

# A FROBENIUS SPLITTING AND COHOMOLOGY VANISHING FOR THE COTANGENT BUNDLES OF THE FLAG VARIETIES OF $GL_n$

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ABSTRACT. Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , let  $G = GL_n$  be the general linear group over  $k$ , let  $P$  be a parabolic subgroup of  $G$ , and let  $\mathfrak{u}_P$  be the Lie algebra of its unipotent radical. We show that the Kumar-Lauritzen-Thomsen splitting of the cotangent bundle  $G \times^P \mathfrak{u}_P$  of  $G/P$  has top degree  $(p-1)\dim(G/P)$ . The component of that degree is therefore given by the  $(p-1)$ -th power of a function  $f$ . We give a formula for  $f$  and deduce that it vanishes on the exceptional locus of the resolution  $G \times^P \mathfrak{u}_P \rightarrow \overline{O}$  where  $\overline{O}$  is the closure of the Richardson orbit of  $P$ . As a consequence we obtain that the higher cohomology groups of a line bundle on  $G \times^P \mathfrak{u}_P$  associated to a dominant weight are zero. The splitting of  $G \times^P \mathfrak{u}_P$  given by  $f^{p-1}$  can be seen as a generalisation of the Mehta-Van der Kallen splitting of  $G \times^B \mathfrak{u}$ .

## INTRODUCTION

Let  $G$  be a reductive group over an algebraically closed field  $k$  of positive characteristic  $p$ . For a parabolic  $P$  containing the positive Borel and  $P$ -module  $M$ , we denote by  $H^i(G/P, M)$  the  $i$ -th cohomology group of the sheaf  $\mathcal{L}_{G/P}(M)$  on  $G/P$  associated to  $M$ . It is an open problem whether we have for all parabolic subgroups  $P$  and all dominant characters  $\lambda$  of  $P$  that

$$H^i(G/P, S(\mathfrak{u}_P^*) \otimes k_{-\lambda}) = 0 \quad \text{for all } i > 0, \quad (*)$$

where the most important case is  $\lambda = 0$ , see e.g. [2, Introduction to Ch 5]. In characteristic 0 this is an easy consequence of the Grauert-Riemenschneider Theorem, see [4, Thm 2.2]. In characteristic  $p$  (\*) is known for  $P = B$ , for arbitrary  $P$  and “ $P$ -regular” dominant  $\lambda$ , see [12], and for  $P$  corresponding to sets of pairwise orthogonal short simple roots and  $\lambda = 0$ , see [15].

It is easy to write a formula for the Euler character

$$\sum_{i \geq 0} (-1)^i \operatorname{ch} H^i(G/P, S(\mathfrak{u}_P^*) \otimes k_{-\lambda}),$$

see [10, Sect 8.14-8.16] and [3, Prop 2.1], so if (8) holds we get a formula for  $\operatorname{ch} H^0(G/P, S(\mathfrak{u}_P^*) \otimes k_{-\lambda})$ .

When  $\mathcal{L}_{G/P}(\lambda) = \mathcal{L}_{G/P}(k_{-\lambda})$  is ample, i.e.  $\lambda$  “ $P$ -regular” dominant, one gets (\*) from the fact that  $G \times^P \mathfrak{u}_P$  is Frobenius split. One can also use Frobenius

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splittings to prove (\*) for  $\lambda = 0$  via a characteristic  $p$ -version of the Grauert-Riemenschneider Theorem [13, Thm 1.2], since the canonical bundle of  $G \times^P \mathfrak{u}_P$  is trivial. But then the map from  $G \times^P \mathfrak{u}_P$  to the Richardson orbit closure has to be birational and the splitting has to be a  $(p-1)$ -th power of a section  $\sigma$  of the anti-canonical bundle which vanishes on the exceptional locus. This is the approach we will follow.

