The proofs in cases (ii) and (iii) are similar, but different from that in case (i).

(i) We will give the proof for arbitrary $n \geqslant 2$. Take any central element $Z \in \mathfrak{sl}_n\mathbb{C}$. Then $[Z, X] = 0$ for any $X \in \mathfrak{sl}_n\mathbb{C}$. Write $Z = \sum_{i,j=1}^n z_{ij}E_{ij}$ and let $k, l \in \{1, \dots, n\}$ with $k \neq l$. Using the commutation relations

$$[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}$$

for the matrix units, we get $0 = [Z, E_{kl}] = \sum_{i=1}^n z_{ik}E_{il} - \sum_{j=1}^n z_{lj}E_{kj}$. So

$$\sum_{i=1}^n z_{ik}E_{il} = \sum_{j=1}^n z_{lj}E_{kj}\ . \tag{$*$}$$

Comparing the entries on position (k, k) in (*) we get $0 = z_{lk}$. So Z is diagonal. Comparing the entries on position (k, l) in (*) we get $z_{kk} = z_{ll}$. So Z is constant on the diagonal, i.e. a multiple of the identity. Now $nz_{11} = \mathrm{tr}(Z) = 0$, so $Z = 0$.

(ii) We will give the proof for arbitrary $n \geqslant 3$. Let $Z \in \mathfrak{so}_n\mathbb{C}$. Then we have

$$Z = \sum_{i,j=1}^n z_{ij}E_{ij}\ ,$$

where $z_{ji} = -z_{ij}$ for all $i \neq j$ and $z_{ii} = 0$ for all i. Now assume that Z is central in $Z \in \mathfrak{so}_n\mathbb{C}$ Then $[Z, E_{kl} - E_{lk}] = 0$ for all $l \neq k$, since the $E_{kl} - E_{lk}$ are in $\mathfrak{so}_n\mathbb{C}$. It follows that for $k \neq l$

$$\sum_{i=1}^n (z_{ik}E_{il} - z_{il}E_{ik}) = \sum_{j=1}^n (z_{lj}E_{kj} - z_{kj}E_{lj})\,.$$

Now we take the terms involving E_{ij} with $i = l$ or $j = l$ to the left and the terms involving E_{ij} with $i = k$ or $j = k$ to the right. Using $z_{ji} = -z_{ij}$ for all $i \neq j$ and $z_{ii} = 0$ for all i, and changing summation indices we obtain

$$\sum_{i \neq k,l} z_{ik}\,(E_{il} - E_{li}) = \sum_{j \neq k,l} z_{lj}\,(E_{kj} - E_{jk}) = 0\,.$$

Now observe that in the last equation none of the pairs (i, l) and (l, i) coincides with (j, k) or (k, j). Therefore here we actually have two equations

$$\sum_{i \neq k,l} z_{ik}\,(E_{il} - E_{li}) = 0\ , \quad \sum_{j \neq k,l} z_{lj}\,(E_{kj} - E_{jk}) = 0\,.$$

Since $n \geqslant 3$, for any two indices $i \neq k$ we can find one more index l such that $l \neq i, k$. The first of the two equations above then implies that $z_{ik} = 0$.

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 12

Consider the action of the operator $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ on the basis vectors E and F. By definition,

$$\big(\operatorname{ad} X\big)(E) = [\,aE + bF, E\,] = -bE\,,$$
$$\big(\operatorname{ad} X\big)(F) = [\,aE + bF, F\,] = aE\,.$$

Therefore the matrix of the operator $\operatorname{ad} X$ relative to the basis E, F in $\mathfrak{g}$ is

$$\begin{bmatrix} -b & a \\ 0 & 0 \end{bmatrix}.$$