Plan

Logistics

Review

Sketched Regression

Problem

Project

Regression

$A \in \mathbb{R}^{n \times d} \quad x \in \mathbb{R}^d$

$$x^* = \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \| Ax - b \|_2^2$$

Problem:

↳ Show $x^* = (A^T A)^{-1} A^T b$

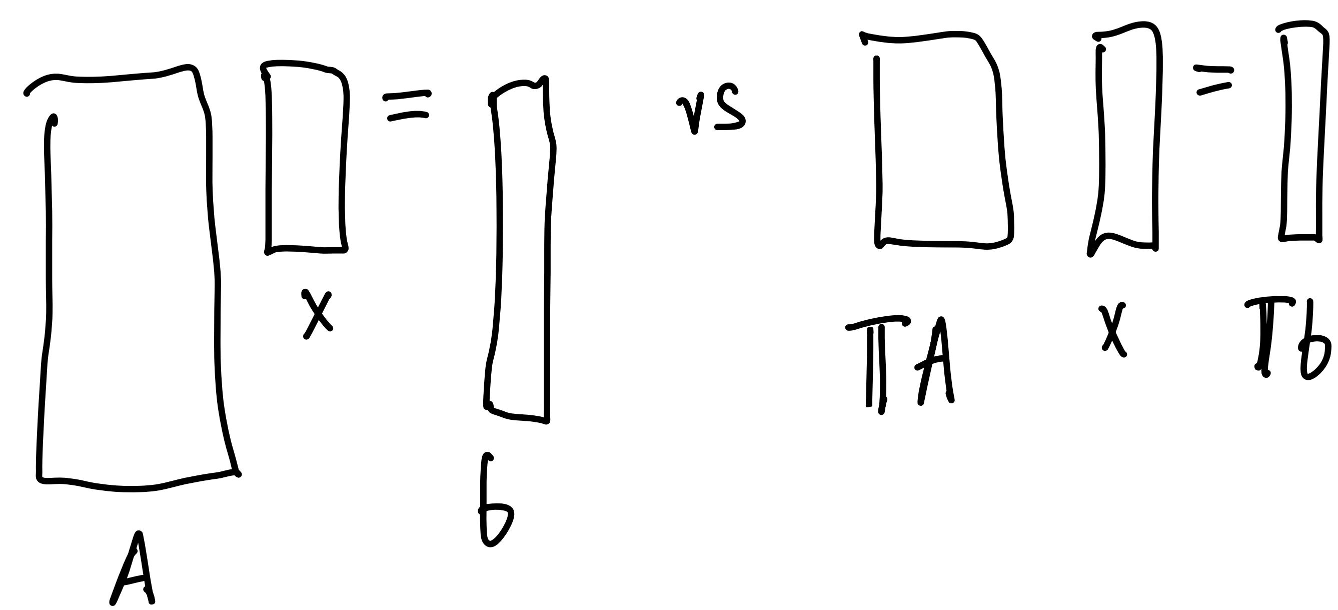
↳ Take $O(nd^2)$ time

Faster?

Sketched Regression

$$\tilde{x} = \underset{x \in \mathbb{R}^{n \times d}}{\operatorname{argmin}} \| \Pi A x - \Pi b \|_2^2$$

$\Pi \in \mathbb{R}^{m \times n} \quad m << d$



$m = O(d/\epsilon^2)$

Theorem 1: $\|A\tilde{x} - b\|_2^2 \le (1+\epsilon)\|Ax^* - b\|_2^2$ wp $9/10$

Claim: $(1-\epsilon)\|Ax - b\|_2^2 \le \|\Pi Ax - \Pi b\|_2^2 \le (1+\epsilon)\|Ax - b\|_2^2 \ \forall x$

Proof of Theorem 1:

$$\|A\tilde{x} - b\|_2^2 \le \frac{1}{1-\epsilon}\|\Pi A\tilde{x} - \Pi b\|_2^2$$

$$\le \frac{1}{1-\epsilon}\|\Pi Ax^* - \Pi b\|_2^2$$

$$\le \frac{(1+\epsilon)}{1-\epsilon}\|Ax^* - \Pi b\|_2^2$$

Distributional JL?