When I asked Thomsen about the case  $G = \mathrm{GL}_n$ , he told me he expected that the pushforward to  $G \times^P \mathfrak{u}_P$  of the splitting of  $G \times^B \mathfrak{u}_P$  induced by the “MVdK-splitting” of  $G \times^B \mathfrak{u}$  from [14] is the homogeneous component of degree  $(p-1) \dim(G/P)$  of the “KLT-splitting”, see Section 1.2, from [12]. Although we can not prove this conjecture, we can show that the above component is in fact the top degree component and therefore a  $(p-1)$ -th power. From this we can then deduce that this homogeneous splitting vanishes on the exceptional locus of the resolution  $\varphi : [g, X] \mapsto gXg^{-1} : G \times^P \mathfrak{u}_P \rightarrow \overline{\mathcal{O}}$ , where  $\overline{\mathcal{O}}$  is the closure of the Richardson orbit corresponding to  $P$ , see Theorem 1 in Section 2. Finally, we then deduce that (\*) holds in type  $A$ , see Theorem 2. In fact we can formulate this as a result for arbitrary reductive groups.

The main idea of the proof is as follows. The “KLT-splitting” from [12] is the  $(p-1)$ -th power of the pullback along  $\varphi$  of the function which maps an  $n \times n$  matrix  $X$  to

$$\prod_{i=1}^{n-1} \det((I_n + X)_{\leq i, \leq i}), \quad (1)$$

where  $Y_{\leq i, \leq i}$  denotes the submatrix of  $Y$  given by the first  $i$  rows and columns, see [2, Example 5.1.15].<sup>1</sup> Unlike in the case  $P = B$ , the degree of the  $i$ -th factor may be less than  $i$ . In Lemma’s 1(ii) and 3 we determine the degree of the  $i$ -th factor and from that it follows that the product (1) has degree  $\dim(G/P)$ .

## 1. PRELIMINARIES

**1.1. Notation.** Let  $k$  be an algebraically closed field of characteristic  $p > 0$  and let  $G$  be a reductive group over  $k$ . We fix a Borel subgroup  $B \leq G$  and maximal torus  $T \leq B$ . We denote by  $R$  the set of roots of  $T$  in the Lie algebra  $\mathfrak{g} = \mathrm{Lie}(G)$  of  $G$ , and we denote the unipotent radical of  $B$  by  $U$ . We call the roots of  $T$  in  $\mathfrak{u} = \mathrm{Lie}(U)$  positive and we denote the corresponding set of simple roots by  $S$ . For a subset  $I$  of  $S$  we denote the root system spanned by  $I$  by  $R_I$ . Furthermore, we denote the corresponding parabolic subgroup containing  $B$  and its Levi subgroup containing  $T$  by  $P_I$  and  $L_I$ . Denote the character group of an algebraic group  $H$  by  $X(H)$ . For  $I \subseteq S$  we identify  $X(P_I)$  and  $X(L_I)$  with  $\{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I\}$ .

For  $P$  a parabolic of  $G$  and  $M$  a  $P$ -module we write  $\mathcal{L}(M)$  for the  $G$ -linearised sheaf on  $G/P$  associated to  $M$ . For  $\lambda \in X(P) \leq X(T)$  we put  $\mathcal{L}(\lambda) = \mathcal{L}(k_{-\lambda})$ , it is the sheaf of sections of the line bundle  $G \times^P k_{-\lambda}$  on  $G/P$ . We use the same symbol  $\mathcal{L}(\lambda)$  to denote the sheaf of sections of the pullback of this line bundle

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<sup>1</sup>Apart from the degree computation, the arguments there work for any parabolic.

to  $G \times^P V$  for any  $P$ -variety  $V$ . We also write  $H^i(G/P, M)$  for

$$H^i(G/P, \mathcal{L}(M)) \simeq R^i \mathrm{ind}_P^G(M),$$

see [9, I.5.12]. We have that

$$H^i(G \times^P \mathfrak{u}_P, \mathcal{L}(\lambda)) = H^i(G/P, k[\mathfrak{u}_P] \otimes k_{-\lambda}),$$

see [2, Lem 5.2.2].

If  $p = \mathrm{char} k$  is good for  $G$ , then we have  $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{u}_P^*$  as  $P$ -modules and  $G \times^P \mathfrak{u}_P$  is the cotangent bundle  $T^\vee(G/P)$  of  $G/P$ , see [2, 5.1.8-11].

