Assignment 4

Aerospace Computational Engineering

Aerospace Technology Engineering Degree

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Contents

1 Task 1. Adaptation of the Matlab program

The code used in this assignment is the one from Assignment 3, with some modifications. The mass matrix must first be calculated and has the following expression:

$$\mathbf{M}^e := \int_{\Omega^e} \rho \mathbf{N}^{eT} \mathbf{N}^e d\Omega \tag{1}$$

Using Gauss quadrature, the latter expression can be rewritten as:

$$\mathbf{M}^e = \sum_{g=1}^{m} w_g (\mathbf{J}^e \mathbf{N}^{eT} \rho \mathbf{N}^e)_{\xi=\xi_g} \tag{2}$$

The Gauss points vector and Jacobian determinant are computed as in Assignment 3. The density is an input to the code, with a value of $\rho = 0.0027 Mkg/m^3$ to be consistent with the units: the Young modulus is given in MPa, and the forces in MN (because $MPa = MN/m^2$), so the density has to be given in Mkg/m^3 (since $N = kg \cdot m/s^2 \rightarrow MN = Mkg \cdot m/s^2$). Then, to compute the matrix $\mathbf{N}^e$, it's necessary to compute the matrix of shape functions for scalar-value fields, which is computed by evaluating the matrix of shape functions at each Gauss point.

The next step is to transform this matrix of shape functions for scalar-value fields into the matrix of shape functions for vector-valued fields, the function *StransfN* is used (the function used to compute the matrix $\mathbf{B}^e$ was the *QtransfB*).

Finally, the assembly of the mass matrix $\mathbf{M}$ follows the same procedure used to assembly the $\mathbf{K}$ matrix.

2 Task 2. Natural frequencies and modes analysis

In this section, a Matlab code is developed to compute the natural frequencies and modes of a generic structure. The objective is to solve the following equation:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{D}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F} \tag{3}$$

To obtain the natural frequencies and its associated modes, the **undamped free vibration** case is analyzed, where $\mathbf{D} = \mathbf{0}$ and $\mathbf{F} = \mathbf{0}$. So

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{0} \tag{4}$$

The solution has the form as it follows:

$$\mathbf{d} = \boldsymbol{\phi}(A\cos(\omega t) + B\sin\omega t) = \boldsymbol{\phi} q(t) \tag{5}$$

Then,

$$\mathbf{M}\left(\boldsymbol{\phi}\frac{d^2 q(t)}{dt^2}\right) + \boldsymbol{K}\boldsymbol{\phi} q(t) = \mathbf{0} \tag{6}$$

Since $\frac{d^2 q(t)}{dt^2} = -\omega^2 q(t)$, one obtains:

$$(-\omega^2\mathbf{M} + \mathbf{K})\boldsymbol{\phi} q(t) = \mathbf{0} \Rightarrow (-\omega^2\mathbf{M} + \mathbf{K})\boldsymbol{\Phi} = \mathbf{0} \tag{7}$$

Mathematically, the preceding equation is called an "eigenvalue problem." It has the trivial solution $\boldsymbol{\phi} = \mathbf{0}$, but the interesting solutions are the nontrivial ones, of which there are as many as the number of degrees of freedom (n). These solutions give the natural frequencies and their associated modes of vibration.

A natural frequency may also be called a **resonant frequency**, and a mode may also be called an eigenvector, mode shape, normal mode, **natural mode**, characteristic mode or principal node. The smallest nonzero ω_i is called the **fundamental frequency of vibration**.

A Matlab code has been developed to compute the 25 first natural frequencies and their respective vibration modes. It uses the function *UndampedFREQ(M,K,neig)* where M and K are the M and K values from the unrestricted (or free) nodes, and *neig* is the number of modes to be calculated, which is 25.

2.1 Natural frequencies

The natural frequencies obtained with the code are the following:

$$\begin{array}{lll}
\omega_1 = 380.1048 \text{ rad/s} & \omega_{10} = 8963.868 \text{ rad/s} & \omega_{18} = 13799.79 \text{ rad/s} \\
\omega_2 = 380.1048 \text{ rad/s} & \omega_{11} = 8963.913 \text{ rad/s} & \omega_{19} = 15293.07 \text{ rad/s} \\
\omega_3 = 2125.365 \text{ rad/s} & \omega_{12} = 11398.14 \text{ rad/s} & \omega_{20} = 16071.02 \text{ rad/s} \\
\omega_4 = 2125.369 \text{ rad/s} & \omega_{13} = 12061.05 \text{ rad/s} & \omega_{21} = 17444.94 \text{ rad/s} \\
\omega_5 = 2263.585 \text{ rad/s} & \omega_{14} = 13015.70 \text{ rad/s} & \omega_{22} = 17445.08 \text{ rad/s} \\
\omega_6 = 4022.063 \text{ rad/s} & \omega_{15} = 13085.87 \text{ rad/s} & \omega_{23} = 17816.95 \text{ rad/s} \\
\omega_7 = 5232.483 \text{ rad/s} & \omega_{16} = 13085.96 \text{ rad/s} & \omega_{24} = 18455.18 \text{ rad/s} \\
\omega_8 = 5232.502 \text{ rad/s} & \omega_{17} = 13172.07 \text{ rad/s} & \omega_{25} = 18772.11 \text{ rad/s} \\
\omega_9 = 6806.736 \text{ rad/s} & &
\end{array}$$

