# Cover Times of Random Walks

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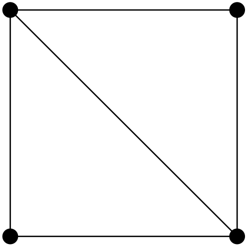
# Overview

# A Graph



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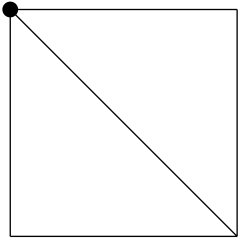

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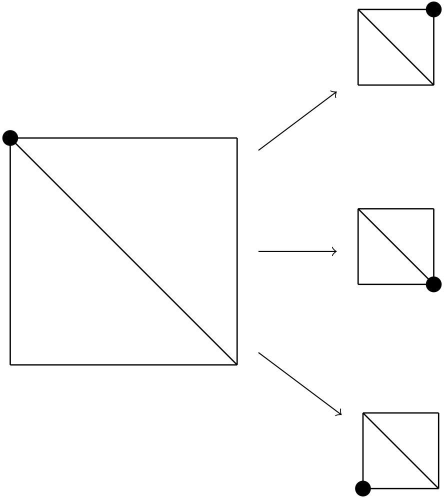

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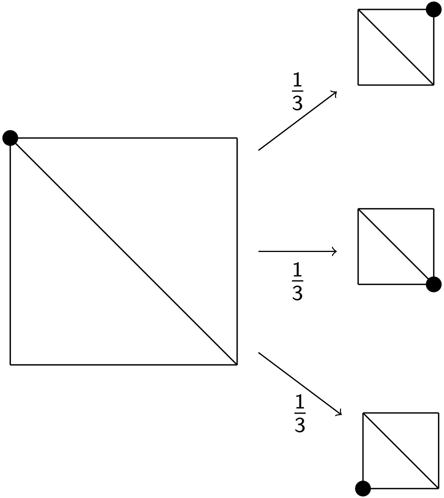


# Cover Times

How many steps does a random walk take to visit every vertex?

# Coupon Collector

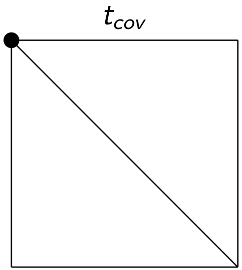
A collector receives one of $n$ possible coupons in the mail everyday. How long until all $n$ coupons have been collected?

# Birthday Problem

How many people do we need for each birthday (excluding February 29) to be represented?

# Overview

# A Small Graph



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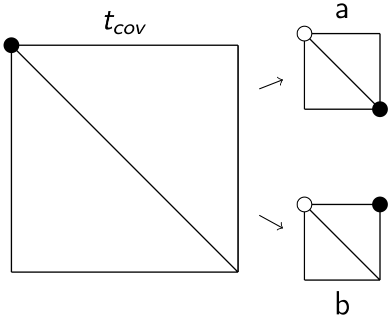

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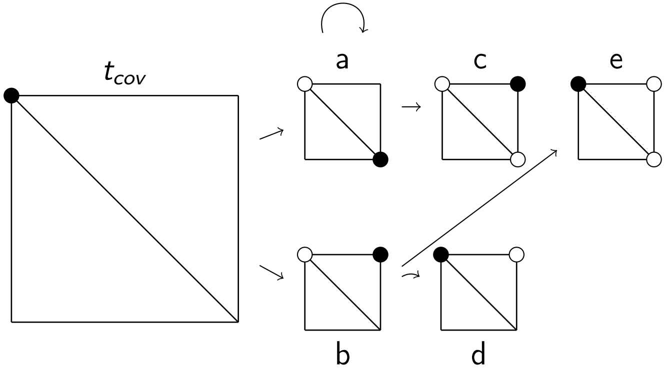

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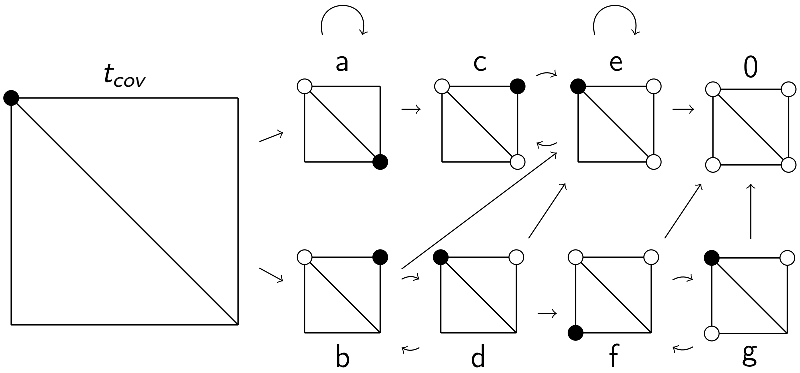

