# A FROBENIUS SPLITTING AND COHOMOLOGY VANISHING FOR THE COTANGENT BUNDLES OF THE FLAG VARIETIES OF $\mathrm{GL}_n$

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Abstract. Let $k$ be an algebraically closed field of characteristic $p>0$, let $G=\mathrm{GL}_n$ be the general linear group over $k$, let $P$ be a parabolic subgroup of $G$, and let $\mathfrak{u}_P$ be the Lie algebra of its unipotent radical. We show that the Kumar-Lauritzen-Thomsen splitting of the cotangent bundle $G\times^P\mathfrak{u}_P$ of $G/P$ has top degree $(p-1)\dim(G/P)$. The component of that degree is therefore given by the $(p-1)$-th power of a function $f$. We give a formula for $f$ and deduce that it vanishes on the exceptional locus of the resolution $G\times^P\mathfrak{u}_P\to\overline{\mathcal{O}}$ where $\overline{\mathcal{O}}$ is the closure of the Richardson orbit of $P$. As a consequence we obtain that the higher cohomology groups of a line bundle on $G\times^P\mathfrak{u}_P$ associated to a dominant weight are zero. The splitting of $G\times^P\mathfrak{u}_P$ given by $f^{p-1}$ can be seen as a generalisation of the Mehta-Van der Kallen splitting of $G\times^B\mathfrak{u}$.

## Introduction

Let $G$ be a reductive group over an algebraically closed field $k$ of positive characteristic $p$. For a parabolic $P$ containing the positive Borel and $P$-module $M$, we denote by $H^i(G/P,M)$ the $i$-th chomology group of the sheaf $\mathcal{L}_{G/P}(M)$ on $G/P$ associated to $M$. It is an open problem whether we have for all parabolic subgroups $P$ and all dominant characters $\lambda$ of $P$ that

$$H^i(G/P,S(\mathfrak{u}_P^*)\otimes k_{-\lambda})=0\quad\text{for all } i>0\,, \tag{$*$}$$

where the most important case is $\lambda=0$, see e.g. [2, Introduction to Ch 5]. In characteristic 0 this is an easy consequence of the Grauert-Riemenschneider Theorem, see [4, Thm 2.2]. In characteristic $p$ (*) is known for $P=B$, for arbitrary $P$ and "$P$-regular" dominant $\lambda$, see [12], and for $P$ corresponding to sets of pairwise orthogonal short simple roots and $\lambda=0$, see [16].[1]

It is easy to write a formula for the Euler character

$$\sum_{i\geq0}(-1)^i\mathrm{ch}\,H^i(G/P,S(\mathfrak{u}_P^*)\otimes k_{-\lambda})\,,$$

see [10, Sect 8.14-8.16] and [3, Prop 2.1], so if (*) holds we get a formula for $\mathrm{ch}\,H^0(G/P,S(\mathfrak{u}^*)\otimes k_{-\lambda})$.

For computing cohomology of Frobenius kernels of $G$, (*), or a special case of it, is often used, see [9, II.12.12-15], [12, Thm 8], [1, Sect 7] and [13, Sect 7].

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[1]The case $P=B$ and $\lambda=0$ is [9, Lem II.12.12].

When $\mathcal{L}_{G/P}(\lambda) = \mathcal{L}_{G/P}(k_{-\lambda})$ is ample, i.e. $\lambda$ "$P$-regular" dominant, one gets (*) from the fact that $G \times^P \mathfrak{u}_P$ is Frobenius split. One can also use Frobenius splittings to prove (*) for $\lambda = 0$ via a characteristic $p$-version of of the Grauert-Riemenschneider Theorem [14, Thm 1.2], since the canonical bundle of $G \times^P \mathfrak{u}_P$ is trivial. But then the map from $G \times^P \mathfrak{u}_P$ to the Richardson orbit closure has to be birational and the splitting has to be a $(p-1)$-th power of a section $\sigma$ of the anti-canonical bundle which vanishes on the excptional locus. This is the approach we will follow.