**1.2. Frobenius splittings.** By [2, Lem 5.1.1] the canonical bundle of  $G \times^P \mathfrak{u}_P$  is trivial, so we can choose a nowhere zero global section: a volume form. It is easy to see that such a section is unique up to a scalar multiple, see [2, 5.1.2]. This means that we can think of Frobenius splittings (up to a scalar multiple) of  $G \times^P \mathfrak{u}_P$  as certain regular functions on  $G \times^P \mathfrak{u}_P$ .

In [12, Thm 1] it was proved that, when  $p$  is good for  $G$ , the cotangent bundle  $T^\vee(G/P)$  of  $G/P$  is Frobenius split, see also [2, Thm 5.1.3]. We will refer to the  $B$ -canonical splitting  $\psi_P(f_- \otimes f_+)$  as the “KLT-splitting” of  $T^\vee(G/P)$ , where  $\psi_P, f_-, f_+$  are as defined in [2, Ch 5]. Actually this is only a splitting up to a scalar multiple, but in the case  $G = \mathrm{GL}_n$  we assume that the chosen volume form on  $T^\vee(G/P)$  is such that the pullback along  $\varphi$  of the function given by (1),  $\varphi$  defined as in the introduction, defines a splitting. That formula is all we need to know about the KLT-splitting in this paper.

The standard grading of  $k[\mathfrak{u}_P] = S(\mathfrak{u}_P^*)$  gives a grading on  $k[G \times^P \mathfrak{u}_P]$ , and in [2, 5.1.14] it is explained that the homogeneous component of degree  $(p-1)\dim(G/P)$  of a splitting  $\sigma$  of  $G \times^P \mathfrak{u}_P$  is again a splitting of  $G \times^P \mathfrak{u}_P$ . This component is  $B$ -canonical if  $\sigma$  is  $B$ -canonical.

**1.3. A result on cohomology vanishing.** The following result is probably well-known, see e.g. [1, Sect 7.2], but for lack of reference we give a proof.

**Proposition 1.** *Assume  $p$  is good for  $G$ , let  $P$  be a parabolic of  $G$ , let  $\lambda \in X(P)$  be dominant, let  $Q$  be the parabolic of  $G$  containing  $P$  such that  $\lambda \in X(Q)$  and  $\mathcal{L}_{G/Q}(\lambda)$  is ample, and let  $L$  be the Levi subgroup of  $Q$  containing  $T$ . If  $H^i(L/L \cap P, S(\mathfrak{l}/\mathfrak{l} \cap \mathfrak{p})) = 0$  for all  $i > 0$ , then  $H^i(G/P, S(\mathfrak{g}/\mathfrak{p}) \otimes k_{-\lambda}) = 0$  for all  $i > 0$ .*

*Proof.* By [12, Cor 3 to Thm 4] or [2, Thm 5.3]  $G \times^P \mathfrak{u}_P$  is Frobenius split, so by [2, Lemma 1.2.7(i)] it is enough to show the vanishing for  $m\lambda$ ,  $m \gg 0$  (in fact we only need it for  $p^m\lambda$  and some  $m \geq 0$ ).

Some of the arguments below are adaptations of arguments from the proof of [2, Lem 5.2.7].

Each  $S^j(\mathfrak{g}/\mathfrak{p})$  has a filtration with sections  $S^r(\mathfrak{q}/\mathfrak{p}) \otimes S^s(\mathfrak{g}/\mathfrak{q})$ ,  $r + s = j$ , so it is enough to show that  $R^i \mathrm{ind}_P^G(S(\mathfrak{q}/\mathfrak{p}) \otimes S(\mathfrak{g}/\mathfrak{q}) \otimes k_{-\lambda}) = 0$  for all  $i > 0$ . We have  $P = (L \cap P)U_Q$  and  $\mathfrak{q}/\mathfrak{p} \simeq \mathfrak{l}/\mathfrak{l} \cap \mathfrak{p}$ . Note that  $U_Q$  acts trivially on  $\mathfrak{q}/\mathfrak{p}$ . Combining [9, I.6.11] and our assumption with a standard spectral sequence argument, we have

$$R^i \mathrm{ind}_P^G(S(\mathfrak{q}/\mathfrak{p}) \otimes S(\mathfrak{g}/\mathfrak{q}) \otimes k_{-\lambda}) \simeq R^i \mathrm{ind}_Q^G(\mathrm{ind}_P^Q S(\mathfrak{q}/\mathfrak{p}) \otimes S(\mathfrak{g}/\mathfrak{q}) \otimes k_{-\lambda}). \quad (2)$$

