6. For each $k = 0, 1, 2, \ldots$ let $\mathfrak{g}_k$ be the vector space of all $n \times n$ matrices $X = [X_{ij}]$ over the field $\mathbb{F}$ such that $X_{ij} = 0$ unless $j - i \geqslant k$. Furthermore, let $\mathfrak{t}$ be the vector space of all $n \times n$ matrices $X = [X_{ij}]$ over the field $\mathbb{F}$ such that $X_{ij} = 0$ unless $i = j$. Note that $\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{g}_1$ and that $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ by definition. Also note that $\mathfrak{g}_n = \mathfrak{g}_{n+1} = \cdots = 0$. Prove that, for $k, l \geqslant 1$, $[\mathfrak{g}_k, \mathfrak{g}_l] = \mathfrak{g}_{k+l}$ for the matrix commutator

$$[X, Y] = XY - YX\,.$$

Deduce that $[\mathfrak{g}_0, \mathfrak{g}_k] = [\mathfrak{t}, \mathfrak{g}_k] = \mathfrak{g}_k$ for $k \geqslant 1$ and that $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{t}, \mathfrak{g}_0] = \mathfrak{g}_1$. Finally, show that each $\mathfrak{g}_k$ is an ideal of $\mathfrak{g}_0$ and that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{gl}_n\mathbb{F}$.

7. Let $D : \mathfrak{g} \to \mathfrak{g}$ be a derivation of an arbitrary Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$. For every $n = 1, 2, \ldots$ the linear operator D can be applied n times, denote the resulting operator by D^n. Prove the following generalization of the *Leibniz rule*:

$$D^n[X, Y] = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \cdot [\,D^k X, D^{n-k} Y\,]\,.$$

8. Let X be a linear operator in an n-dimensional vector space V over the field $\mathbb{C}$. Suppose X has n distinct eigenvalues $a_1, \ldots, a_n$. Show that X is semisimple. Now only assume that X is semisimple with eigenvalues $a_1, \ldots, a_n$. Show that the operator $\operatorname{ad} X : Y \mapsto [X, Y]$ in the n^2-dimensional vector space $\mathfrak{gl}(V)$ is semisimple and has as eigenvalues the n^2 scalars $a_i - a_j$ where $i, j = 1, \ldots, n$.