Plan

Logistics

Review

Singular Value Decomposition

Low-Rank Approximation

Great job on problem set!

↳ Come talk to me with questions or if I left comment

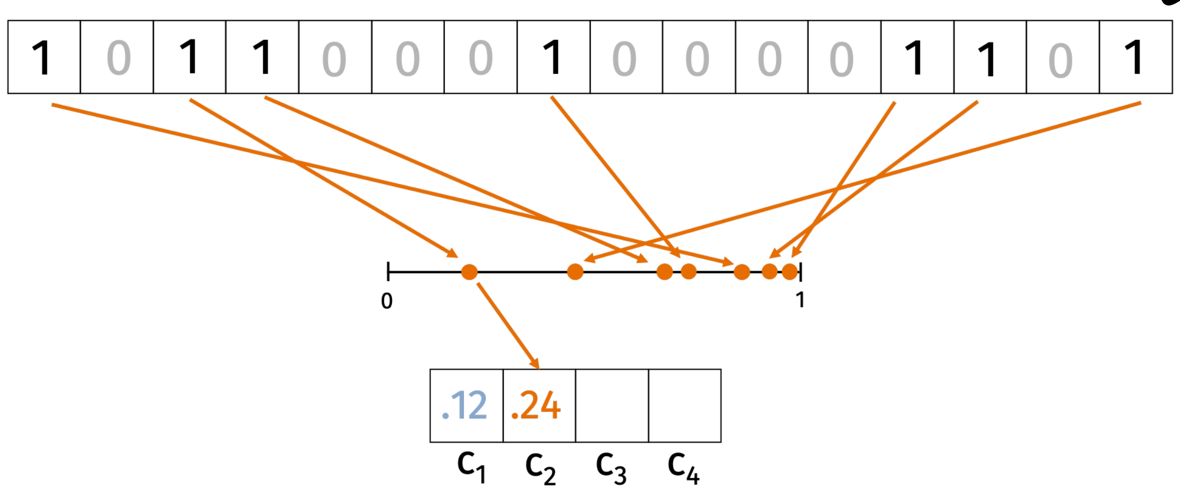
Goal: less problem time outside class

1. I'll provide more guidance
2. Make sure you have *rough* solution before you leave
3. Calibrate to my solutions

Locality Sensitive Hashing

Find "similar" vectors (no dim dependence)

