A new proof of Viazovska's modular form inequalities for sphere packing in dimension 8

Dan Romik
UC Davis

Computer Algebra Workshop
Institut Henri Poincaré

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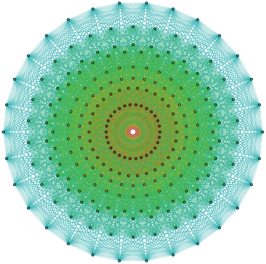

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4. A new proof

Some useful references

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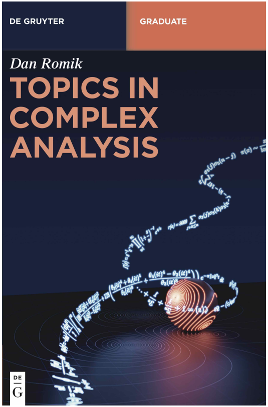
- My paper "On Viazovska's modular form inequalities" (PNAS, 2023).
- Chapter 6 + Appendix of my book "Topics in Complex Analysis"

  `https://www.math.ucdavis.edu/~romik/topics-in-complex-analysis/`



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- The case $d = 24$. Viazovska with Cohn, Kumar, Miller, and Radchenko then proved that for $d = 24$, the densest packing is the **Leech lattice packing**, with packing density $\frac{\pi^{12}}{12!}$.

Background: sphere packings in $\mathbb{R}^d$ (continued)

In other dimensions the problem remains open.

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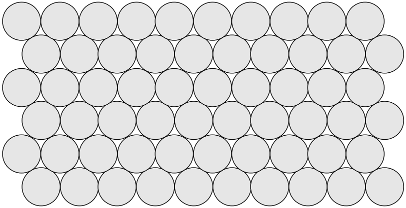

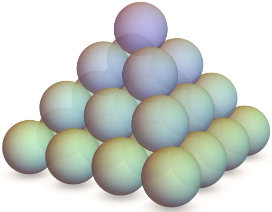

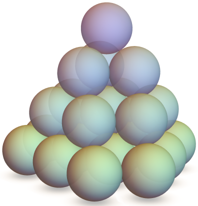

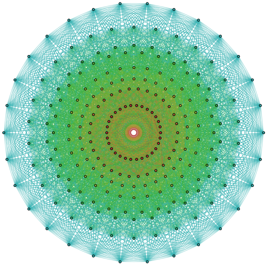

The optimal lattices for sphere packing in dimensions 2, 3, 8

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- One component of the proof makes extensive use of computer calculations.

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- For the case $d = 8$, the sharp bound $\frac{\pi^4}{384}$ is obtained when $\rho = \sqrt{2}$. A function satisfying the conditions of the theorem for that ρ is called a **magic function**.

Applying the Cohn-Elkies bounds in practice

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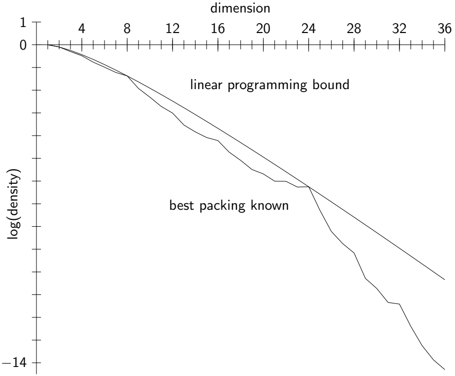
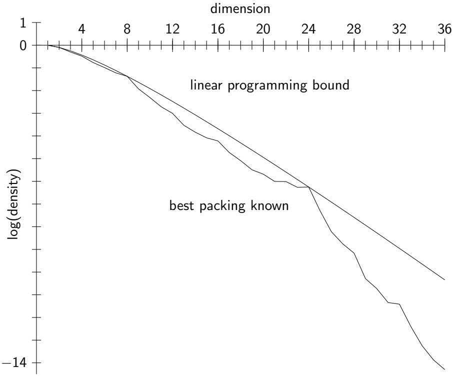


Image source: Henry Cohn, A conceptual breakthrough in sphere packing (Notices of AMS, 2017)

Applying the Cohn-Elkies bounds in practice (continued)

In dimensions 2, 8 and 24, their bounds came extremely close to matching the known lower bounds.



