Computer algebra in a combinatorialist's life

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In this talk

Computer algebra in the solution of a counting problem

I. From objects to numbers

II. Guess

III. Prove

IV. Simplify

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I. From objects to numbers

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Examples

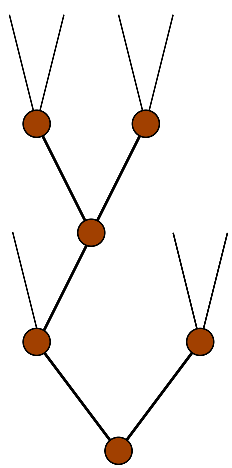
Questions

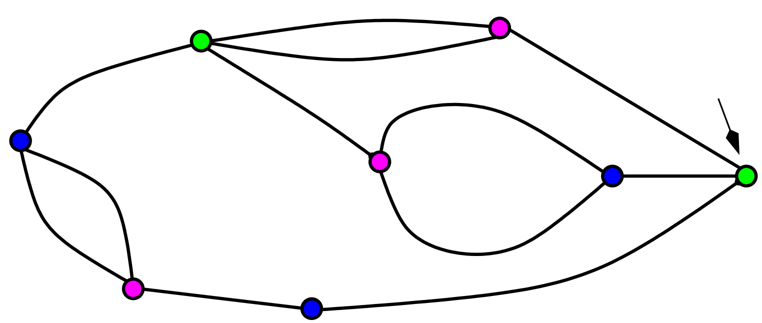
Three objectives

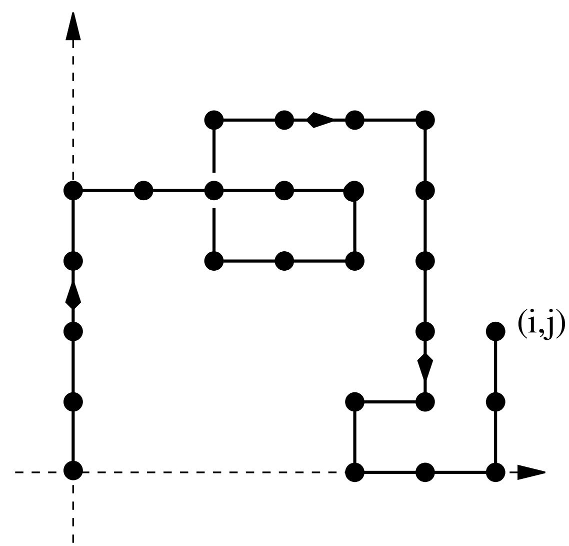
I. From objects to numbers

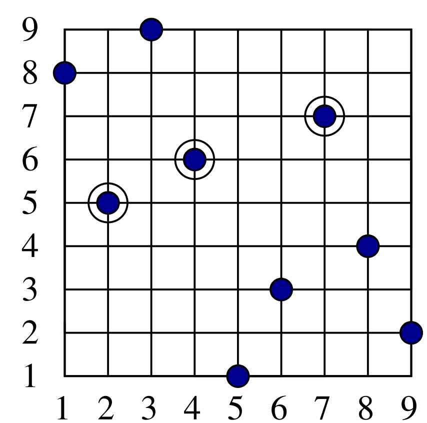
Setting

Let A be a set of discrete objects, equipped with an integer size such that the number a(n) of objects of size n is **finite** for any n.





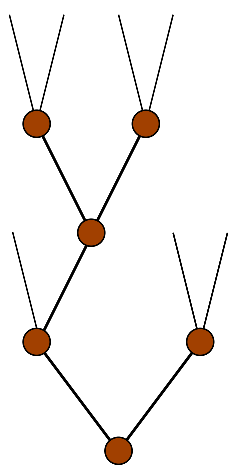


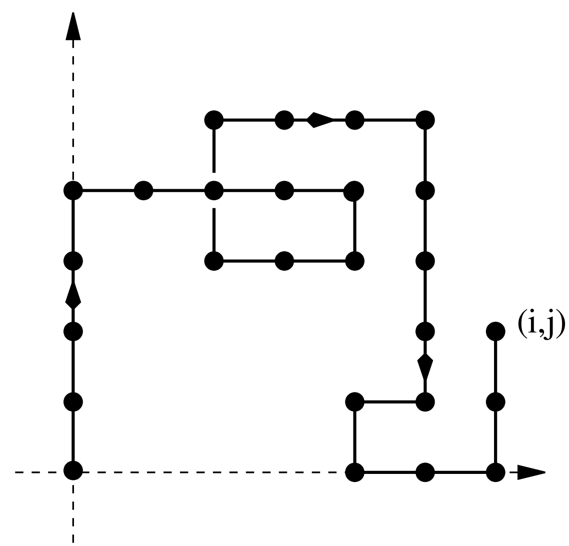


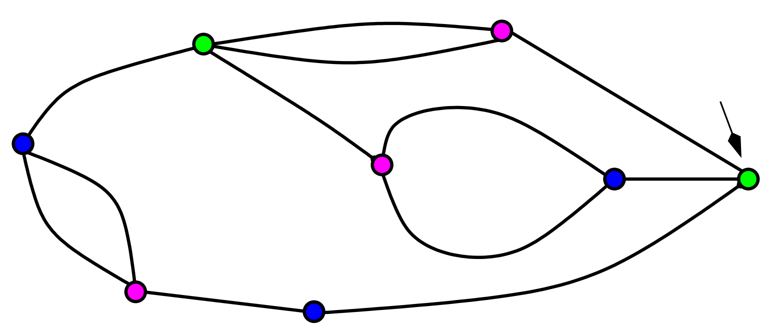
Setting

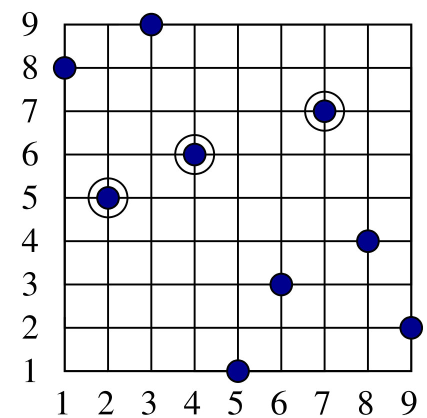
Let A be a set of discrete objects, equipped with an integer size such that the number a(n) of objects of size n is finite for any n.

Objective: generate a(1), a(2), ... , a(N) for N large.





