MA1214 Sheet 7

(1) Let G be a group acting on a set S. Show that the relation $\sim$ on S given by $x \sim y \Leftrightarrow \exists_{g\in G}\, x = g{\cdot}y$ is an equivalence relation and that its equivalence classes are the orbits of G on S.

(2) Let G be a group. An *automorphism of* G is an isomorphism from G to G.

(i) Show that the set $\mathrm{Aut}(G)$ of automorphisms of G endowed with composition is a group.

(ii) Show that for every $g \in G$ the map $\mathrm{inner}(g) = h \mapsto ghg^{-1} : G \to G$ is an automorphism of G. Such automorphisms of G are called *inner automorphisms*.

(iii) Show that $g \mapsto \mathrm{inner}(g) : G \to \mathrm{Aut}(G)$ is a homomorphism. Equivalently: Show that the rule $g \cdot h = ghg^{-1}$ defines an action of G on itself.

This action is called the *conjugation action* of G on itself and the orbits for this action are called the *conjugacy classes* of G. Furthermore $h_1, h_2 \in G$ are called are called *conjugate* if there exists a $g \in G$ such that $h_1 = gh_2g^{-1}$.

(3)(i) Show that two cycles in S_n are conjugate if and only if they have the same length.
Hint. Use that for a cycle $\sigma = (a_1 \dots a_k) \in S_n$, $k \geq 2$, and for $\pi \in S_n$ we have the identity $\pi(a_1 \dots a_k)\pi^{-1} = (\pi(a_1) \dots \pi(a_k))$.

(ii) A *partition* of n is a sequence of positive integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ with $\sum_{i=1}^{r} \lambda_i = n$. A partition λ is often symbolically written as $1^{m_1} \cdots n^{m_n}$ or $n^{m_n} \cdots 1^{m_1}$, where m_i is the number of occurrences of i in λ and i^{m_i} is omitted if $m_i = 0$. Note that $\sum_{i=1}^{n} m_i\, i = n$. Example: $(55522111) = 5^3 2^2 1^3 = 1^3 2^2 5^3$ is a partition of 22.

The *cycle structure* of a permutation π is the partition $n^{m_n} \cdots 1^{m_1}$ of n, where m_i is the number of cycles of length i in the disjoint cycle form of π. Note that we do count 1-cycles (m_1 is the number of fixed points of π). So the cycle structure of $(1\,5)(4\,9)(2\,3\,7) \in S_9$ is 32^21^2.

Prove the following generalisation of (i): Two permutations are conjugate if and only if they have the same cycle structure. So the conjugacy classes of S_n are labelled by the partitions of n.