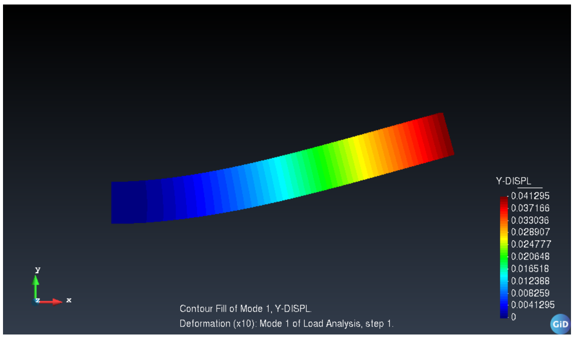
To verify the results, the first natural frequency is computed using an analytical method. This frequency can be obtained from the following expression, obtained from reference [1]:

$$\omega_1 = 1.875^2 \sqrt{\frac{EI}{\rho A L^4}} = 1.875^2 \sqrt{\frac{70 \cdot 10^9 \cdot 2.833 \cdot 10^{-4}}{2700 \cdot (0.25^2 - 0.15^2) \cdot 2^4}} = 376.62 \text{ rad/s} \tag{8}$$

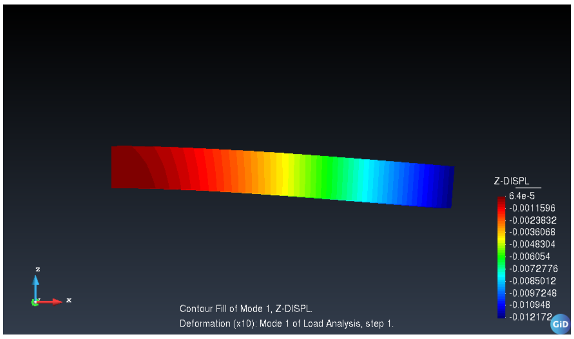
The numerical and analytical results can be seen to be very similar with only a 1% error, leading to the assumption that the Matlab code is accurate. It verifies the good correlation between the units of density, Young modulus and forces.

2.2 Vibration modes

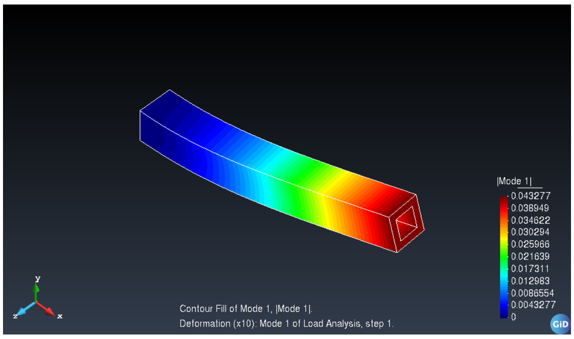
The Matlab code also computes the vibration modes, which are represented by the vector $\boldsymbol{\phi}$. In order to provide a clearer picture of the vibration modes, the results are displayed in GID. Next, the first five modes are represented in detail (with Y and Z displacements and different views) since they are the most frequent ones. In addition, appendix 4.1 contains images of modes 6–25.



(a) Y displacement, XY view.

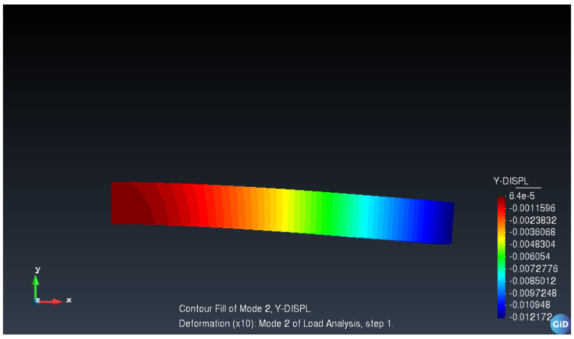


(b) Z displacement, XZ view

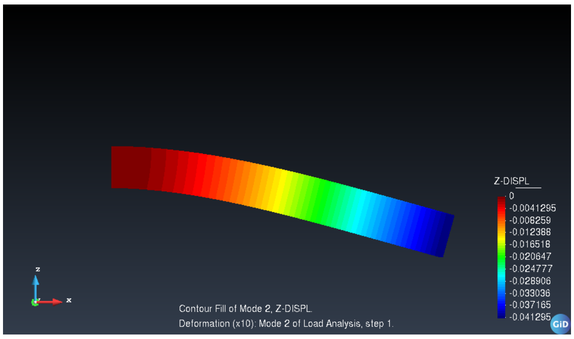


(c) Overall displacements, general view

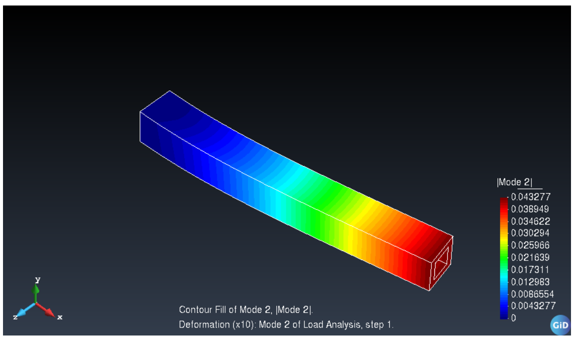
Figure 1: Mode 1



(a) Y displacement, XY view.

