CSCI 145 Problem Set 9

November 4, 2025

Submission Instructions

Please upload *your* work by **11:59pm Monday November 3, 2025.**

- You are encouraged to discuss ideas and work with your classmates. However, you **must acknowledge** your collaborators at the top of each solution on which you collaborated with others and you **must write** your solutions independently.

- Your solutions to theory questions must be written legibly, or typeset in LaTeX or markdown. If you would like to use LaTeX, you can import the source of this document (available from the course webpage) to Overleaf.

- I recommend that you write your solutions to coding questions in a Jupyter notebook using Google Colab.

- You should submit your solutions as a **single PDF** via the assignment on Gradescope.

Grading: The point of the problem set is for *you* to learn. To this end, I hope to disincentivize the use of LLMs by **not** grading your work for correctness. Instead, you will grade your own work by comparing it to my solutions. This self-grade is due the Friday *after* the problem set is due, also on Gradescope.

Problem 1: Autoencoders and Variational Autoencoders

In this problem, we will explore unsupervised learning through autoencoders (AEs) and variational autoencoders (VAEs).

Part A: KL Divergence from First Principles

Let P be the univariate normal distribution $\mathcal{N}(\mu, \sigma^2)$ and Q be the univariate normal distribution $\mathcal{N}(0, 1)$. Starting from the scalar densities

$$p(z) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right), \quad q(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right),$$

derive the KL-divergence

$$\mathbb{E}_{z \sim P}\left[\log \frac{p(z)}{q(z)}\right] = \tfrac{1}{2}\left(\mu^2 + \sigma^2 - 1 - \log\sigma\right).$$

Hint: Take logs, and notice that $\mathbb{E}[(z-\mu)^2] = \sigma^2$ and $\mathbb{E}[z^2] = \sigma^2 + \mu^2$.

What is the KL divergence when we have a *multivariate* distribution where each dimension is independent? That is,

$$p(\mathbf{z}) = \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left(-\frac{(z_i - \mu_i)^2}{2\sigma_i^2}\right), \quad q(\mathbf{z}) = \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_i^2}{2}\right).$$

Part B: Autoencoder

Implement and train an autoencoder with a bottleneck $k = 2$ on MNIST.

Take a random sample of 100 points and embed them into the latent space with your trained encoder. Label the points based on their class, what do you notice about where the points cluster?

Part C: Variational Autoencoder

Implement a VAE, also with bottleneck $k = 2$. In particular, you should produce $\mu_{\mathbf{x}} \in \mathbb{R}^2$ and $\Sigma^2 = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$. Train the VAE with the reconstruction and multivariate KL loss you derived in part A.

Like for the autoencoder, embed 100 points into the latent space. How do those encoded with the VAE compare to those encoded with the autoencoder?

Problem 2: Principal Component Analysis

Part A: Frobenius Norm

Let $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$ be the rank r singular value decomposition of $\mathbf{X}$. Suppose that $\mathbf{X} \in \mathbb{R}^{n \times d}$, $\mathbf{U} \in \mathbb{R}^{n \times r}$, $\boldsymbol{\Sigma} \in \mathbb{R}^{r \times r}$, and $\mathbf{V} \in \mathbb{R}^{d \times r}$. In this problem, you will show that

$$\|\mathbf{X}\|_{\mathrm{F}}^2 = \sum_{i=1}^r \sigma_i^2.$$

Because the singular vectors are orthogonal, recall that $\mathbf{U}^\top\mathbf{U} = \mathbf{I}_{r \times r} = \mathbf{V}^\top\mathbf{V}$. Use this property to show that

$$\|\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top\|_{\mathrm{F}}^2 = \|\boldsymbol{\Sigma}\mathbf{V}^\top\|_{\mathrm{F}}^2 = \|\boldsymbol{\Sigma}\|_{\mathrm{F}}^2.$$

Finally, rewrite the previous equation as the sum of squared singular values.

Part B: PCA Reconstruction

Load an image $\mathbf{X}$ of your choice. Compute its singular value decomposition $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$, and plot the PCA reconstruction $\mathbf{X}_k$ for $k = 1, 10, 50, 100$.

Using the singular values computed in $\boldsymbol{\Sigma}$, plot the singular value *spectrum* (singular value by index) and the reconstruction error you computed in class by k.

Part C: Word Math

Load a text encoder of your choice. Choose n words e.g., young, old, puppy, and dog, etc. Use the text encoder to represent each word in d dimensions. Build the matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$, and find its singular value decomposition. Compute the latent representations of the words with $k = 2$. Plot and label the latent representations of the words. Can you get word math to work?