$m = O\left(\frac{\log 1/\delta}{\epsilon^2}\right)$

$(1-\epsilon)\|y\|_2^2 \le \|\Pi y\|_2^2 \le (1+\epsilon)\|y\|_2^2$ ~~for fixed y~~ wp $1-\delta$

We want it to hold for $Ax - b$ for all $x \in \mathbb{R}^d$

d-dimensional subspace

Goal!! :)

Subspace Embedding Theorem

$m = O\left(\frac{d\log(1/\epsilon) + \log(1/\delta)}{\epsilon^2}\right)$

$U \subset \mathbb{R}^n$ is d-dim subspace

$(1-\epsilon)\|y\|_2^2 \le \|\Pi y\|_2^2 \le (1+\epsilon)\|y\|_2^2$ for all $y \in U$ wp $1-\delta$

Prove Subspace Embedding

Suffices to prove for w on unit sphere in U

$$(1-\epsilon)\|w\|_2 \leq \|\Pi w\|_2 \leq (1+\epsilon)\|w\|_2$$

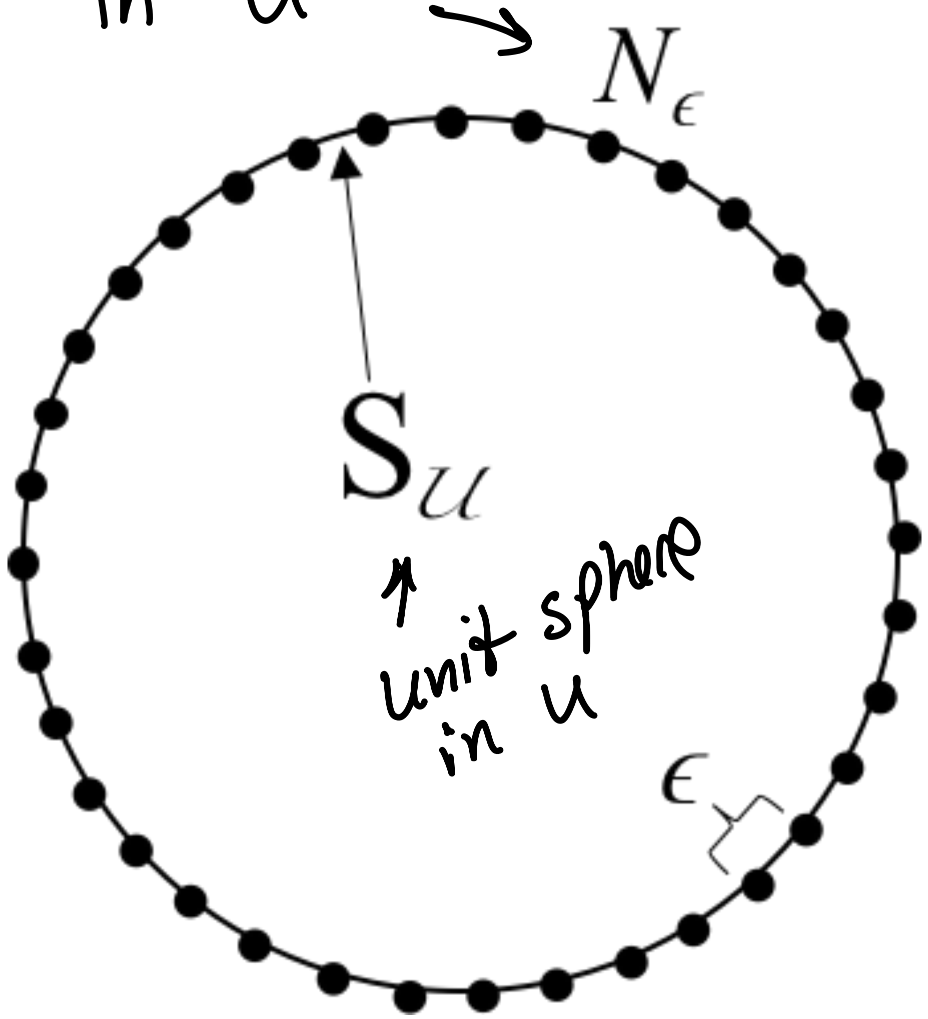
$$c\,(1-\epsilon)\|w\|_2 \leq c\,\|\Pi w\|_2 \leq c\,(1+\epsilon)\|w\|_2$$

