INTRODUCTION TO LIE ALGEBRAS – SOLUTION 9

Denote the Lie algebras from (i) and (ii) by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let us choose the basis in the Lie algebra $\mathfrak{g}_2$ of the three matrices

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can define a bijective linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ by

$$\varphi(E) = E_{13}, \quad \varphi(F) = E_{12}, \quad \varphi(G) = E_{23}.$$

Let us prove that φ is an isomorphism of Lie algebras. Because of Exercise 2, we only have to prove that $[E_{13}, E_{12}] = 0$, $[E_{13}, E_{23}] = 0$ and $[E_{12}, E_{23}] = E_{13}$. But this is clear.

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 10

By definition, for any two derivations $D, D' \in \mathrm{Der}(\mathfrak{g})$ their Lie bracket is the usual commutator:

$$[D, D'] = D\,D' - D'D\,.$$

Now suppose that the derivation D' is inner: $D' = \mathrm{ad}\,X$ for some element $X \in \mathfrak{g}$. This means that for any $Y \in \mathfrak{g}$ we have $D'(Y) = [X, Y]$ in $\mathfrak{g}$. Let us prove that $[D, D'] \in \mathrm{Der}(\mathfrak{g})$ is again an inner derivation. Indeed, for any $Y \in \mathfrak{g}$ we have

$$[D, D'](Y) = \big(D\,D' - D'D\big)(Y) = D\big(D'(Y)\big) - D'\big(D(Y)\big) = D\big([X, Y]\big) - [X, D(Y)]\,.$$

By the Leibniz rule for the derivation D, the right hand side of this equalities coincides with $[D(X), Y]$. This means that $[D, D'] = \mathrm{ad}\big(D(X)\big)$, the inner derivation corresponding to the element $D(X) \in \mathfrak{g}$.