Course 2318 2012

Sheet 3

Due: see www.maths.tcd.ie/∼rtange/teaching/algebraic_geometry/algebraic_geometry.html

Exercise 1

Let p_1, p_2, p_3, p_4 be distinct points of $\mathbb{P}^2$ with no 3 collinear.

(i) Prove that there exists coordinate system, unique upto multiplication by a single nonzero scalar, in which the 4 points are $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ and $(1 : 1 : 1)$.

(ii) Find all conics passing through $p_1, \dots, p_4$ and $p_5 = (a : b : c)$.

Exercise 2

Use the parametrization of the cuspidal cubic $C = \{y^2 = x^3\}$ to show that any polynomial vanishing on C is divisible by $y^2 - x^3$.

Exercise 3

Let C be the curve given by $f(x, y) = 0$ and $p = (a, b) \in C$. Assume that the gradient $\nabla f = (f_x, f_y)$ is nonzero at (a, b).

(i) Recall that the tangent line to C at p is defined by the equation

$$f_x(p)(x - a) + f_y(p)(y - b) = 0\,.$$

Show that L is the unique line through p such that $f|L$ has a multiple root at p.

(ii) Show that the tangent line can be obtained as the limit of the secant line passing through p and another point $q \in C$ as $q \to p$.

Exercise 4

Let C be given by $y^2 = x(x - 1)(x + 1)$.

(i) What is the projectivisation $\widetilde{C}$ of C, i.e. the cubic obtained in $\mathbb{P}^2$ by homogenization of the equation for C? Find the point at infinity of $\widetilde{C}$.

(ii) Show that $\widetilde{C}$ is smooth.

(iii) Choosing the origin at infinity, find all points of order 2 with respect to the group law, i.e. all points $A \in \widetilde{C}$ with $2A = A + A = 0$. Hint. Use Exercise 3.