When I asked Thomsen about the case $G = \mathrm{GL}_n$, he told me he expected that the pushforward to $G \times^P \mathfrak{u}_P$ of the splitting of $G \times^B \mathfrak{u}_P$ induced by the "MVdK-splitting" of $G \times^B \mathfrak{u}$ from [15] is the homogeneous component of degree $(p-1)\dim(G/P)$ of the "KLT-splitting", see Section 1.2, from [12]. Although we can not prove this conjecture, we can show that the above component is in fact the top degree component and therefore a $(p-1)$-th power. From this we can then deduce that this homogeneous splitting vanishes on the exceptional locus of the resolution $\varphi : [g, X] \mapsto gXg^{-1} : G \times^P \mathfrak{u}_P \to \overline{\mathcal{O}}$, where $\overline{\mathcal{O}}$ is the closure of the Richardson orbit corresponding to $P$, see Theorem 1 in Section 2. Finally, we then deduce that (*) holds in type $A$, see Theorem 2. In fact we can formulate this as a result for arbitrary reductive groups.

The main idea of the proof is as follows. The "KLT-splitting" from [12] is the $(p-1)$-th power of the pullback along $\varphi$ of the function which maps an $n \times n$ matrix $X$ to

$$\prod_{i=1}^{n-1} \det\left((I_n + X)_{\leq i, \leq i}\right), \tag{1}$$

where $Y_{\leq i, \leq i}$ denotes the submatrix of $Y$ given by the first $i$ rows and columns, see [2, Example 5.1.15].[2] Unlike in the case $P = B$, the degree of the $i$-th factor may be less than $i$. In Lemma's 1(ii) and 3 we determine the degree of the $i$-th factor and from that it follows that the product (1) has degree $\dim(G/P)$.

## 1. Preliminaries

1.1. **Notation.** Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $G$ be a reductive group over $k$. We fix a Borel subgroup $B \leq G$ and maximal torus $T \leq B$. We denote by $R$ the set of roots of $T$ in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of $G$, and we denote the unipotent radical of $B$ by $U$. We call the roots of $T$ in $\mathfrak{u} = \mathrm{Lie}(U)$ positive and we denote the corresponding set of simple roots by $S$. For a subset $I$ of $S$ we denote the root system spanned by $I$ by $R_I$. Furthermore, we denote the corresponding parabolic subgroup containing $B$ and its Levi subgroup containing $T$ by $P_I$ and $L_I$. Denote the character group of an algebraic group $H$ by $X(H)$. For $I \subseteq S$ we identify $X(P_I)$ and $X(L_I)$ with $\{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I\}$.

For $P$ a parabolic of $G$ and $M$ a $P$-module we write $\mathcal{L}(M)$ for the $G$-linearised sheaf on $G/P$ associated to $M$. For $\lambda \in X(P) \leq X(T)$ we put $\mathcal{L}(\lambda) = \mathcal{L}(k_{-\lambda})$, it is the sheaf of sections of the line bundle $G \times^P k_{-\lambda}$ on $G/P$. We use the same

[2]Apart from the degree computation, the arguments there work for any parabolic.

symbol $\mathcal{L}(\lambda)$ to denote the sheaf of sections of the pullback of this line bundle to $G \times^P V$ for any $P$-variety $V$. We also write $H^i(G/P, M)$ for

$$H^i(G/P, \mathcal{L}(M)) \simeq R^i\mathrm{ind}_P^G(M)\,,$$

see [9, I.5.12]. We have that

$$H^i(G \times^P \mathfrak{u}_P, \mathcal{L}(\lambda)) = H^i(G/P, k[\mathfrak{u}_P] \otimes k_{-\lambda})\,,$$

see [2, Lem 5.2.2].

If $p = \mathrm{char}\, k$ is good for $G$, then we have $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{u}_P^*$ as $P$-modules and $G \times^P \mathfrak{u}_P$ is the cotangent bundle $T^\vee(G/P)$ of $G/P$, see [2, 5.1.8-11].

1.2. **Frobenius splittings.** By [2, Lem 5.1.1] the canonical bundle of $G \times^P \mathfrak{u}_P$ is trivial, so we can choose a nowhere zero global section: a volume form. It is easy to see that such a section is unique up to a scalar multiple, see [2, 5.1.2]. This means that we can think of Frobenius splittings (up to a scalar multiple) of $G \times^P \mathfrak{u}_P$ as certain regular functions on $G \times^P \mathfrak{u}_P$.

