February 21, 2012 Lecturer Rudolf Tange Hilary Term 2012

Course 2318 2012

S h e e t 4

Due: see www.maths.tcd.ie/∼rtange/teaching/algebraic_geometry/algebraic_geometry.html

Exercise 1

For the ideal J determine whether $V(J)$ is irreducible, find its components if not:

(i) $J = (X^2Y^3) \subset \mathbb{C}[X, Y]$, i.e. J is generated by the polynomial X^2Y^3;

(ii) $J = (XY, YZ, XZ) \subset \mathbb{C}[X, Y, Z]$.

Is it true that $J = I(V(J))$? Calculate $\sqrt{J}$.

Exercise 2

Let C be the affine curve given by the irreducible equation $y^2 = \prod_{i=1}^{2g+1}(x - a_i)$, g an integer ≥ 2 and the $a_i \in \mathbb{C}$ distinct. Show that the projectivization of C has a singular point at infinity.

Exercise 3

Let S be the surface in $\mathbb{P}^3$ defined by the (irreducible) equation $XV = YU$ in the homogeneous coordinates (X, Y, U, V).

(i) Consider the rational map $\varphi = X/Y : S - - > \mathbb{A}^1$. What is the domain of φ? What is the domain of φ when we consider it as a rational map $S - - > \mathbb{P}^1$?

(ii) Use the map $\varphi : S - - > \mathbb{P}^1$ and an analogous map to construct an isomorphism $S \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$. This is nothing but the inverse of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$.

Exercise 4

Denote the singular locus of a variety X by $\mathrm{Sing}(X)$. Let $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ be coprime (recall that $\mathbb{C}[X_1, \ldots, X_n]$ is a unique factorisation domain). Show that

$$\mathrm{Sing}(V(fg)) = \mathrm{Sing}(V(f)) \cup \mathrm{Sing}(V(g)) \cup \big(V(f) \cap V(g)\big)\,.$$

Hint. Observe that the statement doesn't change if we replace f and g by their "square-free versions".

Exercise 5

Show that a nonempty open subset of an irreducible affine or projective algebraic set X is dense in X.