INTRODUCTION TO LIE ALGEBRAS – SOLUTION 1

(i) *No.* This bracket is not bilinear: $[ax, y] \neq a[x, y]$ for any $a \in \mathbb{C}$, when $y \neq 0$.

(ii) *No.* This bracket is not bilinear: $[cz, w] \neq c[z, w]$ for any $c \in \mathbb{C} \setminus \mathbb{R}$, when $z, w \neq 0$.

(iii) *No.* If $[X, Y] \neq 0$ (which is possible when $n > 1$), then $[X, Y] \notin \mathfrak{g}$. Indeed, for $X^t = X$ and $Y = Y^t$ we have

$$[X, Y]^t = Y^t X^t - X^t Y^t = YX - XY = -[X, Y] \neq [X, Y]\,.$$

(iv) *Yes.* The bracket $[f, g]$ is antisymmetric and bilinear by definition. We also have $[f, g] \in \mathfrak{g}$. Let us check the Jacobi identity for any three $\mathbb{C}$-valued continuous functions f, g, h on the interval $[0, 1]$. By definition, we have

$$\begin{aligned}
&[f,[g,h]](x) = f(x)\int_0^1 [g,h](t)\,\mathrm{d}t - [g,h](x)\int_0^1 f(t)\,\mathrm{d}t = \\
&f(x)\int_0^1 g(t)\,\mathrm{d}t \int_0^1 h(s)\,\mathrm{d}s - f(x)\int_0^1 h(t)\,\mathrm{d}t \int_0^1 g(s)\,\mathrm{d}s \\
&- g(x)\int_0^1 h(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t + h(x)\int_0^1 g(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t = \\
&- g(x)\int_0^1 h(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t + h(x)\int_0^1 g(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&[f,[g,h]](x) + [g,[h,f]](x) + [h,[f,g]](x) = \\
&- g(x)\int_0^1 h(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t + h(x)\int_0^1 g(s)\,\mathrm{d}s \int_0^1 f(t)\,\mathrm{d}t \\
&- h(x)\int_0^1 f(s)\,\mathrm{d}s \int_0^1 g(t)\,\mathrm{d}t + f(x)\int_0^1 h(s)\,\mathrm{d}s \int_0^1 g(t)\,\mathrm{d}t \\
&- f(x)\int_0^1 g(s)\,\mathrm{d}s \int_0^1 h(t)\,\mathrm{d}t + g(x)\int_0^1 f(s)\,\mathrm{d}s \int_0^1 h(t)\,\mathrm{d}t
\end{aligned}$$

where the six summands cancel each other.

(v) *No.* The function $[f, g](x)$ may be non-differentiable, then $[f, g] \notin \mathfrak{g}$. This happens, for example, when $f(x) = 1$, $g(x)$ is differentiable, but $\mathrm{d}g/\mathrm{d}x$ is not.

(vi) *No.* The Jacobi identity is not satisfied. For example, for the 2×2 matrices

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we get

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq 0\,.$$

Since φ maps a basis to a basis, it is bijective. So we only have to show that it is a homomorphism of Lie algebras. Put differently, we have to show that the map

$$\psi : (X, X') \mapsto \varphi([X, X']) - [\varphi(X), \varphi(X')] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$$

is identically zero. Since ψ is bilinear and "skew-symmetric" (i.e. $\psi(X, X) = 0$ for all $X \in \mathfrak{g}$), it suffices to show that $\psi(X_i, X_j) = 0$ for all $i < j$. We compute

$$\psi(X_i, X_j) = \varphi([X_i, X_j]) - [\varphi(X_i), \varphi(X_j)] = \varphi\big(\sum_{k=1}^{n} c_{ij}^k X_k\big) - \sum_{k=1}^{n} c_{ij}^k Y_k = 0 \ .$$