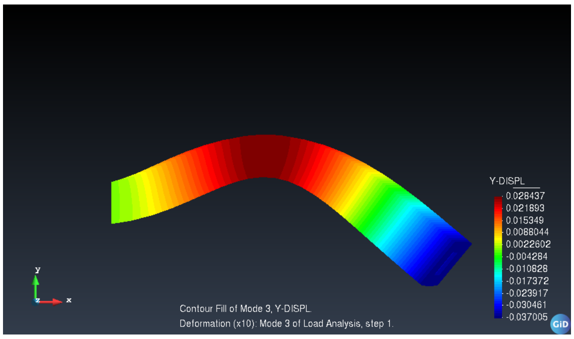


(b) Z displacement, XZ view

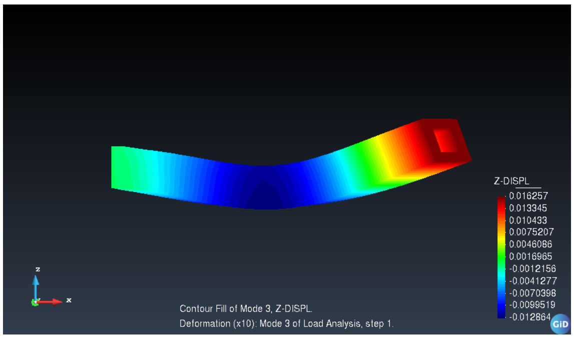


(c) Overall displacements, general view

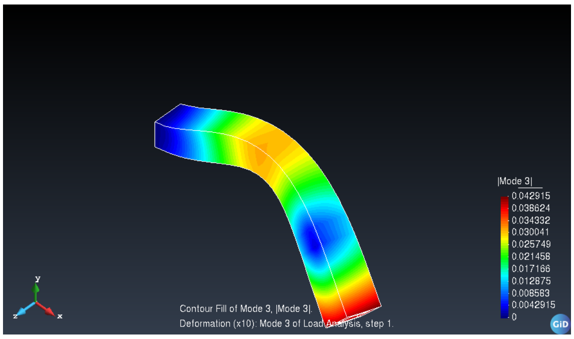
Figure 2: Mode 2



(a) Y displacement, XY view.

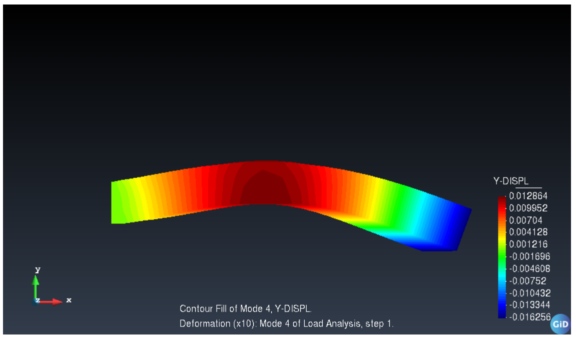


(b) Z displacement, XZ view

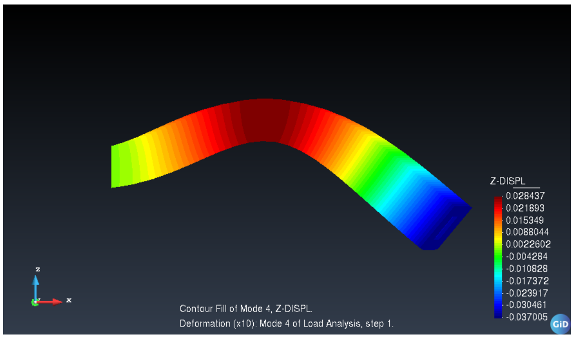


(c) Overall displacements, general view

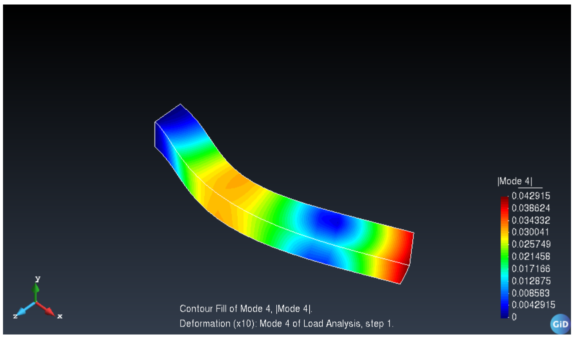
Figure 3: Mode 3



(a) Y displacement, XY view.

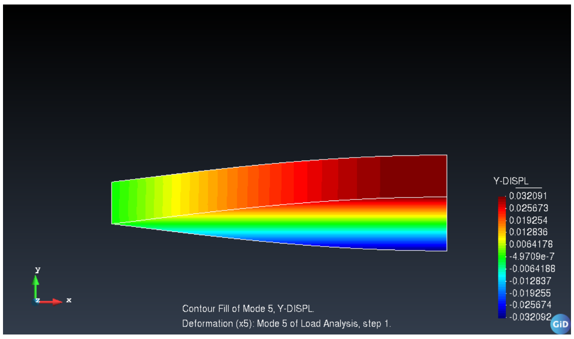


(b) Z displacement, XZ view

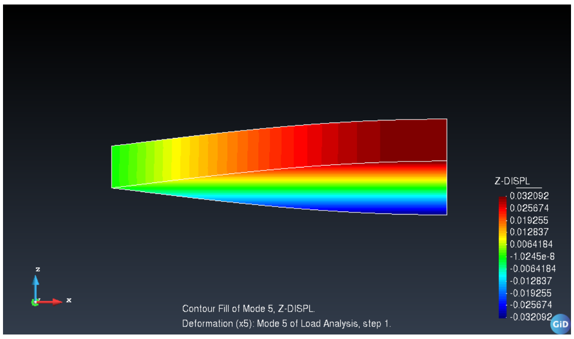


(c) Overall displacements, general view

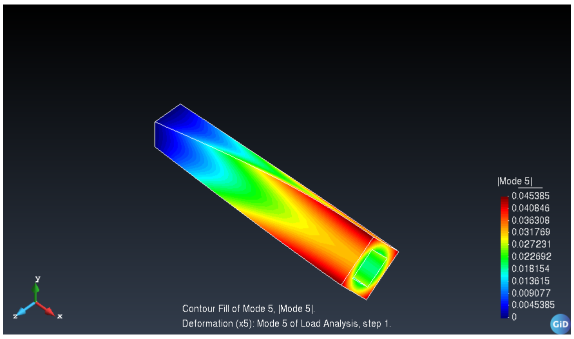
Figure 4: Mode 4



(a) Y displacement, XY view.