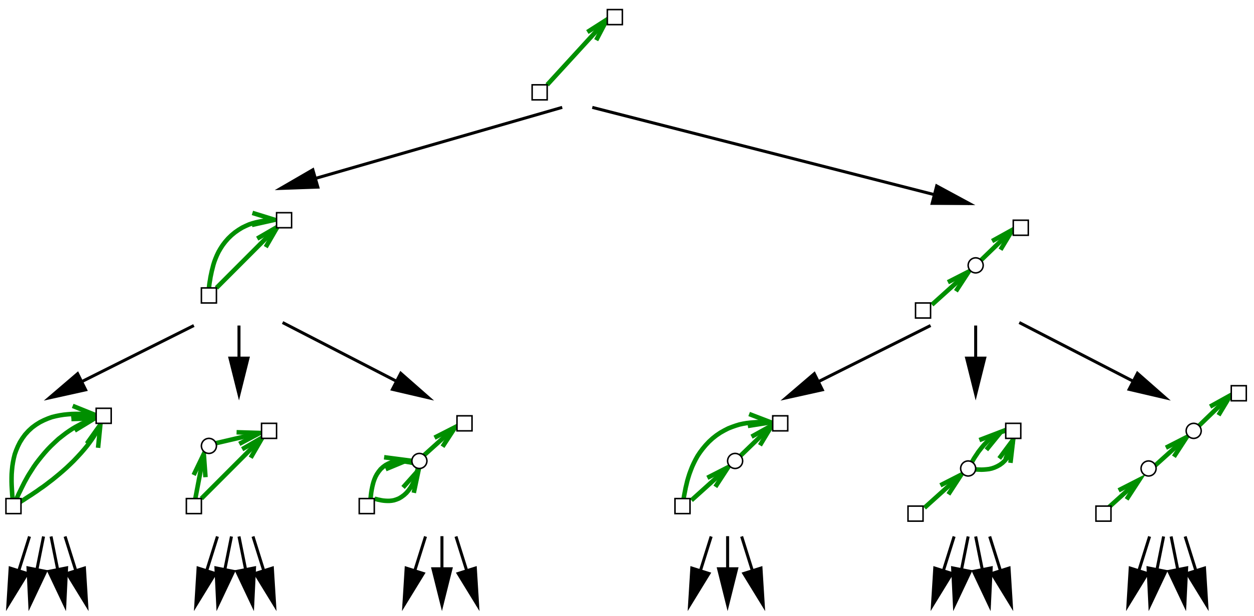


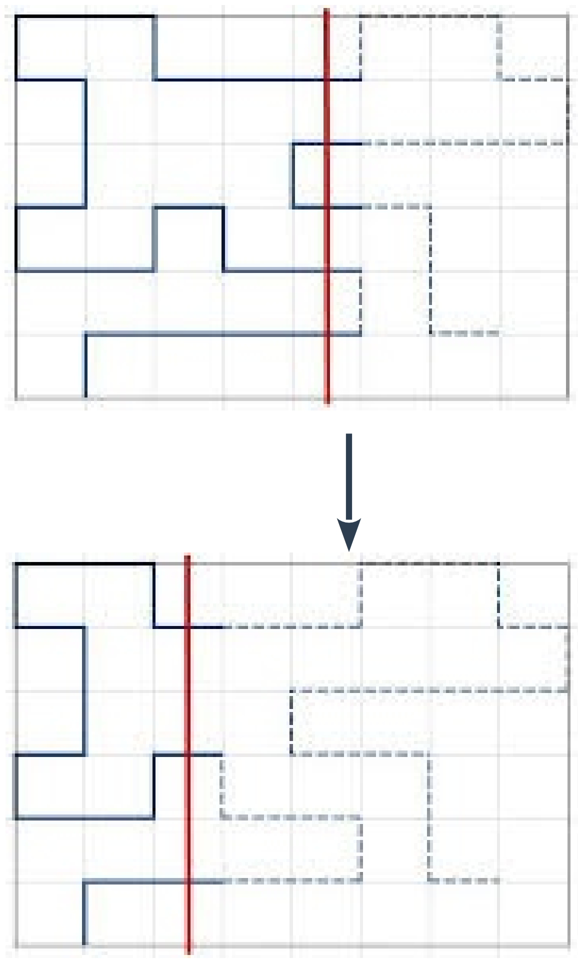


Case 1: when no recurrence relation is known

Generate numbers (and often objects) by any possible recursive construction

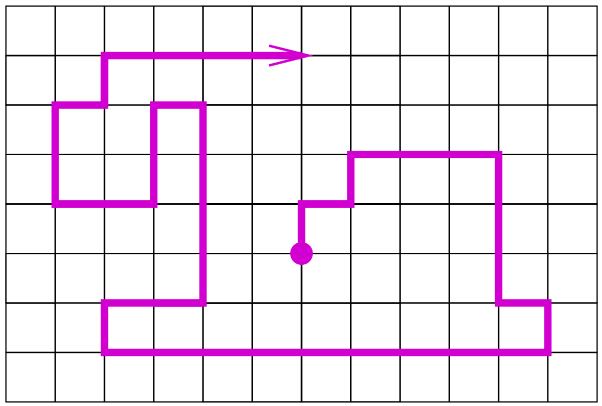
- **Generating trees:** add a step, an edge, a node...
- **Transfer matrices:** add a layer



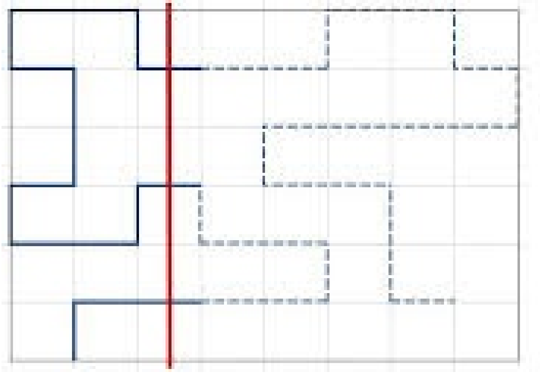


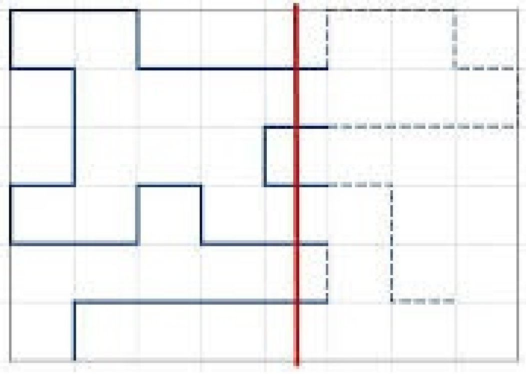
Case 1: when no recurrence relation is known

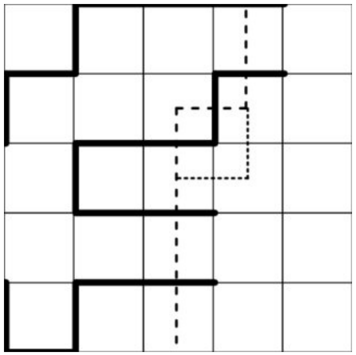
Self-avoiding walks



[Enting, Guttmann]







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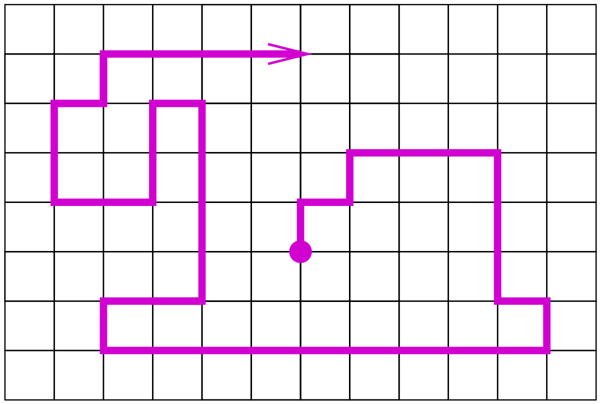
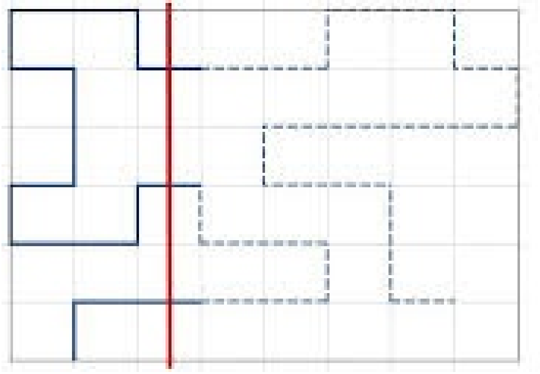
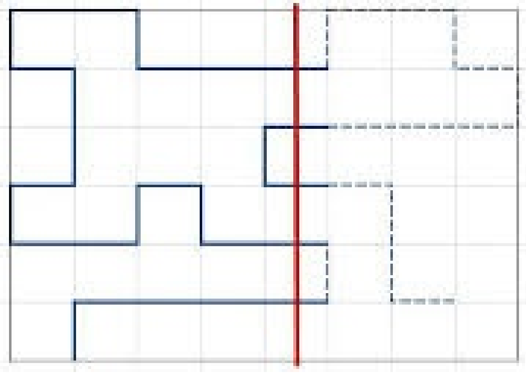


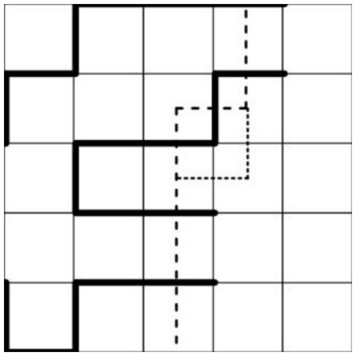
Question: is there a sub-exponential algorithm that computes the number of self-avoiding walks of length n?

