AE/ME/BME 518 - Computational Fluid Dynamics

Spring 2019, Project 1, Due: January 24, 2019

1 Background

For those who have taken ME391 or ME570 from me, this is not a new problem. However, this is simple enough in a graduate level CFD course to get things started. If you already have a code that you had written in the past, feel free to use it. Once we are through this assignment, we will add another governing equation and will turn it into a Navier-Stokes solver and will investigate laminar flow in a lid driven cavity.

Consider the differential cube of solid material as illustrated in the Figure shown below.

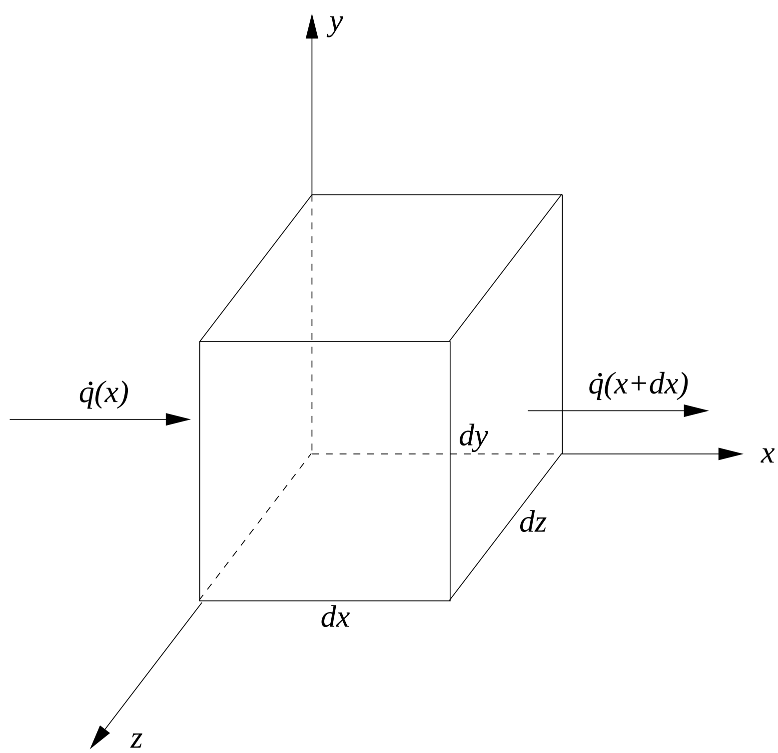

Figure 1: Heat conduction.

Heat flow in a solid is governed by the Fourier's law of conduction, which states that

$$\dot{q} = -kA\frac{dT}{dn} \tag{1}$$

where $\dot{q}$ is the energy transfer per unit time, T is the temperature, A is the area across which the energy flows, dT/dn is the temperature gradient normal to the area A, and k

is the thermal conductivity of the solid. The net rate of energy flow into the solid in the x-direction can be written as

$$\dot{q}_{net,x} = \dot{q}(x) - \dot{q}(x+dx) = \dot{q}(x) - \left(\dot{q}(x) + \frac{\partial \dot{q}(x)}{\partial x}dx\right) = -\frac{\partial \dot{q}(x)}{\partial x}dx \tag{2}$$

Substituting Eq. (1) into Eq. (2) yields

$$\dot{q}_{net,x} = -\frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)dx = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)d\mathcal{V} \tag{3}$$

where $d\mathcal{V} = Adx$ is the volume of the differential cube. Similarly,

$$\dot{q}_{net,y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)d\mathcal{V} \tag{4}$$

$$\dot{q}_{net,z} = \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)d\mathcal{V} \tag{5}$$

For steady heat flow, the net change in the amount of the energy stored is zero. Therefore, the sum of the net rate of enery flow is zero. That is

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0 \tag{6}$$

Equation (6) governs the steady conduction of heat in a solid. Assuming the thermal conductivity to be constant, Eq (6) simplifies to

$$\nabla^2 T = 0 \tag{7}$$

which is the Laplace equation. Similar to steady heat conduction, Laplace equation applies to many problems in engineering such as mass diffusion, electrostatics, and inviscid incompressible fluid flow.