e.g. song



$\in \{0,1\}^d$

$$\Pr(c_i(x) = c_i(y)) = \frac{|X \cap Y|}{|X \cup Y|} = J(x,y)$$

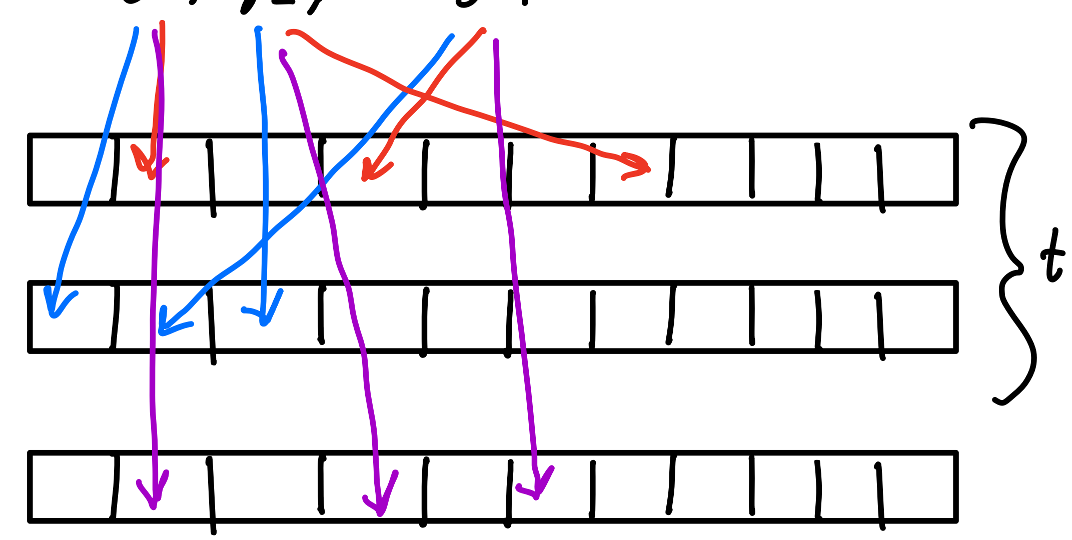
$$c : \{0,1\}^d \to [0,1]^r$$

$$g : [0,1]^r \to \{1, \dots, m\}$$

$$\Pr(g(x) = g(y)) = \Pr(c_i(x) = c_i(y))^r$$

Preprocess

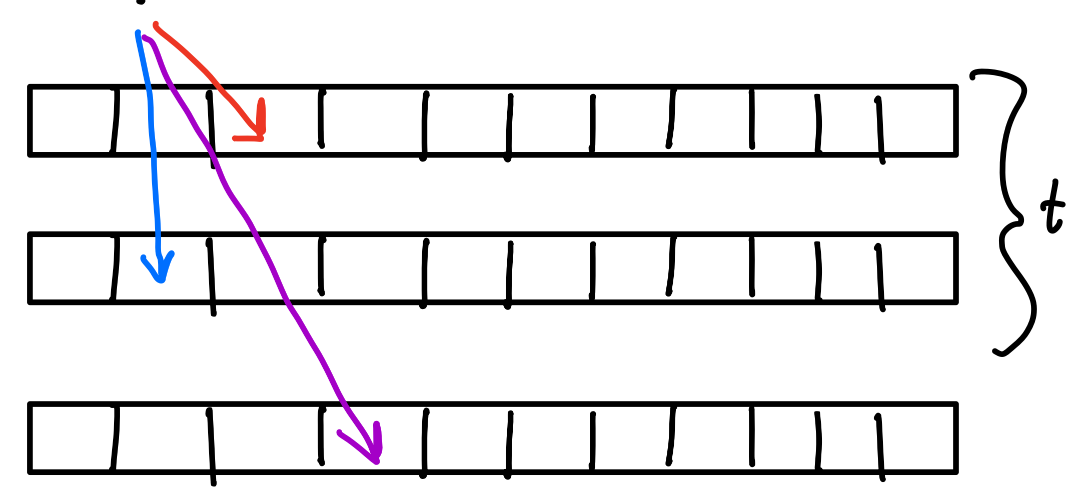
$y_1, y_2, \dots, y_n$



Query

Hits: y_n, y_2

x



$$\Pr(\text{hit}) = 1 - (1 - J(x,y)^r)^t$$

or $\left(1 - \frac{\theta}{\pi}\right)$

Linear Algebra

Consider $X \in \mathbb{R}^{d \times d}$

Eigenvector $v \in \mathbb{R}^d$ $\|v\|_2 = 1$

and eigenvalue $\lambda \in \mathbb{R}$ if

$$Xv = \lambda v$$

Suppose X has d eigenvectors/values

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$$

$$v_1, v_2, \ldots, v_d$$

$$V = \begin{bmatrix} - v_1 - \\ - v_2 - \\ \vdots \\ - v_d - \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{bmatrix}$$

$$X = V \Lambda V^T$$

eigendecomposition

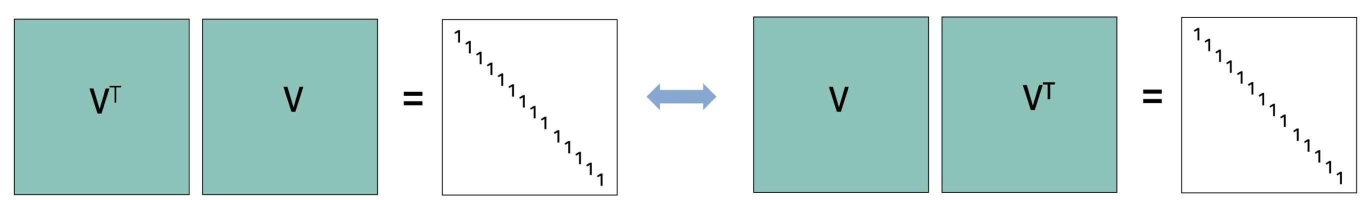
$$V V^T = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ by orthonormal

$$= I$$

$$\Rightarrow V^T = V^{-1}$$

$$I = V^T (V^T)^{-1}$$

$$= V^T (V^{-1})^{-1} = V^T V$$



(1) Show $\|Vx\|_2^2 = \|x\|_2^2$

(2) Show $\|V^T x\|_2^2 = \|x\|_2^2$

By definition, $\|X\|_F^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} X_{i,j}$