(b) Z displacement, XZ view



(c) Overall displacements, general view

Figure 5: Mode 5

3 Task 3. Dynamic response

In this section, the dynamic behavior of the beam is examined (in each case separately) when the force and the torque are no longer applied. For each iteration of time, the following equation is calculated using a Matlab program (called *Modes_and_Dyn*):

$$\mathbf{d} = \sum_{i=1}^{n} \phi_i \left(e^{-\bar{\xi}_i \omega_i t} q_i^0 \cos(\omega_i t) + \frac{\dot{q}_i^0 + \bar{\xi}_i \omega_i q_i^0}{\omega_i} \sin(\omega_i t) \right) \tag{9}$$

The parameters of equation 9 can be defined as follows:

- ϕ_i: vibration modes (computed in the section before).
- $\bar{\xi}_i$: damping factor of each mode. Some values have been tested, but the chosen one is 0.01.
- ω_i: natural frequencies of the different modes (computed in the section before).
- t: time in which the dynamic response of the system is analyzed. This time is assumed to be 40 times the maximum natural period of the system (T_0):

$$T_0 = \frac{2\pi}{\omega_1} = \frac{2\pi}{380.1048} = 0.0165s$$
$$t = 40 \cdot T_0 = 0.6612s$$

- q_i^0 and $\dot{q}_i^0$: generalized coordinates and its derivate. They can be computed with the following expressions (note that $\dot{\mathbf{d}}^{\mathbf{0}} = \mathbf{0}$ since there's no initial speed):

$$q_i^0 = \boldsymbol{\phi_i^T M d^0}$$
$$\dot{q}_i^0 = \boldsymbol{\phi_i^T M \dot{d}^0} = 0 \forall i$$

- The Matlab code needs to know the total time divisions desired by the user. It uses a total of 500 time divisions, so each step is computed every 0.0013 seconds:

$$\Delta t = \frac{0.6612}{500} = 0.0013s$$

3.1 Case 1: Distributed load

The Matlab code used to compute equation 9 for each step of time (*Modes_and_Dyn*) has also been used to draw the amplitude for each vibration mode while the distribution force is abruptly released from the beam. As shown in figure 6, the only modes activated are modes 1, 2, 3, and 4 (the others can be assumed to be non-participant), with the amplitude of modes 1 and 2 being the most significant ones. This makes sense looking at the results of the vibration modes obtained in section 2.

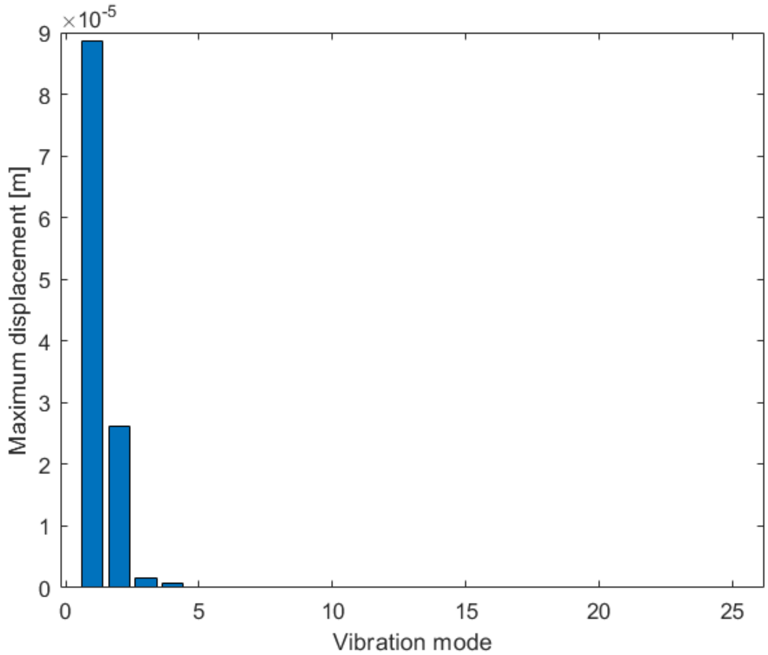


Figure 6: Maximum displacement of each vibration mode for the distributed force.

The dynamic response is recorded in a video to show how the beam moves once the distributed force is removed. This video can be found at the following link: `https://drive.google.com/drive/folders/19bUlv-J_yguyo833V6ubY1fzQBdWV39M?usp=sharing`. Observing the video, it makes sense that the modes 1 and 2 are the most significant ones, since its movements corresponds to the movment of the beam.

3.2 Case 2: Torque

Additionally, the code computes the amplitude for each vibration mode when the torque is abruptly released from the beam.

