Plan

Logistics

Review

Sketched Regression

Project

Problem

Regression

$A \in \mathbb{R}^{n \times d} \quad x \in \mathbb{R}^d$

$$x^* = \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \| Ax - b \|_2^2$$

Problem:

↳ Show $x^* = (A^T A)^{-1} A^T b$

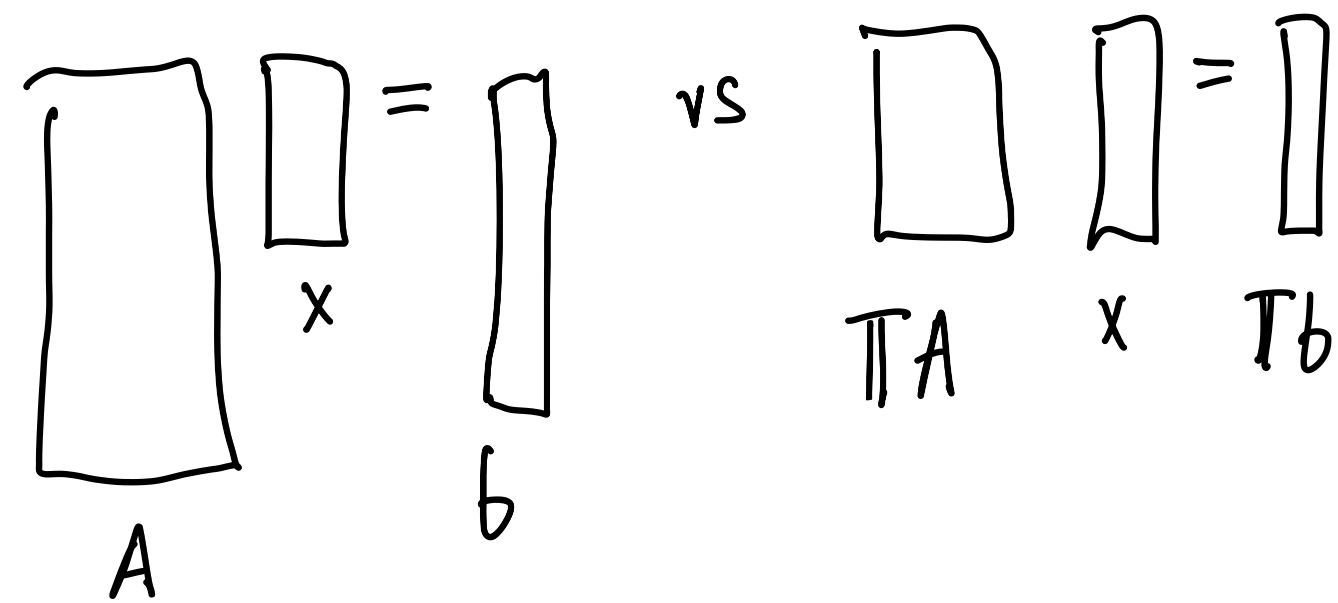
↳ Take $O(nd^2)$ time

Faster?

Sketched Regression

$$\tilde{x} = \underset{x \in \mathbb{R}^{n \times d}}{\operatorname{argmin}} \| \Pi A x - \Pi b \|_2^2$$

$\Pi \in \mathbb{R}^{m \times n} \quad m << d$



$m = O(d/\epsilon^2)$

Theorem 1: $\|A\tilde{x} - b\|_2^2 \le (1+\epsilon)\|Ax^* - b\|_2^2$ wp $9/10$

Claim: $(1-\epsilon)\|Ax - b\|_2^2 \le \|\Pi A x - \Pi b\|_2^2 \le (1+\epsilon)\|Ax - b\|_2^2 \; \forall x$

Proof of Theorem 1:

$$\|A\tilde{x} - b\|_2^2 \le \frac{1}{1-\epsilon}\|\Pi A \tilde{x} - \Pi b\|_2^2$$

$$\le \frac{1}{1-\epsilon}\|\Pi A x^* - \Pi b\|_2^2$$

$$\le \frac{(1+\epsilon)}{1-\epsilon}\|Ax^* - \Pi b\|_2^2$$

Distributional JL?