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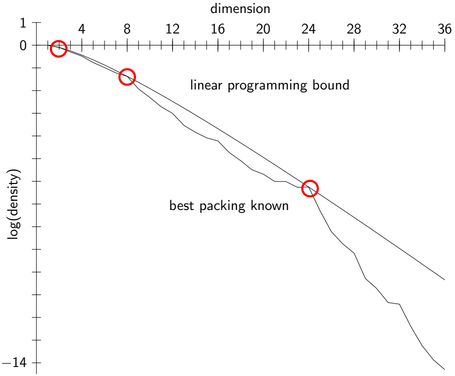

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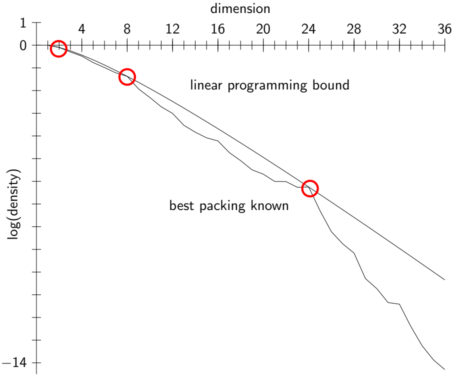

They conjectured that in those dimensions there exists a "magic function" f certifying a *sharp* bound.

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$$\varphi(x) = -4\sin^2\left(\frac{\pi\|x\|^2}{2}\right)$$

$$\times \int_0^\infty e^{-\pi t\|x\|^2}\Bigg[108\frac{(itE_4'(it)+4E_4(it))^2}{E_4(it)^3-E_6(it)^2}$$

$$+128\left(\frac{\theta_3(it)^4+\theta_4(it)^4}{\theta_2(it)^8}+\frac{\theta_4(it)^4-\theta_2(it)^4}{\theta_3(it)^8}\right)\Bigg]\,dt,$$

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where E_4, E_6 are the **Eisenstein series** and $\theta_2, \theta_3, \theta_4$ are the **Jacobi thetanull functions**, defined by

$$E_4(z) = 1 + 240\sum_{n=1}^\infty \sigma_3(n)q^{2n}, \qquad \theta_2(z) = \sum_{n=-\infty}^\infty q^{(n+1/2)^2},$$

$$E_6(z) = 1 - 504\sum_{n=1}^\infty \sigma_5(n)q^{2n}, \qquad \theta_3(z) = \sum_{n=-\infty}^\infty q^{n^2},$$

$$\theta_4(z) = \sum_{n=-\infty}^\infty (-1)^n q^{n^2},$$

(with the standard notation $q = e^{\pi iz}$, $\sigma_\alpha(n) = \sum_{d\,|\,n} d^\alpha$).

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The modular forms in the definition of φ

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$$V(z) = 128 \left(\frac{\theta_3(z)^4 + \theta_4(z)^4}{\theta_2(z)^8} + \frac{\theta_4(z)^4 - \theta_2(z)^4}{\theta_3(z)^8} \right).$$

Define functions $\varphi_\pm : \mathbb{R}^8 \to \mathbb{R}$ by (the analytic continuation of)

$$\varphi_+(x) = -4 \sin^2 \left(\frac{\pi \|x\|^2}{2} \right) \int_0^\infty e^{-\pi t \|x\|^2} U(it)\, dt$$

$$\varphi_-(x) = -4 \sin^2 \left(\frac{\pi \|x\|^2}{2} \right) \int_0^\infty e^{-\pi t \|x\|^2} V(it)\, dt,$$

so that $\varphi = \varphi_+ + \varphi_-$.

Viazovska's modular form inequalities

The definitions of $U(z)$, $V(z)$ were carefully chosen to satisfy several conditions, including, crucially,

$$\widehat{\varphi_+} = \varphi_+, \qquad \widehat{\varphi_-} = -\varphi_-.$$

(That is, $\varphi_\pm$ are eigenfunctions of the Fourier transform with resp. eigenvalues ± 1.)

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Theorem (Viazovska)

The functions U, V satisfy the inequalities

$$U(it) + V(it) \ge 0 \qquad (t > 0) \tag{V1}$$

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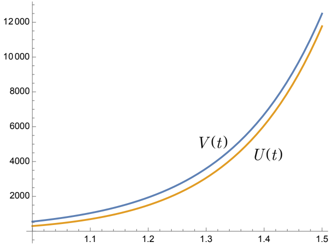

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Then use the facts that $\theta_3(it) > 0$ (trivially), that $\lambda(it) \in (0,1)$ for $t > 0$, and that the map $x \mapsto \frac{(1-x)(2+x+2x^2)}{x^2}$ takes positive values for $x \in (0,1)$.