In [12, Thm 1] it was proved that, when $p$ is good for $G$, the cotangent bundle $T^\vee(G/P)$ of $G/P$ is Frobenius split, see also [2, Thm 5.1.3]. We will refer to the $B$-canonical splitting $\psi_P(f_- \otimes f_+)$ as the "KLT-splitting" of $T^\vee(G/P)$, where $\psi_P, f_-, f_+$ are as defined in [2, Ch 5]. Actually this is only a splitting up to a scalar multiple, but in the case $G = \mathrm{GL}_n$ we assume that the chosen volume form on $T^\vee(G/P)$ is such that the pullback along $\varphi$ of the function given by (1), $\varphi$ defined as in the introduction, defines a splitting. That formula is all we need to know about the KLT-splitting in this paper.

The standard grading of $k[\mathfrak{u}_P] = S(\mathfrak{u}_P^*)$ gives a grading on $k[G \times^P \mathfrak{u}_P]$, and in [2, 5.1.14] it is explained that the homogeneous component of degree $(p-1)\dim(G/P)$ of a splitting $\sigma$ of $G \times^P \mathfrak{u}_P$ is again a splitting of $G \times^P \mathfrak{u}_P$. This component is $B$-canonical if $\sigma$ is $B$-canonical.

1.3. **A result on cohomology vanishing.** The following result may be well-known, but for lack of reference we give a proof.

**Proposition 1.** *Asume $p$ is good for $G$, let $P$ be a parabolic of $G$, let $\lambda \in X(P)$ be dominant, let $Q$ be the parabolic of $G$ containing $P$ such that $\lambda \in X(Q)$ and $\mathcal{L}_{G/Q}(\lambda)$ is ample, and let $L$ be the Levi subgroup of $Q$ containing $T$. If $H^i(L/L \cap P, S(\mathfrak{l}/\mathfrak{l} \cap \mathfrak{p})) = 0$ for all $i > 0$, then $H^i(G/P, S(\mathfrak{g}/\mathfrak{p}) \otimes k_{-\lambda})$ for all $i > 0$.*

*Proof.* By [12, Cor 3 to Thm 4] or [2, Thm 5.3] $G \times^P \mathfrak{u}_P$ is Frobenius split, so by [2, Lemma 1.2.7(i)] it is enough to show the vanishing for $m\lambda$, $m \gg 0$ (in fact we only need it for $p^m\lambda$ and some $m \geq 0$).

Some of the arguments below are adaptations of arguments from the proof of [2, Lem 5.2.7].

Each $S^j(\mathfrak{g}/\mathfrak{p})$ has a filtration with sections $S^r(\mathfrak{q}/\mathfrak{p}) \otimes S^s(\mathfrak{g}/\mathfrak{q})$, $r + s = j$, so it is enough to show that $R^i\mathrm{ind}_P^G(S(\mathfrak{q}/\mathfrak{p}) \otimes S(\mathfrak{g}/\mathfrak{q}) \otimes k_{-\lambda}) = 0$ for all $i > 0$. We have $P = (L \cap P)U_Q$ and $\mathfrak{q}/\mathfrak{p} \simeq \mathfrak{l}/\mathfrak{l} \cap \mathfrak{p}$. Note that $U_Q$ acts trivially on $\mathfrak{q}/\mathfrak{p}$. Combining [9, I.6.11] and our assumption with a standard spectral sequence argument, we have

$$R^i\mathrm{ind}_P^G(S(\mathfrak{q}/\mathfrak{p})\otimes S(\mathfrak{g}/\mathfrak{q})\otimes k_{-\lambda})\simeq R^i\mathrm{ind}_Q^G(\mathrm{ind}_P^Q S(\mathfrak{q}/\mathfrak{p})\otimes S(\mathfrak{g}/\mathfrak{q})\otimes k_{-\lambda})\,. \quad (2)$$

We have $\mathfrak{p}=\mathfrak{l}\cap\mathfrak{p}\oplus\mathfrak{u}_Q$, $\mathfrak{u}_P=\mathfrak{u}_{L\cap P}\oplus\mathfrak{u}_Q$, $(\mathfrak{g}/\mathfrak{p})^*\simeq\mathfrak{u}_P$, $(\mathfrak{g}/\mathfrak{q})^*\simeq\mathfrak{u}_Q$, and $(\mathfrak{q}/\mathfrak{p})^*\simeq\mathfrak{u}_{L\cap P}$. By the arguments of [10, p94] there exists an affine $Q$-variety $V_0$ such that $k[V_0]\simeq k[Q\times^P\mathfrak{u}_{L\cap P}]=\mathrm{ind}_P^Q S(\mathfrak{q}/\mathfrak{p})$, $Q$-equivariantly ($U_Q$ acting trivially). Put $V=V_0\times\mathfrak{u}_Q$. Then $k[V]=\mathrm{ind}_P^Q S(\mathfrak{q}/\mathfrak{p})\otimes S(\mathfrak{g}/\mathfrak{q})$. Now the morphism $G\times^Q V\to G/Q$ is affine, so by [7, Ex III.8.2] the RHS of (2) is isomorphic to