[Enting, Guttmann]

So far, n=79 [Jensen 13(a)]







Case 2: with a recurrence relation

... often encoded as a functional equation for the associated generating function:

$$A(t) \equiv A := \sum_{n \geq 0} a(n) t^n = \sum_{o \in \mathcal{A}} t^{|o|}$$

Multivariate enumeration: record additional statistics

$$A(t; x, y) \equiv A(x, y) := \sum_{n,i,j \geq 0} a(n; i, j) t^n x^i y^j$$

Case 2: with a recurrence relation

… often encoded as a functional equation for the associated **generating function:**

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A rich zoo of equations

Functional equations: our pet animals

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

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Several variables: one DE per variable

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Several variables: one DE per variable

More exotic animals

Substitutions: set partitions

$$A(t) = 1 + \frac{t}{1-t} A\left(\frac{t}{1-t}\right)$$

q-Equations: Dyck paths by length (t) and area (q)

$$A(t;q) = 1 + tqA(tq;q)A(t;q)$$

More exotic animals

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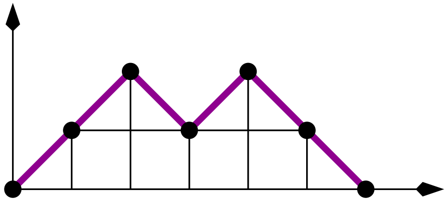

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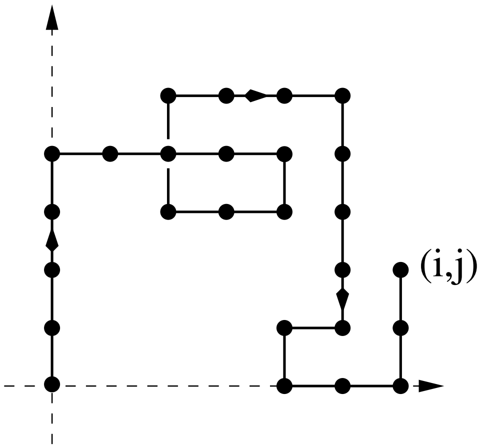

Discrete derivatives: quadrant walks

$$Q(x,y) = 1 + t(x+y)Q(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

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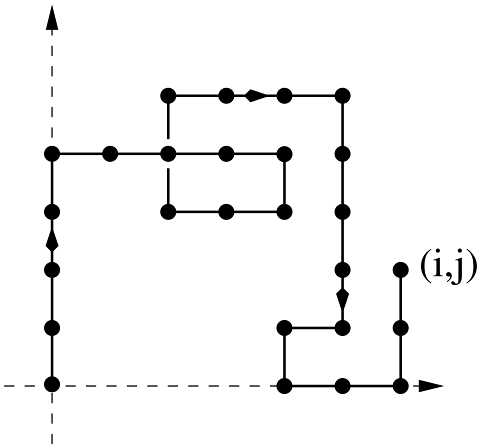

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or $\left(1 - t\left(x + y + \frac{1}{x} + \frac{1}{y}\right)\right) xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$

x, y: *catalytic* variables

Hybrids

Discrete derivatives and q-equations: Tamari intervals on Dyck paths

[mbm, Fusy, Préville-Ratelle 11]

$$A(x,q) = 1 + tqA(x,q)\frac{A(xq,q) - A(1,q)}{xq-1}$$

Hybrids

Discrete derivatives and q-equations: Tamari intervals on Dyck paths

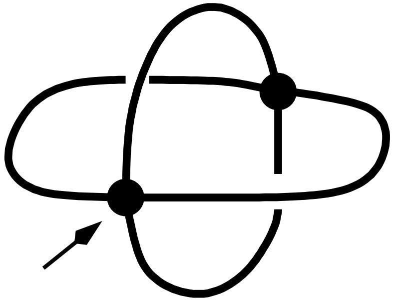
[mbm, Fusy, Préville-Ratelle 11]

$$A(x,q) = 1 + tqA(x,q)\frac{A(xq,q) - A(1,q)}{xq - 1}$$

Substitutions in "catalytic" variables: bipartite quadrangulations by edges (t) and vertices (x), arbitrary genus [Louf 21]

$$2(1+2D)DA(x) = (A(x+1) + A(x-1) - 2A(x) - 2)(1+2D)A(x)$$

where D= t d/dt and A(x)=A(t,x).



With a recurrence relation/fixed point equation

- Coefficients in **polynomial time**
- Newton iteration [Pivoteau, Salvy & Soria 12]
- Work with the recurrence relation? With the functional equation?
- Work modulo primes?

Produce numbers: why?

- Predict asymptotic behaviour

Example: 1324-avoiding permutations [Conway & Guttmann 15]

$$a(n) \sim \kappa \alpha^n \beta^{\sqrt{n}} n^\gamma$$

(50 terms known)

$$\alpha \simeq 11.6 \quad \beta \simeq 0.04 \quad \gamma \simeq -1.1$$

- Conjecture (simpler) recurrence relations or functional equations

Interlude: Combinatorial exploration

An automatized construction of recurrence relations for some combinatorial classes.

"The Combinatorial Exploration framework produces rigorously verified combinatorial specifications for families of combinatorial objects. These specifications then lead to generating functions, counting sequence, polynomial-time counting algorithms, random sampling procedures, and more."

[Albert, Bean, Claesson, Nadeau, Pantone & Ulfarsson 22(a)]

Interlude: Combinatorial exploration

An automatized construction of recurrence relations for some combinatorial classes.

Ex. 1234-avoiding permutations

[Albert, Bean, Claesson, Nadeau, Pantone & Ulfarsson 22(a)]

[PermPAL database]
Permutation Pattern Avoidance Library

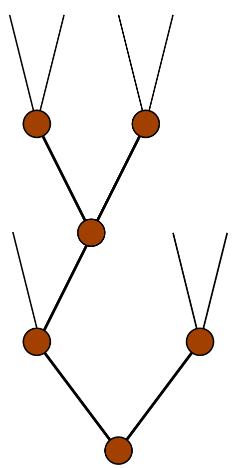
$$
\begin{aligned}
F_0(x) &= F_1(x) + F_2(x)\\
F_1(x) &= 1\\
F_2(x) &= F_{15}(x)\,F_3(x)\\
F_3(x) &= F_4(x,1)\\
F_4(x,y) &= F_1(x) + F_{16}(x,y) + F_5(x,y)\\
F_5(x,y) &= F_{10}(x,y)\,F_6(x,y)\\
F_6(x,y) &= F_7(x,1,y)\\
F_7(x,y,z) &= F_8(x,yz,z)\\
F_8(x,y,z) &= F_1(x) + F_{11}(x,y,z) + F_{13}(x,y,z) +\\
F_9(x,y,z) &= F_{10}(x,y)\,F_8(x,y,z)\\
F_{10}(x,y) &= yx\\
F_{11}(x,y,z) &= F_{10}(x,z)\,F_{12}(x,y,z)\\
F_{12}(x,y,z) &= \frac{-zF_7(x,1,z) + yF_7\left(x,\frac{y}{z},z\right)}{-z+y}\\
F_{13}(x,y,z) &= F_{14}(x,y,z)\,F_{15}(x)\\
F_{14}(x,y,z) &= \frac{zF_8(x,y,z) - F_8(x,y,1)}{-1+z}\\
F_{15}(x) &= x\\
F_{16}(x,y) &= F_{15}(x)\,F_{17}(x,y)\\
F_{17}(x,y) &= \frac{yF_4(x,y) - F_4(x,1)}{-1+y}
\end{aligned}
$$