any scaling

$$(1-\epsilon)\|c\,w\|_2 \leq \|c\,\Pi w\|_2 \leq (1+\epsilon)\|c\,w\|_2$$

Not too many "different" points on sphere

ϵ-net in $\mathcal{U}$



Goal:

JL Lemma with union bound

1. $(1-\epsilon)\,\|w\|_2 \le \|\Pi w\|_2 \le (1+\epsilon)\,\|w\|_2$

   for all $w \in N_\epsilon$

2. $\min_{w \in N_\epsilon} \|v - w\|_2 \le \epsilon$

   for all $v \in S_{\mathcal{U}}$

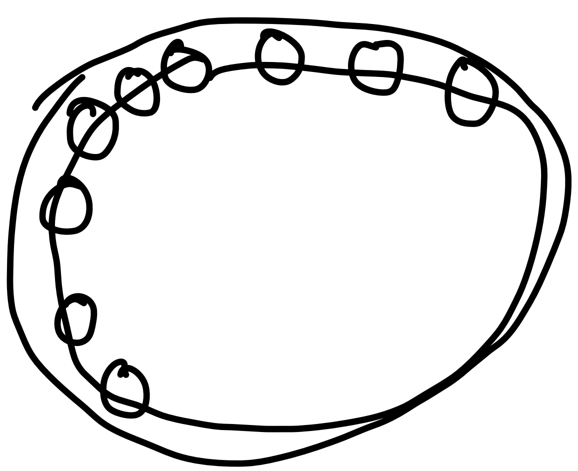
3. $|N_\epsilon| \le (4/\epsilon)^q$

Construct N_ϵ

$N_\epsilon = \{\}$ ← not efficient!

while point in sphere that is more than ϵ away:

add point to N_ϵ



Is it too large?

Every point ϵ away

So each has $\epsilon/2$ ball

All balls in sphere with $1+\epsilon/2$ radius

$\mathrm{vol}(d, r) = c \cdot r^d$ ← d-dimensional ball with radius r

how many can we fit?

$$\mathrm{vol}(d, \epsilon/2)\, |N_\epsilon| \leq \mathrm{vol}(d, 1+\epsilon/2)$$

$$\Rightarrow |N_\epsilon| \leq \frac{\mathrm{vol}(d, 1+\epsilon/2)}{\mathrm{vol}(d, \epsilon/2)}$$

$$= \frac{c \cdot (1+\epsilon/2)^d}{c \cdot \left(\frac{\epsilon}{2}\right)^d} = \left(\frac{2}{\epsilon} + 1\right)^d$$

$$\leq (3/\epsilon)^d$$

Set $\delta' = \frac{1}{|\mathcal{N}_\epsilon|} \cdot \delta \approx \left(\frac{\epsilon}{3}\right)^d \delta$ $\qquad$ $\log 1/\delta' = d \log \frac{3}{\epsilon} + \log 1/\delta$

$m = O\left(\frac{\log (1/\delta')}{\epsilon^2}\right) = O\left(\frac{d \log 1/\epsilon + \log(1/\delta)}{\epsilon^2}\right)$ in distributional JL