$$H^i(G\times^Q V,\mathcal{L}(\lambda))\,. \quad (3)$$

By [6, 5.1.12] $\mathcal{L}(\lambda)$ is ample on $G\times^Q V$, since $G\times^Q V\to G/Q$ is affine. The morphism $V_0\to\overline{Q\cdot\mathfrak{u}_{L\cap P}}$ is finite, see [10, p94], so the same is true for the morphisms $V\to\overline{Q\cdot\mathfrak{u}_{L\cap P}}\times\mathfrak{u}$ and $G\times^Q V\to G\times^Q(\overline{Q\cdot\mathfrak{u}_{L\cap P}}\times\mathfrak{u}_Q)$. So composing the latter with the projective morphism $G\times^Q(\overline{Q\cdot\mathfrak{u}_{L\cap P}}\times\mathfrak{u}_Q)\to\mathfrak{g}$, given by the embedding of $\overline{Q\cdot\mathfrak{u}_{L\cap P}}\times\mathfrak{u}_Q$ in $\mathfrak{g}$, we obtain a proper morphism $G\times^Q V\to\mathfrak{g}$. Now [7, III.5.3] tells us that (3) is 0 if we replace $\lambda$ by $m\lambda$, $m\gg 0$. □

## 2. The main results

Throughout this section, except in Theorem 2 and its proof, $G=\mathrm{GL}_n=\mathrm{GL}(k^n)$ and $T$ is the subgroup of diagonal matrices in $G$. As simple roots we choose the usual characters $\varepsilon_i-\varepsilon_{i+1}$, $1\le i\le n-1$, where we used additive notation for characters, and $\varepsilon_i$ is the $i$-th coordinate function on $T$. Then $B$ consists of the upper triangular matrices in $G$. As is well-known, the conjugacy classes of parabolic subgroups of $G$ are labelled by the compositions of $n$, see e.g.[8, 3.2]. By $\nu$ we denote a composition of $n$ and $P=P_\nu\supseteq B$ is the standard parabolic whose block sizes are given in order by $\nu$. If $A_\nu$ is the set $\{\nu_1,\nu_1+\nu_2,\dots,\sum_{j=1}^{s-1}\nu_j\}$, $s$ the length of $\nu$, then $P_\nu=P_{I_\nu}$, the parabolic associated to the set of simple roots $I_\nu=\{\varepsilon_i-\varepsilon_{i+1}\,|\,i\in\{1,\dots,n-1\}\setminus A_\nu\}$. We denote by $\lambda$ the transposed partition of the weakly descending sorted version of $\nu$. It is well-kown that the Richardson orbit of $P_\nu$ is $\mathcal{O}_\lambda$, the nilpotent orbit whose Jordan block sizes are given by $\lambda$, see e.g. [8, Thm 3.3(a)].

It is well-known that the map $\varphi:[g,X]\mapsto gXg^{-1}:G\times^P\mathfrak{u}_P\to\overline{\mathcal{O}_\lambda}$ is birational. Indeed the group centraliser $G_X$ of any $X\in\mathfrak{g}$ is the set of invertible elements in the Lie algebra centraliser $\mathfrak{g}_X$, so is connected. Now see [10, 4.9 and 8.8 Remark]. It is also well-known that $\overline{\mathcal{O}_\lambda}$ is normal, see e.g. [5] or [15, Sect 4.7].

For $i\in\{1,\dots,n-1\}$ we denote by $d_{\lambda,i}$ the number of nonzero positions on the $(n-i)$-th upper codiagonal of $\mathfrak{u}_P$. So for $\nu=(2,1,2)$ we have $d_{\lambda,1},d_{\lambda,2},d_{\lambda,3},d_{\lambda,4}=1,2,3,2$, see the figure of $\mathfrak{u}_p$ below.

$$\begin{bmatrix}0&0&*&*&*\\0&0&*&*&*\\0&0&0&*&*\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Since each diagonal $j \times j$ block of $P$ takes away $j-i$ nonzero positions from the $i$-th upper codiagonal, we have, if $j$ occurs $m_j$ times in $\nu$,

$$d_{\lambda,n-i} = n-i-\sum_{j>i}(j-i)m_j = n-i-\sum_{j>i}\lambda_j = -\sum_{j=i+1}^{n}(\lambda_j-1) = \sum_{j=1}^{i}(\lambda_j-1)\,.$$

Therefore, $d_{\lambda,i} = i-\sum_{j>n-i}\lambda_j = \sum_{j=1}^{n-i}(\lambda_j-1)$. So indeed the $d_{\lambda,i}$ only depend on $\lambda$, moreover, they determine $\lambda$.