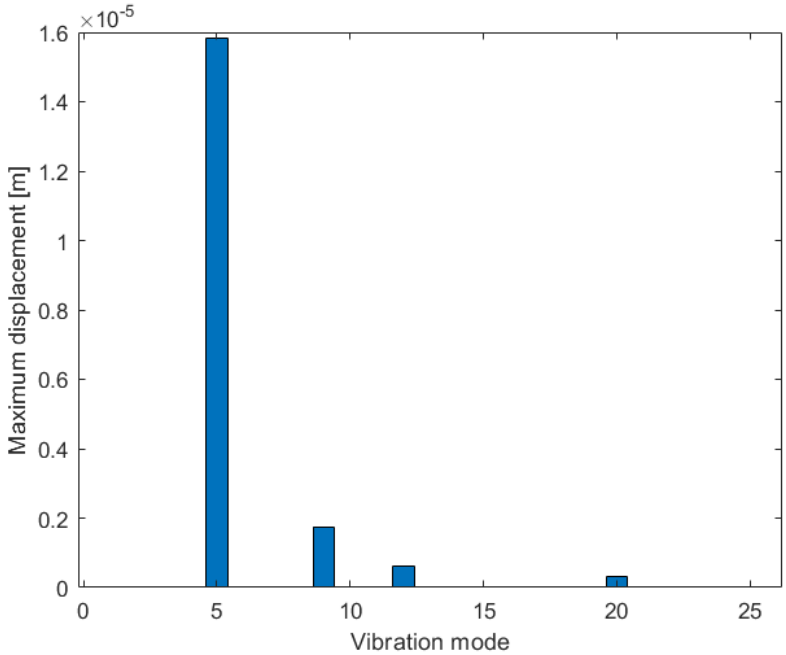
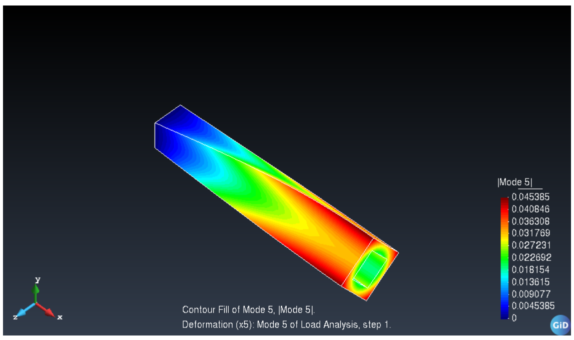
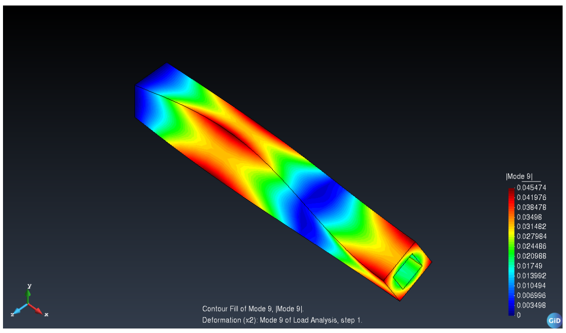


Figure 7: Maximum displacement of each vibration mode for the distributed force.

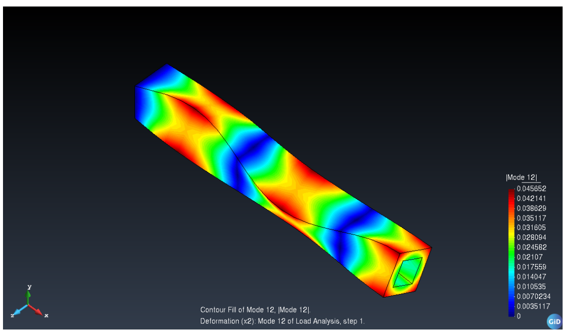
As shown in figure 7, the only modes activated are modes 5, 9, 12 and 20 (the others can be assumed to be non-participant). This make sense since these modes correspond to x torques, as shown in figure 8. The most relevant mode is mode 5, since it corresponds to a pure x torque, followed by the mode 9, which has an amplitude 8 times lower. It makes sense because the external torque is applied to the free end of the beam, so mode 5 is the most representative one.



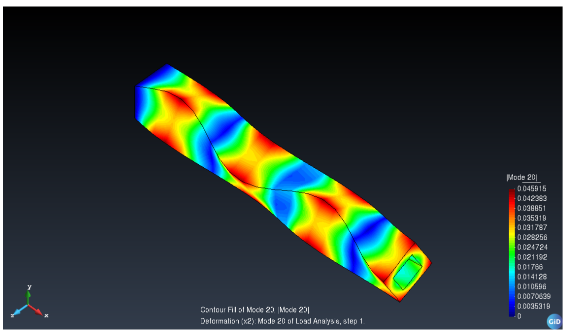
(a) Mode 5



(b) Mode 9



(c) Mode 12



(d) Mode 20

Figure 8: Vibration modes which corresponds to x torques.

The dynamic response is recorded in two videos to show how the beam moves after the torque is removed from it. They can be found at the link: `https://drive.google.com/drive/folders/19bUlv-J_yguyo833V6ubY1fzQBdWV39M?usp=sharing`.

One is saved with a 100 ms delay between steps to see the movement slowly, while the other is saved with a 10 ms delay.

Conclusions

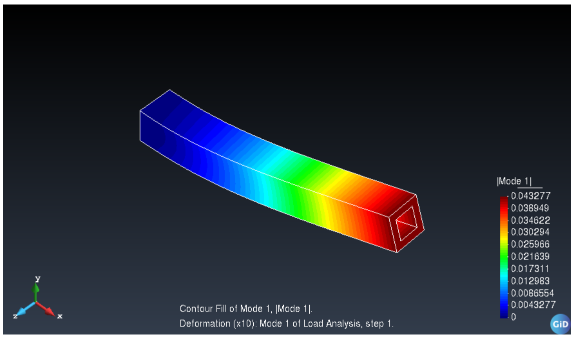
In this assignment, an elasto-dynamic analysis of the beam has been studied. In tasks 1 and 2, a Matlab code has been developed to compute the mass matrix, which will then be followed by the computation of the natural frequencies and associated vibration modes.

