The symplectic ideal and a double centraliser theorem

Rudolf Tange

Abstract

We interpret a result of Oehms as a statement about the symplectic ideal. We use this result to prove a double centraliser theorem for the symplectic group acting on $\bigoplus_{r=0}^{s} \otimes^r V$, where V is the natural module for the symplectic group. This result was obtained in characteristic zero by Weyl. Furthermore, we use this to extend to arbitrary connected reductive groups G with simply connected derived group the earlier result of the author that the algebra $K[G]^{\mathfrak{g}}$ of infinitesimal invariants in the algebra of regular functions on G is a unique factorisation domain.

Introduction

Throughout this note, k denotes an infinite field, K denotes the algebraic closure of k, n and s are positive integers, V is the vector space k^n and $\mathrm{Mat}_n = \mathrm{Mat}_n(k)$ is the k-algebra of $n \times n$ matrices acting on V as vector space endomorphisms. We denote the vector space $\bigoplus_{r=0}^{s} \otimes^r V$ by $T^{\leqslant s}(V)$. A matrix $u \in \mathrm{Mat}_n$ acts on $\otimes^s V$ by $u(x_1 \otimes \ldots \otimes x_s) = u(x_1) \otimes \ldots \otimes u(x_s)$. For a subset X of Mat_n we denote by $\mathcal{E}^s(X)$ and $\mathcal{E}^{\leqslant s}(X)$ the *enveloping algebra* of X in $\mathrm{End}_k(\otimes^s V)$ and $\mathrm{End}_k(T^{\leqslant s}(V))$, respectively, that is, the subalgebra generated by the endomorphisms corresponding to the elements of X. Let A be an associative k-algebra. The *centraliser algebra* (also called the *commutating algebra, commutator algebra* or *commuting algebra*) of a subalgebra B of A is defined as the subalgebra of A that consists of the elements of A that commute with all elements of B. We say that the *double centraliser theorem* holds for a subalgebra B of A if B is equal to the centraliser algebra of its centraliser algebra. We say that the double centraliser theorem holds for a subset X of Mat_n acting on $\otimes^s V$ or $T^{\leqslant s}(V)$ if it holds for the corresponding enveloping algebra.

Now assume that $k = K = \mathbb{C}$, and let G be a subgroup of $\mathrm{GL}_n(\mathbb{C})$. In [3] Brauer showed that, if the double centraliser theorem holds for G, then one can determine defining equations for $\mathcal{E}^s(G)$ by first determining the centraliser algebra. Determining the centraliser algebra turned out to be equivalent to a version of the first fundamental theorem of invariant theory for G. (In the case where $G = \mathrm{GL}_n$ the centraliser algebra was already known and therefore yielded another proof of this version of the first fundamental theorem for GL_n. In characteristic zero, one can deduce from this the general first fundamental theorem for GL_n; see [20, p. 139].) Brauer's methods applied to any semisimple complex Lie subgroup G of $\mathrm{GL}_n(\mathbb{C})$ for which there is a first fundamental theorem of invariant theory, in particular, to the symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$. The point is that in these cases all finite-dimensional Lie group representations of G are semisimple and the double centraliser theorem holds for semisimple subalgebras of $\mathrm{Mat}_n(\mathbb{C})$.

Let I be the ideal of polynomial functions on Mat_{2m} that vanish on Sp_{2m}. In [20] Weyl showed that finding generators $g_1, \ldots, g_r$ of I, which have the property that any $f \in I$ can be expressed as $f = \sum_i a_i g_i$ with $\deg(a_i g_i) \leqslant \deg(f)$ for certain polynomial functions $a_1, \ldots, a_r$ on Mat_{2m}, is equivalent to determining defining equations for $\mathcal{E}^{\leqslant s}(\mathrm{Sp}_{2m})$, for an arbitrary s. Then he used Brauer's method, which also applies to $T^{\leqslant s}(V)$, to determine such generators of I.

For the classical groups the first fundamental theorem of invariant theory has been generalised to positive characteristic by De Concini and Procesi in [6], but the algebras $\mathcal{E}^s(G)$ and $\mathcal{E}^{\leqslant s}(G)$, G classical, are no longer semisimple for all s in positive characteristic. In Sections 1 and 3 of this paper we will reverse Weyl's procedure and show that we can easily derive the double centraliser theorem for Sp_{2m} acting on $T^{\leqslant s}(V)$ once we have generators for the symplectic ideal with the aforementioned property. All this relies on the work of Oehms and De Concini and Procesi.

In Section 2 we derive another consequence from the results on the symplectic ideal in Section 1. We show that for a connected reductive group G with simply connected derived group, the algebra $K[G]^{\mathfrak{g}}$ of infinitesimal invariants in the algebra of regular functions on G is a unique factorisation domain.

1. *The symplectic ideal and the symplectic enveloping algebra*

We begin by considering the following property of a set of generators $g_1, \ldots, g_r$ of an ideal I in the polynomial ring $k[x_1, \ldots, x_n]$.

Any $f \in I$ can be expressed as

$$f = \sum_{i=1}^{r} a_i g_i \quad \text{with } \deg(a_i g_i) \leqslant \deg(f) \tag{1}$$

for certain $a_1, \ldots, a_r \in k[x_1, \ldots, x_n]$.

This property is related to the associated graded ideal, for which we now introduce some notation. Let A be a commutative algebra over k with a filtration $A_0 \subseteq A_1 \subseteq A_2 \ldots$. The associated graded algebra is denoted by $\mathrm{gr}(A)$. If $x \in A_i \setminus A_{i-1}$, then we put $\deg(x) = i$ and $\mathrm{gr}(x) = x + A_{i-1} \in \mathrm{gr}(A)^i = A_i/A_{i-1}$. For an ideal I of A we have $\mathrm{gr}(A/I) \cong \mathrm{gr}(A)/\mathrm{gr}(I)$; see [2, Chapter 3, §2.4]. In particular, when A is graded, we have $\mathrm{gr}(A/I) \cong A/\mathrm{gr}(I)$.

LEMMA 1. *Let I be an ideal of the polynomial ring $k[x_1, \ldots, x_n]$, which is generated by the non-zero elements $g_1, \ldots, g_r$. Then these generators have property (1) if and only if the elements $\mathrm{gr}(g_1), \ldots, \mathrm{gr}(g_r)$ generate the ideal $\mathrm{gr}(I)$.*

Proof. Assume that $g_1, \ldots, g_r$ is a set of generators of I which has property (1), and let $\mathrm{gr}(f)$ be a non-zero homogeneous element of $\mathrm{gr}(I)$. Then $f = \sum_{i=1}^{r} a_i g_i$ with $\deg(a_i g_i) \leqslant \deg(f)$ for certain $a_1, \ldots, a_r \in k[x_1, \ldots, x_n]$. Therefore $\mathrm{gr}(f) = \sum_i \mathrm{gr}(a_i)\mathrm{gr}(g_i)$, where the sum is over all indices i such that $\deg(a_i g_i) = \deg(f)$. Thus the $\mathrm{gr}(g_i)$ generate $\mathrm{gr}(I)$. Now assume that the latter is the case, and let f be a non-zero element of I. We can write $\mathrm{gr}(f) = \sum_{i=1}^{r} a_i \mathrm{gr}(g_i)$, where a_i is homogeneous of degree $\deg(f) - \deg(g_i)$. Then $f - \sum_{i=1}^{r} a_i g_i$ is of strictly lower degree than f and we can complete the proof by induction. $\square$

The symmetric group of degree r, Sym_r, acts on $\otimes^r V$ by

$$\pi \cdot (x_1 \otimes \ldots \otimes x_r) = x_{(\pi^{-1})_1} \otimes \ldots \otimes x_{(\pi^{-1})_r}.$$

Denote the group algebra of Sym_r over k by $k\langle \mathrm{Sym}_r \rangle$. The elements of $\mathrm{End}_{k\langle \mathrm{Sym}_r \rangle}(\otimes^r V)$, that is, the k-linear endomorphisms of $\otimes^r V$ commuting with the action of Sym_r, were called *bisymmetric substitutions* by Weyl. Clearly all enveloping algebras $\mathcal{E}^r(X)$ consist of bisymmetric substitutions.