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 3

The Poisson bracket $\{F,G\}$ is antisymmetric and bilinear by definition. Let us check the Jacobi identity. We have

$$\{F,\{G,H\}\} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial\{G,H\}}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial\{G,H\}}{\partial q_i}\right) =$$

$$\sum_{i,j=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial G}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j}\right) - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial G}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j}\right)\right) =$$

$$\sum_{i,j=1}^{n}\Bigg(\frac{\partial F}{\partial q_i}\frac{\partial^2 G}{\partial p_i\partial q_j}\frac{\partial H}{\partial p_j} + \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial q_j}\frac{\partial^2 H}{\partial p_i\partial p_j} - \frac{\partial F}{\partial q_i}\frac{\partial^2 G}{\partial p_i\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial p_i\partial q_j}$$
$$-\frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial q_i\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_j}\frac{\partial^2 H}{\partial q_i\partial p_j} + \frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial q_i\partial p_j}\frac{\partial H}{\partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_i\partial q_j}\Bigg).$$

Therefore

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} =$$

$$\sum_{i,j=1}^{n}\Bigg(\frac{\partial F}{\partial q_i}\frac{\partial^2 G}{\partial p_i\partial q_j}\frac{\partial H}{\partial p_j} + \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial q_j}\frac{\partial^2 H}{\partial p_i\partial p_j} - \frac{\partial F}{\partial q_i}\frac{\partial^2 G}{\partial p_i\partial p_j}\frac{\partial H}{\partial q_j} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial p_i\partial q_j}$$
$$-\frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial q_i\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_j}\frac{\partial^2 H}{\partial q_i\partial p_j} + \frac{\partial F}{\partial p_i}\frac{\partial^2 G}{\partial q_i\partial p_j}\frac{\partial H}{\partial q_j} + \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}\frac{\partial^2 H}{\partial q_i\partial q_j}$$
$$\frac{\partial G}{\partial q_i}\frac{\partial^2 H}{\partial p_i\partial q_j}\frac{\partial F}{\partial p_j} + \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial q_j}\frac{\partial^2 F}{\partial p_i\partial p_j} - \frac{\partial G}{\partial q_i}\frac{\partial^2 H}{\partial p_i\partial p_j}\frac{\partial F}{\partial q_j} - \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 F}{\partial p_i\partial q_j}$$
$$-\frac{\partial G}{\partial p_i}\frac{\partial^2 H}{\partial q_i\partial q_j}\frac{\partial F}{\partial p_j} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_j}\frac{\partial^2 F}{\partial q_i\partial p_j} + \frac{\partial G}{\partial p_i}\frac{\partial^2 H}{\partial q_i\partial p_j}\frac{\partial F}{\partial q_j} + \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 F}{\partial q_i\partial q_j}$$
$$\frac{\partial H}{\partial q_i}\frac{\partial^2 F}{\partial p_i\partial q_j}\frac{\partial G}{\partial p_j} + \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial q_j}\frac{\partial^2 G}{\partial p_i\partial p_j} - \frac{\partial H}{\partial q_i}\frac{\partial^2 F}{\partial p_i\partial p_j}\frac{\partial G}{\partial q_j} - \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_j}\frac{\partial^2 G}{\partial p_i\partial q_j}$$
$$-\frac{\partial H}{\partial p_i}\frac{\partial^2 F}{\partial q_i\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_j}\frac{\partial^2 G}{\partial q_i\partial p_j} + \frac{\partial H}{\partial p_i}\frac{\partial^2 F}{\partial q_i\partial p_j}\frac{\partial G}{\partial q_j} + \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial p_j}\frac{\partial^2 G}{\partial q_i\partial q_j}\Bigg).$$

Here we are taking the sum over i and j of 24 terms, arranged in 6 rows. Let us exchange the summation indices i and j in the 1st and 3rd terms in the 3rd row, the 1st and 3rd terms in the 4rd row, and in all terms in the 5th and 6th rows. Then all the 24 terms will cancel each other. This proves the Jacobi identity.