For a square matrix $X$ we denote by $X_{\le i,\le i}$ the submatrix of $X$ given by the first $i$ rows and columns. For an $i\times i$ matrix $Y$ we denote by $s_j(Y)$ the trace of the $j$-th exterior power of $Y$, i.e. the sum of the diagonal $j\times j$ minors of $Y$. As is well-known, $\det(aI_i-Y) = a^i+\sum_{j=1}^{i}(-1)^j a^{i-j}s_j(Y)$, where $I_i$ is the $i\times i$ identity matrix. So the largest $j$ with $s_j(Y)\ne 0$ is the number of nonzero eigenvalues of $Y$, counted with (algebraic) multiplicity. This number also equals the rank of $Y^l$ for $l$ sufficiently big. We will call it the *stable rank* of $Y$.

**Lemma 1.** *Let $X\in\mathcal{O}_\lambda$.*

(i) *Any $i$-dimensional subspace $W$ of $V=k^n$ contains an $X$-invariant subspace $U$ of dimension $\ge\sum_{j>n-i}\lambda_j$.*
(ii) *$X_{\le i,\le i}$ has stable rank $\le d_{\lambda,i}$.*

*Proof.* (i). We show this by induction on $n$. It is trivial when $i\le n-r$, $r$ the length of $\lambda$, in particular when $n=0$. Assume $i>n-r$. Then $W$ has nonzero intersection with $\mathrm{Ker}(X)$ for dimension reasons. Pick $v$ nonzero in that intersection. First note that the transformation $\overline{X}$ induced on $V/kv$ by $X$ has partition $\mu$ which is obtained from $\lambda$ by subtracting 1 from one part of $\lambda$ and then sorting the result in weakly descending order. Indeed if we decompose $V$ as a direct sum of $X$-Jordan blocks and we pick a $X$-Jordan block of minimal size with the property that $v$ has nonzero component in it, then we can replace that $X$-Jordan block by an $X$-Jordan block of the same size which contains $v$. Now we apply the induction hypothesis to $V/kv$ and $W/kv$, noting that $(n-1)-(i-1)=n-i$, to obtain an $\overline{X}$-invariant subspace $U/kv$ of $W/kv$ of dimension $\ge\sum_{j>n-i}\mu_j\ge\sum_{j>n-i}\lambda_j-1$. Now $U$ is the $X$-invariant subspace we want.
(ii). The linear map $(X_{\le i,\le i})^i$ coincides with $X^i$ on any $X$-invariant subspace $U$ of $k^i\le k^n$ and therefore kills it. Choosing $U$ as in (i), it induces a linear map $k^i/U\to k^i$ and therefore has rank $\le i-\sum_{j>n-i}\lambda_j = d_{\lambda,i}$. □

Lemma 3 below follows from Lemma 1(ii) and the existence of the KLT-splitting, but we prefer to give a direct proof.

**Lemma 2.** *For any $h\in\{1,\dots,i-1\}$ there exists a regular nilpotent $i\times i$ matrix $X$ such that $X_{\le h,\le h}$ is invertible.*

*Proof.* Let $(e_1,\dots,e_i)$ be the standard basis of $k^i$. Then the regular nilpotent matrix $X$ given by $X(e_j)=e_{j-1}$ for $2<j\le i$, $X(e_2)=e_1+e_{h+1}$ and $X(e_1+e_{h+1})=0$ has the desired property. □

**Remark 2.1.** Of course it follows from Lemma 2 that there exsists a regular $i \times i$ matrix $X$ such that $X_{\leq h, \leq h}$ is invertible for all $h \in \{1, \ldots, i-1\}$, but we won't need this.

