Guidelines for Assignment 4

Enginyeria Aeroespacial Computacional

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1 Theoretical introduction

1.1 Static analysis

In the finite element analysis that you carried out in Assignment 3 (part 1), it was tacitly assumed that the distributed load was applied "very slowly". This implies that the acceleration can be considered negligible for practical purposes. As a consequence, the unknown entries of the nodal displacement vector ($\boldsymbol{d}_{\mathrm{l}}$) can be simply obtained, once the global stiffness matrix $\boldsymbol{K}$ and the global external force vector $\boldsymbol{F}$ have been constructed, by means of the algebraic equation

$$\overbrace{\boldsymbol{K}_{\mathrm{ll}}\boldsymbol{d}_{\mathrm{l}}}^{\text{Nodal internal forces}} = \overbrace{\boldsymbol{F}_{\mathrm{l}}}^{\text{Nodal external forces}} - \overbrace{\boldsymbol{K}_{\mathrm{lr}}\bar{\boldsymbol{u}}}^{\text{Forces due to boundary conditions}} \tag{1.1}$$

(this is Eq. 89 of theory document STATIC.pdf). This is what is called a purely *static analysis*. The left-hand side of the preceding equation represents the *nodal internal forces*. Note that they depend linearly on the displacements $\boldsymbol{d}$ and, through the stiffness matrix[1] $\boldsymbol{K}$, on the elastic properties of the material (in this case Young's Modulus and the Poisson's ratio). On the other hand, $\boldsymbol{F}_{\mathrm{l}}$ is the vector of *nodal external forces*, which in this case are only nonzero at the nodes of the top surface of the beam, where the load is applied. Lastly, $\boldsymbol{K}_{\mathrm{lr}}\bar{\boldsymbol{u}}$ is the external force due to the application of the displacement boundary conditions. In the case at hand, since the nodes with prescribed displacements are those of the fixed end of the beam, we have that $\bar{\boldsymbol{u}} = \boldsymbol{0}$, and thus, expression 1.1 boils down to

$$\overbrace{\boldsymbol{K}_{\mathrm{ll}}\boldsymbol{d}_{\mathrm{l}}}^{\text{Nodal internal forces}} = \overbrace{\boldsymbol{F}_{\mathrm{l}}}^{\text{Nodal external forces}} \tag{1.2}$$

From now on, we shall denote by $\boldsymbol{d}_0$ the solution of this system equations.

1.1.1 Inertial forces

Now suppose that, at a given instant t_0, the applied load is *suddenly* released, which means that, in the above equation, the right-hand side becomes zero for $t > t_0$, i.e., $\boldsymbol{F}_{\mathrm{l}}(t) = \boldsymbol{0}$, for $t > t_0$. To keep the equilibrium of forces, inertia forces must come into play

$$\overbrace{\boldsymbol{K}_{\mathrm{ll}}\boldsymbol{d}_{\mathrm{l}}}^{\text{Nodal internal forces}} = \overbrace{\boldsymbol{F}_{\mathrm{l}}^{iner}}^{\text{Nodal inertial forces}} \tag{1.3}$$

[1] Recall that $\boldsymbol{K} = \int_{\Omega} \boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B}\, d\Omega$, see Eq. 85 in STATIC.pdf, where $\boldsymbol{C} = \boldsymbol{C}(E,\nu)$ is the 6×6 elasticity matrix, defined, for isotropic materials, in Eq. 44 of STATIC.pdf.

By Newton's second law, we know that these forces have to be, somehow, proportional to the acceleration and the mass (i.e., the density ρ). Accordingly, we make

$$\boldsymbol{F}_{\mathrm{l}}^{iner} = -\boldsymbol{M}_{\mathrm{ll}}(\rho)\ddot{\boldsymbol{d}}_{\mathrm{l}} \tag{1.4}$$

where $\boldsymbol{M}(\rho)$ stands for the nodal *mass matrix* of the structure (), and $\ddot{\boldsymbol{d}} = \dfrac{\mathrm{d}^2\boldsymbol{d}}{\mathrm{d}t^2}$ designates the nodal acceleration. Inserting the above equation into Eq.(1.3), we arrive at

$$\boldsymbol{M}_{\mathrm{ll}}\ddot{\boldsymbol{d}}_{\mathrm{l}}(t) + \boldsymbol{K}_{\mathrm{ll}}\boldsymbol{d}_{\mathrm{l}}(t) = \boldsymbol{0}. \tag{1.5}$$