$(1-\epsilon)\,\|w\|_2^2 \leq \|\Pi w\|_2^2 \leq (1+\epsilon)\,\|w\|_2^2$ ~~for all $w \in \mathcal{N}_\epsilon$~~ wp $1-\delta$

What about $v \in S_u$ but not $\mathcal{N}_\epsilon$?

$$w_0 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \|v - w\|_2 \qquad r_0 = v - w_0 \qquad c_1 = \|r_0\|_2$$

$$w_1 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \left\|\frac{r_0}{c_1} - w\right\|_2 \qquad r_1 = v - w_0 - c_1 w_1 \qquad c_2 = \|r_1\|_2$$

$$w_2 = \underset{w \in \mathcal{N}_\epsilon}{\operatorname{argmin}} \left\|\frac{r_1}{c_2} - w\right\|_2 \qquad r_2 = v - w_0 - c_1 w_1 - c_2 w_2 \qquad c_3 = \|r_2\|_2$$

$$\vdots$$

Induction: $\|r_i\|_2 < \epsilon^i$, $\left\|\frac{r_{i-1}}{c_i} - w_i\right\|_2 \le \epsilon$ by $\mathcal{N}_\epsilon$

$$\Rightarrow \|r_i\|_2 = \|r_{i-1} - c_i w_i\|_2 \le \epsilon c_i = \epsilon \|r_{i-1}\|_2 < \epsilon \epsilon^{i-1} = \epsilon^i$$

$\| \Pi v \|_2$

$= \| \Pi w_0 + c_1 \Pi w_1 + c_2 \Pi w_2 + \dots \|_2$ $\quad$ in N_ϵ

$\leq \| \Pi w_0 \|_2 + c_1 \| \Pi w_1 \|_2 + c_2 \| \Pi w_2 \|_2 + \dots$

$\leq (1+\epsilon) \|w_0\|_2 + (1+\epsilon) \cdot \epsilon \|w_1\|_2 + (1+\epsilon) \cdot \epsilon^2 \|w_2\|_2 + \dots$

$= 1 + 2\epsilon + 2\epsilon^2 + 2\epsilon^3 + \dots$

$\leq 1 + \dfrac{2\epsilon}{1-\epsilon} \leq 1 + 4\epsilon$ $\quad$ for $0 < \epsilon < 1/2$

triangle inequality

$\| a + b \|_2 \leq \|a\|_2 + \|b\|_2$

$\| \Pi v \|_2$

$$= \| \Pi w_0 + c_1 \underset{\text{in } N_\epsilon}{\Pi w_1} + c_2 \Pi w_2 + \dots \|_2$$

$$\geq \| \Pi w_0 \|_2 - c_1 \| \Pi w_1 \|_2 - c_2 \| \Pi w_2 \|_2 - \dots$$

$$\geq (1-\epsilon) \| w_0 \|_2 + (1-\epsilon) \cdot \epsilon \| w_1 \|_2 + (1-\epsilon) \cdot \epsilon^2 \| w_2 \|_2 + \dots$$

$$= 1 - 2\epsilon + 2\epsilon^2 - 2\epsilon^3 + \dots$$

$$\geq 1 + \frac{2\epsilon}{1+\epsilon} \geq 1 + 4\epsilon \quad \text{for } 0 < \epsilon < 1/2$$

triangle inequality

$\| a + b \|_2 \leq \| a \|_2 + \| b \|_2$

$\| x + y + -x \|_2 \leq \| x + y \|_2 + \| x \|_2$

$\| y \|_2 \leq \| x + y \|_2 + \| x \|_2$

reverse

$\| x + y \|_2 \geq \| y \|_2 - \| x \|_2$

$$(1 - \epsilon) \| v \|_2 \leq \| \Pi v \|_2 \leq (1 + \epsilon) \| v \|_2 \quad \text{for any } v \in U$$