Plan

Logistics

Review

spectral Graph Thoory

Project

1. Codebase (12)
2. Presentation (6)
   - each person records 2x whole talk
3. Report (6)
   written explanation

Goal: Find top eigenvector fast

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Power Method

$$A = V \Sigma^2 V^T$$

$$A = \sum_{i=1}^{d} v_i v_i^T \sigma_i^2$$

$z^{(0)} \sim \mathcal{N}(0,1)$

$z^{(0)} = z^{(0)} / \|z\|_2$

$$z^{(0)} = \sum_{j=1}^{d} v_j \, c_j^{(0)}$$

for $t = 1, \ldots, q$

$\quad z^{(t)} = A z^{(t-1)}$

$\quad n_t = \|z^{(t)}\|_2$

$\quad z^{(t)} = z^{(t)} / n_t$

Return $z^{(q)}$

$\approx v_1$

$$\frac{A z^{(t)}}{n_t} = \sum_{i=1}^{d} v_i \frac{\sigma_i^2}{n_t} c_i^{(t-1)}$$

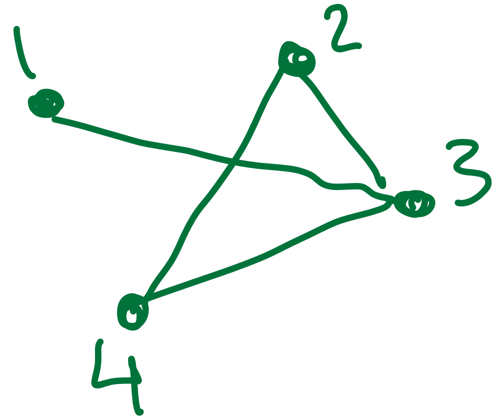
$$= \sum_{i=1}^{d} v_i \sigma_i^2 \frac{\sigma_i^2}{n_t} c_i^{(t-2)}$$

$$= \sum_{i=1}^{d} v_i \frac{\sigma_i^{2t}}{\prod_{l=1}^{t} n_l} \cdot c_i^{(0)}$$

Spectral Graph Theory

$$G = (V, E)$$

n nodes, m edges



- social networks
- citations
- internet
- more!

Adjacency A

$$A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$$

Degree D

$$D_{i,j} = \begin{cases} \sum_{k=1}^{n} A_{i,k} & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$D = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 2 \end{bmatrix}$$

$$L = D - A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Normalized

$$\bar{A} = D^{-1/2} A D^{-1/2}$$

$$L = I - \bar{A}$$

↳ If L has k 0-eigenvalues

then k connected components,

↳ $\sum_{i=1}^{n} \lambda_i^3 = 6 \cdot \#$ triangles

↳ $\sum_{i=1}^{n} \lambda_i^q \simeq \#$ q-cycles

$B \in \mathbb{R}^{m \times n}$ edge-incidence

$$\underbrace{B_{(i,j),k}}_{} = \begin{cases} 1 & \text{if } k=i \\ -1 & \text{if } k=j \\ 0 & \text{else} \end{cases}$$

(i,j) ↑ edge, k ↑ node

$$b_{(i,j)} = \begin{array}{ccccccccc} & & & i & & & j & & \\ [0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0] \end{array}$$

$L =$ ______

(1) Example

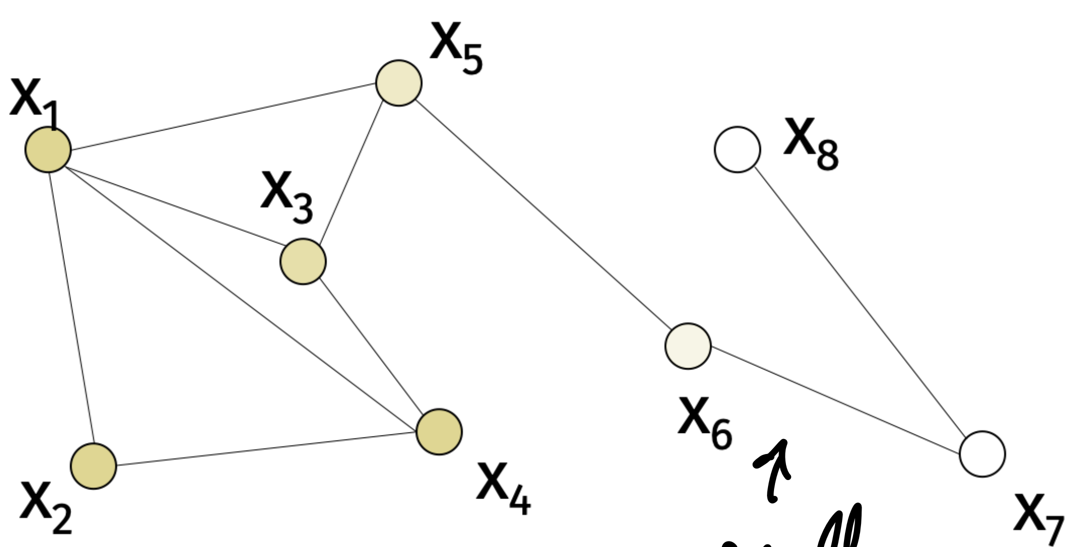
(2) Generality

$x^T L x =$ ______

$L = V \Sigma^2 V^T =$ ______

$$f(x) = x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2$$

$f(x)$ small when x is "smooth"