Moreover, by computing the natural frequency analytically and comparing this value with the numerical result given by the code, the latter can be validated since the results are very similar.

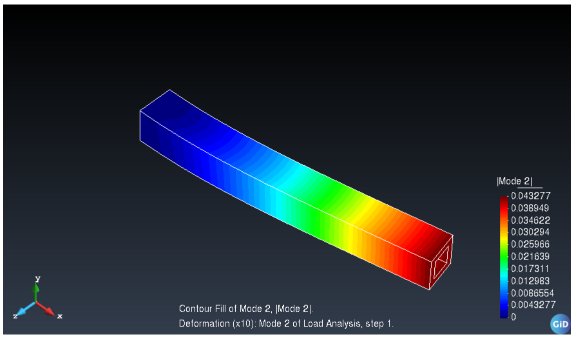
Then, in task 3, the dynamic response of the beam once the external force and torque are released from it will be studied. It's been observed that in each case, the activated modes have a direct relationship with the forces applied. In the torsion case, the activated modes are 5, 9, 12, and 20, which correspond to x torques.

4 Appendix

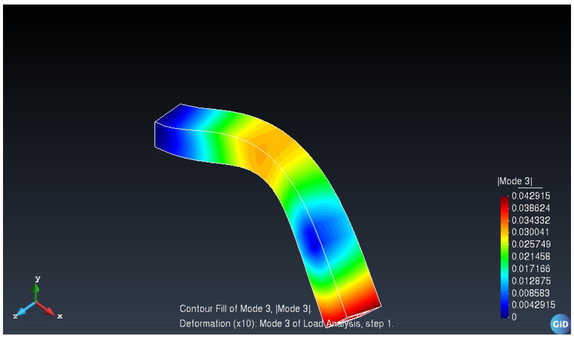
4.1 Representation of the first 25 vibration modes