1.1.2 Free undamped vibration

As in Eq.(1.1), the unknowns in the above equations are the nodal displacements in the unrestricted Degrees of Freedom ($\boldsymbol{d}_{\mathrm{l}}$). The difference now is that each entry of $\boldsymbol{d}_{\mathrm{l}}$ is a *function of time*, and therefore, the above is not an matrix algebraic equation, but rather constitutes a *system of n ordinary differential equations*. Since the system is second order, two initial conditions are to be supplemented: one for the displacement itself at time t_0, and another one for its derivative (the velocity). In the case we are considering, at the initial time t_0, the displacement is $\boldsymbol{d}_{\mathrm{l}}(t_0) = \boldsymbol{d}_0$ (recall that $\boldsymbol{d}_0$ is the solution of Eq.(1.2)), whereas the velocity is zero ($\dot{\boldsymbol{d}}_{\mathrm{l}}(t_0) = 0$).

Equation 1.5 describes the so-called *free, undamped vibration* of the structure. The qualifier "free" refers to the fact that there are no external forces (what causes the motion is the release of elastic energy stored in the structure). On the other hand, the motion is "undamped" because there are no energy dissipation whatsoever: during a cycle, the elastic energy stored in the structure is progressively released and converted into kinetic energy –and then the other way around—, without any loss of energy in the process due to dissipation. As a consequence, the motion will be maintained perpetually.

1.1.3 Damping forces

A more realistic scenario is when a damping force $\boldsymbol{F}_{\mathrm{l}}^{damp}$ is incorporated into equation 1.5. As explained in page 4 of DYNAMIC.pdf, damping forces include the effect of both the *viscoelasticity* of the material and/or external friction with the surrounding medium, and can be written, under the assumption of linearity, as

$$\boldsymbol{F}_{\mathrm{l}}^{damp} = -\boldsymbol{D}\dot{\boldsymbol{d}}_{\mathrm{l}} \tag{1.6}$$

where $\boldsymbol{D}$ is the damping matrix. For simplicity, this matrix is assumed to be a linear combination of the stiffness and mass matrix (see page 14 of DYNAMIC.pdf):

$$\boldsymbol{D} = \bar{\alpha}\boldsymbol{M} + \bar{\beta}\boldsymbol{K} \tag{1.7}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are constants.

1.1.4 Free damped vibration

If we now add the damping force defined in Eq.(1.6) to the balance of external and internal forces, we end up with the system of differential equations that govern the free *damped* vibration of a given structure

$$\boldsymbol{M}_{\mathrm{ll}}\ddot{\boldsymbol{d}}_{\mathrm{l}}(t) + \boldsymbol{D}_{\mathrm{ll}}\dot{\boldsymbol{d}}_{\mathrm{l}} + \boldsymbol{K}_{\mathrm{ll}}\boldsymbol{d}_{\mathrm{l}}(t) = \boldsymbol{0}. \tag{1.8}$$

The *initial conditions* are the same as for the undamped case:

$$\boldsymbol{d}_{\mathrm{l}}(t_0) = \boldsymbol{d}_0 \qquad \dot{\boldsymbol{d}}_{\mathrm{l}}(t_0) = \boldsymbol{0}. \tag{1.9}$$

1.1.5 Expression for the mass matrix

The expression for the nodal mass matrix of a structure is given by (see Eqs. 27 and 117 of DYNAMIC.pdf)

$$\boldsymbol{M} := \int_{\Omega} \rho \boldsymbol{N}^T \boldsymbol{N} \, d\Omega = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \int_{\Omega^e} \rho \boldsymbol{N}^{eT} \boldsymbol{N}^e \, d\Omega \tag{1.10}$$

This expression can be derived from the weak form of the momentum balance equation (this is explained in DYNAMIC.pdf, pages 9 and 13).

1.2 Methods for solving the semi-discrete motion equations

For solving either Eq.(1.8) or Eq.(1.5), two types of methods may be employed: methods based on finite differences, on the one hand, or modal analysis, on the other hand.