We have  $\mathfrak{p} = \mathfrak{l} \cap \mathfrak{p} \oplus \mathfrak{u}_Q$ ,  $\mathfrak{u}_P = \mathfrak{u}_{L \cap P} \oplus \mathfrak{u}_Q$ ,  $(\mathfrak{g}/\mathfrak{p})^* \simeq \mathfrak{u}_P$ ,  $(\mathfrak{g}/\mathfrak{q})^* \simeq \mathfrak{u}_Q$ , and  $(\mathfrak{q}/\mathfrak{p})^* \simeq \mathfrak{u}_{L \cap P}$ . By the arguments of [10, p94] there exists an affine  $Q$ -variety  $V_0$  such that  $k[V_0] \simeq k[Q \times^P \mathfrak{u}_{L \cap P}] = \text{ind}_P^Q S(\mathfrak{q}/\mathfrak{p})$ ,  $Q$ -equivariantly ( $U_Q$  acting trivially). Put  $V = V_0 \times \mathfrak{u}_Q$ . Then  $k[V] = \text{ind}_P^Q S(\mathfrak{q}/\mathfrak{p}) \otimes S(\mathfrak{g}/\mathfrak{q})$ . Now the morphism  $G \times^Q V \rightarrow G/Q$  is affine, so by [7, Ex III.8.2] the RHS of (2) is isomorphic to

$$H^i(G \times^Q V, \mathcal{L}(\lambda)). \quad (3)$$

By [6, 5.1.12]  $\mathcal{L}(\lambda)$  is ample on  $G \times^Q V$ , since  $G \times^Q V \rightarrow G/Q$  is affine. The morphism  $V_0 \rightarrow \overline{Q \cdot \mathfrak{u}_{L \cap P}}$  is finite, see [10, p94], so the same is true for the morphisms  $V \rightarrow \overline{Q \cdot \mathfrak{u}_{L \cap P}} \times \mathfrak{u}$  and  $G \times^Q V \rightarrow G \times^Q (\overline{Q \cdot \mathfrak{u}_{L \cap P}} \times \mathfrak{u}_Q)$ . So composing the latter with the projective morphism  $G \times^Q (\overline{Q \cdot \mathfrak{u}_{L \cap P}} \times \mathfrak{u}_Q) \rightarrow \mathfrak{g}$ , given by the embedding of  $\overline{Q \cdot \mathfrak{u}_{L \cap P}} \times \mathfrak{u}_Q$  in  $\mathfrak{g}$ , we obtain a proper morphism  $G \times^Q V \rightarrow \mathfrak{g}$ . Now [7, III.5.3] tells us that (3) is 0 if we replace  $\lambda$  by  $m\lambda$ ,  $m \gg 0$ .  $\square$

## 2. THE MAIN RESULTS

Throughout this section, except in Theorem 2 and its proof,  $G = \text{GL}_n = \text{GL}(k^n)$  and  $T$  is the subgroup of diagonal matrices in  $G$ . As simple roots we choose the usual characters  $\varepsilon_i - \varepsilon_{i+1}$ ,  $1 \leq i \leq n-1$ , where we used additive notation for characters, and  $\varepsilon_i$  is the  $i$ -th coordinate function on  $T$ . Then  $B$  consists of the upper triangular matrices in  $G$ . As is well-known, the conjugacy classes of parabolic subgroups of  $G$  are labelled by the compositions of  $n$ , see e.g. [8, 3.2]. By  $\nu$  we denote a composition of  $n$  and  $P = P_\nu \supseteq B$  is the standard parabolic whose block sizes are given in order by  $\nu$ . If  $A_\nu$  is the set  $\{\nu_1, \nu_1 + \nu_2, \dots, \sum_{j=1}^{s-1} \nu_j\}$ ,  $s$  the length of  $\nu$ , then  $P_\nu = P_{I_\nu}$ , the parabolic associated to the set of simple roots  $I_\nu = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \dots, n-1\} \setminus A_\nu\}$ . We denote by  $\lambda$  the transposed partition of the weakly descending sorted version of  $\nu$ . It is well-known that the Richardson orbit of  $P_\nu$  is  $\mathcal{O}_\lambda$ , the nilpotent orbit whose Jordan block sizes are given by  $\lambda$ , see e.g. [8, Thm 3.3(a)].

