INTRODUCTION TO LIE ALGEBRAS – SOLUTION 13

(i) Consider the basis in $L(n)$ consisting of the monomials $v_k = x^{n-k}y^k$, $k = 0, \dots, n$. We have

$$H \cdot v_k = \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) x^{n-k}y^k = (n-2k)\, x^{n-k}y^k = (n-2k)v_k$$

for any $k = 0, \dots, n$. Hence our basis consists of eigenvectors of the operator H. The corresponding eigenvalues are the numbers $n, n-2, n-4, \dots, -n$.

E acts as $x \frac{\partial}{\partial y}$ and F acts as $y \frac{\partial}{\partial x}$. So $E \cdot v_0 = 0$, $E \cdot v_k = k\, v_{k-1}$ for $k = 1, \dots, n$, $F \cdot v_n = 0$, $F \cdot v_k = (n-k)v_{k+1}$ for $k = 0, \dots, n-1$

(ii) Since H acts semisimply with integral eigenvalues on the irreducible modules by part (i) and Theorem 3(i) from the lectures, it acts semisimply with integral eigenvalues on any direct sum of these and therefore on any finite dimensional $\mathfrak{sl}_2\mathbb{C}$-module by Theorem 3(ii) from the lectures.

Let $v \in V_i$. Then $H \cdot (E \cdot v) = [H, E] \cdot v + E \cdot (H \cdot v) = 2E \cdot v + iE \cdot v = (i+2)E \cdot v$. So $E \cdot v \in V_{i+2}$. The case of F is completely analogous. It follows that V^e and V^o are stable under the action of E, H and F. So they are submodules of V. Now note that each H-eigenspace V_i is the direct sum of the eigenspaces $L(n_j)_i$ for the $L(n_j)$ in a direct sum decomposition of V into irreducibles. The weights of an irreducible are either all even or all odd. In the first case the 0-weight space is one dimensional, in the second case the 1-weight space is one dimensional. So each summand in a direct sum decomposition of V^e into irreducibles contributes one dimension to V_0 and each summand in a direct sum decomposition of V^o into irreducibles contributes one dimension to V_1.