LIE ALGEBRAS ARE INFINITESIMAL GROUPS

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Throughout k is a field and δ is a "dual number" satisfying $\delta^2 = 0$. Furthermore $\mathfrak{g}$ is a Lie algebra over k which is the Lie algebra of a group G. We denote the identity of G by I.

Whenever $X \in \mathfrak{g} = T_I(G)$, the tangent space of G at I, we consider the corresponding group element $I + \delta X$ of which you should think as "I with an infinitesimal bit added in the direction X". You should think of δ as being a very small number of $k = \mathbb{R}$ and calculate upto the first order.

More formally, δ is the image of δ_0 in $R = R(\delta) := k[\delta_0]/(\delta_0^2)$, the quotient of the polynomial ring $k[\delta_0]$ by the ideal (δ_0^2), and $I + \delta X$ is an element of $G(R)$, the points of G over R.[1]

The commutator

Now we embed G into some GL_n. Our computations below will take place in $\mathrm{Mat}_n(R)$. The inverse of $I + \delta X$ is $I - \delta X$:

$$(I + \delta X)(I - \delta X) = I - \delta X + \delta X - \delta^2 X^2 = I.$$

Note that the elements $I + \delta X$ form a commutative subgroup of $G(R)$. To capture something of the noncommutativity we need to look at second order terms. To do this we add another dual number η to our ring R, that is, we work with the ring $R = R(\delta, \eta) := k[\delta_0, \eta_0]/(\delta_0^2, \eta_0^2)$, where we denote the images of δ_0 and η_0 by δ and η.

Let $X, Y \in \mathfrak{g}$. We compute the group commutator of $I + \delta X$ and $I + \delta Y$:

$$\begin{aligned}
&(I + \delta X)(I + \eta Y)(I - \delta X)(I - \eta Y) \\
=& (I + \eta Y + \delta X + \delta\eta XY)(I - \eta Y - \delta X + \delta\eta XY) \\
=& I - \eta Y - \delta X + \delta\eta XY + \eta Y - \delta\eta YX + \delta X - \delta\eta XY + \delta\eta XY \\
=& I + \delta\eta(XY - YX) \\
=& I + \delta\eta[X, Y].
\end{aligned}$$

So the Lie algebra commutator can be derived from the group commutator. Note that it crucial that $\delta\eta[X, Y]$ is the lowest order term. That's why we can be sure that $[X, Y] \in \mathfrak{g} = T_I(G) \subseteq \mathrm{Mat}_n$.

Exercise. Assume that $k = \mathbb{R}$ or $\mathbb{C}$ and that $G = \mathrm{GL}_n(k)$. In an expression $I + X$ we consider the entries of X as coordinates at I. Compute the commutator of $I + X$ and $I + Y$ upto terms of the second order in the entries of X and Y. Note that you may take $(I + X)^{-1} = I - X + X^2$.

[1] R is sometimes called the ring of dual numbers, see AG.16.2, I.3.20 in Borel's book. I will not explain what "$G(R)$" means in general, but in case $G = \mathrm{GL}_n$ it is the group of invertible $n \times n$ matrices with entries in R.

Infinitesimal action

Let V and W be G-modules and let $X \in \mathfrak{g}$. We consider the action on the tensor product $V \otimes W$. Let $v \in V$ and $w \in W$. Then

$$v \otimes w + \delta X \cdot (v \otimes w) = (I + \delta X) \cdot (v \otimes w) =$$
$$((I + \delta X) \cdot v) \otimes ((I + \delta X) \cdot w) = (v + \delta X \cdot v) \otimes (w + \delta X \cdot w) =$$
$$v \otimes w + \delta\big((X \cdot v) \otimes w + v \otimes (X \cdot w)\big).$$

So $X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w)$ and the action of $\mathfrak{g}$ on the tensor product can be derived from the action of G on the tensor product.[2]

Similarly, we deduce that if G acts on an algebra A by automorphisms, then $\mathfrak{g}$ acts on A by derivations.

One can also easily deduce that the action of $\mathfrak{g}$ on the dual space V^* is given by $(X \cdot f)(v) := f(-X \cdot v)$ for all $v \in V$, using the fact that the group action is given by $(g \cdot f)(v) := f(g^{-1} \cdot v)$ for all $v \in V$.

Finally, we deduce the correct notion of an invariant for $\mathfrak{g}$. Let $v \in V$. Then v is a invariant under $X \in \mathfrak{g}$ if and only if $(I + \delta X) \cdot v = v$, that is, if and only if $X \cdot v = 0$. So v is a $\mathfrak{g}$-invariant if and only if $X \cdot v = 0$ for all $X \in \mathfrak{g}$.

Exercises.

1. Let V be a finite dimensional G-module. Recall that a bilinear form $\beta : V \times V \to k$ is invariant under $g \in G$ if $\beta(g \cdot v, g \cdot w) = \beta(v, w)$ for all $v, w \in V$. When is it invariant under $X \in \mathfrak{g}$? *Hint*: You could use the above results and the fact that the space of bilinear forms is isomorphic to $(V \otimes V)^*$, but you can also use direct calculation.
2. The matrix version of above exercise. Let J be an $n \times n$ matrix over k. Then J defines a form $(v, w) := v^T J w$ on k^n. It is invariant under $S \in \mathrm{GL}_n$ if $S^T J S = J$. When is it invariant under $X \in \mathfrak{g}$?

Final remark

Most notions for (Lie) groups have an analogue for Lie algebras. For example, "subgroup" corresponds to "subalgebra" and "normal subgroup" corresponds to "ideal". Sometimes the corresponding notion has the same name (e.g. "solvable", "nilpotent"). You are encouraged to find some more corresponding notions (with their definitions) and to look for theorems that relate them.

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[2] I have been somewhat sloppy: formally one should use that an algebraic linear action of G on V determines an action of $G(R)$ on $R \otimes_k V = V \oplus \delta V$. But it is probably better to take an intuitive point of view: think of δ as being a very small element of $k = \mathbb{R}$ and calculate upto the first order.