It is well-known that the map  $\varphi : [g, X] \mapsto gXg^{-1} : G \times^P \mathfrak{u}_P \rightarrow \overline{\mathcal{O}_\lambda}$  is birational. Indeed the group centraliser  $G_X$  of any  $X \in \mathfrak{g}$  is the set of invertible elements in the Lie algebra centraliser  $\mathfrak{g}_X$ , so is connected. Now see [10, 4.9 and 8.8 Remark]. It is also well-known that  $\overline{\mathcal{O}_\lambda}$  is normal, see e.g. [5] or [14, Sect 4.7].

For  $i \in \{1, \dots, n-1\}$  we denote by  $d_{\lambda,i}$  the number of nonzero positions on the  $(n-i)$ -th upper codiagonal of  $\mathfrak{u}_P$ . So for  $\nu = (2, 1, 2)$  we have  $d_{\lambda,1}, d_{\lambda,2}, d_{\lambda,3}, d_{\lambda,4} = 1, 2, 3, 2$ , see the figure of  $\mathfrak{u}_P$  below.

$$\begin{bmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since each diagonal  $j \times j$  block of  $P$  takes away  $j - i$  nonzero positions from the  $i$ -th upper codiagonal, we have, if  $j$  occurs  $m_j$  times in  $\nu$ ,

$$d_{\lambda, n-i} = n - i - \sum_{j>i} (j - i)m_j = n - i - \sum_{j>i} \lambda_j = - \sum_{j=i+1}^n (\lambda_j - 1) = \sum_{j=1}^i (\lambda_j - 1).$$

Therefore,  $d_{\lambda, i} = i - \sum_{j>n-i} \lambda_j = \sum_{j=1}^{n-i} (\lambda_j - 1)$ . So indeed the  $d_{\lambda, i}$  only depend on  $\lambda$ , moreover, they determine  $\lambda$ .

For a square matrix  $X$  we denote by  $X_{\leq i, \leq i}$  the submatrix of  $X$  given by the first  $i$  rows and columns. For an  $i \times i$  matrix  $Y$  we denote by  $s_j(Y)$  the trace of the  $j$ -th exterior power of  $Y$ , i.e. the sum of the diagonal  $j \times j$  minors of  $Y$ . As is well-known,  $\det(aI_i - Y) = a^i + \sum_{j=1}^i (-1)^j a^{i-j} s_j(Y)$ , where  $I_i$  is the  $i \times i$  identity matrix. So the largest  $j$  with  $s_j(Y) \neq 0$  is the number of nonzero eigenvalues of  $Y$ , counted with (algebraic) multiplicity. This number also equals the rank of  $Y^l$  for  $l$  sufficiently big. We will call it the *stable rank* of  $Y$ .

**Lemma 1.** *Let  $X \in \mathcal{O}_\lambda$ .*

- (i) *Any  $i$ -dimensional subspace  $W$  of  $V = k^n$  contains an  $X$ -invariant subspace  $U$  of dimension  $\geq \sum_{j>n-i} \lambda_j$ .*
- (ii)  *$X_{\leq i, \leq i}$  has stable rank  $\leq d_{\lambda, i}$ .*

*Proof.* (i). We show this by induction on  $n$ . It is trivial when  $i \leq n - r$ ,  $r$  the length of  $\lambda$ , in particular when  $n = 0$ . Assume  $i > n - r$ . Then  $W$  has nonzero intersection with  $\mathrm{Ker}(X)$  for dimension reasons. Pick  $v$  nonzero in that intersection. First note that the transformation  $\bar{X}$  induced on  $V/kv$  by  $X$  has partition  $\mu$  which is obtained from  $\lambda$  by subtracting 1 from one part of  $\lambda$  and then sorting the result in weakly descending order. Indeed if we decompose  $V$  as a direct sum of  $X$ -Jordan blocks and we pick a  $X$ -Jordan block of minimal size with the property that  $v$  has nonzero component in it, then we can replace that  $X$ -Jordan block by an  $X$ -Jordan block of the same size which contains  $v$ . Now we apply the induction hypothesis to  $V/kv$  and  $W/kv$ , noting that  $(n - 1) - (i - 1) = n - i$ , to obtain an  $\bar{X}$ -invariant subspace  $U/kv$  of  $W/kv$  of dimension  $\geq \sum_{j>n-i} \mu_j \geq \sum_{j>n-i} \lambda_j - 1$ . Now  $U$  is the  $X$ -invariant subspace we want.

