1. For each of the following cases you are given a set $\mathfrak{g}$ and a bracket $[\ ,\ ]$ on $\mathfrak{g}$. Addition of elements of $\mathfrak{g}$ and scalar multiplication by elements of the complex field $\mathbb{C}$ are to be understood in the obvious sense. In which of these cases is $\mathfrak{g}$ a Lie algebra over $\mathbb{C}$? Give reasons for your answers.

(i) $\mathfrak{g} = \mathbb{C}$ and $[x, y] = x - y$.

(ii) $\mathfrak{g} = \mathbb{C}$ and $[z, w] = z\bar{w} - w\bar{z}$.

(iii) $\mathfrak{g} =$ set of all symmetric $n \times n$ matrices ($n > 1$) with complex entries and

$$[X, Y] = XY - YX\,.$$

(iv) $\mathfrak{g} =$ set of all continuous complex-valued functions on the interval $[0, 1]$ and

$$[f, g](x) = f(x)\int_0^1 g(t)\,\mathrm{d}t - g(x)\int_0^1 f(t)\,\mathrm{d}t\,.$$

(v) $\mathfrak{g} =$ set of all differentiable complex-valued functions on the interval $[0, 1]$,

$$[f, g] = f\frac{\mathrm{d}g}{\mathrm{d}x} - g\frac{\mathrm{d}f}{\mathrm{d}x}\,.$$

(vi) $\mathfrak{g} =$ set of all complex $n \times n$ matrices and

$$[X, Y] = XY^T - YX^T\,,$$

where the superscript T indicates matrix transposition.

2. Let n be a natural number and for $i, j, k \in \{1, \dots, n\}$ with $i < j$, let $c_{ij}^k \in \mathbb{F}$ be a scalar. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. Assume that $\mathfrak{g}$ has a basis $(X_1, \dots, X_n)$ such that $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$ for all $i < j$ and assume that $\mathfrak{h}$ has a basis $(Y_1, \dots, Y_n)$ such that $[Y_i, Y_j] = \sum_{k=1}^n c_{ij}^k Y_k$ for all $i < j$. Show that the linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ defined by $\varphi(X_i) = Y_i$ is an isomorphism of Lie algebras.

3. Let $\mathfrak{g}$ be the vector space of all those complex-valued functions of $2n$ variables $q_1, \dots, q_n, p_1, \dots, p_n$ which have partial derivatives of all orders with respect to all variables. Addition of these functions and their scalar multiplication by elements of the field $\mathbb{C}$ are understood in the obvious sense. The *Poisson bracket* $\{F, G\}$ of two functions $F, G \in \mathfrak{g}$ is defined by the formula

$$\{F, G\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right)\,.$$

Prove that $[F, G] = \{F, G\}$ satisfies all axioms of a Lie bracket.