Plan

Logistics

Review

Course Response Forms

Problems

Sorry! I'll show you today...

Games @ 6pm Wednesday

↳ Extra credit (name and point # on paper)

↳ Pizza

Any questions about grades?

Project

↳ work hard tomorrow

↳ play hard tomorrow night

↳ presentation (recorded links!)

↳ Friday:

- problem
- codebase
- report

↳ Monday:

- self-grade

Review

$A \in \mathbb{R}^{n\times d}$, $b \in \mathbb{R}^n$

$$x^* = \underset{x}{\operatorname{argmin}} \|Ax - b\|_2^2$$

$$= (A^T A)^{-1} A^T b$$

$O(nd^2)$ time

$\Pi \in \mathbb{R}^{m\times n}$ $\quad m \approx d$

$$\tilde{x} = \underset{x}{\operatorname{argmin}} \|\Pi A x - \Pi b\|_2^2$$

$$\|A\tilde{x} - b\|_2^2 \le (1+\epsilon)\|Ax^* - b\|_2^2$$

ΠA, Πb fast

$\Pi x = SHDA_i$ for every column A_i

$$\Pi A = \begin{bmatrix} | & | & & | \\ SHDA_1 & SHDA_2 & \cdots & SHD_d \\ | & | & & | \end{bmatrix}$$

$$\Pi b = \underset{\text{sampling}}{S}\,\underset{\text{Hadamard}}{H}\,\underset{\text{diagonal}}{D}\, b$$

- $$Dx = \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & -1 \end{bmatrix} x = \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \end{bmatrix} * x$$

- $$H_k v = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}$$

Course Response Forms

- Important for improving teaching
- Really important for me
  (applying as prof in October, will use anonymized version)

What did I do that you liked? What could I improve?

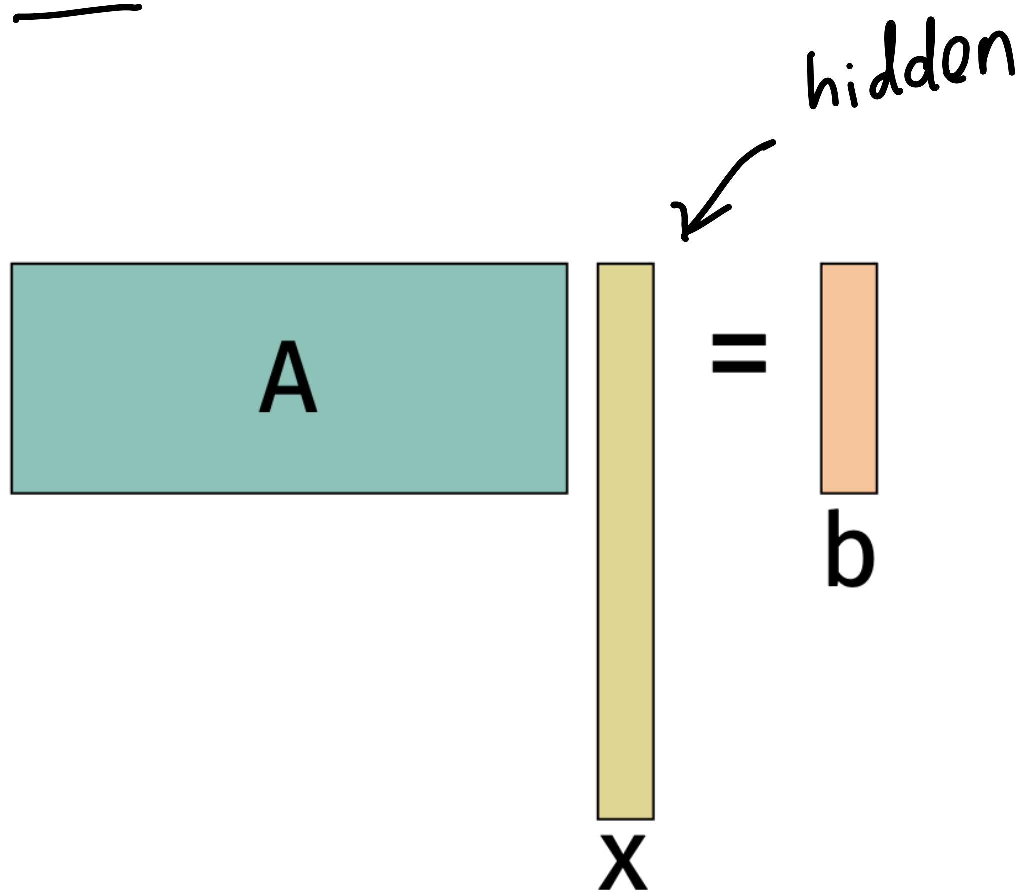
- Content and difficulty
- Group activities in class
- Accessibility/Receptiveness
- Self-grade on problems/LaTeX
- Daily form for questions
- Review the next day
- Afternoon guided problem solving
- Typed notes/slides