(ii). The linear map  $(X_{\leq i, \leq i})^i$  coincides with  $X^i$  on any  $X$ -invariant subspace  $U$  of  $k^i \leq k^n$  and therefore kills it. Choosing  $U$  as in (i), it induces a linear map  $k^i/U \rightarrow k^i$  and therefore has rank  $\leq i - \sum_{j>n-i} \lambda_j = d_{\lambda, i}$ .  $\square$

Lemma 3 below follows from Lemma 1(ii) and the existence of the KLT-splitting, but we prefer to give a direct proof.

**Lemma 2.** *For any  $h \in \{1, \dots, i - 1\}$  there exists a regular nilpotent  $i \times i$  matrix  $X$  such that  $X_{\leq h, \leq h}$  is invertible.*

*Proof.* Let  $(e_1, \dots, e_i)$  be the standard basis of  $k^i$ . Then the regular nilpotent matrix  $X$  given by  $X(e_j) = e_{j-1}$  for  $2 \leq j \leq i$ ,  $X(e_2) = e_1 + e_{h+1}$  and  $X(e_1 + e_{h+1}) = 0$  has the desired property.  $\square$

**Remark 2.1.** Of course it follows from Lemma 2 that there exists a regular  $i \times i$  matrix  $X$  such that  $X_{\leq h, \leq h}$  is invertible for all  $h \in \{1, \dots, i-1\}$ , but we won't need this.

**Lemma 3.** *There exists  $X \in \mathcal{O}_\lambda$  such that  $X_{\leq i, \leq i}$  has stable rank  $d_{\lambda, i}$ .*

*Proof.* First choose any  $Y \in \mathfrak{g}$  nilpotent with partition  $\lambda$  and decompose  $k^n$  into  $Y$ -Jordan blocks with sizes  $\lambda_1, \lambda_2, \dots, \lambda_r$ , where  $r$  is the length of  $\lambda$ . It suffices to find an ordered basis  $\mathcal{B}$  of  $k^n$  such that the upper left  $i \times i$ -block  $Z$  of the matrix of  $Y$  relative to this basis has stable rank  $d_{\lambda, i}$ .

Determine  $s \leq r$  maximal with  $\sum_{j=1}^s (\lambda_j - 1) \leq i$  and put  $h = i - \sum_{j=1}^s (\lambda_j - 1)$ . Using Lemma 2 choose for each  $j \leq s$  a basis of the  $j$ -th block such that the upper left  $(\lambda_j - 1) \times (\lambda_j - 1)$  block of the matrix of  $Y$  relative to this basis is invertible, if  $s < r$  and  $h > 0$  choose a basis of the  $(s+1)$ -th block such that the upper left  $h \times h$  block of the matrix of  $Y$  relative to this basis is invertible, and for the remaining blocks choose any basis.

We now form  $\mathcal{B}$  as follows. First consider the case  $i \leq n - r$ . For each  $j \leq s$  we pick the first  $\lambda_j - 1$  basis vectors from the  $j$ -th block, if  $s < r$  we append the first  $h$  basis vectors from the  $(s+1)$ -th block, and finally we append all remaining  $n - i$  basis vectors. Now  $Z$  is in block diagonal form with invertible diagonal block of sizes  $\lambda_1 - 1, \dots, \lambda_s - 1, h$ , where  $h$  has to be omitted if  $h = 0$ . Now consider the case  $i > n - r$ . For each  $j \leq n - i$  we pick the first  $\lambda_j - 1$  basis vectors from the  $j$ -th block, then we append the basis vectors from the next  $r - (n - i)$  blocks, and finally we append all remaining  $n - i$  basis vectors. Now  $Z$  is in block diagonal form with diagonal block sizes  $\lambda_1 - 1, \dots, \lambda_{n-i} - 1, \lambda_{n-i+1}, \dots, \lambda_r$  where the first  $n - i$  blocks are invertible, and the others nilpotent. In both cases we obtain that  $Z$  has stable rank  $d_{\lambda, i}$  (when  $i \leq n - r$  we have  $d_{\lambda, i} = i$ ).  $\square$

Below we will denote a function  $X \mapsto E(X)$  on a closed subvariety of  $\mathfrak{g}$  just by the expression  $E(X)$ .

