Assignment 1

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Suppose a bar of length L of constant cross-sectional area A and Young's Modulus E. The bar has a prescribed displacement at $x = 0$ equal to $u(0) = -g$, and it is subjected to, on the one hand, an axial force F on the right end ($x = L$), and on the other hand, a distributed axial force (per unit area) given by the expression:

$$q(x) = E\left(\rho u(x) - s\, r(\bar{x})\right)$$

where $u = u(x)$ is the unknown displacement function, $r = r(\bar{x})$ is an input function, $\bar{x} = x/L$, and

$$\rho = \frac{\pi^3}{L^2}, \quad s = \frac{g\pi^4}{L^2}, \quad \frac{F}{AE} = \frac{g\pi^2}{L}.$$

PART 1 (7 points)

1. Derive the corresponding Boundary Value Problem (BVP) for the displacement field $u : [0, L] \rightarrow \mathbb{R}$ using the equilibrium equation for 1D problems (strong form).
2. Find the *exact solution* of this BVP. Plot in a graph the solution of the problem using the following values of the involved constants and the function $r = r(x/L)$:
$$L = 1\, m, \quad g = 0.01\, m, \quad r(x/L) = (x/L)^2 \tag{0.1}$$
3. Formulate the *Variational (or Weak)* form of the Boundary Value Problem.
4. Derive the corresponding matrix equation in terms of a generic matrix of basis functions $\boldsymbol{N}$ and their corresponding derivatives $\boldsymbol{B} = \frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}x}$.
5. Seek an approximation to the solution of this weak form by using basis polynomial basis functions of increasing order (up to order 6), i.e.:
$$\boldsymbol{N} = [1, x], \quad \boldsymbol{N} = [1, x, x^2], \quad \ldots \quad \boldsymbol{N} = [1, x, \ldots x^i], \cdots \quad \boldsymbol{N} = [1, x, \ldots x^6] \tag{0.2}$$
Plot the approximate solutions together with the exact solution.
OBSERVATION: You can use the *Symbolic* Math Toolbox in Matlab (or its counterpart in Python) for determining both the exact solution and the polynomial approximations.
6. Develop a program in either Matlab or Python able to solve the weak form using *linear* finite element basis functions. Employ equally sized finite elements. The size of the element —or, equivalently, the number of finite elements n —should be an input of the program.
7. Solve the problem for increasing number of finite elements and plot the corresponding approximate solutions (for at least four discretizations, of 5, 10, 20 and 40 elements) and compare them with the exact solution.

PART 2 (3 points)

- Implement a function able to calculate the approximation error for both u and its derivative u'. Plot the error versus the element size on a log-log plot. The approximation error for a given solution u^h is given by

$$\|e^h\|_{L_2} = \left(\int_0^1 \left(u(x) - u^h(x)\right)^2\, dx\right)^{\frac{1}{2}}$$

and for its derivative

$$\|e^{h'}\|_{L_2} = \left(\int_0^1 \left(u'(x) - u'^h(x) \right)^2 \, dx \right)^{\frac{1}{2}}$$

OBSERVATION: The integrals must be computed using Gauss integration rule of appropriate order.

What is the slope of the convergence plot in each case ?

"Advanced" assignment: Nonlinear 1D equilibrium

A bar of length L and varying cross-sec'tional area

$$A(x) = A_0 \left(1 + 2\frac{x}{L}(\frac{x}{L} - 1)\right) \tag{0.3}$$

where $A_0 > 0$, is fixed at one end while the other end is subjected to a linearly increasing displacement $u_L(t) = u_m \frac{t}{T}$, where $u_m > 0$ and $t \in [0, T]$, $T > 0$ being the interval of time to analyze —this interval is considered sufficiently large so as to ignore inertial effects. On the other hand, the material of the bar obeys the following (exponential) constitutive equation (relation between stress σ and strain ε)

$$\sigma = \sigma_0 \left(1 - e^{-\dfrac{E}{\sigma_0}\varepsilon}\right). \tag{0.4}$$

Using the Finite Element method, find the displacement solution $u = u(x, t)$ as a function of $t \in [0, T]$ and $x \in [0, X]$. To this end, follow the steps outlined below.

1. Formulate the strong form of the boundary value problem for the 1D equilibrium of a bar with varying cross-sectional area.

2. Determine the weak form of the problem formulated above.

3. Formulate the corresponding matrix equations.

4. Solve the resulting system of nonlinear equations by means of a *Newton-Raphson algorithm.*

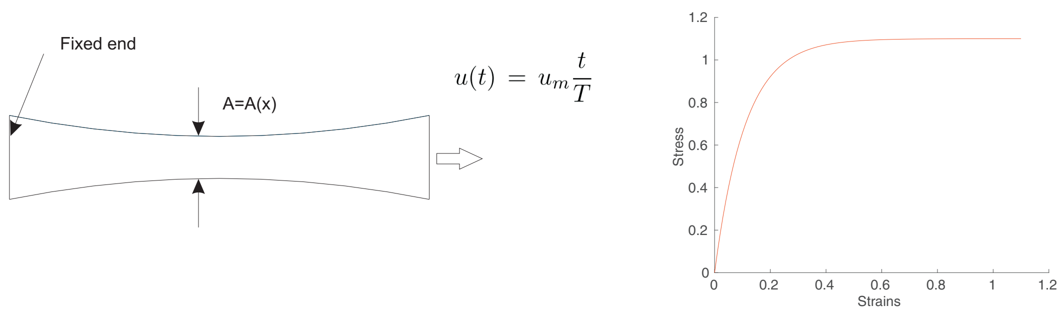

Figure 1 *Geometry and constitutive equation*

5. Check that the solution converges upon increasing the number of elements and time steps.

HINT: Use the matlab script ProgramNonlinear.m as starting point for the required code (if you use Python, you may translate this file to Python language using AI (Chat GPT, for instance)). In *ProgramNonlinear.m*, you will notice that there are two functions missing (one for computing internal forces, and another one for assemblying the stiffness matrix).

DATA: $E = 10\ MPa$; $\sigma_0 = 1\ MPa$; $L = 1$ m; $A_0 = 10^{-2}\ m^2$; $u_m = 0.2L$