2 Problem

The governing equation for two-dimensional steady heat conduction is given by

$$\nabla^2 T = T_{xx} + T_{yy} = 0 \tag{8}$$

Equation (8) is an elliptic partial differential equation for which the solution $T(x, y)$ is governed by **boundary conditions** specified at each point on the boundary of the physical domain. The boundary conditions may be of Dirichlet type (i.e., specified temperature),

$$T(x, y) = G(x, y) \qquad (\textit{on the boundaries}) \tag{9}$$

the Neumann type (i.e., specified temperature gradient),

$$T_n(x, y) = G(x, y) \qquad (\textit{on the boundaries}) \tag{10}$$

or the mixed type (i.e., some specified combination of temperature and temperature gradient),

$$aT(x, y) + bT_n(x, y) = G(x, y) \qquad (\textit{on the boundaries}) \tag{11}$$

Consider a rectangular flat plate of height $h = 20$ cm and width $w = 15$ cm as shown in the figure below. The temperature on the top edge of the plate is held at 100 C, and the temperatures on the left, right, and bottom edges of the plate are held at 0 C (i.e., Dirichlet boundary conditions are to be employed on all four sides of the physical domain). For this

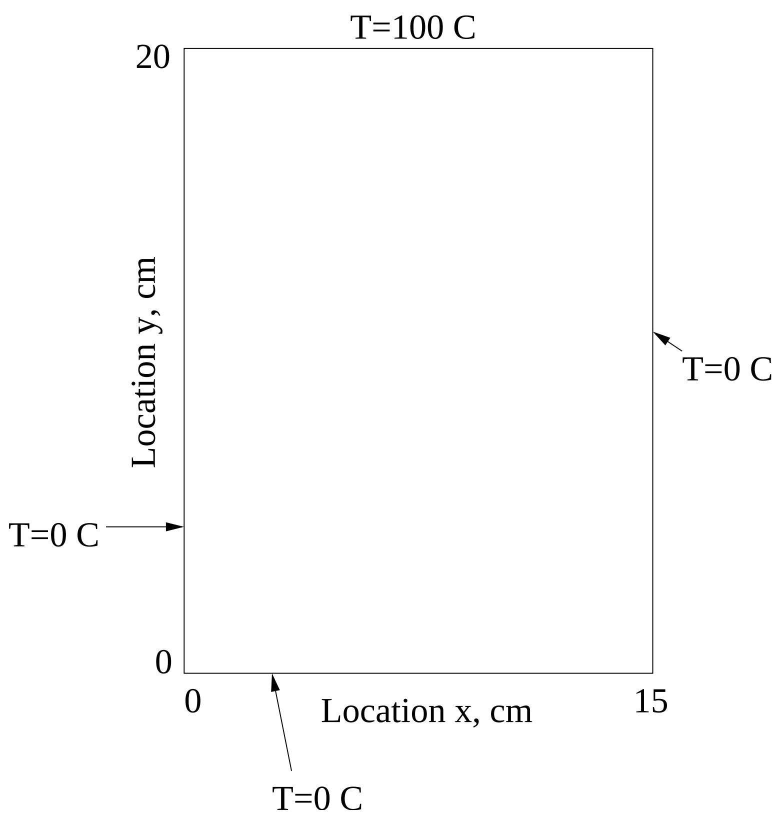

Figure 2: Two-dimensional heat conduction problem.

problem, the exact solution of the linear partial differential equation can be obtained by assuming $T(x, y) = X(x)Y(y)$ and using separation of variables, which yields

$$T_{exact}(x, y) = \frac{400}{\pi} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{n \sinh(n\pi h/w)} \sinh(\frac{n\pi y}{w}) \sin(\frac{n\pi x}{w}) \tag{12}$$

Your task for this project is to solve the Laplace equation numerically (to obtain $T(x, y)$) for the given domain and the boundary conditions. Specifically you are asked to perform the following

- Obtain a finite difference equation (FDE) using second order central-difference finite difference approximations of the partial derivatives for the Laplace's equation. Assume that the rectangular domain is equally spaced in x and y directions with increments of Δx and Δy, respectively. **For now also assume that:** $\Delta x \neq \Delta y$. Write your finite difference equation in a compact form defining a *grid aspect ratio*, $\beta = \Delta x/\Delta y$..

- Discuss the nature of the algebraic FDE you have obtained and propose an iterative solution technique for the solution of the FDE.

- Apply the given boundary conditions and determine the numerical solution of the temperature values using a:
  - 4×5 grid
  - 31×41 grid
  - 121×161 grid

- For each numerical solution plot the lines of constant temperature over the computational domain (use contour plot capability of Matlab) together with the exact solution and observe how the grid density affects the numerical solution.

- For each numerical solution calculate the global error defined by

$$Error = \frac{\sqrt{\sum_{i=1}^{I} \sum_{j=1}^{J} \left[(T_{exact} - T_{numerical})_{ij} \right]^2}}{I \times J} \tag{13}$$

  where I and J are the number of grid points in x and y directions. Compare the error for each grid used.

- Determine the order of your FDE.

- For each numerical solution compute the global residual at each iteration and plot log(Global Residual) vs. iteration number to investigate the efficiency of the iterative solution procedure you have picked.

- Discuss how the **efficiency** and the **accuracy** of your numerical solution can be improved.

Please include your plots, tables and discussion in your report.