To state the proposition below we need some auxiliary notation. For a subset S of a fixed finite-dimensional k-vector space W, we denote by $k[S]$ the k-algebra of *polynomial functions* on S, that is, functions on S that are restrictions of polynomial functions on W. The algebra

$k[W]$ is a polynomial algebra in the elements of any basis of W^*, because k is infinite. In particular, $k[\mathrm{Mat}_n]$ is a polynomial algebra in the matrix entry functions.

Now take $W = \mathrm{Mat}_n$. Clearly $k[S]$ is isomorphic to $k[\mathrm{Mat}_n]/I$, where I is the ideal of polynomial functions on Mat_n that vanish on S. The algebra $k[S]$ inherits a filtration from $k[\mathrm{Mat}_n]$. We denote the filtration subspace of index s by $k[S]^{\leqslant s}$. If S is closed under multiplication by non-zero scalars, then the ideal I will be homogeneous and $k[S]$ is a graded algebra. We denote the graded subspace of index s by $k[S]^s$. If S is a submonoid of Mat_n, then I is a biideal and $k[S]$ is a k-bialgebra. The subspace $k[S]^{\leqslant s}$ is then a subcoalgebra of $k[S]$, and if S contains the non-zero multiples of the identity, then $k[S]^s$ is a subcoalgebra of $k[S]$. We denote the closure of a subset S of Mat_n under multiplication by non-zero scalars by $k^{\times}S$. Note that in this notation $k[\mathrm{GL}_n]$ is equal to $k[\mathrm{Mat}_n]$ and not to $k[\mathrm{Mat}_n][\det^{-1}]$.

After Proposition 1 we will only use this notation in the following situation, where it is in accordance with the standard notation. If X is a k-defined closed subvariety of $W(K) := K \otimes_k W$ such that the set S of k-defined points in X is dense in X, then $k[S]$ in the notation here is naturally isomorphic to the algebra $k[X]$ of regular functions on X that are defined over k. This applies, for example, to the symplectic group Sp_n as we shall see later. The next proposition is essentially due to Schur and Weyl.

PROPOSITION 1. *Let M be a submonoid of Mat_n. Then the following assertions hold:*
  (i) *the natural map $\mathcal{E}^s(M)^* \to k[k^{\times}M]^s$ is an isomorphism of coalgebras;*
 (ii) *the natural map $\mathcal{E}^{\leqslant s}(M)^* \to k[M]^{\leqslant s}$ is an isomorphism of coalgebras;*
(iii) $\mathcal{E}^s(\mathrm{GL}_n) = \mathrm{End}_{k\langle\mathrm{Sym}_s\rangle}(\otimes^s V)$;
(iv) $\mathcal{E}^{\leqslant s}(\mathrm{GL}_n) = \mathrm{End}_{\mathfrak{S}}(T^{\leqslant s}(V))$, *where* $\mathfrak{S} = \bigoplus_{r=0}^s k\langle\mathrm{Sym}_r\rangle$.

Proof. The natural map in (ii) composes $f \in \mathcal{E}^{\leqslant s}(M)^*$ with the monoid homomorphism $M \to \mathcal{E}^{\leqslant s}(M)$. The result is clearly a polynomial function on M of degree at most s. That the natural map is injective follows from the fact that endomorphisms of $T^{\leqslant s}(V)$ that represent the elements of M span $\mathcal{E}^{\leqslant s}(M)$, since M is a submonoid of Mat_n.

The algebra $\mathcal{E}^s(M)$ is equal to $\mathcal{E}^s(k^{\times}M)$, since $\alpha \cdot \mathrm{id} \in \mathrm{Mat}_n$ is represented by $\alpha^s \cdot \mathrm{id}$ on $\otimes^s V$. Therefore, in the case of (i) we may assume that M contains the non-zero multiples of the identity. The injectivity of the natural map, which composes $f \in \mathcal{E}^s(M)^*$ with $M \to \mathcal{E}^s(M)$, now follows as in (ii).

By considering the commutative diagram below and its graded version in the case where M contains the multiples of the identity, we see that it is now sufficient to prove (i) and (ii) for $M = \mathrm{Mat}_n$, since the two vertical arrows are surjective morphisms of coalgebras.

$$\begin{array}{ccc} \mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)^* & \longrightarrow & k[\mathrm{Mat}_n]^{\leqslant s} \\ \downarrow & & \downarrow \\ \mathcal{E}^{\leqslant s}(M)^* & \longrightarrow & k[M]^{\leqslant s} \end{array} \qquad (2)$$

We first consider the natural map of (i) for $M = \mathrm{GL}_n$. To avoid confusion, we will write $k[\mathrm{Mat}_n]$ instead of $k[\mathrm{GL}_n]$, since these algebras are canonically isomorphic. The map comes from the natural map $\mathrm{End}_k(\otimes^s V)^* \to k[\mathrm{Mat}_n]^s$. Let $I = I(n,s) = \{1,\dots,n\}^s$ as in [**12**, 2.1]. We have a standard basis $(E_{i,j})_{(i,j)\in I\times I}$ of $\mathrm{End}_k(\otimes^s V)$. Our map sends the element $E_{i,j}^*$ of the dual basis to Green's $c_{i,j}$. The $c_{i,j}$ form a basis when we pass to $(I \times I)/\sim$, where $\sim$ is the equivalence relation given by the action of the Sym_s on $I \times I$ defined in [**12**, 2.1]. Now we form the dual basis $c^*_{\overline{(i,j)}}$ and define $c^*_{i,j} = c^*_{\overline{(i,j)}}$. Here $\overline{(i,j)}$ denotes the canonical image of (i,j) in $(I \times I)/\sim$. Then, in the transpose of our map, $c^*_{i,j}$ is mapped to $\sum_{(k,l)\sim(i,j)} E_{k,l}$, which is precisely what Green's map $(k[\mathrm{Mat}_n]^s)^* \to \mathrm{End}_k(\otimes^s V)$ does. Assertion (i) for GL_n and assertion (iii) now follow from [**12**, 2.6]. Note that (iii) shows that $\mathcal{E}^s(\mathrm{Mat}_n) = \mathcal{E}^s(\mathrm{GL}_n)$.

Now we consider the natural map of (ii) for $M = \mathrm{GL}_n$. We have seen that it is injective. It is also surjective, since for any polynomial function f on GL_n of degree at most s we can find an element of $\mathrm{End}(T^{\leqslant s}(V))^* \cong T^{\leqslant s}(\mathrm{Mat}_n^*)$ that is mapped to f. We have

$$\mathcal{E}^{\leqslant s}(\mathrm{GL}_n) \subseteq \bigoplus_{r=0}^{s} \mathcal{E}^r(\mathrm{GL}_n).$$

Since the natural maps of (i) and (ii) for $M = \mathrm{GL}_n$ are bijective, we must have equality. Also, now it is clear that (ii) is an isomorphism of coalgebras.

Since the image of $\mathfrak{S}$ in $\mathrm{End}_k(T^{\leqslant s}(V))$ contains the projections given by the direct sum decomposition of $T^{\leqslant s}(V)$, we see that $\mathrm{End}_{\mathfrak{S}}(T^{\leqslant s}(V))$ is equal to $\bigoplus_{r=0}^{s} \mathrm{End}_{k\langle \mathrm{Sym}_r\rangle}(\otimes^r V)$: it consists of $(s+1)$-tuples of 'bisymmetric substitutions'. Therefore, (iv) now follows from (iii) and it is also clear that $\mathcal{E}^{\leqslant s}(\mathrm{GL}_n) = \mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)$. □

REMARKS 1. (1) Assertion (iv) of Proposition 1 was proved in a different way by Weyl; see [**20**, Theorem 4.4.E].

