),$$

where $(df)^\sharp$ denotes the vector field that corresponds to the covector field df by means of ω: $d_p f(v) = \omega_p(((df)^\sharp)_p, v)$ for all $p \in M$ and $v \in T_p(M)$.

References

Lie algebras:

[1] K. Erdmann, M. J. Wildon, *Introduction to Lie algebras*, Springer Undergraduate Mathematics Series, Springer-Verlag London, London, 2006.

[2] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York-Berlin, 1972.

[3] N. Jacobson, *Lie Algebras*, Interscience, New York, 1962.

[4] J.-P. Serre, *Lie algebras and Lie groups*, W. A. Benjamin, New York, 1965.

Lie groups and algebraic groups:

[5] J. F. Adams, *Lectures on Lie groups*, W. A. Benjamin, New York-Amsterdam, 1969.

[6] A. Borel, *Linear algebraic groups*, Second edition, Graduate Texts in Mathematics 126, Springer-Verlag, New York, 1991.

[7] P. M. Cohn, *Lie groups*, Cambridge University Press, New York, 1957.

[8] J. J. Duistermaat, J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

Further reading:

[9] R. Abraham, J. E. Marsden, T. Ratiu, *Manifolds, tensor analysis, and applications*, Third edition.

[10] N. Bourbaki, *Groupes et Algèbres de Lie*, Ch. 1-9.

[11] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics 98, Springer-Verlag, New York, 1985.

[12] W. Fulton, J. Harris, *Representation theory A first course*, Graduate Texts in Mathematics 129, Readings in Mathematics, Springer-Verlag, New York, 1991.

[13] Helgason, Sigurdur *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics 80, Academic Press, New York-London, 1978.

[14] P. J. Olver, *Applications of Lie groups to differential equations*, Second edition, Graduate Texts in Mathematics, 107. Springer-Verlag, New York, 1993.

[15] H. Strade, R. Farnsteiner, *Modular Lie algebras and their representations*, Marcel Dekker, Inc., New York, 1988.

[16] P. Tauvel, R. W. T. Yu, *Lie algebras and algebraic groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.