Sparse Recovery

$A \in \mathbb{R}^{n \times d}$ $\quad x \in \mathbb{R}^d$ $\quad b \in \mathbb{R}^n$

$Ax = b$

Now $\quad n << d$



Goal: Recover x by choosing A

Trivial solution?

Additional assumption:

x is k-sparse

i.e. $\|x\|_0 \leq k$

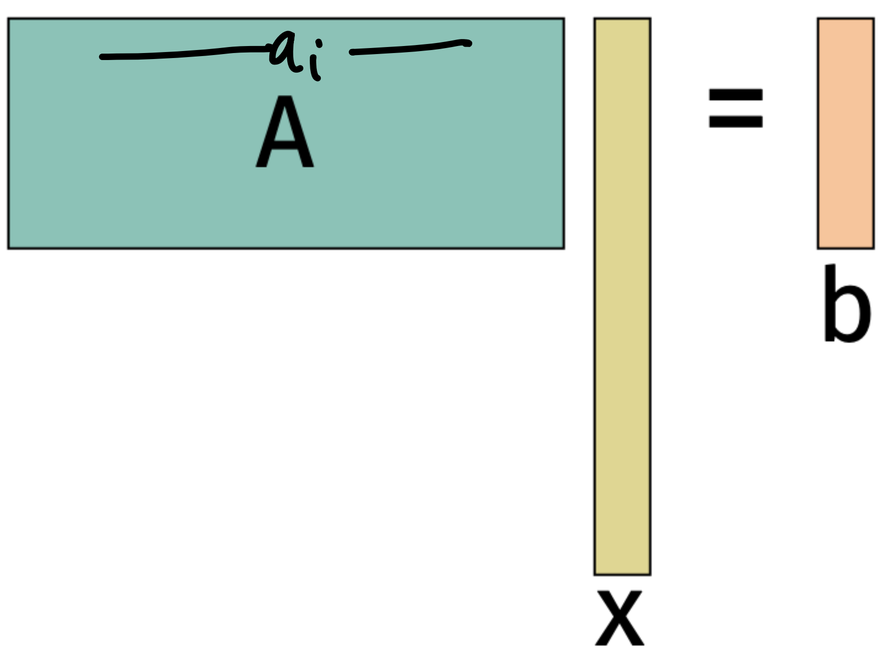
$$\|x\|_p^p = \sum_{i=1}^{d} x_i^p$$

$$\|x\|_0 = \sum_{i=1}^{d} x_i^0 = \#\text{ non-zero}$$

Goal: Recover k-sparse x with only a few measurements

$m = O(k \log k)$ measurements

$\langle a_i, x \rangle = b_i$

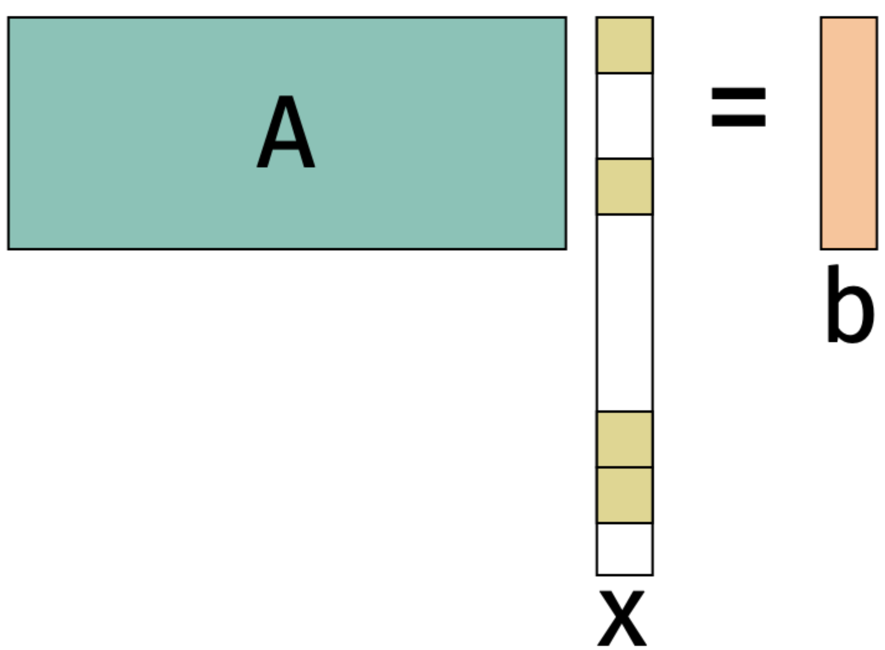


Applications

- Compress images because frequency basis is sparse
- Parameters that fit model which "achieve" Occam's razor