(2) Proposition 1 shows that $\mathcal{E}^{\leqslant s}(M)$ depends only on the ideal I of polynomial functions that vanish on M and that $\mathcal{E}^s(M)$ depends only on the largest homogeneous ideal contained in I, that is, the ideal generated by the homogeneous polynomial functions on Mat_n that vanish on M.

(3) Assume that $k = K$ is algebraically closed. Let G be a closed subgroup of GL_n, and let A be the algebra of polynomial functions on G. Then the representations of the algebra $\mathcal{E}^{\leqslant s}(G) = (A^{\leqslant s})^*$ are precisely the rational representations of G, with coefficients the polynomial functions on G that are of filtration degree at most s. If G contains the non-zero multiples of the identity, then the representations of the algebra $\mathcal{E}^s(G) = (A^s)^*$ are precisely the rational representations of G with coefficients the homogeneous polynomial functions on G that are of degree s. In the case $G = \mathrm{GL}_n$, we see that $(A^s)^*$ is the well-known *Schur algebra* denoted by $S_K(n,s)$ in [**12**]. Similar remarks apply to a closed submonoid M of Mat_n. Then the coefficients of a rational representation of M are automatically polynomial, since all regular functions on M are polynomial.

We now come to the problem of finding defining equations for the enveloping algebras $\mathcal{E}^s(M)$ and $\mathcal{E}^{\leqslant s}(M)$. We are only interested in defining equations within $\mathcal{E}^s(\mathrm{Mat}_n)$ and $\mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)$. By Proposition 1, parts (iii) and (iv), one can then obtain a complete set of defining equations (within $\mathrm{End}_k(\otimes^s V)$ and $\bigoplus_{r=0}^{s} \mathrm{End}_k(\otimes^r V)$) by adding the equations of 'bisymmetry'. See [**3**, (30)] or [**20**, III (6.8)]. Statement (i) of the next corollary is a formalisation of an idea of Weyl; see [**20**, p. 142].

COROLLARY. *Let M be a submonoid of Mat_n, let I be the ideal of polynomial functions on Mat_n that vanish on M, and let I_{hom} be the largest homogeneous ideal contained in I. Furthermore, let $g_1, \ldots, g_k$ be non-zero elements of I, and let $h_1, \ldots, h_l$ be non-zero homogeneous elements of I_{hom}. Denote the isomorphism $k[\mathrm{Mat}_n]^{\leqslant s} \to \mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)^*$ by η and the isomorphism $k[\mathrm{Mat}_n]^s \to \mathcal{E}^s(\mathrm{Mat}_n)^*$ by θ. Then we have the following assertions.*

(i) *The elements $g_1, \ldots, g_k$ are generators of I with property (1) if and only if for each $s \geqslant 1$ the functionals $\eta(g_i m_i)$, where the m_i are arbitrary monomials in the matrix entries of degree at most $s - \deg(g_i)$, define the algebra $\mathcal{E}^{\leqslant s}(M)$.*

(ii) *The elements $h_1, \ldots, h_k$ are generators of I_{hom} if and only if for each $s \geqslant 1$ the functionals $\theta(h_i m_i)$, where the m_i are arbitrary monomials in the matrix entries of degree $s - \deg(h_i)$, define the algebra $\mathcal{E}^s(M)$.*

Proof. Note that I is a proper ideal, since M is nonempty. Denote the subspace of I that consists of the elements of I of degree at most s by $I^{\leqslant s}$. Then the g_i are generators of I with property (1) if and only if for each $s \geqslant 1$ the polynomials $g_i m_i$, where the m_i are arbitrary monomials in the matrix entries of degree at most $s - \deg(g_i)$, span $I^{\leqslant s}$. Assertion (i) now follows from the fact that, because of the commutativity of (2), η maps $I^{\leqslant s}$ to the space of linear functionals on $\mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)$ that vanish on $\mathcal{E}^{\leqslant s}(M)$. Assertion (ii) is proved in a similar way. $\square$

REMARK. In [**10**, Theorems 9.5(a) and 10.5(a)] Doty determines generators for the defining ideals of the symplectic and orthogonal monoids in characteristic zero by a method which is essentially assertion (ii) of the above corollary. The point is that in characteristic zero one can use the defining equations of the symplectic and orthogonal enveloping algebras in $\mathrm{End}_k(\otimes^s V)$ as obtained by Brauer in [**3**, (46),(47)].

In the remainder of this paper, $n = 2m$ is an even integer 2 or higher. We first introduce some notation for the symplectic group, which closely follows that of [**17**]. Let $i \mapsto i'$ be the involution of $\{1, \ldots, n\}$ defined by $i' := n + 1 - i$. Set $\epsilon_i = 1$ if $i \leqslant m$ and $\epsilon_i = -1$ if $i > m$ and define the $n \times n$ matrix J with coefficients in k by $J_{ij} = \delta_{ij'}\epsilon_i$. Therefore

$$
J = \begin{bmatrix}
 & & & & & 1 \\
 & 0 & & & \cdot^{\displaystyle\cdot^{\displaystyle\cdot}} & \\
 & & & 1 & & \\
 & & -1 & & & \\
 & \cdot^{\displaystyle\cdot^{\displaystyle\cdot}} & & & 0 & \\
-1 & & & & &
\end{bmatrix}.
$$

On V we define the nondegenerate symplectic form $\langle\ ,\ \rangle$ by

$$
\langle u, v \rangle := u^{\mathrm{T}} J v = \sum_{i=1}^{n} \epsilon_i u_i v_{i'} \ .
$$

The *symplectic group* $\mathrm{Sp}_n = \mathrm{Sp}_n(k)$ is defined as the set of $n \times n$ matrices over k that satisfy $A^{\mathrm{T}} J A = J$, that is, the matrices for which the corresponding endomorphism of V preserves the form $\langle\ ,\ \rangle$. Clearly those matrices are invertible and Sp_n is a subgroup of GL_n. Furthermore, $\mathrm{Sp}_n(K)$ is an algebraic group which is connected, semisimple and defined and split over the prime field. This implies that all root subgroups $U_\alpha(K)$ with respect to some k-split maximal torus are defined over k. Since clearly $U_\alpha(k)$ is dense in $U_\alpha(K)$, necessarily Sp_n is dense in $\mathrm{Sp}_n(K)$.

Note that $A^{\mathrm{T}} J A$ consists of the scalar products of the columns of A with each other and that $A J A^{\mathrm{T}}$ consists of the scalar products of the rows of A with each other. An easy calculation shows that the condition $A^{\mathrm{T}} J A = J$ is equivalent to the condition $A J A^{\mathrm{T}} = J$. We denote the (i, j)th entry function on Mat_n by x_{ij}. Define

$$
g_{ij} := \sum_{l=1}^{n} \epsilon_l x_{li} x_{l'j} \quad \text{and} \quad \overline{g}_{ij} := \sum_{l=1}^{n} \epsilon_l x_{il} x_{jl'}. \tag{3}
$$

The condition $A^{\mathrm{T}} J A = J$ means that we require the functions $g_{ij} - \delta_{ij'}\epsilon_i$, $i < j$, to vanish on A. The condition $A J A^{\mathrm{T}} = J$ means that we require the functions $\overline{g}_{ij} - \delta_{ij'}\epsilon_i$, $i < j$, to vanish on A. We call the ideal of polynomial functions on Mat_n that vanish on Sp_n the *symplectic ideal*. It is well known that each set of functions separately generates the symplectic ideal in the case where k is algebraically closed. In view of the density of Sp_n in $\mathrm{Sp}_n(K)$ this must also hold for an arbitrary infinite field k.