$m = O\left(\frac{\log 1/\delta}{\epsilon^2}\right)$

$(1-\epsilon)\|y\|_2^2 \le \|\Pi y\|_2^2 \le (1+\epsilon)\|y\|_2^2$ ~~for fixed y~~ wp $1-\delta$

We want it to hold for $Ax-b$ for all $x \in \mathbb{R}^d$

d-dimensional subspace

Goal!! :)

Subspace Embedding Theorem

$m = O\left(\frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon^2}\right)$ $\mathcal{U} \subset \mathbb{R}^n$ is d-dim subspace

$(1-\epsilon)\|y\|_2^2 \le \|\Pi y\|_2^2 \le (1+\epsilon)\|y\|_2^2$ for all $y \in \mathcal{U}$ wp $1-\delta$

Prove Subspace Embedding

Suffices to prove for w on unit sphere in $\mathcal{U}$

$$(1-\epsilon)\|w\|_2 \leq \|\Pi w\|_2 \leq (1+\epsilon)\|w\|_2$$

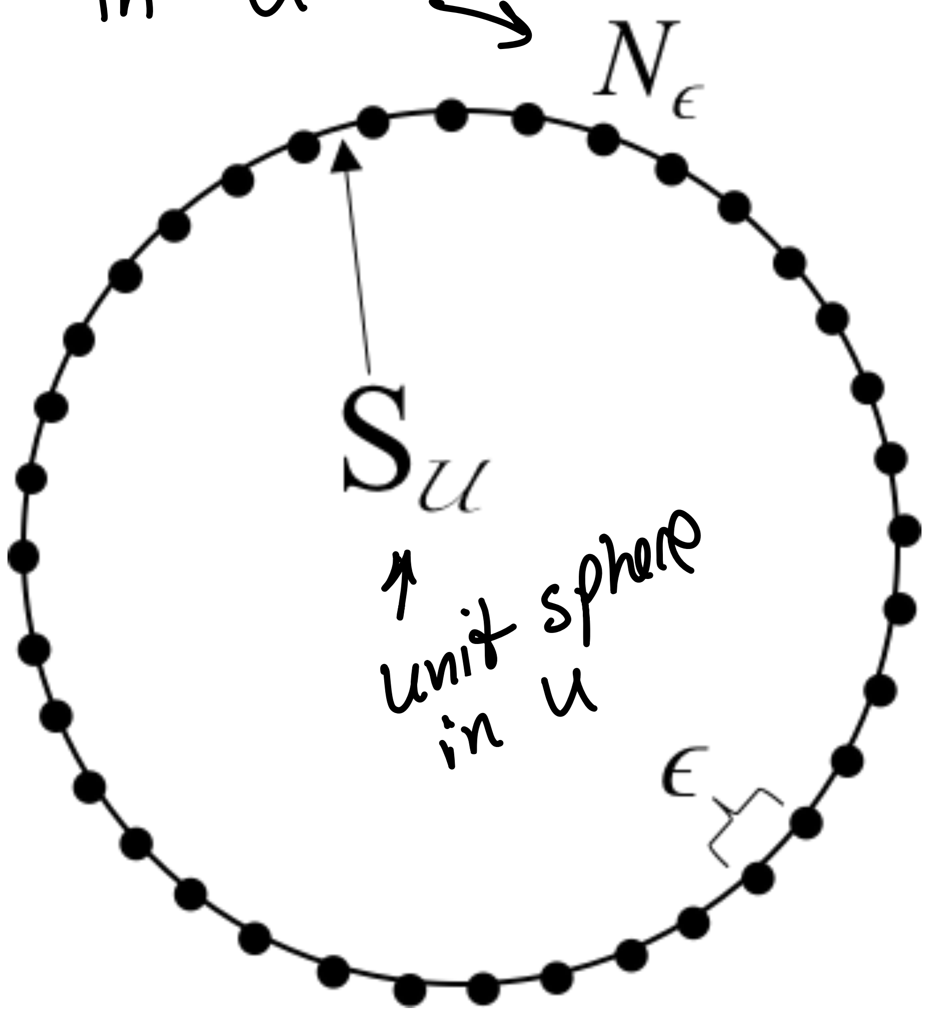
$$C(1-\epsilon)\|w\|_2 \leq C\|\Pi w\|_2 \leq C(1+\epsilon)\|w\|_2$$