small changes between adjacent nodes

$v_1, \ldots, v_n$ eigenvectors of L

Courant-Fischer min-max

$$v_n = \underset{\|v\|_2 = 1}{\arg\min} \; v^T L v$$

$$v_{n-1} = \underset{\|v\|_2 = 1,\; v \perp v_{n-1}}{\arg\min} \; v^T L v$$

$$\vdots$$

$$v_1 = \underset{\|v\|_2 = 1,\; v \perp v_2 \perp v_3 \perp \cdots \perp v_n}{\arg\min} \; v^T L v$$

Spectral Clustering

Partition

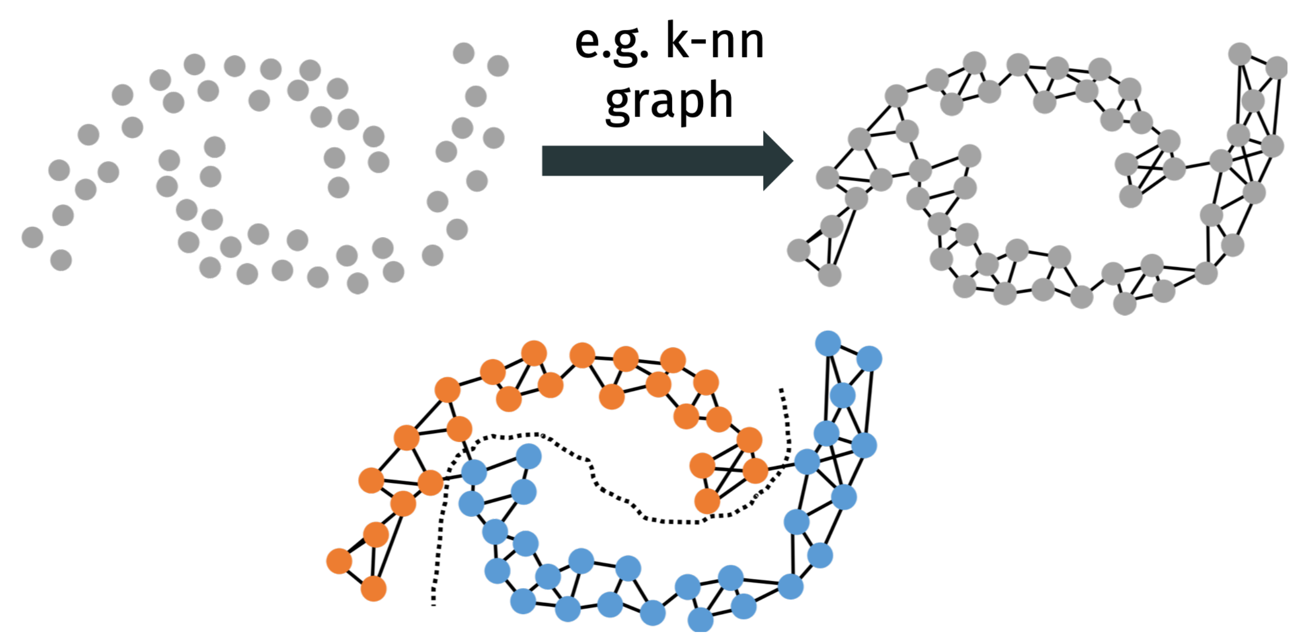
- ↳ social networks
- ↳ machine learning
- ↳ graph visualization

$c \in \{-1,1\}^n$ cut indicator

$$\begin{aligned} c^T L c &= c^T B^T B c \\ &= \sum_{(i,j)\in E} (c_i - c_j)^2 \\ &= 4\,\mathrm{cut}(S, S^c) \end{aligned}$$

$\propto$ quality of cut

Goal: Separate between S, S^c



Example with karate graph

Different ways to formalize best cut

β- Balanced

$\beta \in [0, 1/2]$

$$\underset{S \subseteq V}{\operatorname{argmin}} \; cut(S, S^c) \quad \text{s.t. } \min(|S|, |S^c|) \geq \beta |V|$$

Sparsest

$$\underset{S \subseteq}{\operatorname{argmin}} \; \frac{cut(S, S^c)}{\min(|S|, |S^c|)}$$

Algorithm

1. Compute second smallest eigenvector v_{n-1} of graph
2. Define S as nodes with positive entries
3. Return S

Laplacian view

$$c \in \left\{ -\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}} \right\}^n$$

$$c^T L c = 4 \cdot \mathrm{cut}(S, S^c) \qquad \underline{\text{cut size}}$$

$$|c^T \mathbb{1}| = \big| |S| - |S^c| \big| \cdot \frac{1}{\sqrt{n}} \qquad \underline{\text{imbalance}}$$

Balanced Cut

$$\min \quad c^T L c \quad \text{s.t. } c^T \mathbb{1} = 0$$

$$c \in \{-1/\sqrt{n}, -1/\sqrt{n}\}^n$$

Familiar?