A new proof of (V1)–(V2), part II: proof of (V2)

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Step 1: A bit of cleanup

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Define functions

$$F(z) = \frac{1}{108}(E_4^3 - E_6^2)U(z) = (E_4')^2 z^2 + 8E_4E_4'z + 16E_4^2,$$
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(making use of the modular transformation properties).

Trivially, the inequality (V2) is equivalent to the pair of inequalities

$$-\widetilde{F}(it) < -\widetilde{G}(it) \qquad (t \geq 1), \tag{V2-I}$$
$$F(it) < G(it) \qquad (t \geq 1). \tag{V2-II}$$

Step 2: Understanding the behavior at $t = 1$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 2: Understanding the behavior at $t = 1$

Theorem (Gauss, Ramanujan, folklore)

We have the explicit evaluations

$$\begin{aligned} E_4(i) &= \frac{3\Gamma(1/4)^8}{64\pi^6}, & \theta_2(i) &= \frac{\Gamma(1/4)}{(2\pi)^{3/4}}, \\ E_4'(i) &= \frac{3\Gamma(1/4)^8}{32\pi^6} i, & \theta_3(i) &= \frac{\Gamma(1/4)}{\sqrt{2}\pi^{3/4}}, \\ & & \theta_4(i) &= \frac{\Gamma(1/4)}{(2\pi)^{3/4}}, \end{aligned}$$

(where $\Gamma(\cdot)$ denotes the Euler gamma function).

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See p. 257 of my book for a proof sketch and references.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 3: Leveraging monotonicity

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Proof of (V2-I)**.** Observe that

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Note that $q^4 = e^{-4\pi t} \ll e^{-3\pi t} = q^3$ for t large, so the inequality $-\widetilde{F}(it) < -\widetilde{G}(it)$ holds asymptotically. To prove the stronger claim that it holds for $t \geq 1$, note that the Fourier coefficients in both series are positive.*

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&\quad + 3199467520 q^9 + 19472547840 q^{11} + \dots.
\end{aligned}$$

Note that $q^4 = e^{-4\pi t} \ll e^{-3\pi t} = q^3$ for t large, so the inequality $-\widetilde{F}(it) < -\widetilde{G}(it)$ holds asymptotically. To prove the stronger claim that it holds for $t \geq 1$, note that the Fourier coefficients in both series are positive.*

*This is easy to prove from the definitions.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 3: Leveraging monotonicity

Proof of (V2-I)**.** Observe that

$$\begin{aligned}-\widetilde{F}(z) &= 230400\pi^2 q^4 + 8294400\pi^2 q^6 + 113356800\pi^2 q^8 \\ &\qquad + 831283200\pi^2 q^{10} + 4337971200\pi^2 q^{12} + \dots, \\ -\widetilde{G}(z) &= 163840 q^3 + 16121856 q^5 + 333250560 q^7 + \\ &\qquad + 3199467520 q^9 + 19472547840 q^{11} + \dots.\end{aligned}$$

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A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 3: Leveraging monotonicity

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Step 3: Leveraging monotonicity

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Step 3: Leveraging monotonicity

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$$-e^{3\pi t}\widetilde{F}(it) \ \leq e^{3\pi}\widetilde{F}(i) = -e^{3\pi}E_4'(i)^2 = e^{3\pi}\frac{9\Gamma(1/4)^{16}}{1024\pi^{12}}$$

*This is easy to prove from the definitions.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 3: Leveraging monotonicity

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A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Step 3: Leveraging monotonicity

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$$
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$$

This in turn is < 163840, which is a lower bound for $-e^{3\pi t}\widetilde{G}(it)$.