**Lemma 3.** *There exists $X \in \mathcal{O}_\lambda$ such that $X_{\leq i, \leq i}$ has stable rank $d_{\lambda,i}$.*

*Proof.* First choose any $Y \in \mathfrak{g}$ nilpotent with partition $\lambda$ and decompose $k^n$ into $Y$-Jordan blocks with sizes $\lambda_1, \lambda_2, \ldots, \lambda_r$, where $r$ is the length of $\lambda$. It suffices to find an ordered basis $\mathcal{B}$ of $k^n$ such that the upper left $i \times i$-block $Z$ of the matrix of $Y$ relative to this basis has stable rank $d_{\lambda,i}$.

Determine $s \leq r$ maximal with $\sum_{j=1}^{s}(\lambda_j - 1) \leq i$ and put $h = i - \sum_{j=1}^{s}(\lambda_j - 1)$. Using Lemma 2 choose for each $j \leq s$ a basis of the $j$-th block such that the upper left $(\lambda_j - 1) \times (\lambda_j - 1)$ block of the matrix of $Y$ relative to this basis is invertible, if $s < r$ and $h > 0$ choose a basis of the $(s+1)$-th block such that the upper left $h \times h$ block of the matrix of $Y$ relative to this basis is invertible, and for the remaining blocks choose any basis.

We now form $\mathcal{B}$ as follows. First consider the case $i \leq n - r$. For each $j \leq s$ we pick the first $\lambda_j - 1$ basis vectors from the $j$-th block, if $s < r$ we append the first $h$ basis vectors from the $(s+1)$-th block, and finally we append all remaining $n - i$ basis vectors. Now $Z$ is in block diagonal form with invertible diagonal block of sizes $\lambda_1 - 1, \ldots, \lambda_s - 1, h$, where $h$ has to be omitted if $h = 0$. Now consider the case $i > n - r$. For each $j \leq n - i$ we pick the first $\lambda_j - 1$ basis vectors from the $j$-th block, then we append the basis vectors from the next $r - (n - i)$ blocks, and finally we append all remaining $n - i$ basis vectors. Now $Z$ is in block diagonal form with diagonal block sizes $\lambda_1 - 1, \ldots, \lambda_{n-i} - 1, \lambda_{n-i+1}, \ldots, \lambda_r$ where the first $n - i$ blocks are invertible, and the others nilpotent. In both cases we obtain that $Z$ has stable rank $d_{\lambda,i}$ (when $i \leq n - r$ we have $d_{\lambda,i} = i$). $\square$

Below we will denote a function $X \mapsto E(X)$ on a closed subvariety of $\mathfrak{g}$ just by the expression $E(X)$.

**Theorem 1.** *The degree $(p-1)\dim(G/P)$ component of the KLT splitting of $G \times^P \mathfrak{u}_P$ is the top degree component and equals the $(p-1)$-th power of the pullback of $\prod_{i=1}^{n-1} s_{d_{\lambda,i}}(X_{\leq i, \leq i}) \in k[\overline{\mathcal{O}_\lambda}]$ along the resolution $\varphi : G \times^P \mathfrak{u}_P \to \overline{\mathcal{O}_\lambda}$. This pullback vanishes on the exceptional locus of $\varphi$.*

*Proof.* The KLT splitting is the pullback along $\varphi$ of the function given by (1). Furthermore, we have $\det(I_i + Y) = \sum_{j=0}^{i} s_j(Y)$ for any $i \times i$ matrix $Y$, and, of course, $s_j(X_{\leq i, \leq i}) \neq 0$ on $\mathcal{O}_\lambda \iff s_j(X_{\leq i, \leq i}) \neq 0$ on $\overline{\mathcal{O}_\lambda}$. So by Lemma's 1(ii) and 3 the top degree component of the $i$-th factor in (1) is $s_{d_{\lambda,i}}(X_{\leq i, \leq i})$. So the KLT-splitting has top degree $p-1$ times $\sum_{i=1}^{n-1} d_{\lambda,i} = \dim \mathfrak{u}_P = \dim(G/P)$, and the top degree component is the $(p-1)$-th power of the pullback along $\varphi$ of the function given by the stated formula.

To prove he second assertion, put $f_{\lambda,i}(X) = s_{d_{\lambda,i}}(X_{\leq i, \leq i})$ and $f_\lambda = \prod_{i=1}^{n-1} f_{\lambda,i}$. The exceptional locus is $\varphi^{-1}(\overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda)$, so it suffices to show that $f_\lambda$ vanishes on any $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda$. We have $\dim(\mathfrak{u}_Q) = \frac{1}{2}\dim(\mathcal{O}_\mu) < \frac{1}{2}\dim(\mathcal{O}_\lambda) = \dim(\mathfrak{u}_P)$, where $Q$ is a standard parabolic whose Richardson orbit is $\mathcal{O}_\mu$, see [10, 4.9].

