

# Cover Times of Random Walks

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# Overview

Introduction

Expected Cover Time of a Small Graph

Expected Cover Time of a Complete Graph with Self-edges

Expected Cover Time of a General Graph

Cover Time Distribution of the a Complete Graph with Self-edges

# Overview

## Introduction

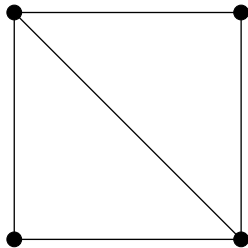
Expected Cover Time of a Small Graph

Expected Cover Time of a Complete Graph with Self-edges

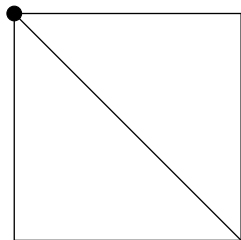
Expected Cover Time of a General Graph

Cover Time Distribution of the a Complete Graph with Self-edges

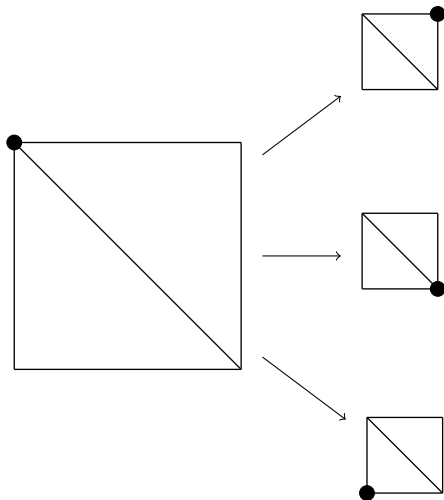
# A Graph



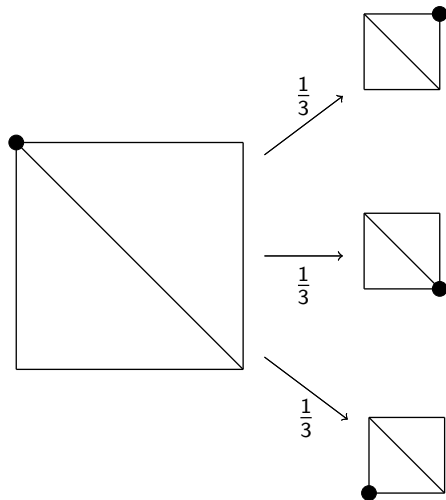
# Random Walks



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# Cover Times

How many steps does a random walk take to visit every vertex?



# Coupon Collector

A collector receives one of  $n$  possible coupons in the mail everyday. How long until all  $n$  coupons have been collected?

# Birthday Problem

How many people do we need for each birthday (excluding February 29) to be represented?

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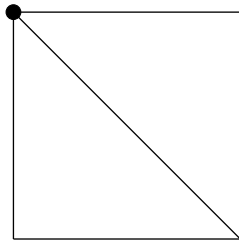
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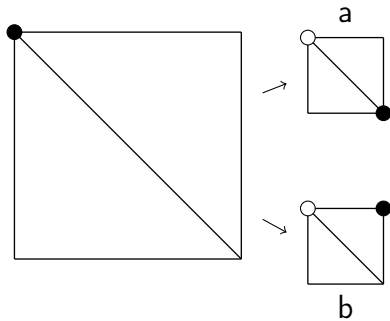
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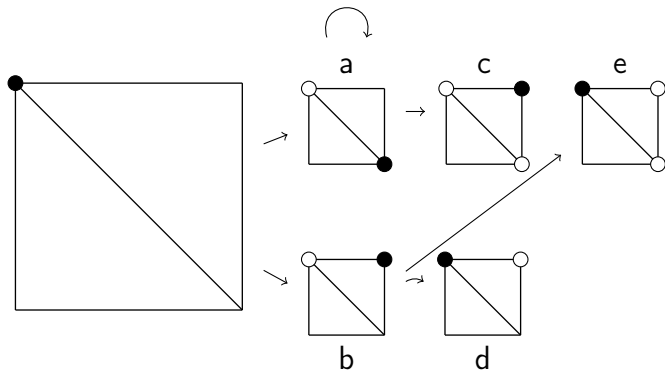
# A Small Graph



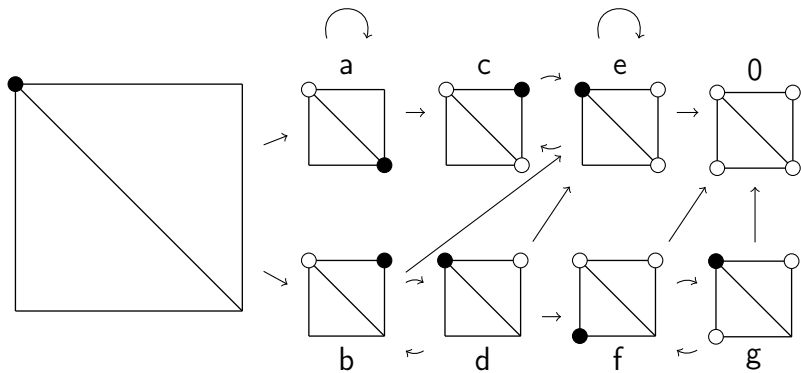
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$$E[t_{\text{cov}}] = 1 + \frac{1}{3}a + \frac{2}{3}b$$

$$a = 1 + \frac{1}{3}a + \frac{2}{3}c$$

$$b = 1 + \frac{1}{2}b + \frac{1}{2}e$$

$$c = 1 + 1e$$

$$d = 1 + \frac{1}{3}a + \frac{1}{3}e + \frac{1}{3}f$$

$$e = 1 + \frac{1}{3}c + \frac{1}{3}e + \frac{1}{3}0$$

$$f = 1 + \frac{1}{2}g + \frac{1}{2}0$$

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# A Small Graph

$$E[t_{\text{cov}}] = 1 + \frac{1}{3}a + \frac{2}{3}b = \frac{43}{6}$$

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# Complete Graph with Self-edges

