INTRODUCTION TO LIE ALGEBRAS – SOLUTION 16

Let us choose the standard basis in the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the operator $\operatorname{ad} H : \mathfrak{g} \to \mathfrak{g}$. By definition, we have $\big(\operatorname{ad} H\big)(Y) = [H, Y]$ for any vector $Y \in \mathfrak{g}$. In particular,

$$\big(\operatorname{ad} H\big)(H) = 0, \quad \big(\operatorname{ad} H\big)(E) = 2E, \quad \big(\operatorname{ad} H\big)(F) = -2F,$$

Thus our chosen basis in $\mathfrak{g}$ consists of eigenvectors of the operator $\operatorname{ad} H$. Note that the three corresponding eigenvalues $0, 2, -2$ are different from each other. Now let $\mathfrak{h}$ is any non-zero ideal in $\mathfrak{g}$. We will prove that then $\mathfrak{h} = \mathfrak{g}$. By the definition of an ideal $[X, Y] \in \mathfrak{h}$ for any $X \in \mathfrak{g}$ and any $Y \in \mathfrak{h}$. Choosing $X = H$ here, we obtain that the operator $\operatorname{ad} H$ preserves the subspace $\mathfrak{h} \subset \mathfrak{g}$. As $\mathfrak{h} \neq \{0\}$, the subspace $\mathfrak{h}$ contains an eigenvector of $\operatorname{ad} H$. This eigenvector must be one of H, E, F up to a non-zero factor from $\mathbb{C}$. We can assume that $\mathfrak{h}$ contains one of the vectors H, E, F exactly. Let us now consider all the three possibilities.

(i) If $H \in \mathfrak{h}$, then $E = -[E, H]/2 \in \mathfrak{h}$ and $F = [F, H]/2 \in \mathfrak{h}$ also.
(ii) If $E \in \mathfrak{h}$, then $H = -[F, E] \in \mathfrak{h}$. Then also $F = [F, H]/2 \in \mathfrak{h}$.
(iii) If $F \in \mathfrak{h}$, then $H = [E, F] \in \mathfrak{h}$. Then also $E = -[E, H]/2 \in \mathfrak{h}$.

In each of these three cases we have $H, E, F \in \mathfrak{h}$, so that $\mathfrak{h} = \mathfrak{g}$.

Note that with the above method one can also show that the irreducible modules $L(n)$, $n \geqslant 2$, from Section 2 are indeed irreducible. In fact we have shown above that $L(2)$ (recall that $L(2)$ is isomorphic to the adjoint representation) is irreducible.

INTRODUCTION TO LIE ALGEBRAS – SOLUTION 17

(i). The commutator of two matrices from the vector space $\mathfrak{g}$

$$A = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} X' & Z' \\ 0 & Y' \end{bmatrix}$$

is

$$AA' - A'A = \begin{bmatrix} XX' - X'X & XZ' + ZY' - X'Z - Z'Y \\ 0 & YY' - Y'Y \end{bmatrix}.$$

Each of the 2×2 matrices $XX'-X'X$ and $YY'-Y'Y$ has zero trace, so $AA'-A'A \in \mathfrak{g}$. This proves that (i) $\mathfrak{g}$ is a Lie subalgebra in $\mathfrak{gl}_4\mathbb{C}$.

(ii). If above $X' = Y' = 0$ then

$$AA' - A'A = \begin{bmatrix} 0 & XZ' - Z'Y \\ 0 & 0 \end{bmatrix}.$$

Therefore all the matrices

$$\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix}$$

form an ideal $\mathfrak{h}$ of $\mathfrak{g}$. The ideal $\mathfrak{h}$ is Abelian, and in particular solvable. Indeed, if $X = Y = 0$ and $X' = Y' = 0$, then $AA' - A'A = 0$.

It is easy to check that the map

$$\varphi : \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \mapsto (X, Y) : \mathfrak{g} \to \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$$

is a surjective homomorphism of Lie algebras and that $\mathrm{Ker}(\varphi) = \mathfrak{h}$. Now let $\mathfrak{a}$ be a solvable ideal of $\mathfrak{g}$. Then $\varphi(\mathfrak{a})$ is an ideal of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, since φ is surjective. Furthermore, it is solvable by Theorem B from the notes. Since $\mathfrak{sl}_2(\mathbb{C})$ is simple by Question 16, $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ is semisimple, by Prop. 7 from the notes. So $\varphi(\mathfrak{a}) = \{0\}$ and $\mathfrak{a} \subseteq \mathrm{Ker}(\varphi) = \mathfrak{h}$. It follows that $\mathfrak{h}$ is the greatest solvable ideal of $\mathfrak{g}$, i.e. $\mathfrak{h} = \mathcal{R}(\mathfrak{g})$.