In what follows, we will also need the *symplectic monoid* SpM_n and the *symplectic similitude group* GSp_n as introduced in [**8**] and [**10**]; see also [**17**] and [**7**]. The symplectic monoid SpM_n is defined as the set of matrices A for which there exists a scalar $d(A) \in k$ such that $A^{\mathrm{T}}JA = AJA^{\mathrm{T}} = d(A)J$. Note that if $d(A) \neq 0$, then $A^{\mathrm{T}}JA = d(A)J$ is equivalent to $AJA^{\mathrm{T}} = d(A)J$. Clearly SpM_n is a submonoid of Mat_n; in fact, it is the set of k-defined points of $\mathrm{SpM}_n(K)$, which is a k-defined closed submonoid of $\mathrm{Mat}_n(K)$. This follows from [**17**, Corollary 6.2]. The function d is a polynomial function on SpM_n and is called the *coefficient of dilation*. We have, for all $i \in \{1, \ldots, n\}$,

$$d = \epsilon_i g_{ii'} = \epsilon_i \overline{g}_{ii'} \quad \text{on } \mathrm{SpM}_n.$$

The *symplectic similitude group* GSp_n is defined as the group of invertible elements in SpM_n. We have $\mathrm{GSp}_n(K) = K^{\times}\mathrm{Sp}_n(K)$ and GSp_n is Zariski dense in $\mathrm{GSp}_n(K)$ and $\mathrm{SpM}_n(K)$. Hence a polynomial function on Mat_n vanishes on GSp_n if and only if it vanishes on SpM_n, and the ideal of $k[\mathrm{Mat}_n]$ that consists of these functions is the largest homogeneous ideal contained in the symplectic ideal.

We will now give defining equations for $\mathcal{E}^{\leqslant s}(\mathrm{Sp}_n)$. As stated before the corollary to Proposition 1, we are only interested in defining equations within the algebra $\mathcal{E}^{\leqslant s}(\mathrm{Mat}_n)$. This algebra consists of $(s+1)$-tuples $(A_0, \ldots, A_s)$ of endomorphisms with $A_r \in \mathrm{End}_k(\otimes^r V)$ bisymmetric for all $r \in \{0, \ldots, s\}$. The standard basis $(e_1, \ldots, e_n)$ of $V = k^n$ gives a basis for the vector space $\otimes^r V$. The entry of index $((i_1, \ldots, i_r), (j_1, \ldots, j_r))$ of the matrix of A_r with respect to this basis is denoted by $a_{i_1 \ldots i_r, j_1 \ldots j_r}$.

The first statement in (i) of the theorem below was already remarked by Oehms [**17**, p. 38].

THEOREM 1. *The following statements hold.*

(i) *The ideal* $\mathrm{gr}(I)$ *is generated by the elements* g_{ij} *and* $\overline{g}_{ij}$, $i < j$. *Furthermore,* $\mathrm{gr}(k[\mathrm{Sp}_n])$ *is isomorphic to the algebra of polynomial functions on the set of* $n \times n$ *matrices over* k *with row and column spaces totally singular, and this algebra is an integral domain.*

(ii) *The sets of elements* $g_{ij} - \delta_{ij'}\epsilon_i$ *and* $\overline{g}_{ij} - \delta_{ij'}\epsilon_i$, $i < j$, *form together a set of generators of the symplectic ideal with property* (1).

(iii) *The symplectic enveloping algebra in* $\mathrm{End}_k(T^{\leqslant s}(V))$ *is defined by the following equations for* $r = 2, \ldots, s$:

$$\sum_{l=1}^{n} \epsilon_l a_{ll'i_3 \ldots i_r, j_1 \ldots j_r} = \delta_{j_1, j_2'}\epsilon_{j_1} a_{i_3 \ldots i_r, j_3 \ldots j_r}, \tag{4}$$

$$\sum_{l=1}^{n} \epsilon_l a_{i_1 \ldots i_r, ll'j_3 \ldots j_r} = \delta_{i_1, i_2'}\epsilon_{i_1} a_{i_3 \ldots i_r, j_3 \ldots j_r}. \tag{5}$$

Proof. (i) Let J be the ideal of $k[\mathrm{Mat}_n]$ generated by the elements g_{ij} and $\overline{g}_{ij}$, $i < j$. Then the zero set of J is the set of $n \times n$ matrices over k with row and column spaces totally singular. It is also the zero set of d in SpM_n. In fact it follows from [**17**, Corollary 6.2] that $k[\mathrm{SpM}_n]/(d) \cong k[\mathrm{Mat}_n]/J$. That result also implies that

$$k[\mathrm{SpM}_n]/(d-1) \cong k[\mathrm{Mat}_n]/I = k[\mathrm{Sp}_n].$$

Since d is non-zero and homogeneous of degree $2 > 0$, we have $\mathrm{gr}(d-1) = d$ and $\mathrm{gr}((d-1)) = (d)$. Therefore

$$k[\mathrm{Mat}_n]/J \cong k[\mathrm{SpM}_n]/(d) \cong \mathrm{gr}(k[\mathrm{Sp}_n]) \cong k[\mathrm{Mat}_n]/\mathrm{gr}(I).$$

Since the composite is the epimorphism given by the inclusion $J \subseteq \mathrm{gr}(I)$, we must have $J = \mathrm{gr}(I)$.

Let $\mathcal{V}$ be the variety of $n \times n$ matrices over K with row and column spaces totally singular. Then we have an epimorphism $K[\mathrm{SpM}_n]/(d) \twoheadrightarrow K[\mathcal{V}]$ of graded K-algebras. To show that it is an isomorphism, it suffices to show that for each l the dimensions of the graded pieces of degree l of both algebras are equal. By [17, Theorem 6.1], the lth graded piece of $K[\mathrm{SpM}_n]/(d)$ has a basis consisting of all bideterminants $(T_1 \mid T_2)$, where T_1 and T_2 are symplectic standard tableaux in the sense of King of the same shape λ, which is a partition of l with at most m parts (notation: $\lambda \vdash_m l$). Hence its dimension is $\sum_{\lambda \vdash_m l} N_\lambda^2$, where N_λ is the number of symplectic standard tableaux of shape λ in the sense of King. By [5, Theorem 6.1], the lth graded piece of $K[\mathcal{V}]$ has a basis consisting of all bideterminants $(T_1 \mid T_2)$, where T_1 and T_2 are symplectic standard tableaux in the sense of De Concini, of the same shape $\lambda \vdash_m l$. (De Concini uses different terminology, but it is clear that his condition of symplectic standardness is imposed on T_1 and T_2 separately.) Therefore its dimension is $\sum_{\lambda \vdash_m l} N_\lambda'^2$, where N_λ' is the number of symplectic standard tableaux of shape λ in the sense of De Concini. Let $Y(\lambda)$ be the irreducible $\mathrm{Sp}_n(\mathbb{C})$-module associated to $\lambda \vdash_m l$. Then $N_\lambda = \dim(Y(\lambda)) = N_\lambda'$, by [15] (or [1] or [8]) and [5, Theorems 4.8, 4.11, Proposition 4.10]. Note that we now also know that $\mathcal{V}$ is defined over k.

We will now show that $\mathcal{V}$ is irreducible. Let $\mathcal{V}_0$ be the set of $n \times n$ matrices with row and column space lying in the K-span V_0 of $e_1, \ldots, e_m$. Note that V_0 is a maximal totally singular subspace of $V(K) = K^n$ and that $\mathcal{V}_0$ is a k-defined vector subspace of $\mathrm{Mat}_n(K)$, where $\mathrm{Mat}_n(K)$ has the standard k-structure. Now consider the morphism $\mathrm{Sp}_n(K) \times \mathcal{V}_0 \times \mathrm{Sp}_n(K) \to \mathrm{Mat}_n(K)$ given by $(A, S, B) \mapsto ASB$. Clearly its image lies in $\mathcal{V}$ and by Witt's lemma its image is equal to $\mathcal{V}$. Since $\mathrm{Sp}_n(K)$ and $\mathcal{V}_0$ are irreducible, necessarily $\mathcal{V}$ is irreducible. Furthermore the set of k-defined points is dense in $\mathcal{V}$, since this holds for $\mathrm{Sp}_n(K)$ and $\mathcal{V}_0$ and our morphism is defined over k. Note that, since $K[\mathcal{V}]$ is an integral domain, d generates a prime ideal in $K[\mathrm{SpM}_n]$ and therefore also in $k[\mathrm{SpM}_n]$.