# A Small Graph

$$
\begin{aligned}
\mathrm{E}[t_{cov}] &= 1 + \frac{1}{3}\mathrm{a} + \frac{2}{3}\mathrm{b} \\
\mathrm{a} &= 1 + \frac{1}{3}\mathrm{a} + \frac{2}{3}\mathrm{c} \\
\mathrm{b} &= 1 + \frac{1}{2}\mathrm{b} + \frac{1}{2}\mathrm{e} \\
\mathrm{c} &= 1 + 1\mathrm{e} \\
\mathrm{d} &= 1 + \frac{1}{3}\mathrm{a} + \frac{1}{3}\mathrm{e} + \frac{1}{3}\mathrm{f} \\
\mathrm{e} &= 1 + \frac{1}{3}\mathrm{c} + \frac{1}{3}\mathrm{e} + \frac{1}{3}\dot{0} \\
\mathrm{f} &= 1 + \frac{1}{2}\mathrm{g} + \frac{1}{2}\dot{0} \\
\mathrm{g} &= 1 + \frac{2}{3}\mathrm{f} + \frac{1}{3}\dot{0}
\end{aligned}
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# A Small Graph

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\begin{aligned}
\mathrm{E}[t_{cov}] &= 1 + \frac{1}{3}\mathrm{a} + \frac{2}{3}\mathrm{b} = \frac{43}{6} \\
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# Overview

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Let $X_i$ be the number of steps from $i-1$ to $i$ distinct vertices.

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$X_i \sim \mathrm{Geom}(\frac{n-i+1}{n})$ so $\mathrm{E}[X_i] = \frac{n}{n-i+1}$ and

$$\mathrm{E}[t_{cov}] = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right)$$

# Coupon Collector

A collector receives one of $n$ possible coupons in the mail everyday. How long until all coupons have been collected?

On average, we need about $n \log n$ days.

# Birthday Problem

How many people do we need for each birthday (excluding February 29) to be represented?

On average, we need about 2365 people.

# Overview

# Matthews Method

## Theorem

*Consider a random walk on an arbitrary connected graph with n vertices. Then*

$$\mathrm{E}[t_{cov}] \leq t_{hit}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right).$$

# Matthews Method Proof

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Let $T_k$ be the first time vertices $\sigma_1, \sigma_2, \ldots, \sigma_k$ have all been visited.

$$\begin{aligned}
\mathrm{E}[t_{cov}] &= \mathrm{E}_n[T_{n-1}] \\
&= \mathrm{E}_n[T_1 + (T_2 - T_1) + \cdots + (T_{n-1} - T_{n-2})].
\end{aligned}$$

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$$\mathrm{E}_n[T_k - T_{k-1}] = \sum_{i=1}^{k} \mathrm{E}_n[T_k - T_{k-1} | L(k) = \sigma_i] \, \mathrm{P}(L_k = \sigma_i).$$

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&= 0 + \mathrm{E}_n[T_k - T_{k-1} | L(k) = \sigma_k] \, \mathrm{P}(L_k = \sigma_k). \\
&\leq t_{hit} \frac{1}{k}
\end{aligned}$$

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# Poisson Distribution

After $r$ steps of a random walk, let $A_i$ be the probability that vertex $i$ is unvisited.

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I promise that

$$N_n/n \xrightarrow{p} \mathrm{E}[N_n/n]$$

# Poisson Distribution

Let the number of unvisited vertices be $r = n \log n + nx$ (trust me).

$$N_n = ne^{-r/n} = ne^{-\log n - x} = nn^{-1}e^{-x} \to e^{-x}$$

## Theorem

*If $ne^{-r/n} \to \lambda \in [0, \infty)$, then the number of unvisited vertices approaches a Poisson distribution with mean $\lambda$.*

# Poisson Distribution

Uppertail of a Poisson with mean $\lambda$

$$\mathrm{P}(N_n > 0) \to 1 - e^{\lambda}$$

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The first time $N_n = 0$ is $t_{cov}$

$$\mathrm{P}(t_{cov} \leq r) \to e^{-e^{-x}}$$
$$\mathrm{P}(t_{cov} - n \log n \leq nx) \to e^{-e^{-x}}$$