INTRODUCTION TO LIE ALGEBRAS – SOLUTION 18

(i) The map $\varphi : \lambda \mapsto \sum_{i=1}^n \lambda_i \varepsilon_i : \mathbb{C}^n \to \mathfrak{t}^*$ is surjective, so its kernel has to be one dimensional. So it must be equal to $\mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is the all-one vector. Since $\mathbb{C}^n = \{\lambda \in \mathbb{C}^n \,|\, \sum_{i=1}^n \lambda_i = 0\} \oplus \mathbb{C}\mathbf{1}$, we get that the restriction of φ to $\{\lambda \in \mathbb{C}^n \,|\, \sum_{i=1}^n \lambda_i = 0\}$ is an isomorphism. So every $\lambda \in \mathfrak{t}^*$ can uniquely be written in the form $\sum_{i=1}^n \lambda_i \varepsilon_i$ with $\sum_{i=1}^n \lambda_i = 0$.
Recall that $\Phi = \{\varepsilon_i - \varepsilon_j \,|\, i \neq j\}$. It is easily checked that $\lambda \in \mathfrak{t}^*$ occurs in the real/rational span of Φ if and only if its coordinates λ_i are real/rational.

(ii) For $C = (c_{ij})_{ij}, D = (d_{ij})_{ij} \in \mathfrak{t}$ we have $\mathrm{tr}(CD) = \sum_{i=1}^n c_{ii} d_{ii}$. It follows that for $\lambda \in \mathfrak{t}^*$ written as above we have $G_\lambda = \sum_{i=1}^n \lambda_i E_{ii}$. Indeed

$$\lambda(D) = \sum_{i=1}^n \lambda_i d_{ii} = \mathrm{tr}(G_\lambda D).$$

So for the form $(-,-)$ on $\mathfrak{t}^*$ given by the trace form on $\mathfrak{t}$ we have

$$(\lambda, \mu) = \mathrm{tr}(G_\lambda G_\mu) = \sum_{i=1}^n \lambda_i \mu_i,$$

when λ and μ are written as above.

(iii) We have $(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = 1 + 1 = 2$ for all $i \neq j$.

(iv) By (iii) we have $(\varepsilon_i - \varepsilon_j)^\vee = \varepsilon_i - \varepsilon_j$. So $r_{\varepsilon_i - \varepsilon_j}(\lambda) = \lambda - (\lambda_i - \lambda_j)(\varepsilon_i - \varepsilon_j)$ which is λ with the i-th and the j-th coordinate swapped.