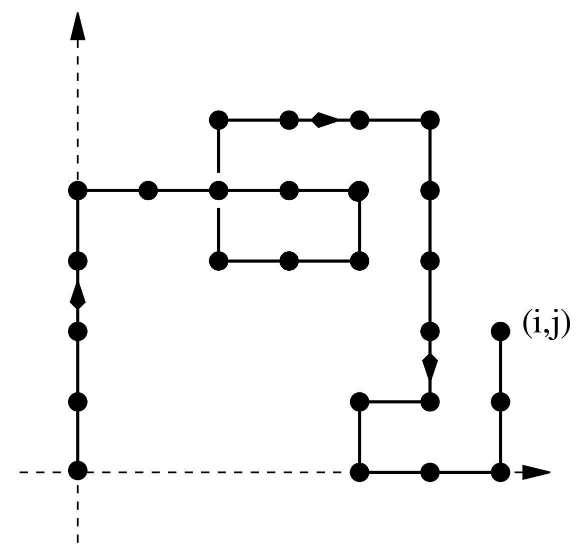
II. Guess

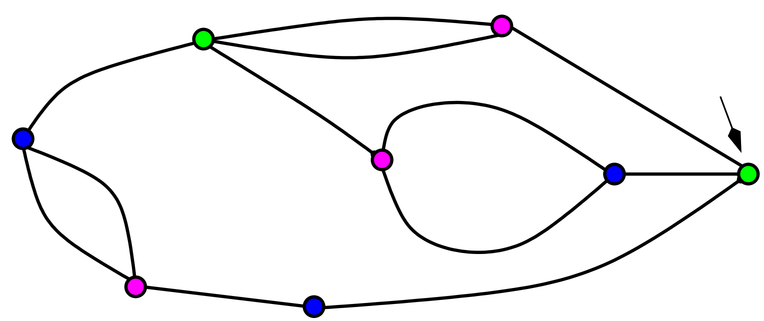
Setting

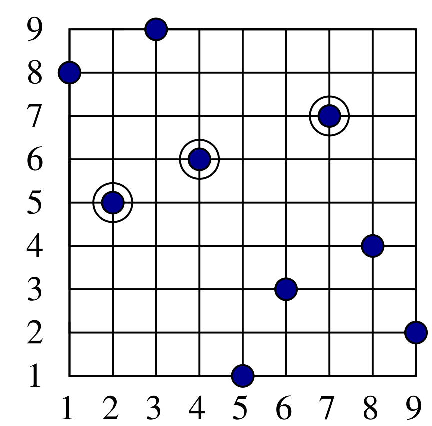
Let a(n) be the number of objects of size n in the set $\mathcal{A}$.

Objective: guess a recurrence relation for a(n) from the knowledge of a(1), a(2), ... , a(N).









Hermite-Padé approximants for linear relations

Given the first coefficients $a_i(0)$, $a_i(1)$, ..., $a_i(n)$ of k series $A_i(t)$, i=1, ..., k, find polynomials $P_1(t)$, ..., $P_k(t)$ of small degree such that

$$P_1 A_1 + \cdots + P_k A_k = \mathcal{O}(t^{n+1})$$

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⇒ Needs about n = kd coefficients in each series to guess an equation with $\deg(P_i) < d$.

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Example: a quadratic q-equation of order 2 corresponds to k=10 series

$$1, A(t), A(tq), A(tq^2),$$
$$A(t)^2, A(tq)^2, A(tq^2)^2, A(t)A(tq), A(t)A(t^2q), A(tq)A(t^2q).$$

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Same for an ODE of order e and degree δ.

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numapprox[hermite_pade]

Special types of functional equations

- Guess **polynomial equations** (degree δ): linear relation between

$$1, A, \ldots, A^{\delta}$$

gfun[seriestoalgeq] [Salvy 94 →]

- Guess **linear differential equations** (order e): linear relation between

$$1, A, A', \ldots, A^{(e)}$$

gfun[seriestodiffeq]

- Guess **polynomial differential equations** (order e, degree δ):

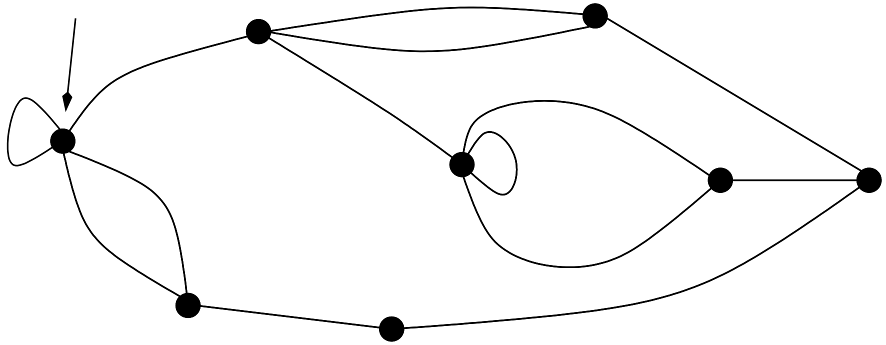
requires $\binom{\delta+e+1}{\delta}$ series.

FPS[delta2guess] [Teguia 23, Pantone 24+]

Example 1: in the 60's, Tutte and planar maps

Equation with a discrete derivative: planar maps by edges (t) and degree of the root vertex (x):

$$A(x) = 1 + tx^2 A(x)^2 + tx \frac{A(x) - A(1)}{x - 1}.$$



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Algebraic guess for A(1):

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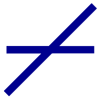
⇒ a guess for A(x) as an algebraic series of degree 4:

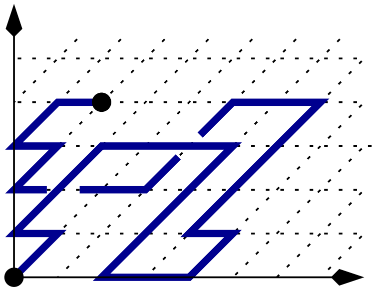
$$\bar{A}(x) = 1 + tx^2 \bar{A}(x)^2 + tx\frac{\bar{A}(x) - \bar{A}_1}{x-1}.$$

Example 2: Gessel's quadrant walks

Equation with two discrete derivatives:

$$Q(x,y) = 1 + t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right) Q(x,y) - t\left(\frac{1}{x} + \frac{1}{xy}\right) Q(0,y) - \frac{t}{xy}\left(Q(x,0) - Q(0,0)\right)$$



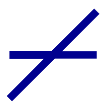


Example 2: Gessel's quadrant walks

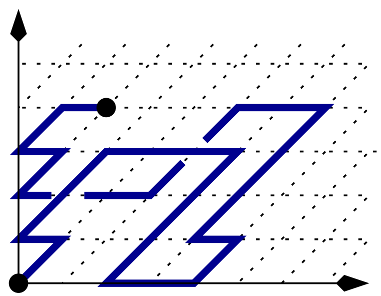
Equation with two discrete derivatives:

$$Q(x,y) = 1 + t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right)Q(x,y)$$
$$-t\left(\frac{1}{x} + \frac{1}{xy}\right)Q(0,y) - \frac{t}{xy}\left(Q(x,0) - Q(0,0)\right)$$

Gessel's ex-conjecture (~2000)



$$Q(0,0) = \sum_{n\geq 0} 16^n \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} t^{2n}$$



with $(a)_n = a(a+1)\cdots(a+n-1)$.