**Theorem 1.** *The degree  $(p-1) \dim(G/P)$  component of the KLT splitting of  $G \times^P \mathfrak{u}_P$  is the top degree component and equals the  $(p-1)$ -th power of the pullback of  $\prod_{i=1}^{n-1} s_{d_{\lambda, i}}(X_{\leq i, \leq i}) \in k[\overline{\mathcal{O}_\lambda}]$  along the resolution  $\varphi : G \times^P \mathfrak{u}_P \rightarrow \overline{\mathcal{O}_\lambda}$ . This pullback vanishes on the exceptional locus of  $\varphi$ .*

*Proof.* The KLT splitting is the pullback along  $\varphi$  of the function given by (1). Furthermore, we have  $\det(I_i + Y) = \sum_{j=0}^i s_j(Y)$  for any  $i \times i$  matrix  $Y$ , and, of course,  $s_j(X_{\leq i, \leq i}) \neq 0$  on  $\mathcal{O}_\lambda \iff s_j(X_{\leq i, \leq i}) \neq 0$  on  $\overline{\mathcal{O}_\lambda}$ . So by Lemma's 1(ii) and 3 the top degree component of the  $i$ -th factor in (1) is  $s_{d_{\lambda, i}}(X_{\leq i, \leq i})$ . So the KLT-splitting has top degree  $p-1$  times  $\sum_{i=1}^{n-1} d_{\lambda, i} = \dim \mathfrak{u}_P = \dim(G/P)$ , and the top degree component is the  $(p-1)$ -th power of the pullback along  $\varphi$  of the function given by the stated formula.

To prove the second assertion, put  $f_{\lambda, i}(X) = s_{d_{\lambda, i}}(X_{\leq i, \leq i})$  and  $f_\lambda = \prod_{i=1}^{n-1} f_{\lambda, i}$ . The exceptional locus is  $\varphi^{-1}(\overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda)$ , so it suffices to show that  $f_\lambda$  vanishes on any  $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda$ . We have  $\dim(\mathfrak{u}_Q) = \frac{1}{2} \dim(\mathcal{O}_\mu) < \frac{1}{2} \dim(\mathcal{O}_\lambda) = \dim(\mathfrak{u}_P)$ , where  $Q$  is a standard parabolic whose Richardson orbit is  $\mathcal{O}_\mu$ , see [10, 4.9].

So for some  $i$  we must have  $d_{\mu,i} < d_{\lambda,i}$  which means that  $f_{\lambda,i}$  and therefore  $f_\lambda$  vanishes on  $\mathcal{O}_\mu$ .  $\square$

**Theorem 2.** *Let  $G$  be any reductive group for which  $p$  is good, let  $\lambda \in X(T)$  be dominant, put  $I = \{\alpha \in S \mid \langle \lambda, \alpha^\vee \rangle = 0\}$ . Then  $H^i(T^\vee(G/P_J), \mathcal{L}(\lambda)) = 0$  for all  $J \subseteq I$  such that  $R_J$  contains all irreducible components of  $R_I$  not of type  $A$ .*

*Proof.* By Proposition 1 we may assume that  $\lambda = 0$  and that all irreducible components of  $R$  have type  $A$ . Since we are dealing with cotangent bundles we may assume that  $G$  is semisimple and simply connected. By the Künneth formula [11, Prop 9.2.4] we may then assume  $G = \mathrm{SL}_n$  and finally we may assume  $G = \mathrm{GL}_n$ . Now the result follows from Theorem 1 and [13, Thm 1.2], bearing in mind that the canonical bundle of  $T^\vee(G/P)$  is trivial, see [2, Lem 5.1.1], and that  $R^i\varphi_*(\mathcal{O}_{T^\vee(G/P)})$  is the sheaf associated with the cohomology group  $H^i(T^\vee(G/P), \mathcal{O}_{T^\vee(G/P)})$ , since  $\varphi$  is affine.  $\square$

We remind the reader that a proper birational morphism  $\psi : X \rightarrow Y$  is called a *rational resolution* if  $\psi_*\mathcal{O}_X = \mathcal{O}_Y$  and the higher direct images of  $\mathcal{O}_X$  and  $\omega_X$  are 0, see [2, Def 3.4.1].