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$X_i \sim \text{Geom}(\frac{n-i+1}{n})$  so  $E[X_i] = \frac{n}{n-i+1}$  and

$$E[t_{\text{cov}}] = n \left( \frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1 \right)$$

# Coupon Collector

A collector receives one of  $n$  possible coupons in the mail everyday. How long until all coupons have been collected?

On average, we need about  $n \log n$  days.



# Birthday Problem

How many people do we need for each birthday (excluding February 29) to be represented?

On average, we need about 2365 people.

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# Matthews Method

## Theorem

*Consider a random walk on an arbitrary connected graph with  $n$  vertices. Then*

$$E[t_{\text{cov}}] \leq t_{\text{hit}} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right).$$

# Matthews Method Proof

Let  $\sigma$  be a uniform permutation of unvisited vertices from (worst-case) vertex  $n$ .

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Let  $T_k$  be the first time vertices  $\sigma_1, \sigma_2, \dots, \sigma_k$  have all been visited.

$$\begin{aligned} \mathbb{E}[t_{\text{cov}}] &= \mathbb{E}_n[T_{n-1}] \\ &\leq \mathbb{E}_n[T_1 + (T_2 - T_1) + \dots + (T_{n-1} - T_{n-2})] \end{aligned}$$

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$$\mathbb{E}_n[T_k - T_{k-1}] = \sum_{i=1}^k \mathbb{E}_n[T_k - T_{k-1} | L(k) = \sigma_i] \mathbb{P}(L_k = \sigma_i)$$



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# Cover Time Distribution

After  $r$  steps of a random walk, let  $A_i$  be the event that vertex  $i$  is unvisited.

$$P(A_i) = P(\text{not visited in one step})^r = (1 - 1/n)^r$$

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Let  $N_n$  be the number of unvisited vertices.

$$\begin{aligned} E[N_n] &= \sum_{i=1}^n P(A_i) = n(1 - 1/n)^r \\ &= n[(1 - 1/n)^n]^{r/n} \sim ne^{-r/n} \end{aligned}$$

# Cover Time Distribution

Introduce  $x$  and set  $r = n \log n + nx$ .

$$N_n \sim ne^{-r/n} = ne^{-\log n - x} = nn^{-1}e^{-x} \rightarrow e^{-x}$$

## Theorem

*If  $ne^{-r/n} \rightarrow \lambda \in [0, \infty)$ , then the number of unvisited vertices approaches a Poisson distribution with mean  $\lambda$ .*

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*If  $ne^{-r/n} \rightarrow \lambda \in [0, \infty)$ , then the number of unvisited vertices approaches a Poisson distribution with mean  $\lambda$ .*

$$P(N_n = 0) = P(t_{\text{cov}} \leq r) = P(t_{\text{cov}} - n \log n \leq nx) \rightarrow e^{-e^{-x}}$$

# Birthday Problem

What is the probability all birthdays are covered in a town of 1825? How about a town of 2190?



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$$\begin{aligned} P(t_{\text{cov}} \leq 1825) &= P\left(\frac{t_{\text{cov}} - 2153}{365} \leq \frac{-328}{365}\right) \\ &\approx e^{-e^{-0.89863}} \approx 0.08576 \end{aligned}$$

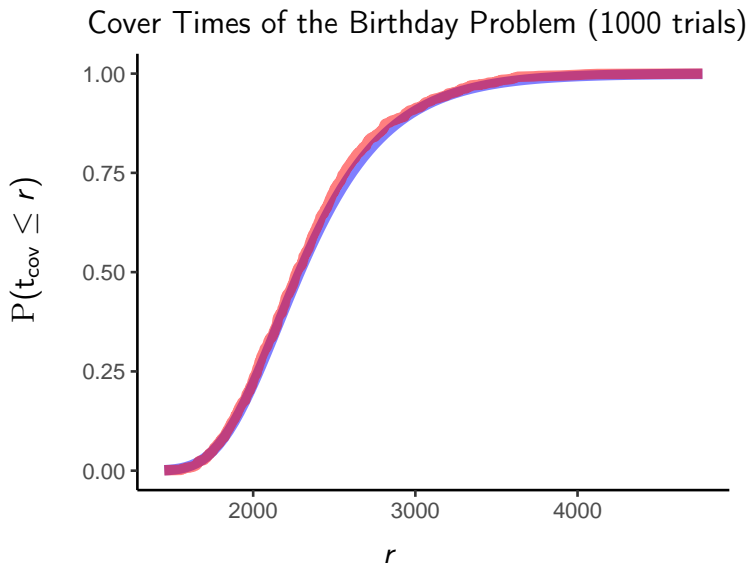
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$$P(t_{\text{cov}} \leq 2190) = P\left(\frac{t_{\text{cov}} - 2153}{365} \leq \frac{37}{365}\right) \\ \approx e^{-e^{-0.10137}} \approx 0.40511$$

# Birthday Problem



# Thank you!

[github.com/rtealw/Cover-Times](https://github.com/rtealw/Cover-Times)

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