Why Zeilberger's Algorithm Works

Try telescoper of order r on a hg. input $f(n,k)$, i.e., apply Gosper's algorithm to $\bar{f}(n,k) := \sum_{i=0}^{r} p_i(n)\, f(n+i,k)$.

$$\frac{\bar{f}(n,k+1)}{\bar{f}(n,k)} =$$

Let $\dfrac{f(n+1,k)}{f(n,k)} = u(n,k) = \dfrac{u_1(n,k)}{u_2(n,k)}$

and $\dfrac{f(n,k+1)}{f(n,k)} = v(n,k) = \dfrac{v_1(n,k)}{v_2(n,k)}$

$$= \frac{\sum_{i=0}^{r} p_i(n)\cdot f(n+i,k+1)}{\sum_{i=0}^{r} p_i(n)\cdot f(n+i,k)}$$

$$= \frac{\sum_{i=0}^{r} p_i(n)\left(\prod_{j=0}^{i-1} u(n+j,k+1)\right)\cdot f(n,k+1)}{\sum_{i=0}^{r} p_i(n)\left(\prod_{j=0}^{i-1} u(n+j,k)\right)\cdot f(n,k)}$$

$$= \frac{\sum_{i=0}^{r} p_i(n)\left(\prod_{j=0}^{i-1} u_1(n+j,k+1)\right)\left(\prod_{j=i}^{r-1} u_2(n+j,k+1)\right)\left(\prod_{j=0}^{r-1} u_2(n+j,k)\right) v_1(n,k)}{\sum_{i=0}^{r} p_i(n)\left(\prod_{j=0}^{i-1} u_1(n+j,k)\right)\left(\prod_{j=i}^{r-1} u_2(n+j,k)\right)\left(\prod_{j=0}^{r-1} u_2(n+j,k+1)\right) v_2(n,k)}$$

$$= \frac{c_0(k+1)}{c_0(k)}\cdot w(k) \qquad \text{Note: } w(k) \text{ has no } p_i.$$

Compute Gosper form: $w(k) = \dfrac{a(k)\, c_1(k+1)}{b(k)\, c_1(k)}$

Let $c(k) = c_0(k)\cdot c_1(k)$, then $\dfrac{a(k)\, c(k+1)}{b(k)\, c(k)}$ is a Gosper form for $\bar{f}(n,k)$.

Gosper equation: $a(k)\, x(k+1) - b(k-1)\cdot x(k) = c(k)$

Proof of Apagodu-Zeilberger

Let $\bar{h}(n,k) = \dfrac{\left(\prod_{j=1}^{A} (\alpha_j)_{a_j' n + a_j k}\right)\left(\prod_{j=1}^{B} (\beta_j)_{b_j' n - b_j k}\right)}{\left(\prod_{j=1}^{C} (\gamma_j)_{c_j'(n+L)+c_j k}\right)\left(\prod_{j=1}^{D} (\delta_j)_{d_j'(n+L)-d_j k}\right)} \cdot z^k$

Now $\dfrac{\bar{h}(n,k+1)}{\bar{h}(n,k)} = \dfrac{\left(\prod_{j=1}^{A} (\alpha_j + a_j' n + a_j k)_{a_j}\right)\left(\prod_{j=1}^{D} (\delta_j + d_j(n+L) - d_j k - d_j)_{d_j}\right) \cdot z}{\left(\prod_{j=1}^{B} (\beta_j + b_j' n - b_j k - b_j)_{b_j}\right) \cdot \left(\prod_{j=1}^{C} (\gamma_j + c_j'(n+L) + c_j k)_{c_j}\right)}$

$=: \dfrac{u(n,k)}{v(n,k)}$

Let $g(n,k) = v(n,k-1) \cdot x(k) \cdot \bar{h}(n,k)$ and plug it into the telescopic rel.:

$$\sum_{i=0}^{r} p_i(n)\, p(n+i,k)\, h(n+i,k) = v(n,k)\, x(k+1)\, \bar{h}(n,k+1) - v(n,k-1)\, x(k)\, \bar{h}(n,k)$$

$$\xRightarrow[\text{by } \bar{h}(n,k)]{\text{divide}} \underbrace{\sum_{i=0}^{r} p_i(n)\, p(n+i,k)\, \frac{h(n+i,k)}{\bar{h}(n,k)}}_{=:\, w(n,k)} = u(n,k) \cdot x(k+1) - v(n,k-1) \cdot x(k)$$

Note that $w(n,k)$ is a polynomial, since

$$\frac{h(n+i,k)}{\bar{h}(n,k)} = \left(\prod_{j=1}^{A} (\alpha_j + a_j' n + a_j k)_{i \cdot a_j'}\right) \cdot \left(\prod_{j=1}^{B} (\beta_j + b_j' n - b_j k)_{i \cdot b_j'}\right)$$
$$\times \left(\prod_{j=1}^{C} (\gamma_j + c_j'(n+i) + c_j k)_{(L-i) c_j'}\right) \cdot \left(\prod_{j=1}^{D} (\delta_j + d_j'(n+i) - d_j k)_{(L-i) d_j'}\right)$$

Make an ansatz for $x(k) = \sum_{i=0}^{s} x_i k^i$ where

$$s := \deg_k(w) - \max(\deg_k(u), \deg_k(v))$$

Coefficient comparison w.r.t. k yields

- $\deg_k(w) + 1$ equations in the
- $(r+1) + (s+1)$ unknowns

The condition #unknowns > #equations yields

$$r + s + 2 \geq \deg_k(w) + 2 \implies r \geq \max(\deg_k(u), \deg_k(v)).$$

But note that

$$\deg_k(u) = \sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j \quad \text{and} \quad \deg_k(v) = \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j$$