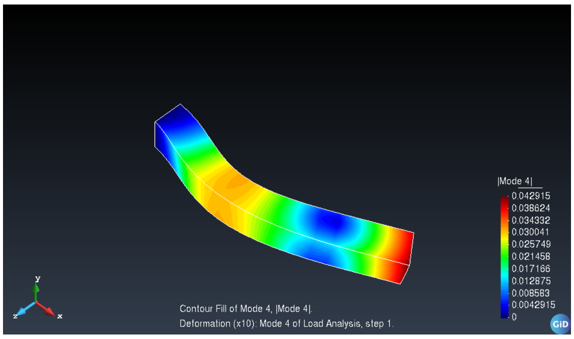
(a) Mode 1



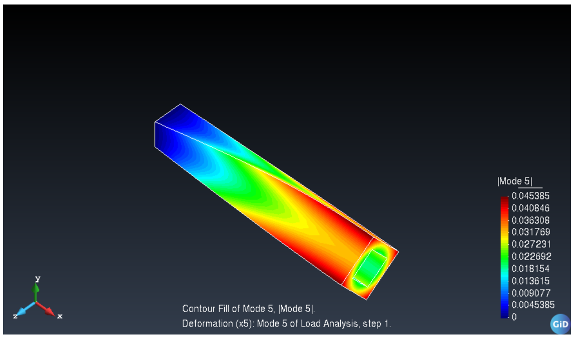
(b) Mode 2



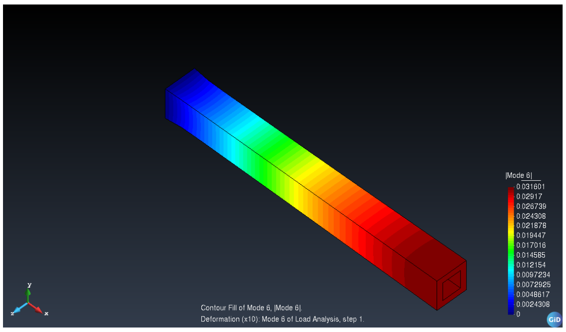
(c) Mode 3



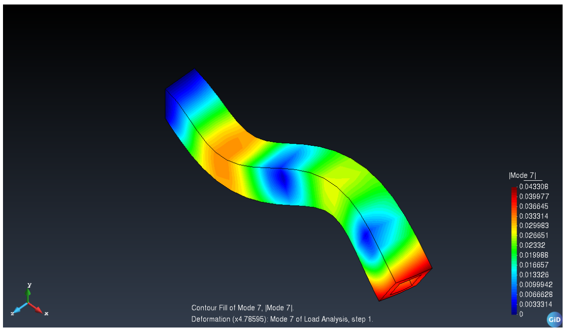
(d) Mode 4



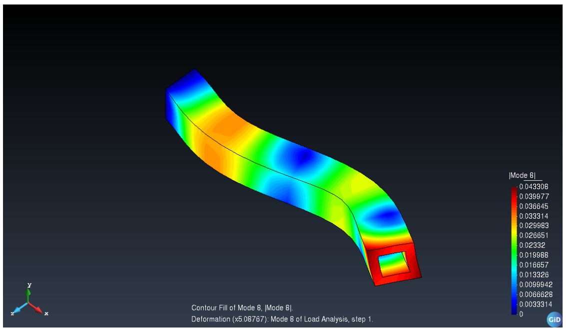
(e) Mode 5



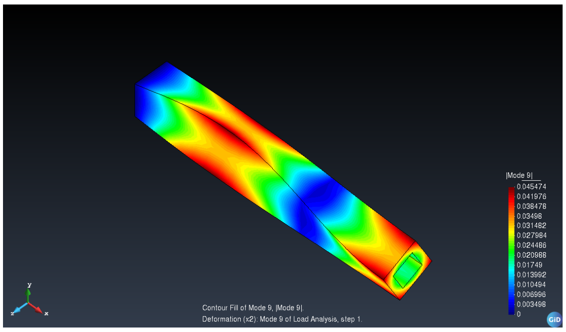
(f) Mode 6



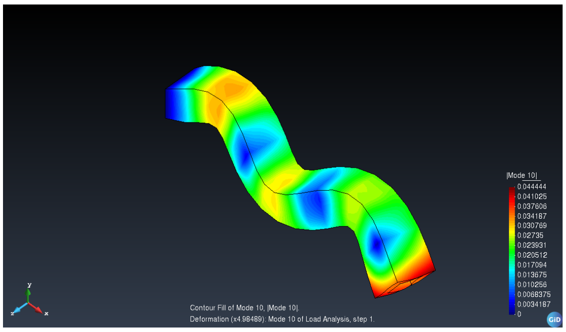
(g) Mode 7



(h) Mode 8



(i) Mode 9



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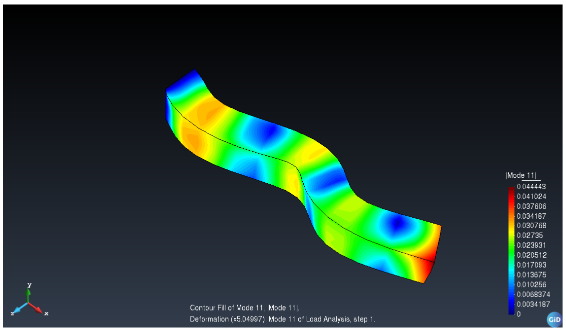

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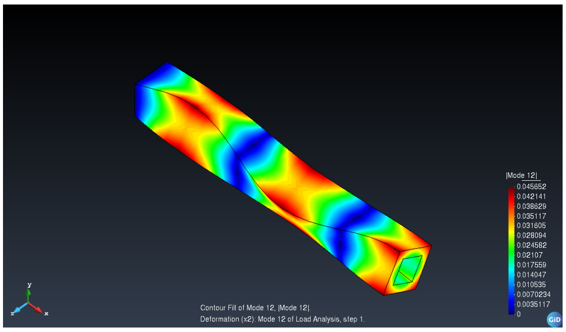

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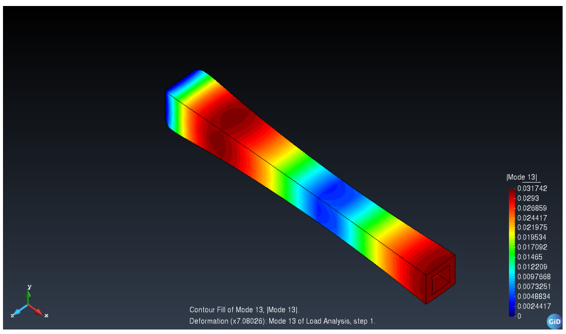

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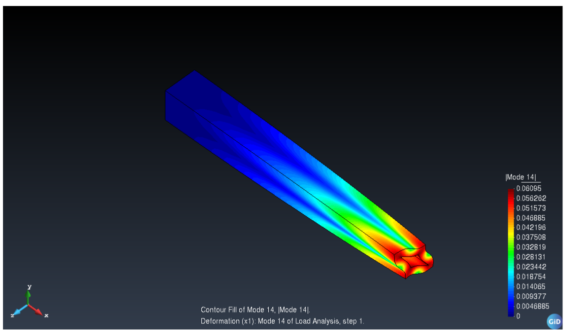

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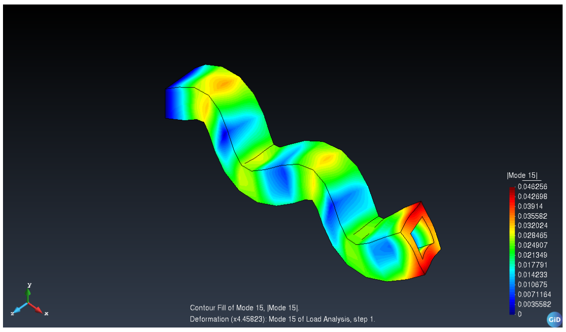
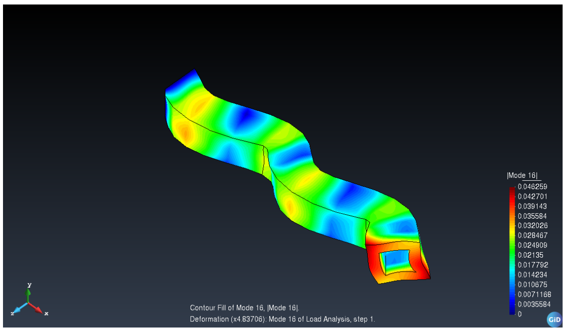


(o) Mode 15



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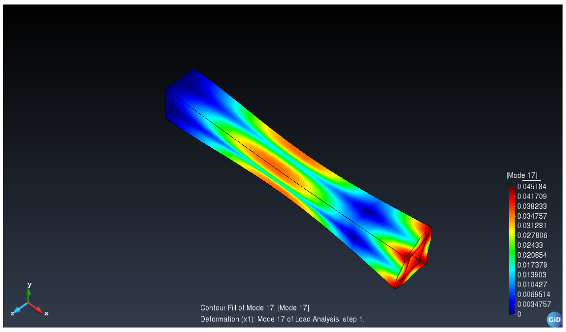