any scaling

$$(1-\epsilon)\|Cw\|_2 \leq \|C\Pi w\|_2 \leq (1+\epsilon)\|Cw\|_2$$

Not too many "different" points on sphere

ϵ-net in $\mathcal{U}$



Goal:

JL Lemma with union bound

1. $(1-\epsilon)\,\|w\|_2 \le \|\Pi w\|_2 \le (1+\epsilon)\,\|w\|_2$

   for all $w \in N_\epsilon$

2. $\min_{w \in N_\epsilon} \|v - w\|_2 \le \epsilon$

   for all $v \in S_\mathcal{U}$

3. $|N_\epsilon| \le (4/\epsilon)^d$

Set $\delta' = \frac{1}{|N_\epsilon|} \cdot \delta \approx \left(\frac{\epsilon}{3}\right)^d \delta$ $\quad$ $\log 1/\delta' = d \log \frac{3}{\epsilon} + \log 1/\delta$

$$m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right) = O\left(\frac{d \log 1/\epsilon + \log(1/\delta)}{\epsilon^2}\right)$$ in distributional JL

$$(1-\epsilon)\|w\|_2^2 \le \|\Pi w\|_2^2 \le (1+\epsilon)\|w\|_2^2$$ ~~for all $w \in N_\epsilon$~~ wp $1-\delta$

What about $v \in S_u$ but not N_ϵ?

$$w_0 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \|v - w\|_2 \qquad r_0 = v - w_0 \qquad c_1 = \|r_0\|_2$$

$$w_1 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \left\|\frac{r_0}{c_1} - w\right\|_2 \qquad r_1 = v - w_0 - c_1 w_1 \qquad c_2 = \|r_1\|_2$$

$$w_2 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \left\|\frac{r_1}{c_2} - w\right\|_2 \qquad r_2 = v - w_0 - c_1 w_1 - c_2 w_2 \qquad c_3 = \|r_2\|_2$$

$$\vdots$$

Induction: $\|r_i\|_2 < \epsilon^i$, $\left\|\frac{r_{i-1}}{c_i} - w_i\right\|_2 \leq \epsilon$ by $\mathcal{N}_\epsilon$

$$\Rightarrow \|r_i\|_2 = \|r_{i-1} - c_i w_i\|_2 \leq \epsilon c_i = \epsilon \|r_{i-1}\|_2 < \epsilon \epsilon^{i-1} = \epsilon^i$$

$\| \Pi v \|_2$

in N_ϵ

$= \| \Pi w_0 + c_1 \Pi w_1 + c_2 \Pi w_2 + \dots \|_2$

$\le \| \Pi w_0 \|_2 + c_1 \| \Pi w_1 \|_2 + c_2 \| \Pi w_2 \|_2 + \dots$

$\le (1+\epsilon) \|w_0\|_2 + (1+\epsilon) \cdot \epsilon \|w_1\|_2 + (1+\epsilon) \cdot \epsilon^2 \|w_2\|_2 + \dots$

$= 1 + 2\epsilon + 2\epsilon^2 + 2\epsilon^3 + \dots$

$\le 1 + \dfrac{2\epsilon}{1-\epsilon} \le 1 + 4\epsilon$ for $0 < \epsilon < 1/2$

triangle inequality

$\| a + b \|_2 \le \| a \|_2 + \| b \|_2$