- X-rays where we measure body by putting frequency through and measuring output
- Earth exploration

Which A work?

Which definitely do not?



A has kruskal rank r if all sets of r columns are linearly independent

(k, ϵ) - Restricted Isometry Property

For all x with $\|x\|_0 \le k$,

$$(1-\epsilon)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\epsilon)\|x\|_2^2$$

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↳ Like JL

↳ Preserves norm of all k-sparse

(rather than set or subspace)

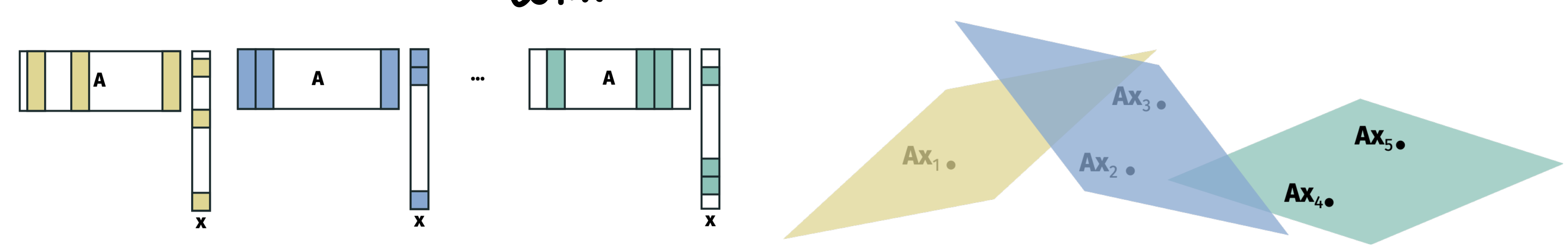
$$m = O\left(\frac{k \log(n) + \log(1/\delta)}{\epsilon^2}\right)$$