(ii) This follows from assertion (i) and Lemma 1.

(iii) This follows immediately from parts (ii) and (i) of the corollary to Proposition 1. $\square$

REMARKS 2. (1) The equations in part (iii) are the same as those obtained by Weyl [20, p. 174] in characteristic zero.

(2) The equations for $\mathcal{E}^s(\mathrm{Sp}_n)$ that Brauer [3, (47)] found in characteristic zero also define this algebra in positive characteristic. This can be shown as follows. Let I be the symplectic ideal. Then [17, Corollary 6.2] gives homogeneous generators of I_{hom}. (In [17, (14)], one can replace m by n and $f_{ll'} - \overline{f}_{kk'}$ by $\epsilon_k f_{ll'} - \epsilon_l \overline{f}_{kk'}$ (or by $\epsilon_l f_{ll'} - \epsilon_k \overline{f}_{kk'}$).) Here one can avoid the FRT-construction by simply taking [17, (13)] as the definition of $A_R^s(n)$. The stability under base change is then trivial and the proofs of [17, Theorem 6.1 and Corollary 6.2] still apply. Now assertion (ii) of the Corollary to Proposition 1 gives the following equations for $\mathcal{E}^s(\mathrm{Sp}_n)$ within $\mathcal{E}^s(\mathrm{Mat}_n)$:

$$\delta_{i_1, i_2'} \epsilon_{i_1} \sum_{l=1}^{n} \epsilon_l a_{ll' i_3 \ldots i_r, j_1 \ldots j_r} = \delta_{j_1, j_2'} \epsilon_{j_1} \sum_{l=1}^{n} \epsilon_l a_{i_1 \ldots i_r, ll' j_3 \ldots j_r}.$$

2. Infinitesimal invariants in $K[G]$

In this section we assume that $k = K$ is algebraically closed. Furthermore, G is a connected reductive algebraic group over K and $\mathfrak{g} = \mathrm{Lie}(G)$ is its Lie algebra. The conjugation action of G on itself induces an action of G on $K[G]$, the algebra of regular functions on G. We will refer to this action and its derived $\mathfrak{g}$-action as *conjugation* actions. The conjugation action of G on $K[G]$ is by algebra automorphisms, and so the conjugation action of $\mathfrak{g}$ on $K[G]$ is by derivations. This implies that the space $K[G]^{\mathfrak{g}} = \{f \in K[G] \mid x \cdot f = 0 \text{ for all } x \in \mathfrak{g}\}$ of *infinitesimal invariants* is a subalgebra of $K[G]$. Note that $K[G]^G \subseteq K[G]^{\mathfrak{g}}$ and that $K[G]^p := \{f^p \mid f \in K[G]\} \subseteq K[G]^{\mathfrak{g}}$ if K is of characteristic $p > 0$.

THEOREM 2.   *Assume that K is of characteristic $p > 0$ and that the derived group of G is simply connected. Then the invariant algebra $K[G]^{\mathfrak{g}}$ is a unique factorisation domain. Its irreducible elements are the irreducible elements of $K[G]$ that are invariant under $\mathfrak{g}$ and the pth powers of the irreducible elements of $K[G]$ that are not invariant under $\mathfrak{g}$.*

Proof.   By [**18**, Remark 3] we have to prove Proposition 1 in that paper for Sp_n. In fact we may assume that $n \geqslant 4$ and that K is of characteristic 2, but we will not use this. The statement of [**18**, Proposition 1] is:

Let $f \in K[G]$ be a regular function. If the ideal $K[G]f$ is stable under the conjugation action of $\mathfrak{g}$ on $K[G]$, then f is $\mathfrak{g}$-invariant.

By [**18**, Proposition 2 and Lemma 2] it is sufficient to show that, for the filtration that $K[\mathrm{Sp}_n]$ inherits from $K[\mathrm{Mat}_n]$, the associated graded algebra is an integral domain. The point is that one can then use a (filtration) degree argument. The result now follows immediately from Theorem 1(ii). □

PROPOSITION 2.   *The algebras $K[\mathrm{Sp}_n]$, $K[\mathrm{GSp}_n]$ and $K[\mathrm{SpM}_n]$ are unique factorisation domains.*

Proof.   Since the derived group $D\mathrm{GSp}_n$ of GSp_n equals Sp_n, which is simply connected, we have by [**18**, corollary to Theorem 1] that $K[\mathrm{Sp}_n]$ and $K[\mathrm{GSp}_n]$ are UFDs. We have $K[\mathrm{SpM}_n][d^{-1}] = K[\mathrm{GSp}_n]$ which is a UFD. Furthermore d generates a prime ideal in $K[\mathrm{SpM}_n]$ by the proof of Theorem 1(i). Therefore $K[\mathrm{SpM}_n]$ is a UFD by Nagata's lemma; see, for example, [**11**, Lemma 19.20]. □

3. *A double centraliser theorem for the symplectic group*

We begin by defining a version of the symplectic Brauer algebra. For each integer $r \leqslant s$ we fix r vector symbols $x_1, \ldots, x_r$ and r covector symbols $y_1, \ldots, y_r$. (Elements of V are called *vectors* and elements of V^* are called *covectors*.) To ease notation we did not give these symbols the extra index r that indicates to which integer $r \leqslant s$ they are associated. In fact one may, as the notation suggests, take the r-symbols as the first r of the s-symbols. Let u and v be non-negative integers s or lower such that $u \equiv v \pmod 2$, that is, such that $u + v$ is even. A (u, v)-*diagram* is a matching of the $u + v$ symbols $y_1, \ldots, y_u, x_1, \ldots, x_v$ in pairs. Such a diagram is depicted as a graph with its vertices arranged in two rows. The top row consists of u vertices representing, from left to right, the u covector symbols $y_1, \ldots, y_u$ and the bottom row consists of v vertices representing the v vector symbols $x_1, \ldots, x_v$. An empty row is represented by a $\emptyset$. Two vertices are joined if the corresponding symbols are matched. So, for example,

                        and $\emptyset$ (6)
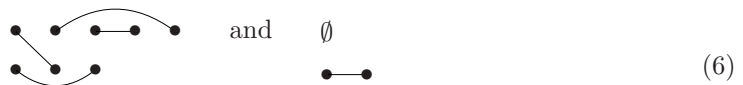
are a $(5, 3)$ and a $(0, 2)$-diagram.

If an edge e in a (u, v)-diagram is horizontal, then its left endpoint is called its *initial point* and its right endpoint is called its *terminal point*. If e is not horizontal, then its endpoint in the top row is called its initial point and its endpoint in the bottom row is called its terminal point.

Let $t \in k$. We now define the k-algebra $\mathfrak{B}_{\leqslant s}(t)$. It has all (u, v)-diagrams, $0 \leqslant u, v \leqslant s$, $u \equiv v \pmod 2$, as a k-basis and its dimension is $\sum_{u=1}^{s} \sum_{v=1}^{s} N_{uv}$, where

$$N_{uv} = \begin{cases} (u+v-1)(u+v-3) \cdot \ldots \cdot 3 \cdot 1 = \frac{(u+v)!}{2^{(u+v)/2}((u+v)/2)!} & \text{if } u \equiv v \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

The multiplication of diagrams is defined as follows. Let D_{uv} be a (u, v)-diagram, and let $D_{u'v'}$ be a (u', v')-diagram. Then we define

$$D_{uv}D_{u'v'} = \begin{cases} \mathrm{sgn}(D_{uv}, D_{u'v'})t^{\gamma(U)}D_{uv'} & \text{if } v = u', \\ 0 & \text{if } v \neq u'. \end{cases}$$

Here, as in [**3**, **13**], U is the graph which is obtained by putting D_{uv} on top of $D_{u'v'}$ and identifying the vertices of the bottom row of D_{uv} with those of the top row of $D_{u'v'}$, $\gamma(U)$ is the number of cycles in U and $D_{uv'}$ is the (u, v')-diagram obtained from U by taking the vertices from top and bottom rows in U and matching two vertices if there is a path between them in U.