Example 2: Gessel's quadrant walks

Equation with two discrete derivatives:

$$Q(x,y) = 1 + t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right)Q(x,y) - t\left(\frac{1}{x} + \frac{1}{xy}\right)Q(0,y) - \frac{t}{xy}\left(Q(x,0) - Q(0,0)\right)$$

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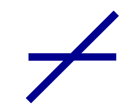

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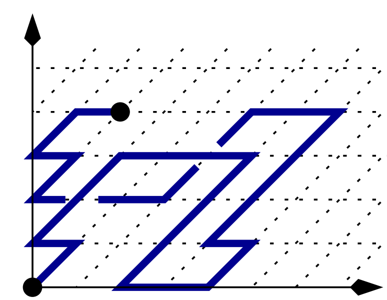


with $(a)_n = a(a+1)\cdots(a+n-1)$.

Later... Q(0,0) satisfies an **polynomial equation** Pol(t,Q)= 0,

of bidegree (7,8) [Bostan & Kauers 10]

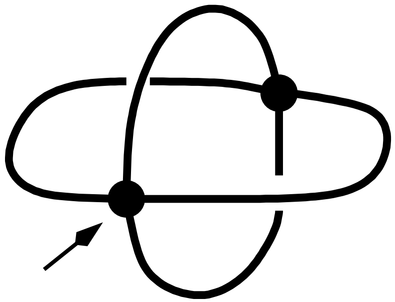
(+ Proof of the algebraicity of Q(x,y))

Example 3: bipartite quadrangulations, any genus

Substitutions in "catalytic" variables: [Louf 21]

$$2(1+2D)DA(x) = (A(x+1)+A(x-1)-2A(x)-2)(1+2D)A(x)$$

where D= t d/dt (plus value at x=1).



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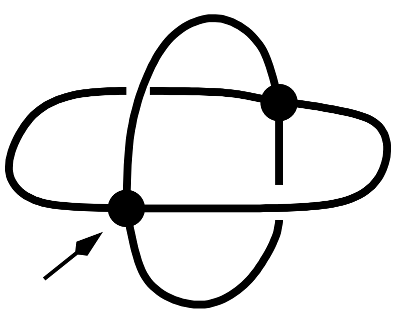
Guess: a quadratic, third order ODE in t

$$(1+D)A = t\,(3t+4x)\,A + t\,(11t+8x)\,DA + 12t^2D^{(2)}A + 4t^2D^{(3)}A$$
$$+3t^2A^2 + 12t^2A(DA) + 12t^2(DA)^2 + x^2$$

Proof [Carrell & Chapuy 15]



III. Prove

Setting

So far: a functional equation (E_1) for $A(t,x,y\ldots)$, possibly wild

Guessed: a simpler equation (E_2) for $A(t,x,y\ldots)$

Two ingredients:

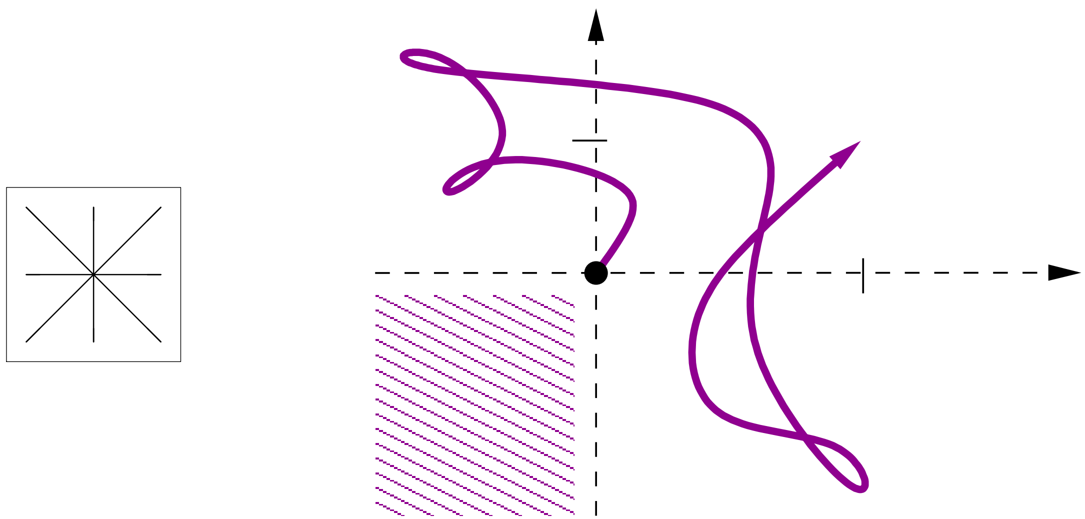
- **Uniqueness** of solution in (E_1)
- **Closure properties** of a class containing (E_2)

Example 1: a big algebraic system

King walks avoiding the negative quadrant [mbm & Wallner 23]

(E_1) A system of **4 polynomial equations** in 4 series R_0, R_1, B_1, B_2

Degree in	R_0	R_1	B_1	B_2	t	Number of terms
Eq. 1	5	3	1	1	7	72
Eq. 2	6	4	2	2	7	132
Eq. 3	5	5	2	2	9	192
Eq. 4	6	6	3	3	10	276



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(E_2) **Guessed minimal polynomials** for all four series, and **rational expressions** in terms of two "simple" series T and U (deg. 12, 24).

Generating function	Degree in GF	Degree in t	Number of terms
R_0	24	36	323
R_1	24	36	623
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thanks to Mark van Hoeij !

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Plug in (E_1) and check by reduction mod minimal polynomials of T and U.

Example 2: in the 60's, Tutte and planar maps

Planar maps by edges (t) and degree of the root vertex (x):

$$A(x) = 1 + tx^2 A(x)^2 + tx \frac{A(x) - A(1)}{x - 1}.$$

Uniqueness: there exists a unique solution $A(x)$ that is a formal power series in t. Its coefficients are polynomials in x.

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$$A(1) = \bar{A}_1 := \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}.$$

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To do: prove that $\bar{A}(x)$ has polynomial coeffs. in x, so that $\bar{A}_1 = \bar{A}(1)$.

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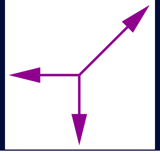
$$\bar{A}(x) = 1 + tx^2 \bar{A}(x)^2 + tx \frac{\bar{A}(x) - \bar{A}_1}{x-1},$$

or

$$(x-1)\left(\bar{A}(x) - 1 - tx^2 \bar{A}(x)^2\right) = tx\left(\bar{A}(x) - \bar{A}_1\right).$$

To do: prove that $\bar{A}(x)$ has **polynomial coeffs**. in x, so that $\bar{A}_1 = \bar{A}(1)$.

