SHEET 4

1) (i) $X^2Y^3(p) = 0 \Leftrightarrow X(p) = 0$ or $Y(p) = 0$. So $V(J) \subseteq \mathbb{A}^2$ is the union of the two coordinate axes. We have $\sqrt{J} = V(I(J)) = (XY) \neq J$.

these are the 2 irr. components

(ii) Let $p \in \mathbb{A}^3$ and assume $XY(p) = YZ(p) = XZ(p) = 0$. If $X(p) \neq 0$, then we must have $Y(p) = Z(p) = 0$. If $X(p) = 0$, then we still have $Y(p) = 0$ or $Z(p) = 0$. So $V(J) = V((Y,Z)) \cup V((X,Z)) \cup V((X,Y))$ the union of the coordinate axes and these are the 3 irreducible components of $V(J)$.

Now let $f = \sum_{r,s,t \geq 0} a_{rst} X^r Y^s Z^t \in I(V(J))$. Restricting f to the X-axis gives $\sum_{r \geq 0} a_{r00} X^r = 0$, so $a_{r00} = 0 \ \forall r$. Restricting f to the Y-axis and to the Z-axis, we get $a_{0s0} = 0 \ \forall s$ and $a_{00t} = 0 \ \forall t$.

finitely many coeff's nonzero

So every monomial occurring in f (with nonzero coefficient) contains at least two variables and is therefore divisible by YZ or by XZ or by XY. So $f \in (XY, XZ, YZ)$. So $\sqrt{J} = I(V(J)) = J$.

2) The curve $\hat{C} \subseteq \mathbb{P}^2_{\mathbb{C}}$ is given by the equation $F = Y^2Z^{2g-1} - \prod_{i=1}^{2g+1}(X - a_i Z) = 0$. Clearly $P_\infty = (0,1,0)$ is a point of $\hat{C}$. Moreover,

$$d_{P_\infty}F = \left(\frac{\partial F}{\partial Y} Y^2Z^{2g-1}\right)(P_\infty)\,Y + \left(\frac{\partial F}{\partial Z} Y^2 Z^{2g-1}\right)(P_\infty)\,Z = (2YZ^{2g-1})(P_\infty)\,Y + (2g-1)Y^2Z^{2g-2}_{(P_\infty)}Z$$

$= 0$, since $2g-2 > 0$. So P_∞ is a singular point.

3) (i) On S we have $X/Y = U/V$. So the domain of φ contains the complement in S of the line $L = \{Y = V = 0\}$ on S. Now let $f, g \in \mathbb{C}[X,Y,U,V]$ be homogeneous of the same degree such that $g|_S \neq 0$ and $X/Y = f/g$ on S. Then $gX = fY$ on the cone $\hat{S} \subseteq \mathbb{A}^4_{\mathbb{C}}$ corresponding to S.
So $g = 0$ on the cone $\hat{L} \subseteq \mathbb{A}^4_{\mathbb{C}}$ corresponding to L ($X|\hat{L} \neq 0$ and $\hat{L}$ is irreducible).
So $\mathrm{Dom}(f) = S \setminus L$.
Clearly $\varphi = (X:Y) = (U:V)$ as a rational map: $S \to \mathbb{P}^1_{\mathbb{C}}$ is defined everywhere on S

3(ii) Put $\psi = (X:U) = (Y:V) : S \to \mathbb{P}^1$. Then we see by the same arguments as for φ that ψ is defined everywhere on S.
So we obtain a morphism $\theta : p \mapsto (\varphi(p), \psi(p)) : S \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$
This morphism is clearly an inverse to the Segre embedding
$((Z:W),(Z':W')) \mapsto (ZZ' : WZ' : ZW' : WW') : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to S$
and therefore an isomorphism.

4) As observed we may assume that f and g are squarefree.
Since $\gcd(f,g) = 1$, fg will then also be squarefree.
So $I(V(f)) = (f)$, $I(V(g)) = (g)$, $I(V(fg)) = (fg)$.
For $p \in \mathbb{A}^n_{\mathbb{C}}$ we have $d_p(fg) = f(p)\, d_p g + g(p)\, d_p f$
Now assume that $p \in V(fg)$, i.e. $f(p) = 0$ or $g(p) = 0$
Then $d_p(fg) = \begin{cases} g(p)\, d_p f & \text{if } f(p) = 0 \\ f(p)\, d_p g & \text{if } g(p) = 0 \end{cases}$ $(*)$
Now the inclusion $\supseteq$ is a straightforward observation. So assume $p \in \mathrm{Sing}(V(fg))$. Then $f(p)g(p) = 0$ and $d_p(fg) = 0$. If $f(p) \neq 0$, then $g(p) = 0$ and we deduce from $(*)$ that $d_p g = 0$, so $p \in \mathrm{Sing}(V(g))$. Similarly, if $g(p) \neq 0$, then $p \in \mathrm{Sing}(V(f))$. Finally, if $f(p) = g(p) = 0$, then, of course, $p \in V(f) \cap V(g)$.

5) Let X be an irreducible topological space (i.e. $X \neq \emptyset$ and cannot be written as the union of two proper closed subsets). If $O \subseteq X$ is open, then $X = \overline{O} \cup X \setminus O$, a union of two closed sets. So we must have closure of O
$\overline{O} = X$ or $X \setminus O = X$. If $O \neq \emptyset$, then the latter is impossible, so $\overline{O} = X$, i.e. O is dense in X.