(3) Show $\|VX\|_F^2 = \|X\|_F^2$

Singular Value Decomposition

$X \in \mathbb{R}^{n \times d} \quad n \geq d \quad$ WLOG

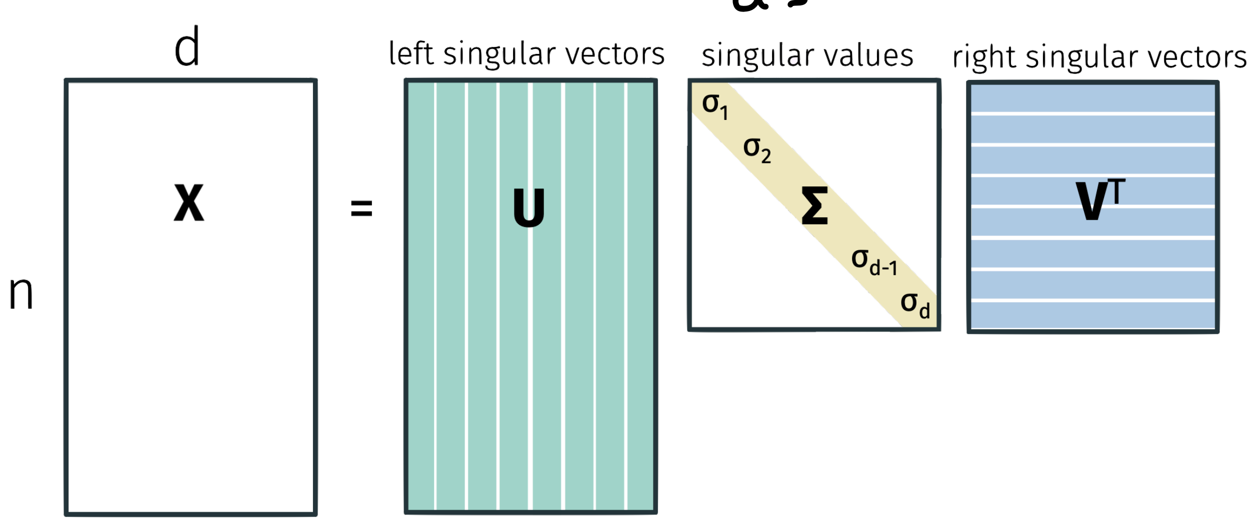
$X = U \Sigma V^T$

$U \in \mathbb{R}^{n \times d} \quad \Sigma \in \mathbb{R}^{d \times d} \quad V \in \mathbb{R}^{d \times d}$

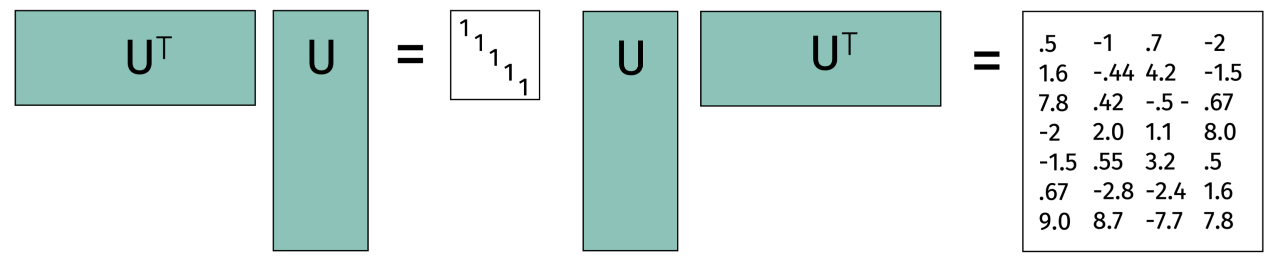
$U^T U = I \qquad V^T V = I$

Why?