Then $\Pi \in \mathbb{R}^{m \times d}$ satisfies

proof: Consider $S_k = \{x : \|x\|_0 \le k\}$

$$S_k = U_1 \cup U_2 \cup U_3 \ldots \cup U_T$$

subspace with these coordinates



$$T = \binom{n}{k} \approx n^k$$

Set $\delta = \frac{\delta}{n^k}$ in subspace

Theorem: If A is $(2k, \epsilon)$-RIP for $\epsilon < 1$

then x is the unique minimizer

$$\min_{z} \|z\|_0 \quad \text{s.t.} \quad Az = b$$

$O(n^k)$ time to solve

Proof: Suppose for contradiction there exist

y with $\|y\|_0 \le \|x\|_0 = k$ and $Ay = b$

Theorem: If A is $(3k, \epsilon)$-RIP for $\epsilon < .17$

then x is the unique minimizer

$$\min_{z} \|z\|_1 \quad \text{s.t.} \quad Az = b$$

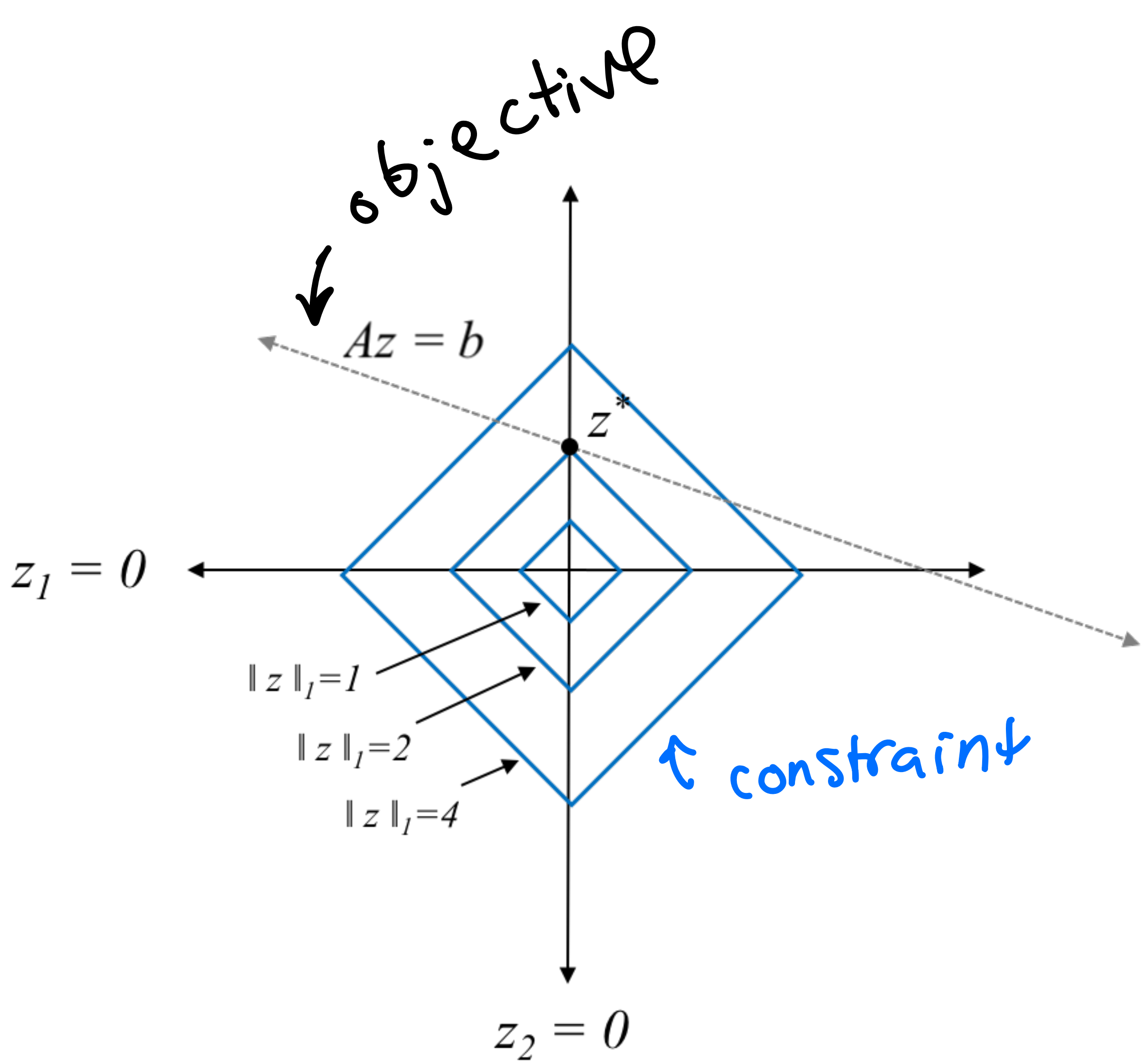
(one of 5 topics we skipped)

