Week 2 Tuesday

- Self-grade due Friday
- Thoughts on "working smart not hard"
  ↳ focus is a muscle
  (fast dopamine weakens it)

  ↳ even 15min is enough to start
  a hard task

Plan

- Review eigendecomposition
- Page Rank
- Power Method

Eigen decomposition

$v_1, \dots, v_r \in \mathbb{R}^d$ orthonormal eigenvectors

$\langle v_i, v_j \rangle = 0$ if $i \neq j$

$\langle v_i, v_i \rangle = 1$

$\lambda_1 \geq \lambda_2 \dots \geq \lambda_r$ eigenvalues

$$A = \sum_{i=1}^{r} \lambda_i v_i v_i^T$$

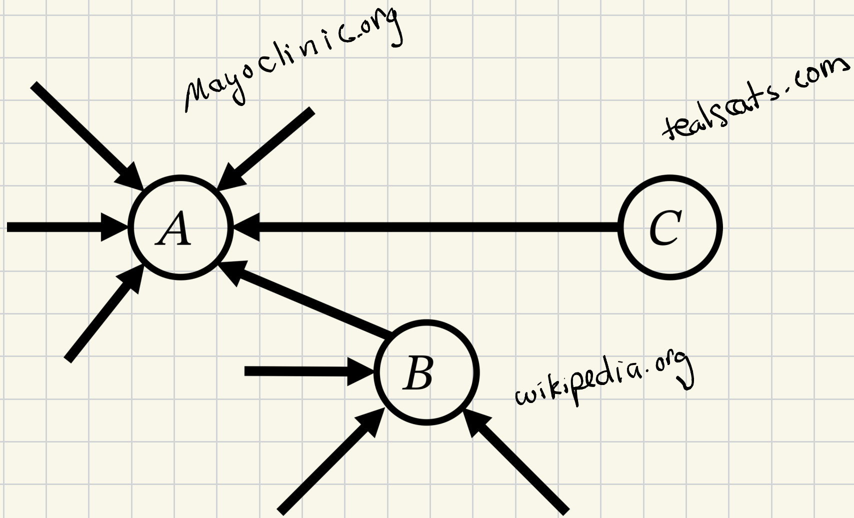
$$A v_j =$$

$$A A =$$

Larry webpages

Page Rank

How do we find important webpages?



Important if:

- lots of incoming links
- incoming links from important pages

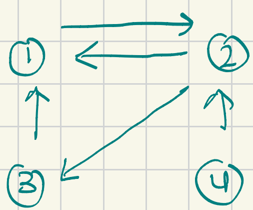
Page Rank Algorithm

iterative process of computing importance

$p^{(0)} \in \mathbb{R}^n$ where $p_i^{(0)} = 1/n$

$$p_i^{(t+1)} = \sum_{j=1}^{n} \mathbb{1}[j \text{ links to } i] \frac{p_j^{(t)}}{d_j}$$

d_j ← # outgoing edges from j



$$p^{(0)} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$p^{(1)} =$

Matrix View

$$p^{(t+1)} = A\, p^{(t)} \qquad A \in \mathbb{R}^{n \times n}$$

$$A_{i,j} = \begin{cases} 1/d_j & \text{if } j \text{ links to } i \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that columns sum to 1!

$$\sum_{i=1}^{n} p_i^{(t+1)} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} [A]_{ij}\, p_j^{(t)} \right)$$

$$= \sum_{j=1}^{n} p_j^{(t)} \sum_{i=1}^{n} [A]_{ij}$$

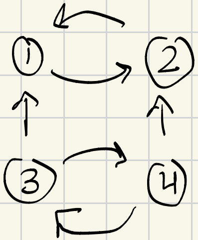
$$= \sum_{j=1}^{n} p_j^{(t)}$$

So, if $\sum_{i=1}^{n} p_i^{(0)} = 1$, then $\sum_{i=1}^{n} p_i^{(t)}$

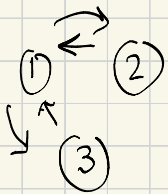
A probability distribution!

$p_i^{(t)} = \Pr($ random clicker at page i in time step $t)$

Problem Matrices



Reducible



Periodic

Solution: Teleporting! $0<\alpha<1$

$$p^{(t+1)} = \alpha A p^{(t)} + (1-\alpha) \mathbb{1} \frac{1}{n}$$

$$= \left(\alpha A + (1-\alpha) \frac{1}{n} \mathbb{1}\mathbb{1}^T\right) p^{(t)}$$

$$\left(\text{since} \quad \mathbb{1}^T p^{(t)} = \sum_{i=1}^{n} p_i^{(t)} = 1 \right)$$

$$= M p^{(t)}$$

$$= M M p^{(t-1)}$$

$$= M M \ldots M \; p^{(0)}$$

$$= M^t p^{(0)}$$

Eigendecomposition

Non-symmetric, square matrix $M \in \mathbb{R}^{n \times n}$

$$M = \sum_{i=1}^{r} \lambda_i v_i w_i^T$$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ eigenvalues

$w_1, w_2, \dots, w_r \in \mathbb{R}^d$ left eigenvectors $\quad w_i^T M =$

$v_1, v_2, \dots, v_r \in \mathbb{R}^d$ right eigenvectors $\quad M v_i =$

---

bi-orthonormal

$$\langle v_i, w_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Proof:

$$w_i^T M v_j = w_i^T \lambda_j v_j = w_i^T \lambda_i v_j$$

$$\Leftrightarrow (w_i^T \lambda_j - w_i \lambda_i) v_j = 0$$

$$\Leftrightarrow w_i^T v_j (\lambda_j - \lambda_i) = 0$$

Power Method

$$M = \sum_{i=1}^{r} \lambda_i v_i w_i^T$$

$$MM =$$

$$M^t =$$

$$M^t p^{(0)} =$$