$\boxed{15}$

(1) (i) Clearly $e \in C$. If $x, y \in C$ and $z \in G$, then $xyz = xzy = zxy$, so
$\boxed{2}$ $xy \in C$. If $x \in C$ and $y \in G$, then $x^{-1}y = (y^{-1}x)^{-1} = (xy^{-1})^{-1} = yx^{-1}$, so $x^{-1} \in C$. If $y \in C$ and $x, z \in G$, then $xyx^{-1} = xx^{-1}y = y$, so $xyx^{-1}z = yz = zy = zxyx^{-1}$. So $xyx^{-1} \in C$.

(ii) Assume π commutes with σ. If we use the first hint,
$\boxed{2}$ then $(\pi(a_1), \dots, \pi(a_k)) = (a_1, \dots, a_k)$. So certainly $\{\pi(a_1), \dots, \pi(a_k)\} = \{a_1, \dots, a_k\}$ and we are done. Now we give the direct proof. Consider the identity $\pi^{-1}(a_1, \dots, a_k)\pi = (a_1, \dots, a_k)$ $(*)$. Let $i \in \{1, \dots, k\}$ and assume $\pi(a_i) \notin \{a_1, \dots, a_k\}$. Applying the LHS of $(*)$ to a_i we get a_i, but applying the RHS of $(*)$ to a_i we get something $\neq a_i$ (a_{i+1} if $i<k$ and a_1 otherwise). Contradiction.

(iii) Assume π is in the centre and let $i \in \{1, \dots, n\}$. Pick $j, j' \in \{1, \dots, n\} \setminus \{i\}$ distinct. This is
$\boxed{2}$ possible, since $n \geq 3$. Then, by (ii), $\pi(\{i,j\}) \subseteq \{i,j\}$ and $\pi(\{i,j'\}) \subseteq \{i,j'\}$. So π maps $\{i\} = \{i,j\} \cap \{i,j'\}$ into itself, i.e. $\pi(i) = i$. So $\pi = id$ and the centre is trivial.

(2) Clearly $e' = f(e) \in f(N)$. If $x' = f(x), y' = f(y) \in f(N)$ $(x, y \in N)$, then
$\boxed{3}$ $x'y' = f(xy) \in f(N)$. If $x' = f(x) \in f(N)$ $(x \in N)$, then $(x')^{-1} = f(x^{-1}) \in f(N)$. If $x' = f(x) \in G'$, $y' = f(y) \in f(N)$ $(x \in G, y \in N)$, then $x'y'(x')^{-1} = f(xyx^{-1}) \in f(N)$. So $f(N) \trianglelefteq G'$.

(3) (i) Let $x \in H$, $y \in K$. Put $z = xyx^{-1}y^{-1}$. Then $z = x(yx^{-1}y^{-1}) \in H$, since
$\boxed{2}$ $H \trianglelefteq G$ and $z = (xyx^{-1})y^{-1} \in K$, since $K \trianglelefteq G$. So $z \in H \cap K = \{e\}$, i.e. $z = e$. So $xy = zyx = yx$.

(ii) Consider the map $\theta : H \times K \to G$ given by $\theta(x,y) = xy$ for
$\boxed{2}$ $(x,y) \in H \cap K$. Let $(x,y), (x',y') \in H \times K$. Then $\theta((x,y)(x',y')) = \theta(xx', yy') = xx'yy' = xyx'y' = \theta(x,y)\theta(x',y')$, since the elements of H commute with those of K. So θ is a homomorphism. Now let $(x,y) \in H \times K$ and assume $\theta(x,y) = xy = e$. Then $x = y^{-1} \in H \cap K = \{e\}$. So $(x,y) = (e,e)$. So $\ker(\theta) = \{(e,e)\}$ and θ is injective. Furthermore θ is surjective, since H and K generate G. So $H \times K \simeq G$.

$\boxed{2}$ Now define $\varphi : K \to G/H$ by $\varphi(x) = xH$. Then φ is a homomorphism, since it is the restriction of the canonical homomorphism to the subgroup H. It's kernel is $H \cap K = \{e\}$, so it's injective, and it is surjective, since H and K generate G.