1.2.1 Methods based on finite differences

This type of techniques consists in replacing the time derivatives in the differential equations by finite differences, resulting in a sequence of discrete equations, one for each time step (in page 17 of DYNAMIC.pdf, we summarize the discrete equations corresponding to one of such methods: the Newmark β-scheme). These methods are rather general, for they can be used also in inelastic problems, and furthermore, they are independent of the type of external actions applied to the body.

1.2.2 Method based on modal decomposition analysis

The drawback of the above explained methods is that they require solving a system of equations at each time step of the discretization. For the particular problem we are addressing here (linear elastic vibration), it is far more efficient to use the so-called *modal decomposition analysis*. This technique is nothing but a generalization (to arbitrary number of unknowns) of the typical method employed to solve the oscillatory equation of 1 single degree of freedom. Modal decomposition is thoroughly described in pages 19 to 35 DYNAMIC.pdf.

2 Tasks

1. Adapt the matlab program *mainELASTOSTATIC.m* (developed in assignment 3) so that it can also compute the mass matrix of the studied mesh (see video Vid01_mass.avi).

    - Add in the input data file the nodal densities of the body under study ($\rho = 2.7$ g/cm^3):
    - The function computing the mass matrix is to be invoked from function SolveElasFE.m (hint: follow the steps made for computing the stiffness matrix).

2. Develop a matlab program able to calculate the **natural frequencies and modes** of a given structure (see video Vid02_modes.avi).

    - The input data need for running this program is to be generated by the modified script *mainELASTOSTATIC.m*. More specifically, the required data are
        - Stiffness matrix: $\boldsymbol{K} \in \mathbb{R}^{3n_{pt} \times 3n_{pt}}$
        - Mass matrix: $\boldsymbol{M} \in \mathbb{R}^{3n_{pt} \times 3n_{pt}}$
        - List of unrestricted degrees of freedom ($\boldsymbol{l}$)

    For post-processing purposes (with GID), the following variables are also needed:

– Coordinate matrix: *COOR*
– Connectivity matrix: *CN*
– Type of element: *TypeElement*
– Position of the Gauss points: *posgp*
– Name of mesh file generated by gid (for instance, "malla1.msh")

The other input of the program is the number of modes to be analyzed.

- For plotting the deformed shapes associated to each node, use the file: GidPostProcessModes.m
  (note: copy all the files contained in folder NEWFUNCTIONS in the folder in which you have the program of Assignment 3.)
- In order to avoid running the script "mainELASTOSTATIC.m" each time you need the stiffness and mass matrices, store these and the other required variables in a binary matlab file ".mat" (by using the *save* function). The information saved in the corresponding binary file can be recovered by just employing the *load* function.
- Assess the performance of the developed code by computing the natural frequencies and modes of the boxed beam studied in assignment 3 .

3. Apply the program to calculate the dynamic response of the cantilever boxed beam studied in Assignment 3, Part 1, (using MESH 4) (see video Vid03_damp.avi).

   In particular, analyze the dynamic response when the initial displacement is:

   (a) The one caused by the distributed load on the surface $y = y_{max}$
   (b) The one caused by the torque applied on the free end of the cantilever.

4. Examine which are the natural modes that are more dominant in each case —by plotting the initial amplitude of the temporal response for each mode.

5. The damping ratios $\bar{\xi}_i$ for each natural frequency should be an input of the program.

6. The interval of time to be studied should be m times the maximum natural period of the system ($T_1 = \dfrac{2\pi}{\omega_1}$). Choose, for instance, $m = 40$.

7. Include in the modal approximation the first $n = 25$ modes.

8. To postprocess the computed solution, use the function

```
GidPostProcessDynamic(COOR,CN,TypeElement,DISP,NAME_INPUT_DATA,posgp,
    NameFileMesh,t);
```

   where

   - $t \;\Rightarrow\;\; nstep \times 1$ vector containing the discretization of the time interval under study (use 500 steps).
   - *DISP*: $3n_{pt} \times nstep$ matrix formed by the nodal displacement solutions at each time step.
   - NAME_INPUT_DATA: Name for identifying the corresponding results file (for instance "DYN-sol")

9. Generate videos using GID of the simulations corresponding to the case of $\bar{\xi}_i = 0.01$ for $i = 1, 2 \ldots n$ (for the two initial conditions). Attach a LINK to these videos (you can place the videos in Google Drive, for instance) in the report. Alternatively, you can generate a sequence of snapshots showing the temporal evolution of the motion of the structure.