1(i) We take the line L to be given by $y=-1$ and the point a to be $(0,1)$.
For a point p with coördinates (x,y) we then have:

$$d(p,L) = d(p,a) \Leftrightarrow |y+1| = \sqrt{x^2+(y-1)^2} \Leftrightarrow 4y = x^2$$

This is clearly a parabola

(ii) We take $a=(0,0)$ and $b=(1,0)$. Then $C > d(a,b) = 1$.
For p with coordinates (x,y) we have

$$d(p,a)+d(p,b) = C \Leftrightarrow \sqrt{x^2+y^2} + \sqrt{(x-1)^2+y^2} = C \Leftrightarrow$$

$$\sqrt{(x-1)^2+y^2} = C - \sqrt{x^2+y^2}.$$

The latter implies

$$(x-1)^2+y^2 = C^2+x^2+y^2-2C\sqrt{x^2+y^2} \Leftrightarrow$$

$$2C\sqrt{x^2+y^2} = 2x+(C^2-1)$$

This implies

$$4C^2(x^2+y^2) = 4x^2+4(C^2-1)x+(C^2-1)^2 \Leftrightarrow$$

$$4(C^2-1)x^2-4(C^2-1)x+4C^2y^2 = (C^2-1)^2 \Leftrightarrow$$

$$x^2-x+\frac{C^2}{C^2-1}y^2 = \frac{C^2-1}{4} \Leftrightarrow$$

$$\boxed{(x-\tfrac{1}{2})^2+\frac{C^2}{C^2-1}y^2 = \frac{C^2}{4}}$$ the equation for an ellipse, since $C>1$.

Assume now that the latter holds. Then

$$x^2+y^2+(x-1)^2+y^2 = 2x^2-2x+2y^2+1 < 2x^2-2x+2\underbrace{\frac{C^2}{C^2-1}}_{>1}y^2+1 = \frac{C^2-1}{2}+1 < C^2.$$

So $2x+(C^2-1) > 2x^2+2y^2 \geq 0$ and $x^2+y^2 < C^2$, i.e. $\sqrt{x^2+y^2} < C$.
So the above two implications were equivalences.

(iii) Now $C < d(a,b) = 1$ and we assume $C>0$ (The case $C=0$ is very easy). Then
$|d(p,a)-d(p,b)| = C \Leftrightarrow \sqrt{(x-1)^2+y^2} = \pm C + \sqrt{x^2+y^2}$. This implies
$(x-1)^2+y^2 = C^2+x^2+y^2 \pm 2C\sqrt{x^2+y^2} \Leftrightarrow \pm 2C\sqrt{x^2+y^2} = -(2x+(C^2-1))$. As before this is
equivalent to $\boxed{(x-\tfrac{1}{2})^2+\frac{C^2}{C^2-1}y^2 = \frac{C^2}{4}}$ which is now the equation for a hyperbola, since $C<1$.
Finally, assume $-2C\sqrt{x^2+y^2} = -(2x+(C^2-1))$. Then $2C\sqrt{x^2+y^2} = 2x+(C^2-1) \leq 2\sqrt{x^2+y^2}+(C^2-1)$.
So $2(C-1)\sqrt{x^2+y^2} \leq C^2-1$ and, since $C-1<0$, $2\sqrt{x^2+y^2} \geq C+1 > 2C$. So $\sqrt{x^2+y^2} > C$.
So the above implication was an equivalence.

2. This I leave to you.

3(i). Let $p \in \mathbb{P}^2_{\mathbb{C}}$ and let $l_p \subseteq \mathbb{C}^3$ be the 1-dimensional subspace of $\mathbb{C}^3$ corresponding to p. Then the lines in $\mathbb{P}^2_{\mathbb{C}}$ through p are in 1-1 correspondence with the 2-dimensional subspaces of $\mathbb{C}^3$ that contain l_p, and these are in 1-1 correspondence with the 1-dimensional subspaces of the quotient vector space $\mathbb{C}^3/l_p$. So the lines in $\mathbb{P}^2_{\mathbb{C}}$ through p are in 1-1 correspondence with the points of the projective line $\mathbb{P}(\mathbb{C}^3/l_p)$.

Here I used that one can, of course, associate with any $(d+1)$-dimensional vector space V over $\mathbb{C}$ a d-dimensional projective space $\mathbb{P}(V)$. Its points are the 1-dimensional subspaces of V.

If you don't like quotient vector spaces, you can take a direct complement U for l_p in $\mathbb{C}^3$ and replace $\mathbb{C}^3/l_p$ above by U.

(ii) The argument is the same as in the first special case of Bezout's Theorem (Lecture 5). If $C = \{F=0\}$ is the curve and L is the line, then $F|_L$ must at least have one zero, since the number of zeros is $\deg F$, counted with multiplicity, see Lecture 4.

(iii) Let p be a point outside the curve C and let S be the set of lines through p. Let $f: C \to S$ be the map that assigns to $q \in C$ the line through p and q. Then f is surjective by (ii). Since S is infinite by (i), we get that C has to be infinite.