So for some $i$ we must have $d_{\mu,i} < d_{\lambda,i}$ which means that $f_{\lambda,i}$ and therefore $f_\lambda$ vanishes on $\mathcal{O}_\mu$. □

**Theorem 2.** *Let $G$ be any reductive group for which $p$ is good, let $\lambda \in X(T)$ be dominant, put $I = \{\alpha \in S \mid \langle\lambda, \alpha^\vee\rangle = 0\}$. Then $H^i(T^\vee(G/P_J), \mathcal{L}(\lambda)) = 0$ for all $J \subseteq I$ such that $R_J$ contains all irreducible components of $R_I$ not of type $A$.*

*Proof.* By Proposition 1 we may assume that $\lambda = 0$ and that all irreducible components of $R$ have type $A$. Since we are dealing with cotangent bundles we may assume that $G$ is semisimple and simply connected. By the Künneth formula [11, Prop 9.2.4] we may then assume $G = \mathrm{SL}_n$ and finally we may assume $G = \mathrm{GL}_n$. Now the result follows from Theorem 1 and [14, Thm 1.2], bearing in mind that the canonical bundle of $T^\vee(G/P)$ is trivial, see [2, Lem 5.1.1], and that $R^i\varphi_*(\mathcal{O}_{T^\vee(G/P)})$ is the sheaf associated with the cohomology group $H^i(T^\vee(G/P), \mathcal{O}_{T^\vee(G/P)})$, since $\varphi$ is affine. □

We remind the reader that a proper birational morphism $\psi : X \to Y$ is called a *rational resolution* if $\psi_*\mathcal{O}_X = \mathcal{O}_Y$ and the higher direct images of $\mathcal{O}_X$ and $\omega_X$ are 0, see [2, Def 3.4.1]. We assume again that $G = \mathrm{GL}_n$.

**Corollary.** *The resolution $\varphi : G \times^P \mathfrak{u}_P \to \overline{\mathcal{O}_\lambda}$ is rational.*

*Proof.* This follows from a standard argument, see e.g. [9, Lem 14.5], and Theorem 2. □

**Remarks 2.2.** 1. If $P = B$, then $d_{(n),i} = i$ for all $i$, so the splitting from Theorem 1 equals the $(p-1)$-th power of the pullback of $\prod_{i=1}^{n-1} \det(X_{\le i,\le i})$ along $\varphi : G \times^B \mathfrak{u} \to \mathcal{N}$. This is the MVdK splitting of $G \times^B \mathfrak{u}$, see [15].
2. Thomsen mentioned to me another proof of Lemma 1(ii): One can easily deduce it from the following result which can be proved by induction on $n$. For $X \in \mathfrak{u}_P$ let $y_{ij} = \delta_{ij} + x_{ij}$ be the $(i,j)$-th entry of $I_n + X$. Then any monomial $y_{i_1j_1}y_{i_2j_2}\cdots y_{i_sj_s}$ with the $i_l$ all distinct and the $j_l$ all distinct has degree $\le d_{\lambda,s}$ in the $x_{ij}$.
3. In [15, Sect 4.9] there is also a proof of the above corollary for certain parabolics, but that relies on the existence of a principal effective divisor $D$ which is a subdivisor of $(\sigma)$ ($\sigma^{p-1}$ is the MVdK splitting) and contains the exceptional locus. This is claimed in [15, Prop 4.5], but the proof of that result is incomplete and it seems rather unlikely that such a divisor exists for the set of parabolics in question. The proof of the above corollary sketched in [2, Exercise 5.3.E(b)] is also problematic: after pushing the splitting of Exercise 5.1.E.6 forward from $G \times^B \mathfrak{u}_P$ to $G \times^P \mathfrak{u}_P$ it's no longer clear that the splitting is a $(p-1)$-th power, so one can't apply [2, Thm 1.3.14].

**Conjecture** (Thomsen)**.** *The pushforward to $G\times^P\mathfrak{u}_P$ of the splitting of $G\times^B\mathfrak{u}_P$ induced by the MVdK splitting is the top degree component of the KLT splitting.*

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