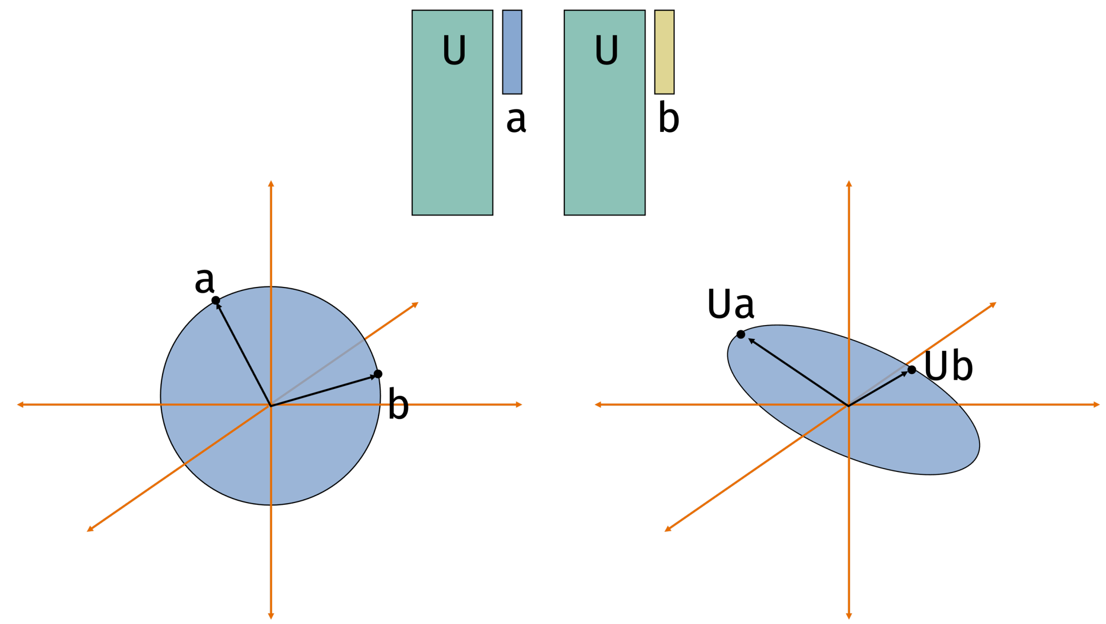
$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix} \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_d \geq 0$$



$U^T U = I \quad$ but $\quad U U^T \neq I$

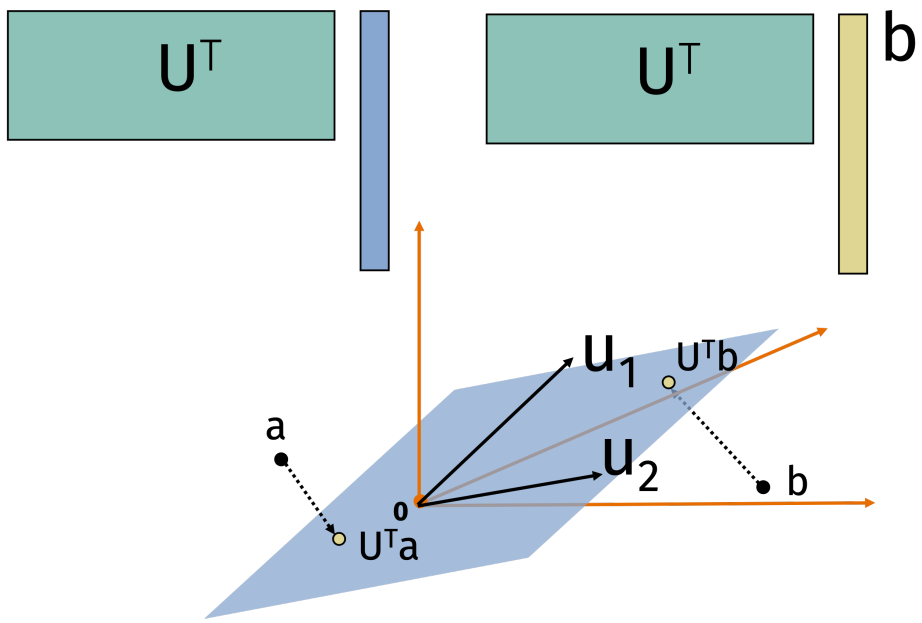


Multiplying by U rotates into higher dimensions



$$\|Ux\|_2^2 = \|x\|_2^2$$

Multiplying by U^T projects down



$$\|U^T x\|_2^2 \leq \|x\|_2^2$$

So $Xa = U(\Sigma(V^T a))$

1. Rotate again
2. Scale coordinates
3. Rotate again

Eigendecomposition vs SVD

$X \in \mathbb{R}^{d \times d}$	$X \in \mathbb{R}^{d \times k}$
λ_i	$\sigma_i \geq 0$
V ortho cols iff —	U, V ortho cols