Example 3: Kreweras' walks in the quadrant



Two discrete derivatives:

$$\left(xy - t\left(x + y + x^2y^2\right)\right)Q(x,y) = xy - A(x) - A(y)$$

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[Bostan & Kauers 10]

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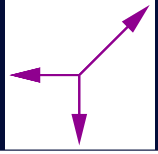

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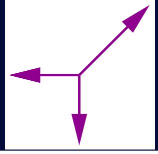

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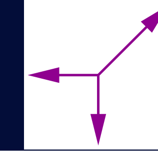

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To do:

- Prove that $\bar{A}(x)$ has **polynomial coefficients** in x.
- Prove that (E_1) holds for $\bar{A}(x)$ by computing a polynomial annihilating the rhs of (E_1), and checking first coefficients.

[Bostan & Kauers 10]

Example 4: more walks in the quadrant

Two discrete derivatives:

$$K(x,y)Q(x,y) = xy - A(x) - B(y)$$

where $A(x) \approx Q(x,0)$ and $B(y) \approx Q(0,y)$.

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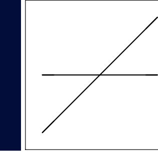

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Uniqueness: $A(x)$, $B(y)$ are the series in t with **polynomial coefficients** solving

$$\begin{aligned} 0 &= X(y)y - A(X(y)) - B(y), \\ 0 &= xY(x) - A(x) - B(Y(x)), \end{aligned} \qquad (E_1)$$

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- Prove that the guessed solutions have **polynomial coefficients**
- Prove that (E_1) holds for the guessed series by **polynomial elimination** and checking first coefficients.

[Bostan & Kauers 10]

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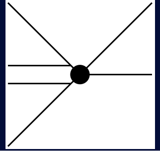

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Uniqueness: $A(x)$, $B(y)$ are the series in t with **polynomial coefficients** solving

$$0 = X(y)y - A(X(y)) - B(y), \qquad (E_1)$$
$$0 = xY(x) - A(x) - B(Y(x)),$$

where $X(y)$ (resp. $Y(x)$) is the only root of K that is a formal series in t.

Guess (E_2): **differential ideals** (in ∂t and ∂x, resp. ∂y) for $A(x)$ and $B(y)$

To do:

- Prove that the guessed solutions have **polynomial coefficients**
- Prove that (E_1) holds for the guessed series by **differential elimination** and checking first coefficients.

[Bostan, mbm, Kauers & Melczer 16]

IV. Simplify

Setting

Given a series A(t,x,y...) and a defining functional equation (algebraic, D-finite, D-algebraic), get a **better understanding** of A.

- Find a simple description of A
- Understand the properties of A
- Determine singularities, asymptotics
- ...

Simplifying in the algebraic world

Classical tools: polynomial factorization, resultants, Gröbner bases...

Given a minimal polynomial P(t,A)=0:

- genus, **rational parametrization** (if genus 0), Weierstrass form for (hyper)elliptic solutions (algcurves)

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Question: Given an algebraic series A(t;x,y...) given by its minimal polynomial over $K=\mathbb{Q}(t,x,y...)$, find a "simple" series generating K(A).

Same question for the subfields between K and K(A).

A small example: properly 3-coloured planar maps

How does one go from this **polynomial of bidegree** (6, 4) in (t,A):

$$-12500A^4t^6+24t^4\left(1000t-71\right)A^3-2t^2\left(3600t^3+7216t^2-1020t+39\right)A^2$$
$$-\left(864t^5-9040t^4-1712t^3+536t^2-42t+1\right)A-40t+540t^2-2720t^3+432t^4+1=0$$

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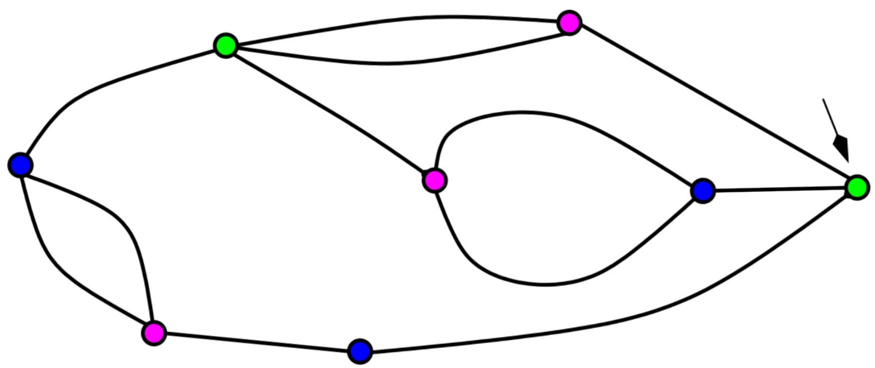

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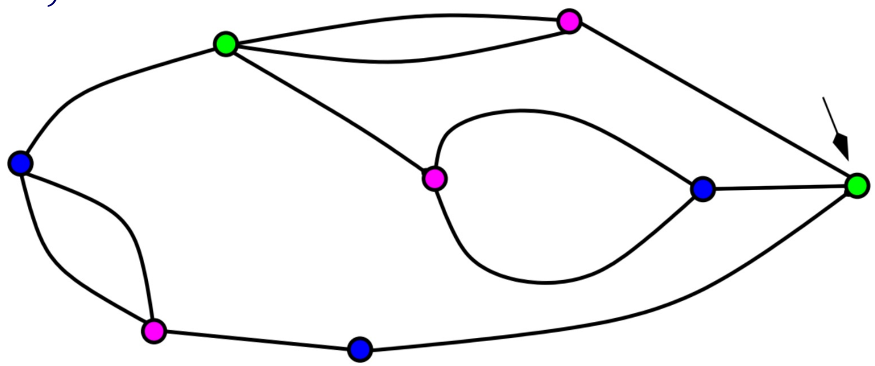
to...

$$A=2T-\frac{T^2(1+2T)(1+2T^2+2T^4)}{(1-2T^3)^3}$$

with $T=t\dfrac{(1+2T)^3}{1-2T^3}$?



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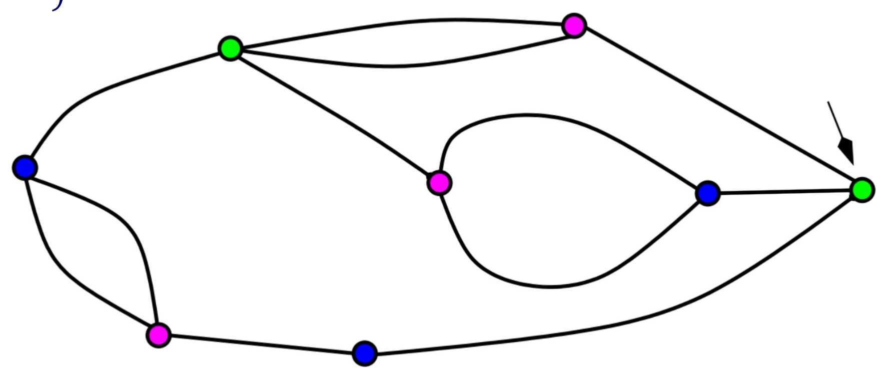

algcurves[parametrization] gives *some* parametrization

$$t=\frac{S^3-6S^2+12S-10}{S^3\left(S-2\right)}$$

(genus 0)

[Bernardi & mbm 09]

A bigger example: king walks avoiding a quadrant

How does one go from this **polynomial of bidegree (24, 12)** in (t,A):

$$
\begin{aligned}
&(154468234973274264443289 6t^6 + 285995642970319677731635 2t^5 + 137174721006404628076953 6t^4 \\
&+261868606648367056551936t^3 + 206859122755182935064576t^2 + 98613397010845517414 4t + 6559233932686417 92)\, A^{12} \\
&+(1190883818143791028843315 2t^8 + 274918428694845126192660 48t^7 + 220661689984043449667420 16t^6 \\
&+945637884496995200040960 0t^5 + 357731710624347699231129 6t^4 + 725362067373633286668288t^3 \\
&+123324842335532119326720t^2 + 42616279894082612428 8t + 2498755783880540 16)\, A^{11} \\
&+ [\cdots] \\
&-2\,(1099511627776t^{16} + 4947802324992t^{15} + 8908835913728t^{14} + 8010919313408t^{13} + 3551066587136t^{12} \\
&+601824952320t^{11} + 128619544576t^{10} + 260050427904t^9 + 187250317568t^8 + 66799107968t^7 + 13529493584t^6 \\
&+1545216528t^5 + 86381746t^4 + 1570596t^3 + 920t^2 + 38t - 1)\,(4t+1)^4\,(8t-1)^4\,A \\
&+ 3t^2\,(t+1)^2\,(4t+1)^6\,(8t-1)^{10} = 0
\end{aligned}
$$

to...

