13. Consider the special linear Lie algebra $\mathfrak{sl}_2\mathbb{C}$. It consists of all 2×2 matrices of trace zero with complex entries. The Lie bracket is the matrix commutator: $[X,Y] = XY - YX$ for any $X, Y \in \mathfrak{sl}_2\mathbb{C}$. Let $L(n)$ be the irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ introduced in Section 2. Recall that it is the complex vector space consisting of all homogeneous polynomials in x and y

$$P(x,y) = \sum_{k=0}^{n} p_k\, x^{n-k} y^k$$

of degree n with complex coefficients and that $X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathfrak{sl}_2\mathbb{C}$ acts on $P \in L(n)$ by

$$X \cdot P = (ax + cy)\frac{\partial P}{\partial x} + (bx - ay)\frac{\partial P}{\partial y}.$$

Let $(H, E, F) = (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$ be the standard basis of $\mathfrak{sl}_2\mathbb{C}$.

(i) Find a basis in $L(n)$ consisting of eigenvectors of the operator H. Specify the eigenvalues of the operator H corresponding to the basis vectors. How do E and F act on your basis vectors?

(ii) Let V be a finite dimensional $\mathfrak{sl}_2\mathbb{C}$-module. Using the theorem on complete reducibility, show that H acts semisimply on V with integral eigenvalues, i.e. show that $V = \bigoplus_{i \in \mathbb{Z}} V_i$, where $V_i = \{v \in V \mid H \cdot v = iv\}$.

Show that $E \cdot V_i \subseteq V_{i+2}$ and $F \cdot V_i \subseteq V_{i-2}$. Put $V^e = \bigoplus_{i \in \mathbb{Z}} V_{2i}$ and $V^o = \bigoplus_{i \in \mathbb{Z}} V_{2i+1}$. So $V = V^e \oplus V^o$ as vector spaces. Show that V^e and V^o are submodules of V. Deduce that the number of summands in a direct sum decomposition of V^e resp. V^o into irreducibles is $\dim V_0$ resp. $\dim V_1$.