Our definition of the sign $\mathrm{sgn}(D_{uv}, D_{u'v'})$ of D_{uv} over $D_{u'v'}$ does not agree with [**13**, p. 411]. The graph U consists of $(u + v')/2$ paths and $\gamma(U)$ cycles that are of even length and consist entirely of edges in the middle row of U. A path in U either consists of one horizontal edge in the top or bottom row of U or it has precisely two non-horizontal edges and all its horizontal edges are in the middle row of U. A path of even length has one endpoint in the top row of U and the other endpoint in the bottom row of U. A path of odd length has its endpoints both in the top row of U or both in the bottom row of U.

Every path P has a 'natural' orientation. If P has even length, then the endpoint in the top row is the initial point and the endpoint in the bottom row is the terminal point. If P has odd length, then the leftmost endpoint of P is the initial point and the other endpoint is the terminal point. For a path P or an oriented cycle P we denote the number of edges of P in the middle row of U that are traversed from left to right by p_{lr} and the number of edges of P in the middle row of U that are traversed from right to left by p_{rl}. For a cycle or a path P we define

$$\mathrm{sgn}(P) = \begin{cases} (-1)^{|p_{\mathrm{lr}}-p_{\mathrm{rl}}|/2} & \text{if } p_{\mathrm{lr}} + p_{\mathrm{rl}} \text{ is even}, \\ (-1)^{|p_{\mathrm{lr}}-p_{\mathrm{rl}}-1|/2} & \text{if } p_{\mathrm{lr}} + p_{\mathrm{rl}} \text{ is odd}. \end{cases}$$

Note that for a cycle this does not depend on the orientation. We now define $\mathrm{sgn}(D_{uv}, D_{u'v'})$ as the product of the signs of all paths and cycles in U.

Below we will define a natural representation of $\mathfrak{B}_{\leqslant s}(n)$ which is the motivation for the definition of the multiplication. We will see later that the multiplication of $\mathfrak{B}_{\leqslant s}(t)$ is associative.

With each (u, v)-diagram D we can associate a $(u + v)$-multilinear function $F(D)$ on $\oplus^u V^* \oplus \oplus^v V$ as follows. First we observe that the form $\langle\ ,\ \rangle$ defines an isomorphism $V \cong V^*$, unique up to sign, and therefore a unique symplectic form on V^* which we also denote by $\langle\ ,\ \rangle$. Furthermore we put $\langle y, x \rangle = y(x) = -\langle x, y \rangle$ for $y \in V^*$ and $x \in V$. If e is an edge in D with its initial point labelled z_1 and terminal point labelled z_2, then we put $\langle e \rangle = \langle z_1, z_2 \rangle$. Now we define

$$F(D) = \prod_{e \in D} \langle e \rangle. \tag{7}$$

For example, for the first diagram D in (6) we have

$$F(D) = \langle y_1, x_2 \rangle \langle y_2, y_5 \rangle \langle y_3, y_4 \rangle \langle x_1, x_3 \rangle.$$

Using the k-vector space isomorphisms

$$\mathrm{End}_k(T^{\leqslant s}(V)) \cong \bigoplus_{0 \leqslant u,v \leqslant s} \mathrm{Hom}_k\big(\otimes^v V, \otimes^u V\big) \tag{8}$$

and

$$\mathrm{Hom}_k\big(\otimes^v V, \otimes^u V\big) \cong \big(\otimes^u V\big) \otimes \big(\otimes^v V\big)^* \cong \big(\big(\otimes^u V^*\big) \otimes \big(\otimes^v V\big)\big)^*, \tag{9}$$

we can now associate to each (u, v)-diagram D an endomorphism $E(D)$ of $T^{\leqslant s}(V)$ as follows. We take $E(D)$ to be the endomorphism of $T^{\leqslant s}(V)$ corresponding to $F(D) \in \big((\otimes^u V^*) \otimes (\otimes^v V)\big)^*$.

Let $(e_1, \ldots, e_n)$ be the standard basis of $V = k^n$. Then we have a dual basis $(e_1^*, \ldots, e_n^*)$ and also bases of the vector spaces $\otimes^r V$. The matrix $M(D)$ of $E(D) \in \mathrm{Hom}_k(\otimes^v V, \otimes^u V)$ with respect to these bases is then given by

$$M(D)_{(i_1,\ldots,i_u),(j_1,\ldots,j_v)} = F(D)(e_{i_1}^*, \ldots, e_{i_u}^*, e_{j_1}, \ldots, e_{j_v}). \tag{10}$$

The linear map $E : \mathfrak{B}_{\leqslant s}(n) \to \mathrm{End}_k(T^{\leqslant s}(V))$ given by the assignment $D \mapsto E(D)$ is a homomorphism of algebras. We indicate a proof.

The isomorphism $\varphi : V^* \to V$ with $\langle \varphi(y), x \rangle = y(x)$ for all $x \in V$ and $y \in V^*$ maps e_i^* to $\epsilon_{i'} e_{i'}$. Therefore we have

$$\langle e_i^*, e_j^* \rangle = \langle e_i, e_j \rangle = \epsilon_i \delta_{ij'}.$$

Now one easily checks that

$$\sum_{l_2,\ldots,l_p=1}^{n} \langle e_{l_1}, e_{l_2} \rangle \langle e_{l_2}, e_{l_3} \rangle \cdot \ldots \cdot \langle e_{l_p}, e_{l_{p+1}} \rangle = \begin{cases} (-1)^{p/2} \langle e_{l_1}^*, e_{l_{p+1}} \rangle & \text{if } p \text{ is even,} \\ (-1)^{(p-1)/2} \langle e_{l_1}, e_{l_{p+1}} \rangle & \text{if } p \text{ is odd.} \end{cases}$$

and that for p even

$$\sum_{l_1,\ldots,l_p=1}^{n} \langle e_{l_1}, e_{l_2} \rangle \langle e_{l_2}, e_{l_3} \rangle \cdot \ldots \cdot \langle e_{l_{p-1}}, e_{l_p} \rangle \langle e_{l_p}, e_{l_1} \rangle = (-1)^{p/2} \cdot n \, .$$

If we evaluate an entry of the product $M(D_{uv})M(D_{u'v'})$, $v = u'$, then this entry will be a sum over v indices corresponding to the vertices in the middle row of U. Here we use (7) and (10). Now we can distribute these v sums over the paths and cycles in U. Each path or cycle gets the sums corresponding to the vertices of the middle row of U that it contains. Then we obtain a product of sums of the above two types, where we have to swap the arguments of certain scalar products $\langle e_{l_{i+1}}, e_{l_{i+2}} \rangle$ according to the position of the corresponding vertices in the middle row of U. Here an extra sign $(-1)^{p_{\mathrm{lr}}}$ comes in. Now it follows that the entry of $M(D_{uv})M(D_{u'v'})$ is equal to the corresponding entry of $M(D_{uv}D_{u'v'})$.