*This is easy to prove from the definitions.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Summarizing this argument:

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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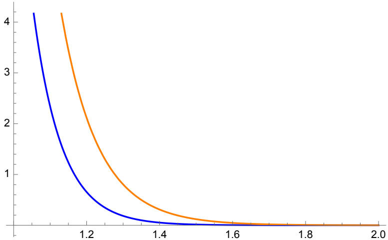

Plots of $-\tilde{F}(it)$, $-\tilde{G}(it)$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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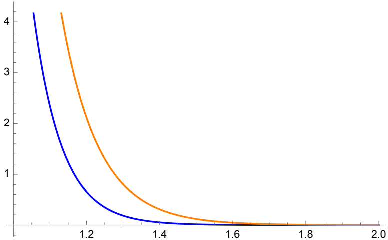

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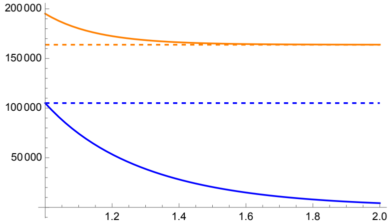

Plots of $-e^{3\pi t}\tilde{F}(it)$, $-e^{3\pi t}\tilde{G}(it)$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Proof of (V2-II)**.** Imitating the approach for (V2-I), note that

$$\begin{aligned}
F(it) = 16 &+ (-3840\pi t + 7680)q^2 \\
&+ (230400\pi^2 t^2 - 990720\pi t + 990720)q^4 \\
&+ (8294400\pi^2 t^2 - 25205760\pi t + 16803840)q^6 + \dots, \\
G(it) = 16 &+ 1920q^2 - 81920q^3 + 1077120q^4 - 8060928q^5 \\
&+ 41725440q^6 - 166625280q^7 + 553054080q^8 + \dots,
\end{aligned}$$

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\end{aligned}
$$

Define renormalized functions

$$
\begin{aligned}
K(z) &= -\frac{F(z)-16}{q^2} = -q^{-2}(E_4')^2 z^2 - 8q^{-2}E_4' E_4 z - 16q^{-2}(E_4^2 - 1), \\
L(z) &= -\frac{G(z)-16}{q^2} = -8q^{-2}\left[\theta_4^8(\theta_3^{12} + \theta_4^4\theta_3^8 + \theta_2^8\theta_4^4 - \theta_2^{12}) - 2\right],
\end{aligned}
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A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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Define renormalized functions

$$K(z) = -\frac{F(z) - 16}{q^2} = -q^{-2}(E_4')^2 z^2 - 8q^{-2}E_4'E_4 z - 16q^{-2}(E_4^2 - 1),$$

$$L(z) = -\frac{G(z) - 16}{q^2} = -8q^{-2}\left[\theta_4^8(\theta_3^{12} + \theta_4^4\theta_3^8 + \theta_2^8\theta_4^4 - \theta_2^{12}) - 2\right],$$

The inequality (V2-II) is thus equivalent to the inequality

$$K(it) > L(it) \qquad (t \geq 1).$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

As in the earlier proof, we will bound each of $K(it)$ and $L(it)$ separately, obtaining the inequality (V2-II) from the combination of the following two lemmas:

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

As in the earlier proof, we will bound each of $K(it)$ and $L(it)$ separately, obtaining the inequality (V2-II) from the combination of the following two lemmas:

Lemma (1)

$L(it) \leq 2297$ for $t \geq 1$.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

As in the earlier proof, we will bound each of $K(it)$ and $L(it)$ separately, obtaining the inequality (V2-II) from the combination of the following two lemmas:

Lemma (1)

$L(it) \leq 2297$ for $t \geq 1$.

Lemma (2)

$K(it) \geq 3747$ for $t \geq 1$.

Proof of Lemma (1). Again the idea is to leverage monotonicity.

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Proof of Lemma (1). Again the idea is to leverage monotonicity. Define

$$H(z) = \frac{L(z+1) - L(z)}{2} = \ldots = 4q^{-2}\left(\theta_2^8(\theta_3^{12} - \theta_4^{12}) + \theta_2^{12}(\theta_3^8 + \theta_4^8)\right)$$
$$= 81920q + 8060928q^3 + 166625280q^5 + 1599733760q^7 + \ldots.$$

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Then for $t \geq 1$,

$$\begin{aligned} L(it) \; = \; & -1920 + 81920q - 1077120q^2 + 8060928q^3 \\ & -41725440q^4 + 166625280q^5 - 553054080q^6 + \dots \end{aligned}$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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(assuming alternating coefficients — need to justify)