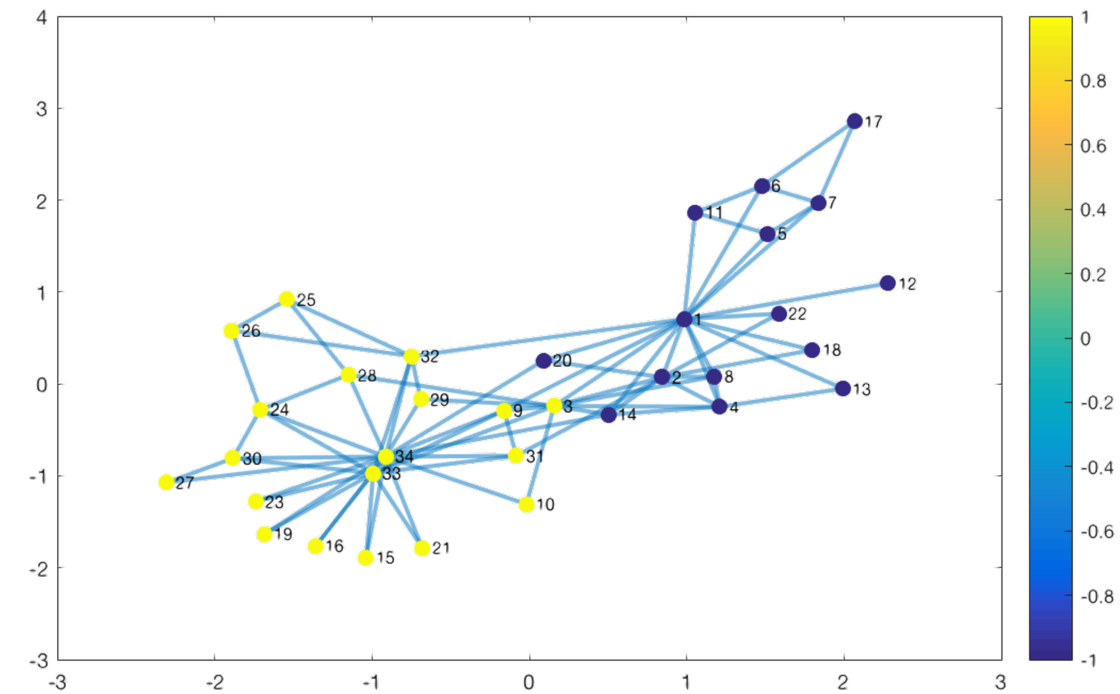
Relaxed Balanced Cut

$$\min \quad c^T L c \quad \text{s.t. } c^T \mathbb{1} = 0$$

$$\|c\|_2^2 = 1$$

Relaxed Balanced Cut

$$\min \; c^T L c \quad \text{s.t.} \quad c^T \mathbb{1} = 0$$
$$\|c\|_2^2 = 1$$



$$v_n = \operatorname*{argmin}_{v:\|v\|_2 = 1} v^T L v$$

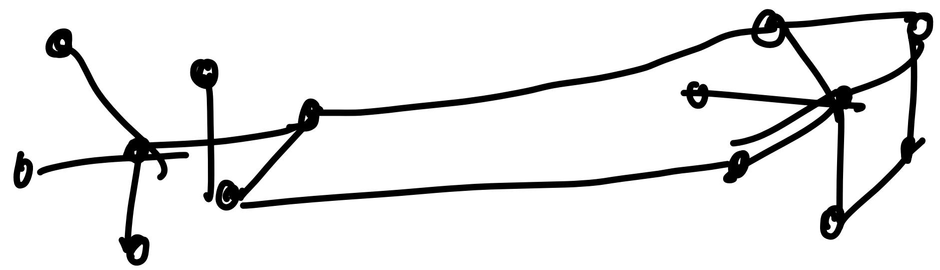
$$v_n = \mathbb{1} \quad \text{since} \quad L\mathbb{1} = 0 \cdot \mathbb{1}$$

HW connection!

$$v_{n-1} = \operatorname*{argmin}_{v:\|v\|_2 = 1,\; v \perp v_{n-1}} v^T L v$$

$$\langle v, v_{n-1} \rangle = \langle \mathbb{1}, v_{n-1} \rangle = 0$$

Stochastic Block Model



Pr(edge between) = q

Pr(edge within) = p

$$\mathbb{E}[A] = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ \\ B \end{array} & \begin{bmatrix} & \\ & \\ & \end{bmatrix} \end{array}$$

$$\mathbb{E}[A] = V \Lambda V^T$$