We now define some special elements of $\mathfrak{B}_{\leqslant s}(t)$.

$$c_r = \underbrace{\begin{array}{c} \bullet \ \cdots \ \bullet \\[-2pt] \,\vdots \ \cdots \ \vdots\, \\[-2pt] \bullet \ \cdots \ \bullet \end{array}}_{r \text{ vertices}} \,, \quad b_r = \bullet\!\!-\!\!\bullet \ \overbrace{\begin{array}{c} \bullet \ \cdots \ \bullet \\[-2pt] \bullet \ \cdots \ \bullet \end{array}}^{r \text{ vertices}} \quad \text{and} \quad \overline{b}_r = \ \begin{array}{c}\bullet\!\!-\!\!\bullet\end{array}\ \underbrace{\begin{array}{c} \bullet \ \cdots \ \bullet \\[-2pt] \bullet \ \cdots \ \bullet \end{array}}_{r \text{ vertices}} \,. \tag{11}$$

Here c_r is defined for $0 \leqslant r \leqslant s$ and b_r and $\overline{b}_r$ are defined for $0 \leqslant r \leqslant s - 2$. The unit element of $\mathfrak{B}_{\leqslant s}(t)$ is $\sum_{r=0}^{s} c_r$. The subspace $c_r \mathfrak{B}_{\leqslant s}(t) c_r$ is the span of all (r,r)-diagrams and is closed under multiplication. It has c_r as a unit element and as an algebra it is isomorphic to the symplectic Brauer algebra $\mathfrak{B}_r(t)$ which was introduced for $t = n$ in [3]. We identify $\mathfrak{B}_r(t)$ with this subspace of $\mathfrak{B}_{\leqslant s}(t)$. Then $\bigoplus_{r=0}^{s} \mathfrak{B}_r(t)$ is identified with a subalgebra with unit of $\mathfrak{B}_{\leqslant s}(t)$. We obtain a natural embedding $k\langle \mathrm{Sym}_r \rangle \subseteq \mathfrak{B}_r(t)$ by assigning to each $\pi \in \mathrm{Sym}_r$ the (r,r)-diagram in which x_i is matched with y_{π_i}. The action of Sym_r on $\otimes^r V$ that it inherits from $\mathfrak{B}_r(n)$ is the same as the action mentioned in Section 1.

The algebra $\mathfrak{B}_{\leqslant s}(t)$ is generated by the elements $b_r, \overline{b}_r, 0 \leqslant r \leqslant s - 2$, c_0, c_1 and the elements of each $\mathfrak{B}_r(t)$, $2 \leqslant r \leqslant s$, that correspond to the elementary transpositions in Sym_r.

In the theorem below we will consider the representation of $\mathfrak{B}_{\leqslant s}(n)$ on $T^{\leqslant s}(V)$ as given by the homomorphism E defined above. First we make a preliminary observation. Let

$B \in \mathrm{Hom}_k(V \otimes V, k) \subseteq \mathrm{End}_k(T^{\leqslant s}(V))$ be the endomorphism that maps $x_1 \otimes x_2$ to $\langle x_1, x_2 \rangle$, and let $\overline{B} \in \mathrm{Hom}_k(k, V \otimes V) \subseteq \mathrm{End}_k(T^{\leqslant s}(V))$ be the endomorphism that maps 1 to $\sum_{i=1}^{n} \epsilon_i e_i \otimes e_{i'}$. Then we have

$$E(b_r) = B \otimes \mathrm{id}_{\otimes^r V} \quad \text{and} \quad E(\overline{b}_r) = \overline{B} \otimes \mathrm{id}_{\otimes^r V}. \tag{12}$$

Note furthermore that $E(c_r)$ is just the projection of $T^{\leqslant s}(V)$ onto $\otimes^r V$.

THEOREM 3.   *The following statements hold.*
  (i) $\mathrm{End}_{\mathrm{Sp}_n}(T^{\leqslant s}(V))$ *coincides with the image of* $\mathfrak{B}_{\leqslant s}(n)$ *in* $\mathrm{End}_k(T^{\leqslant s}(V))$.
  (ii) $\mathrm{End}_{\mathfrak{B}_{\leqslant s}(n)}(T^{\leqslant s}(V))$ *is the enveloping algebra of* Sp_n *in* $\mathrm{End}_k(T^{\leqslant s}(V))$.
  (iii) *If* $m \geqslant s$, *then the homomorphism* $\mathfrak{B}_{\leqslant s}(n) \to \mathrm{End}_k(T^{\leqslant s}(V))$ *is injective.*
  (iv) *If* $m < s$, *then the homomorphism* $\mathfrak{B}_s(n) \to \mathrm{End}_k(\otimes^s V)$ *is not injective.*

Proof.   (i) We follow Brauer's method as adjusted to our situation by Weyl; see [20, V.2, pp. 141–142]. There is a natural action of Sp_n on $\mathrm{End}_k(T^{\leqslant s}(V))$, and it is clear that $\mathrm{End}_{\mathrm{Sp}_n}(T^{\leqslant s}(V))$ consists of the Sp_n-invariant elements of $\mathrm{End}_k(T^{\leqslant s}(V))$. The vector spaces in (8) and (9) all have a natural Sp_n-action, and the isomorphisms there are Sp_n-equivariant. It therefore suffices to show that for $u, v \in \{1, \ldots, s\}$, $\left((\otimes^u V^*) \otimes (\otimes^v V) \right)^{*\,\mathrm{Sp}_n}$ is spanned by the multilinear functions $F(D)$, where D is a (u, v)-diagram and F is given by (7).

Let $u, v \in \{1, \ldots, s\}$ and put $w = u + v$. We will identify V^* with V by means of the isomorphism $\varphi : V^* \xrightarrow{\sim} V$. This means that the y_i are now vector variables. We put $z_i = y_i$ for $i \in \{1, \ldots, u\}$ and $z_i = x_{i-u}$ for $i \in \{u+1, \ldots, w\}$. Since k is infinite, $k[\oplus^w V]$ can be identified with the polynomial ring in the components of the z_i. If $f \in k[\oplus^w V]$ is Sp_n-invariant, then it is also $\mathrm{Sp}_n(K)$-invariant as an element of $K \otimes_k k[\oplus^w V]$, since Sp_n is dense in $\mathrm{Sp}_n(K)$. However, then f is a formal invariant in the definition of [6, §2]; see for example [14, Remark I.2.8]. We can now apply the first fundamental theorem of invariant theory for the symplectic group, [6, Theorem 6.6]. This gives us that $k[\oplus^w V]^{\mathrm{Sp}_n}$ is generated as a k-algebra by the scalar products $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$. It follows immediately that $(\otimes^w V)^{*\,\mathrm{Sp}_n}$ is spanned by the monomials in the $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$, in which each z_i occurs exactly once. These monomials are precisely the $F(D)$, where D is a (u, v)-diagram.

(ii) We have $\mathfrak{S} \subseteq \mathfrak{B}_{\leqslant s}(n)$, where $\mathfrak{S}$ is as in Proposition 1(iv). Hence, as in the proof of that result, we see that $\mathrm{End}_{\mathfrak{B}_{\leqslant s}(n)}(T^{\leqslant s}(V))$ consists of $(s+1)$-tuples of bisymmetric substitutions. Using (12) one easily checks that the condition of commuting with $E(b_r)$ is given by equation (4) and that (5) gives the condition of commuting with $E(\overline{b}_r)$. By Theorem 1(iii) these are precisely the equations that define $\mathcal{E}^{\leqslant s}(\mathrm{Sp}_n)$.

The arguments in the proofs of parts (iii) and (iv) below are very similar to those for the orthogonal group in [20, p. 149]. Recall that the *Pfaffian* of an alternating $2r \times 2r$ matrix A is defined by

$$\mathrm{Pf}(A) = \sum_{\pi} \mathrm{sgn}(\pi) \prod_{i=1}^{r} a_{\pi_{2i-1}, \pi_{2i}}, \tag{13}$$

where the sum is over all permutations π of $\{1, 2, \ldots, 2r\}$ with $\pi_1 < \pi_3 < \ldots < \pi_{2r-1}$ and $\pi_{2i-1} < \pi_{2i}$ for all $i \in \{1, \ldots, r\}$.

(iii) By the isomorphisms (8) and (9) it is sufficient to show that for each $u, v \in 1, \ldots, s$ with $u \equiv v \pmod 2$, the multilinear functions $F(D)$, D a (u, v)-diagram, are linearly independent. We use the identification $V^* \cong V$ and the notation of (i). As we have seen in (i) the $F(D)$ are the monomials in the $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$, in which each z_i occurs exactly once. The second fundamental theorem for the symplectic group [6, Theorem 6.7] says that the ideal of relations between the $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$, is generated by the Pfaffians of the principal submatrices of size $n + 1$ of the $w \times w$ alternating matrix $\langle z_i, z_j \rangle_{1 \leqslant i, j \leqslant w}$. (A principal submatrix is obtained by

choosing rows and columns from the same index set.) Since, by assumption, $w \leqslant 2s \leqslant n$, there are no such submatrices; that is, the $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$, are algebraically independent. This means that the monomials in the $\langle z_i, z_j \rangle$, $1 \leqslant i < j \leqslant w$, are linearly independent.