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$$\begin{aligned} &\leq \ -1920 + 81920q + 8060928q^3 + 166625280q^5 + \ldots \\ &= \ -1920 + \frac{L(it+1) - L(it)}{2} \end{aligned}$$

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(assuming alternating coefficients — need to justify)

$$\begin{aligned} &\leq -1920 + 81920q + 8060928q^3 + 166625280q^5 + \ldots \\ &= -1920 + \frac{L(it+1) - L(it)}{2} = -1920 + H(it) \end{aligned}$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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Then for $t \geq 1$,

$$\begin{aligned} L(it) \;&=\; -1920 + 81920q - 1077120q^2 + 8060928q^3 \\ &\qquad -41725440q^4 + 166625280q^5 - 553054080q^6 + \ldots \\ &\qquad\qquad \text{(assuming alternating coefficients — need to justify)} \\ &\leq\; -1920 + 81920q + 8060928q^3 + 166625280q^5 + \ldots \\ &=\; -1920 + \frac{L(it+1) - L(it)}{2} = -1920 + H(it) \\ &\leq\; -1920 + H(i) = \ldots \end{aligned}$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Proof of Lemma (1). Again the idea is to leverage monotonicity. Define

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\begin{aligned}
H(z) &= \frac{L(z+1)-L(z)}{2} = \ldots = 4q^{-2}\left(\theta_2^8(\theta_3^{12}-\theta_4^{12})+\theta_2^{12}(\theta_3^8+\theta_4^8)\right) \\
&= 81920q + 8060928q^3 + 166625280q^5 + 1599733760q^7 + \ldots .
\end{aligned}
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Then for $t \geq 1$,

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&\qquad \textit{(assuming alternating coefficients — need to justify)} \\
&\leq\; -1920 + 81920q + 8060928q^3 + 166625280q^5 + \ldots \\
&=\; -1920 + \frac{L(it+1)-L(it)}{2} = -1920 + H(it) \\
&\leq\; -1920 + H(i) = \ldots = -1920 + 3e^{2\pi}\frac{\Gamma(1/4)^{20}}{2048\pi^{15}}
\end{aligned}
$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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$$H(z) = \frac{L(z+1) - L(z)}{2} = \dots = 4q^{-2}\left(\theta_2^8(\theta_3^{12} - \theta_4^{12}) + \theta_2^{12}(\theta_3^8 + \theta_4^8)\right)$$
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(assuming alternating coefficients — need to justify)

$$\begin{aligned} &\leq -1920 + 81920q + 8060928q^3 + 166625280q^5 + \dots \\ &= -1920 + \frac{L(it+1) - L(it)}{2} = -1920 + H(it) \\ &\leq -1920 + H(i) = \dots = -1920 + 3e^{2\pi}\frac{\Gamma(1/4)^{20}}{2048\pi^{15}} \approx 2296.16 \end{aligned}$$

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(assuming alternating coefficients — need to justify)

$$\begin{aligned} &\leq -1920 + 81920q + 8060928q^3 + 166625280q^5 + \ldots \\ &= -1920 + \frac{L(it+1) - L(it)}{2} = -1920 + H(it) \\ &\leq -1920 + H(i) = \ldots = -1920 + 3e^{2\pi}\frac{\Gamma(1/4)^{20}}{2048\pi^{15}} \approx 2296.16 \\ &\leq 2297, \quad \text{which is what we wanted.} \end{aligned}$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

Justification of the assumption about alternating coefficients: define

$$W(z) = \theta_3^{12}\theta_2^8 + \theta_3^8\theta_2^{12} + \theta_3^{12}\theta_4^8 + \theta_3^8\theta_4^{12}.$$

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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By simple algebra, $-L(z+1) = 8q^{-2}(W(z)-2)$, so the claim is equivalent to the statement that the Fourier expansion of $W(z)$ has nonnegative coefficients.*

A new proof of (V1)–(V2), part II: proof of (V2) (cont'd)

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** I discovered this identity using computer algebra + a linear program solver.

Proof of Lemma (2).

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Proof of Lemma (2). The asymptotic expansion of $K(it)$ is

$$\begin{aligned} K(it) = (3840\pi t - 7680) &+ (-230400\pi^2 t^2 + 990720\pi t - 990720)q^2 \\ &+ (-8294400\pi^2 t^2 + 25205760\pi t - 16803840)q^4 + \dots . \end{aligned}$$

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That's all — thank you!