[mbm & Wallner 23]

A bigger example: king walks avoiding a quadrant

How does one go from this **polynomial of bidegree (24, 12)** in (t,A):

$$\begin{aligned}
&\left(15446823497327426444432896t^6 + 28599564297031967773163 52t^5 + 137174721006404628076953 6t^4\right.\\
&\left.+2618686066483670565519 36t^3 + 2068591227551829350645 76t^2 + 98613397010845517414 4t + 655923393268641792\right) A^{12}\\
&+\left(11908838181437910288433152t^8 + 27491842869484512619266048t^7 + 22066168998404344966742016t^6\right.\\
&+9456378844969952000409600t^5 + 3577317106243476992311296t^4 + 725362067373633286668288t^3\\
&\left.+123324842335532119326720t^2 + 4261627989408261242 88t + 2498755783880540 16\right) A^{11}\\
&+ [\cdots]\\
&-2\left(1099511627776t^{16} + 4947802324992t^{15} + 8908835913728t^{14} + 8010919313408t^{13} + 3551066587136t^{12}\right.\\
&+601824952320t^{11} + 128619544576t^{10} + 260050427904t^9 + 187250317568t^8 + 66799107968t^7 + 13529493584t^6\\
&\left.+1545216528t^5 + 86381746t^4 + 1570596t^3 + 920t^2 + 38t - 1\right)(4t+1)^4(8t-1)^4 A\\
&+ 3t^2(t+1)^2(4t+1)^6(8t-1)^{10} = 0
\end{aligned}$$

to...

$$A = 3(1-8t)\,\frac{T^2(1+4T+T^2)(T^2-1)(1+2T)}{2(1-3T^2-4T^3)^3(1+4T-2T^3)},$$

with

$$\frac{T\left(T^2+T+1\right)\left(1+3T-T^3\right)^3}{\left(T^2+4T+1\right)\left(1-3T^2-4T^3\right)^3} = \frac{t(1+t)}{1-8t}.$$

(genus 4)

[mbm & Wallner 23]

A recurrent question: dependence on parameters

Subfields. If $P(t,A)=0$, what are the subfields of $\mathbb{Q}(t,A)$?

Example. Starting from $P(t,a)$ of bidegree $(24, 12)$, the command

evala(Subfields(subs(t=10^k, P(t,a)),4))

yields a subfield of degree 4 over $\mathbb{Q}(t)$ for each value of t.

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E.g, for t=10,

$$RootOf\left(59059089842541\_Z^4 + 40291825844958\_Z^3 - 14363433497042654706\_Z^2 + 3848807433734406268482\_Z - 290439563039835597485204\right)$$

but the coefficients need not be polynomials in t.

⇒ **Reconstruction?**

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$$\frac{Z}{(1+Z)(1-3Z)^3} = \frac{t(1+t)}{1-8t}$$

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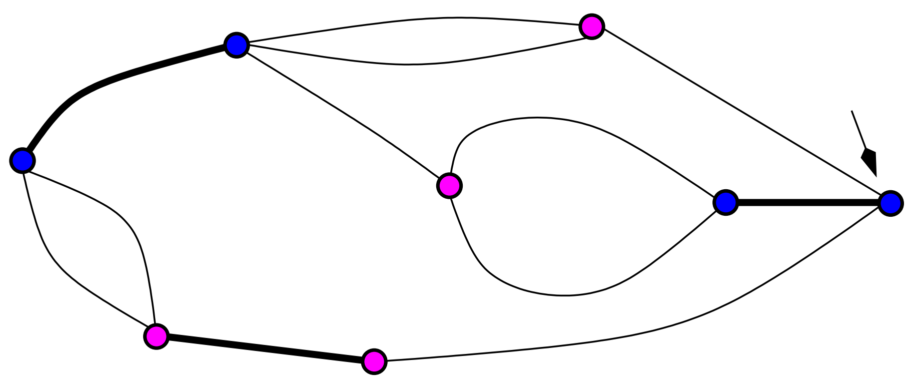
$$314928x^7t^9(x+1)^6A^6 - 34992t^7x^5(x+1)^4\left(36tx^3+54tx^2-x^2+18tx-1\right)A^5 + [\cdots$$



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[mbm & Bernardi 09]

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$$\frac{t}{1782\,(22T-41229)} = \frac{234256T^4 - 1793975040T^3 + 5149664707176T^2 - 6542185481249616T + 309152728386271l2}{\left(10648T^3 - 33989868T^2 + 13112460306T + 24152458116951\right)^2}$$

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$$T = t\,\frac{\left(1+3\,x\,T-3\,x\,T^2-x^2T^3\right)^2}{1-2\,T+2\,x^2T^3-x^2T^4}.$$

(genus 0)

[mbm & Bernardi 09]

Simplifying in the D-finite world

Classical tools for linear ODEs

- Closure properties [Gfun]
- Factorisation of differential operators
- ODE of minimal order satisfied by a D-finite series
- Singular expansions

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Steve Melczer's talk

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[Petkovsek, Wilf & Zeilberger 96]

Gessel's quadrant walks ending on the y-axis

- Start from the polynomial equation for A=Q(0,1):

$$
\begin{gathered}
1090491731185059590307 84A^8t^6 + 1211657479094510655 8976t^4\left(16t+1\right)A^6 \\
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- Convert into a linear DE (gfun[algeqtodiffeq])

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- The solution (LREtools[hypergeomsols])

$$a(n) = \frac{4\sqrt{3}\,\Gamma\left(\frac{5}{6}\right) 16^n \Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{7}{6}\right)}{9\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)\Gamma(n+2)\,\Gamma\left(n+\frac{4}{3}\right)} + \frac{2\Gamma\left(\frac{2}{3}\right) 16^n \Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{2}\right)}{9\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)\Gamma(n+2)\,\Gamma\left(n+\frac{5}{3}\right)}$$

Simplifying in the D-finite world

Question: decide whether a given D-finite series is algebraic
[Bostan 17, Bostan, Caruso & Roques 23(a), Singer 80]

Simplifying in the D-algebraic world

Classical tools for polynomial ODEs

DifferentialAlgebra

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- Differential elimination
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Consider the recursion given by $a(2) = \alpha$ and for n>0:

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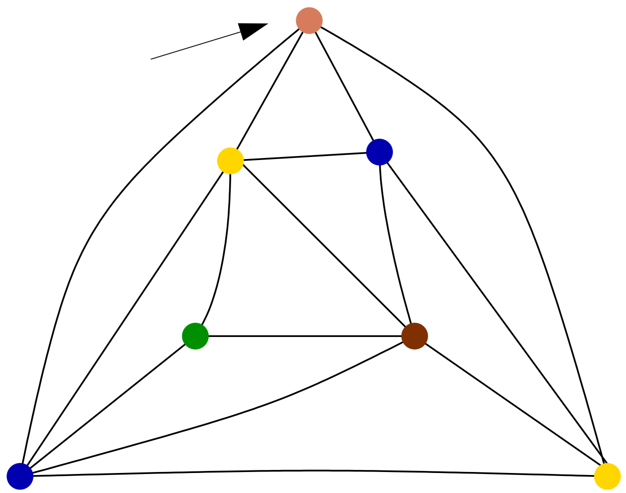