(iv) From (13) it is clear that $\mathrm{Pf}(A)$ is a signed sum of distinct monomials in the a_{ij}, $1 \leqslant i < j \leqslant 2r$, in which each number $i \in \{1, 2, \ldots, 2r\}$ occurs precisely once either as a row or as a column index. Since $n < 2s$, necessarily the alternating $2s$-multilinear map $\mathrm{Pf}(\langle z_i, z_j \rangle_{1 \leqslant i,j \leqslant 2s})$ on $\oplus^{2s}V$ is zero. This gives us a nontrivial dependence relation between the multilinear functions $F(D)$, D an (s, s)-diagram. $\qquad \square$

COROLLARY.   *The algebra $\mathfrak{B}_{\leqslant s}(t)$ is associative for any $t \in k$.*

Proof.   First we observe that we can define the Brauer algebra over any commutative ring R and then we have $\mathfrak{B}_{\leqslant s}(R, t_0) \cong R \otimes_{\mathbb{Z}[t]} \mathfrak{B}_{\leqslant s}(\mathbb{Z}[t], t)$ for all $t_0 \in R$, where we now consider t as an indeterminate and the homomorphism $\mathbb{Z}[t] \to R$ is given by $t \mapsto t_0$. Therefore it suffices to show that $\mathfrak{B}_{\leqslant s}(\mathbb{Z}[t], t)$ is associative. If $D, D', D'' \in \mathfrak{B}_{\leqslant s}(\mathbb{Z}[t], t)$, then the coefficients of $(DD')D'' - D(D'D'')$ with respect to the diagram basis are polynomials in t. By Theorem 3(iii) applied with $k = \mathbb{Q}$, these polynomials vanish whenever we specialise t to an even integer $2s$ or higher. It follows that these polynomials are identically zero. $\qquad \square$

REMARKS 3.   (1) Theorem 3(iii) was proved in another way in [**7**, Theorem 3.4].

(2) One can also derive the double centraliser theorem for Sp_n acting on $\otimes^s V$ as obtained in [**7**, Proposition 1.3 and Theorem 1.4] by our method. Simply combine Remark 2(2) and the arguments of the proof of Theorem 3.

(3) Drop the assumption that n is even and assume that the characteristic of k is not 2. Orthogonal versions $\mathfrak{A}_{\leqslant s}(t)$ and $\mathfrak{A}_s(t)$ of $\mathfrak{B}_{\leqslant s}(t)$ and $\mathfrak{B}_s(t)$ can also be defined. One just has to omit the sign in the definition of the multiplication. The algebra $\mathfrak{A}_s(t)$ coincides with the orthogonal Brauer algebra defined in [**13**]. Furthermore the algebras $\mathfrak{A}_{\leqslant s}(n)$ and $\mathfrak{A}_s(n)$ have natural representations in $T^{\leqslant s}(V)$ and $\otimes^s V$.

Let I be the orthogonal ideal. In characteristic zero homogeneous generators for I_{hom} are given in [**10**, Theorem 10.5(a)], and generators for I with property (1) are given in [**20**, Theorem 5.2.C]. The corollary to Proposition 1 and the proof of Theorem 3 show that, to prove the double centraliser theorem over k for the orthogonal group acting on $\otimes^s V$ and $T^{\leqslant s}(V)$, it is enough to show that these elements are also generators of I_{hom} and generators for I with property (1) over k.

(4) Assume that $k = \mathbb{C}$. Then it can be shown by easy invariant theoretic arguments as in the proof of Theorem 3(iii) and (iv) that the natural representation $\mathfrak{A}_s(n) \to \mathrm{End}_k(\otimes^s V)$ is injective if $n \geqslant 2s$ and not injective if $n < s$; see [**20**, p. 149]. Therefore, contrary to the symplectic case, these arguments do not give a complete criterion for faithfulness. In [**4**] it is proved that the natural representation of $\mathfrak{A}_s(n)$ is faithful if and only if $n \geqslant s$.

(5) From [**16**, Theorems 2.11, 4.6, 4.3] it follows that $\mathfrak{A}_s(t)$ has the well-known presentation given in, for example, [**7**, p. 3]. From this one easily deduces a presentation for $\mathfrak{B}_s(t)$ and an isomorphism $\mathfrak{B}_s(t) \cong \mathfrak{A}_s(-t)$. This is in accordance with the results in [**13**, **19**].

References

1. A. BERELE, 'Construction of Sp-modules by tableaux', *Linear Multilinear Algebra* 19 (1986) 299–307.

2. N. Bourbaki, *Commutative algebra*, Translated from the French, Reprint of the 1972 edition (Springer, Berlin, 1989) Chapters 1–7.

3. R. Brauer, 'On algebras which are connected with the semisimple continuous groups', *Ann. of Math.* (2) 38 (1937) 857–872.

4. W. P. Brown, 'The semisimplicity of ω_f^n', *Ann. of Math.* (2) 63 (1956) 324–335.

5. C. De Concini, 'Symplectic standard tableaux', *Adv. Math.* 34 (1979) 1–27.

6. C. De Concini and C. Procesi, 'A characteristic free approach to invariant theory', *Adv. Math.* 21 (1976) 330–354.

7. R. Dipper, S. Doty and J. Hu, 'Brauer algbras, symplectic Schur algebras and Schur–Weyl duality', *Trans. Amer. Math. Soc.* 360 (2008) 189–213.

8. S. Donkin, 'Good filtrations of rational modules for reductive groups', *The Arcata conference on representations of finite groups*, Arcata, California, 1986, Proceedings of Symposia Pure Mathematics 47:1 (American Mathematical Society, Providence, RI, 1987) 69–80.

9. S. Donkin, 'Representations of symplectic groups and the symplectic tableaux of R. C. King', *Linear Multilinear Algebra* 29 (1991) 113–124.

10. S. Doty, 'Polynomial representations, algebraic monoids, and Schur algebras of classical type', *J. Pure Appl. Algebra* 123 (1998) 165–199.

11. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics 150 (Springer, New York, 1995).

12. J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics 830 (Springer, Berlin, 1980).

13. P. Hanlon and D. Wales, 'On the decomposition of Brauer's centralizer algebras', *J. Algebra* 121 (1989) 409–445.

14. J. C. Jantzen, *Representations of algebraic groups*, Pure and Applied Mathematics 131 (Academic Press, Boston, 1987).

15. R. C. King, 'Weight multiplicities for the classical groups', *Group theoretical methods in physics*, Fourth Internat. Colloq., Nijmegen, 1975, Lecture Notes in Physics 50 (Springer, Berlin, 1976) 490–499.

16. H. R. Morton and A. J. Wasserman, 'A basis for the Birman–Wenzl algebra', Preprint, 1989, http://citeseer.ist.psu.edu/313132.html.

17. S. Oehms, 'Centralizer coalgebras, FRT-construction, and symplectic monoids', *J. Algebra* 244 (2001) 19–44.

18. R. H. Tange, 'Infinitesimal invariants in a function algebra', *Canad. J. Math.*, to appear, http://arxiv.org/abs/math/0611438.

19. H. Wenzl, 'On the structure of Brauer's centralizer algebras', *Ann. of Math.* (2) 128 (1988) 173–193.

20. H. Weyl, *The classical groups, their invariants and representations*, 2nd edn (Princeton University Press, Princeton, NJ, 1946).

Rudolf Tange
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstrasse 150
D-44780 Bochum
Germany

rudolf.tange@rub.de
rht502@york.ac.uk