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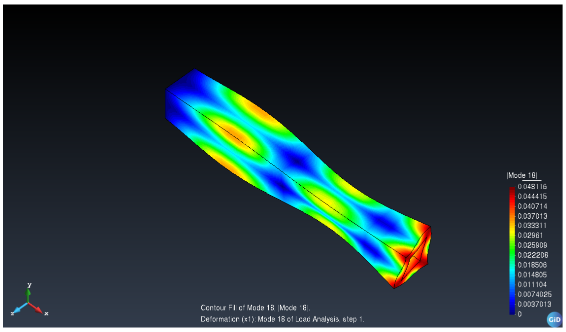

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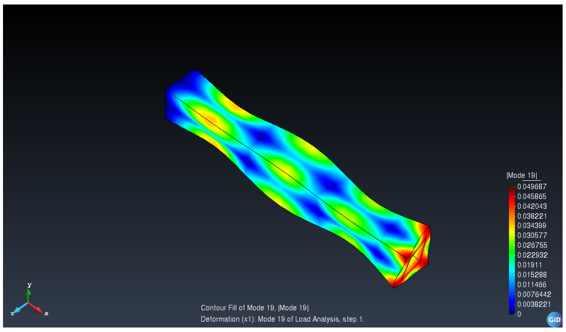

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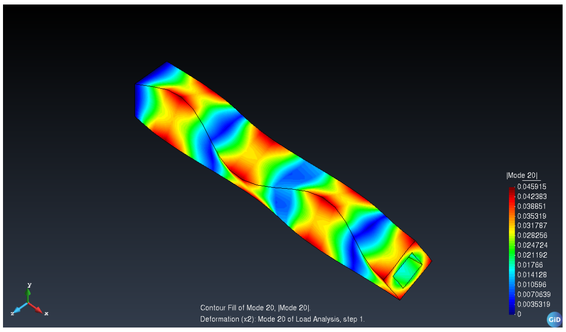

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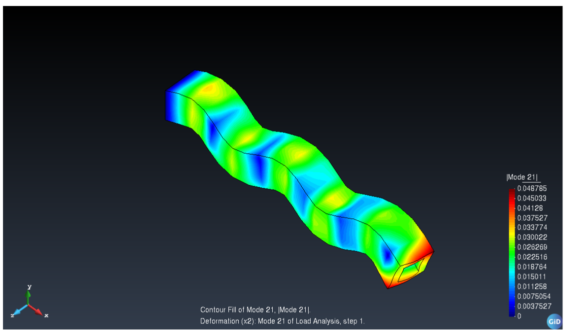

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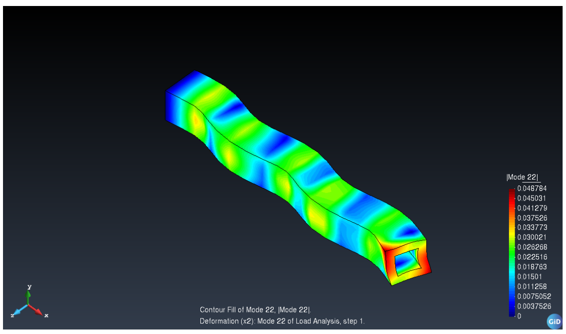

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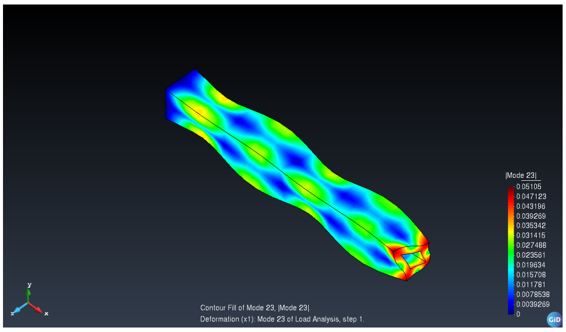

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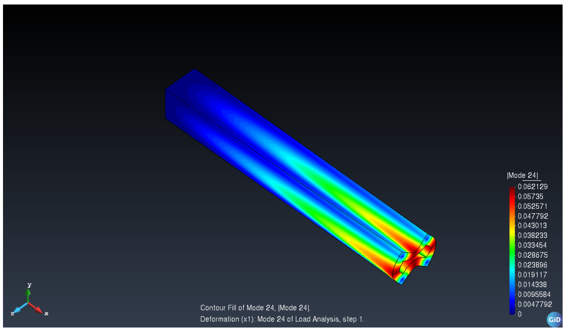
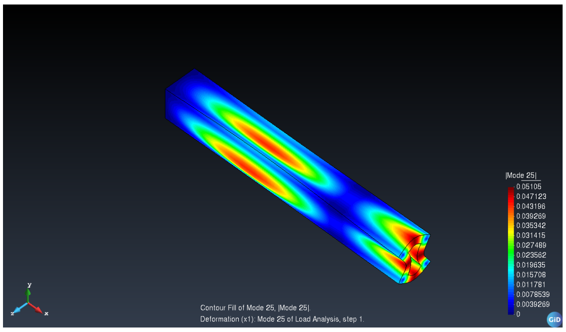


(x) Mode 24



(y) Mode 25

References

[1] Amrita, Value: *Free vibration of a cantilever beam (continuous system)*. `https://vlab.amrita.edu/?sub=3&brch=175&sim=1080&cnt=1`, 2011. Retrieved December, 2022.