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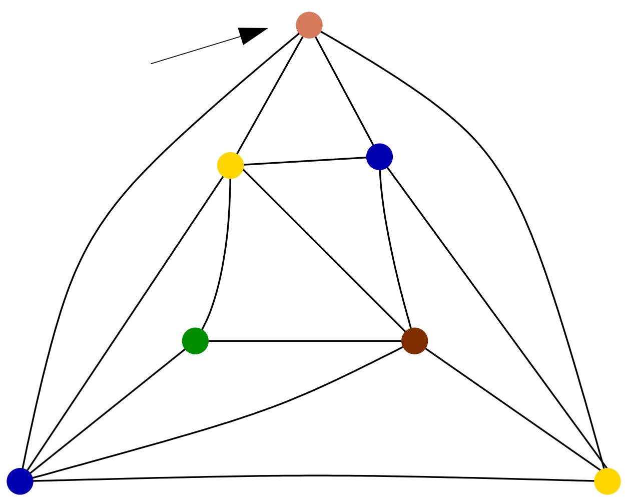

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α = 4. Properly 5-coloured triangulations, probably not D-finite

[Tutte 73-84]
[Bettinelli]

My favourite tool...

My favourite tool...

Ask people !

My favourite tool...

Ask people !

The A≠B team...

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Thanks for your attention

Coefficient extraction

Three-colourings of planar quadrangulations [mbm & Elvey Price 20]

$$P(t,y) = \frac{1}{y}[x^1]C(t,x,y),$$

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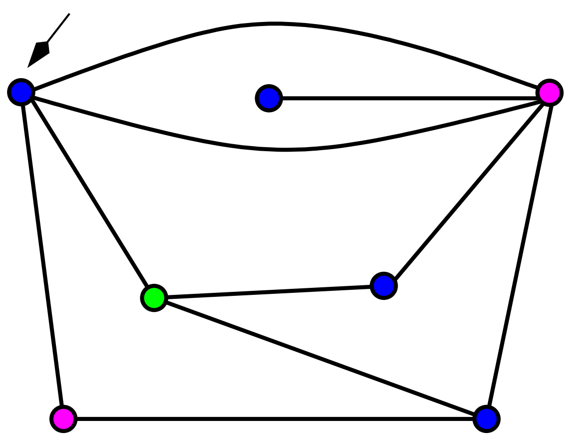

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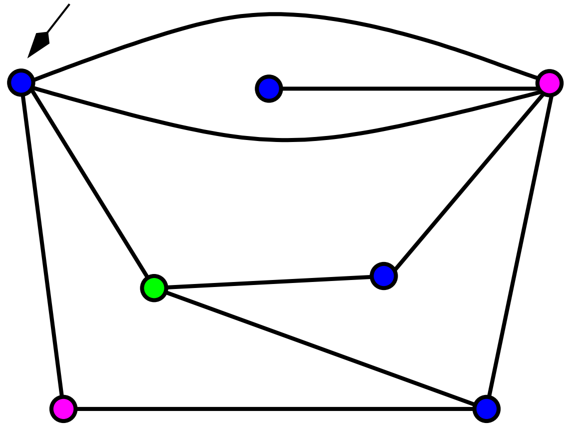
with initial condition

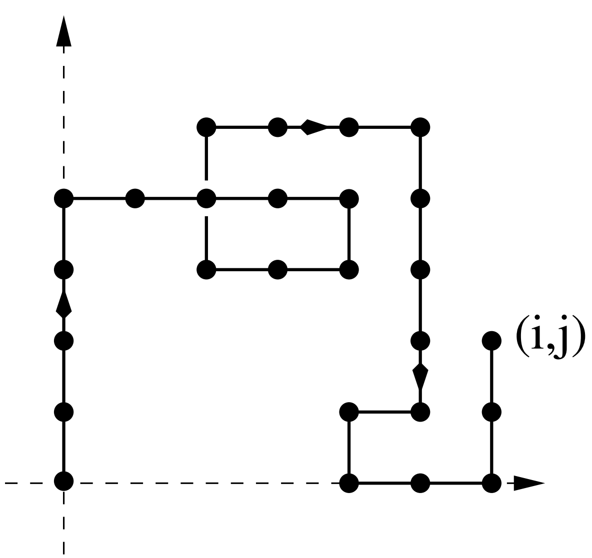
$$[y^1]D(t,x,y) = \frac{1}{1-x}\left(1 + 2t[y^2]D(t,x,y) - t([y^1]D(t,x,y))^2\right).$$



Two discrete derivatives: some examples

- Square lattice **walks confined to a quadrant**: linear equation

$$Q(x,y) = 1 + t(x+y)Q(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$



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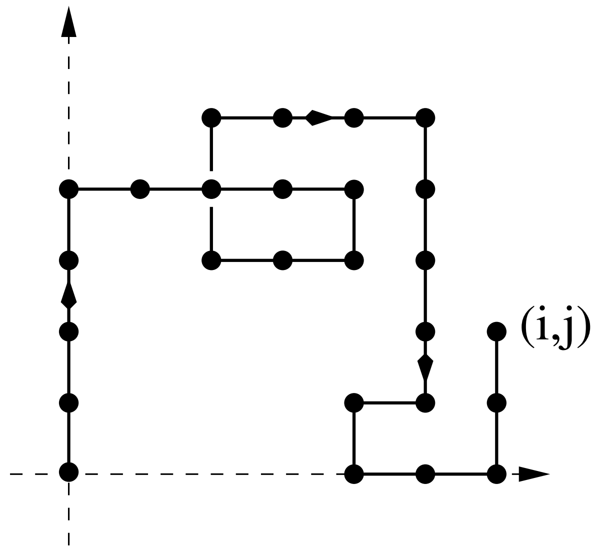
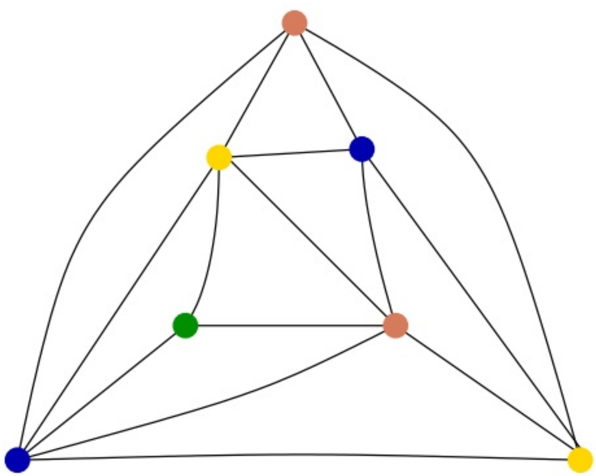
$$T(x,y) = x(q-1) + txyT(1,y)T(x,y) + tx\frac{T(x,y) - T(x,0)}{y} - tx^2y\frac{T(x,y) - T(1,y)}{x-1}.$$





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- Three-stack sortable permutations [Defant, Elvey Price, Guttmann 21]

$$P(x,y) = t(x+1)^2(y+1)^2 + ty(1+x)P(x,y) + t(1+x)\frac{P(x,y)-P(x,0)}{y}\left((1+y)^2 + y\frac{P(x,y)-P(0,y)}{x}\right)$$