convex so we can solve with linear program in $O(n^{3.5})$ time

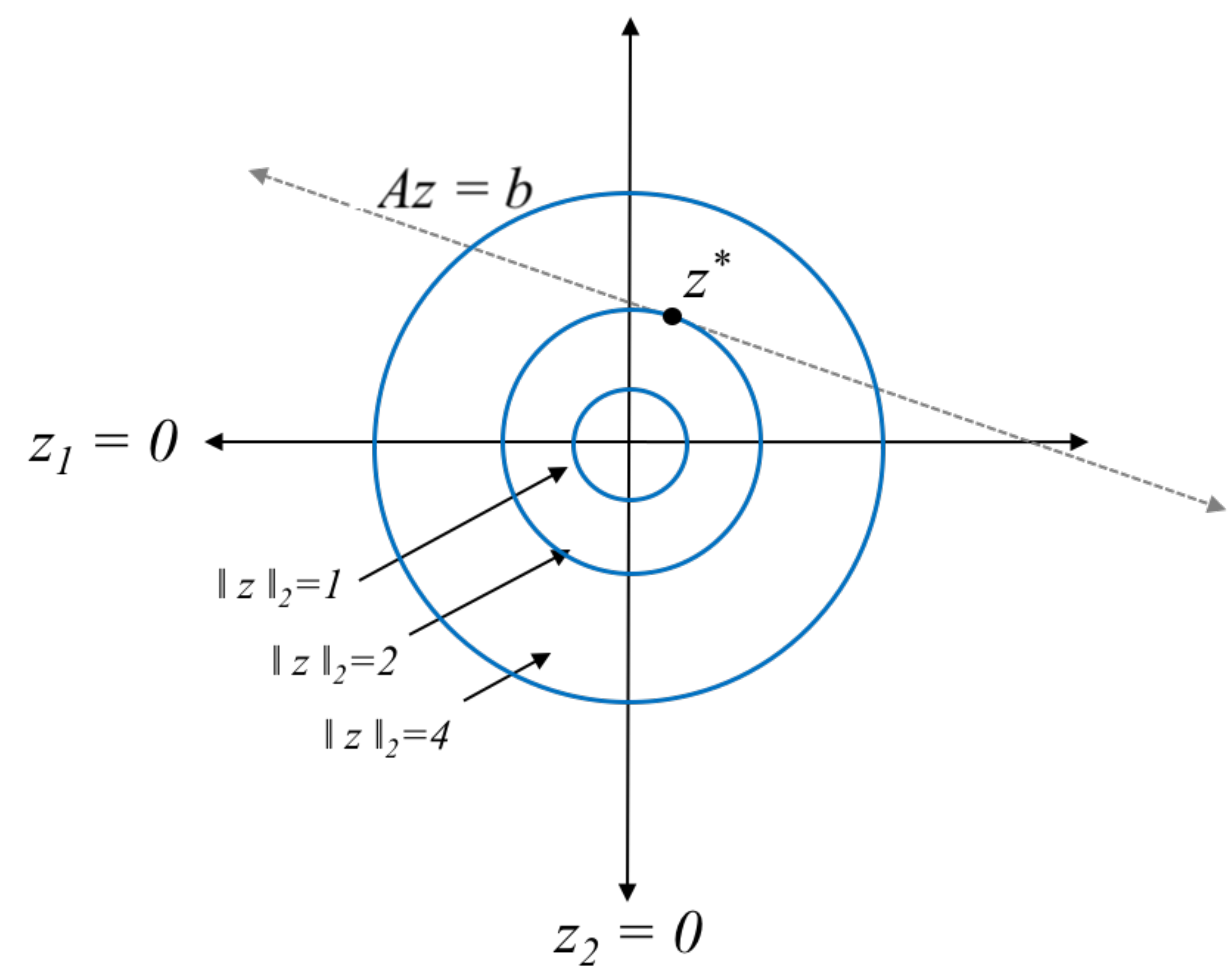
☆ Exponentially faster

☆ Like relaxation of spectral clustering without rounding

Intuition :



ℓ_1 minimization



ℓ_2 minimization

Tools:

(1) $\|a+b\|_2 \geq \|a\|_2 - \|b\|_2$

(2) $\|w\|_2 \leq \|w\|_1 \leq \sqrt{k}\,\|w\|_2$

$$\|w\|_2^2 = \sum_{i=1}^{d} w_i^2 \leq \sum_{i=1}^{d} w_i^2 + \sum_{i=1}^{d}\sum_{j=1}^{d} |w_i||w_j|$$

$$= \left(\sum_{i=1}^{d} |w_i|\right)^2 = \|w\|_1^2$$

$$\|w\|_1 = \langle w, \text{sign}(w) \rangle \leq \|w\|_2 \cdot \|\text{sign}(w)\|_2 = \|w\|_2 \cdot \sqrt{k}$$