(0) (i) $\mathrm{range}(g\circ f) = g(\mathrm{range}(f))$

Proof $\mathrm{range}(g\circ f) = \{(g\circ f)(x) \mid x \in \mathrm{domain}\, g\circ f\}$

$= \{g(f(x)) \mid x \in \mathrm{domain}(f)\}$

$= \{g(y) \mid y \in \mathrm{range}(f)\} = g(\mathrm{range}(f))$

(ii) Assume $g\circ f$ and $h\circ g$ defined. Then $\mathrm{range}(f) \subseteq \mathrm{domain}(g)$ and $\mathrm{range}(g) \subseteq \mathrm{domain}(h)$

So $\mathrm{range}(g\circ f) \underset{\text{by (i)}}{=} g(\mathrm{range}(f)) \underset{\text{obvious}}{\subseteq} \mathrm{range}(g) \underset{\text{by assumption}}{\subseteq} \mathrm{domain}(h)$ and $h\circ(g\circ f)$ is defined.

Furthermore, $\mathrm{range}(f) \underset{\text{by assumption}}{\subseteq} \mathrm{domain}(g) = \mathrm{domain}(h\circ g)$, so $(h\circ g)\circ f$ is defined

We have $\mathrm{domain}(h\circ(g\circ f)) = \mathrm{domain}(g\circ f) = \mathrm{domain}(f) = \mathrm{domain}((h\circ g)\circ f)$ and $\mathrm{codomain}(h\circ(g\circ f)) = \mathrm{codomain}(h) = \mathrm{codomain}(h\circ g) = \mathrm{codomain}((h\circ g)\circ f)$.

Finally, we have $(h\circ(g\circ f))(x) = h((g\circ f)(x)) = h(g(f(x))) = (h\circ g)(f(x)) = ((h\circ g)\circ f)(x)$ for all $x \in \mathrm{domain}(f)$.

(1) sin and cos are both not injective. Both their ranges are equal to the closed interval $[-1,1]$.

(2) $\exp: \mathbb{R} \to \mathbb{R}$ is injective. It's range is $\mathbb{R}^+ = \{x\in\mathbb{R} \mid x>0\}$. When we consider it as a map: $\mathbb{R} \to \mathbb{R}^+$, then it has an inverse: the natural logarithm $\ln: \mathbb{R}^+ \to \mathbb{R}$

(3) • $\exp(2\pi i) = \exp(0) = 1$, so the complex exponential is not injective.

We have $|\exp(x+iy)| = |\exp(x)|\,|\cos(y) + i\sin(y)| = |\exp(x)|$, so $0 \notin \mathrm{range}(\exp)$

Recall: the absolute value of a complex number $z = x+iy$ is given by the length of the corresponding plane vector: $|z| = \sqrt{x^2+y^2}$. Furthermore $|zw| = |z||w|\ \forall z,w\in\mathbb{C}$

Now let $z \in \mathbb{C}\setminus\{0\}$ and write it in polar form $z = r(\cos(\varphi) + i\sin(\varphi))$ (so $r = |z|$ and $\varphi = \arg(z) + k2\pi$ for some $k\in\mathbb{Z}$). Put $w = \ln(r) + i\varphi$. Then $\exp(w) = \exp(\ln(r))\exp(i\varphi) = r(\cos(\varphi) + i\sin(\varphi)) = z$. So $\mathrm{range}(\exp_{\mathbb{C}}) = \mathbb{C}\setminus\{0\}$.

after replacing the codomain $\mathbb{C}$ by $\mathbb{C}\setminus\{0\}$, the complex exponential is still not injective and therefore not invertible

• Since the real sine and cosine are not injective, the same holds for the complex sine and cosine. They are both surjective. We show this for the complex sine. Let $w\in\mathbb{C}$. We want to find $z\in\mathbb{C}$ such that $\sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)) = w$. Put $u = \exp(iz)$. Then $u^{-1} = \exp(-iz)$ and u must satisfy: $\frac{1}{2i}(u - u^{-1}) = w$, or: $u^2 - 2iwu - 1 = 0$ (*)

Since $\mathbb{C}$ is algebraically closed, there always exists a u satisfying (*).

Clearly this u must be nonzero, so, by the surjectivity of $\exp_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}\setminus\{0\}$ there exists a $z_1 \in\mathbb{C}$ with $\exp(z_1) = u$. Put $z = -iz_1$. Then $z_1 = iz$ and $\sin(z) = \frac{1}{2i}(u - u^{-1}) = w$.

(4) This I leave to you.