**Corollary.** *The resolution  $\varphi : G \times^P \mathfrak{u}_P \rightarrow \overline{\mathcal{O}_\lambda}$  is rational.*

*Proof.* This follows from a standard argument, see e.g. [9, Lem 14.5], and Theorem 2.  $\square$

**Remarks 2.2.** 1. If  $P = B$ , then  $d_{(n),i} = i$  for all  $i$ , so the splitting from Theorem 1 equals the  $(p-1)$ -th power of the pullback of  $\prod_{i=1}^{n-1} \det(X_{\leq i, \leq i})$  along  $\varphi : G \times^B \mathfrak{u} \rightarrow \mathcal{N}$ . This is the MVdK splitting of  $G \times^B \mathfrak{u}$ , see [14].  
2. Thomsen mentioned to me another proof of Lemma 1(ii): One can easily deduce it from the following result which can be proved by induction on  $n$ . For  $X \in \mathfrak{u}_P$  let  $y_{ij} = \delta_{ij} + x_{ij}$  be the  $(i, j)$ -th entry of  $I_n + X$ . Then any monomial  $y_{i_1 j_1} y_{i_2 j_2} \cdots y_{i_s j_s}$  with the  $i_l$  all distinct and the  $j_l$  all distinct has degree  $\leq d_{\lambda,s}$  in the  $x_{ij}$ .

**Conjecture (Thomsen).** *The pushforward to  $G \times^P \mathfrak{u}_P$  of the splitting of  $G \times^B \mathfrak{u}_P$  induced by the MVdK splitting is the top degree component of the KLT splitting.*

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#### REFERENCES

- [1] C. Bendel, D. Nakano, B. Parshall, C. Pillen, *Cohomology for quantum groups via the geometry of the nullcone*, Mem. Amer. Math. Soc. **229** (2014), no. 1077.
- [2] M. Brion, S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, 231, Birkhäuser Boston, Inc., Boston, MA, 2005.
- [3] A. Broer, *Line bundles on the cotangent bundle of the flag variety*, Invent. Math. **113** (1993), no. 1, 1-20.
- [4] A. Broer, *Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety*, in “Lie Theory and Geometry”, Prog. Math. **123** (1994), Birkhäuser, Boston, 1-19.

- [5] S. Donkin, *The normality of closures of conjugacy classes of matrices*, Invent. Math. **101** (1990), no. 3, 717-736.
- [6] A. Grothendieck, *Eléments de géométrie algébrique II, Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. No. 8, 1961.
- [7] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag, New York-Heidelberg, 1977.
- [8] W. H. Hesselink, *Polarizations in the classical groups*, Math. Z. **160** (1978), no. 3, 217-234.
- [9] J. C. Jantzen, *Representations of algebraic groups*, Second edition, American Mathematical Society, Providence, RI, 2003.
- [10] J. C. Jantzen, *Nilpotent orbits in representation theory*, “Lie theory”, Progr. Math. **228**, Birkhäuser Boston, 2004, 1-211.
- [11] G. R. Kempf, *Algebraic Varieties*, London Math. Soc. Lecture Note Ser. 172, Cambridge University Press, Cambridge, 1993.
- [12] S. Kumar, N. Lauritzen, J. F. Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), no.3, 603-621.
- [13] V. B. Mehta, W. van der Kallen, *On a Grauert-Riemenschneider vanishing theorem for Frobenius split varieties in characteristic  $p$* , Invent. Math. **108** (1992), no. 1, 11-13.
- [14] V. B. Mehta, W. van der Kallen, *A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices*, Compositio Math. **84** (1992), no. 2, 211-221.
- [15] J. F. Thomsen, *Normality of certain nilpotent varieties in positive characteristic*, J. Algebra **227** (2000), no. 2, 595-613.

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