SVD + Eigendecomposition

$X = U \Sigma V^T$

$X^T X = ?$

Eigenvalues of $X^T X$ are ___ of X?

SVD used for

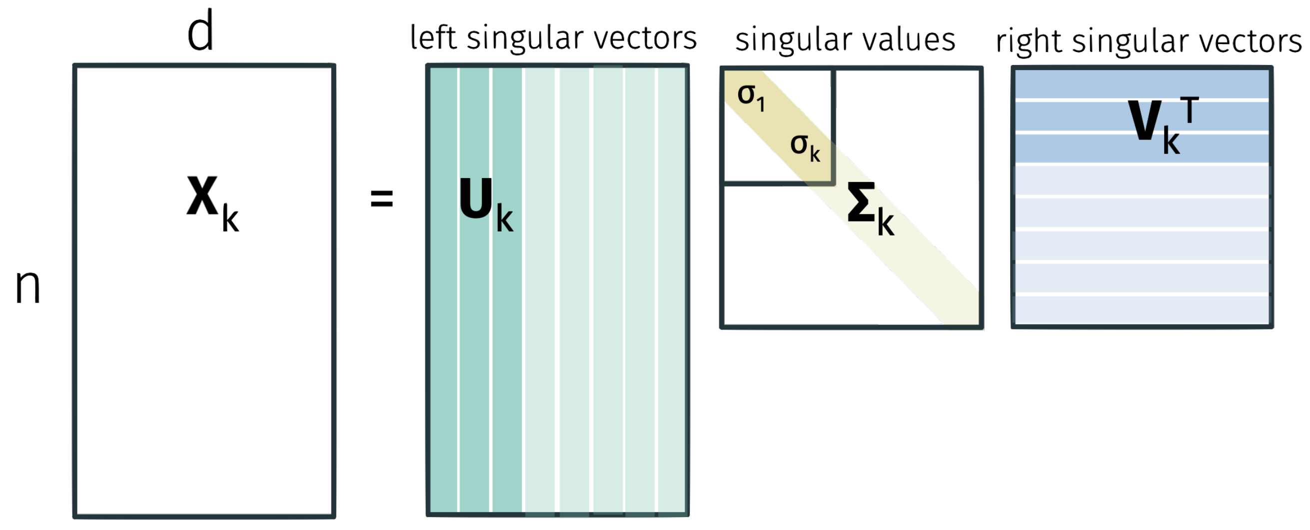
- ↳ Pseudoinverse $V \Sigma^{-1} U^T$
- ↳ Condition number
- ↳ $\|X\|_2 = \sigma_1$
- ↳ $\|X\|_F^2 = \sum_{i=1}^{d} \sigma_i^2$
- ↳ $X^{1/2} = V \Sigma^{1/2} U^T$

Low Rank Approximation

$X \in \mathbb{R}^{n \times d}$, $n \geq d$, space : $O(nd)$

$X_k \approx U_k \Sigma_k V_k^T$ space : $O(nk)$

$n \times k$ $\quad k \times k$ $\quad k \times k$



What is the best rank k approximation?

Tools

① $\|VX\|_F^2 = \|X\|_F^2$ if $V^TV = I$

② $\|X^T\|_F^2 = \|X\|_F^2$

③ $\|X\|_F^2 \overset{\text{show}}{=} \sum_{i=1}^{d} \sigma_i^2$

Best rank k approximation

$$\underset{\text{rank } k \; B}{\operatorname{argmin}} \; \|X - B\|_{\underset{\text{a norm}}{F}}^2 = \operatorname{arg\,min} \; \| U \Sigma V^T - B \|_F^2$$

$$= \operatorname{argmin} \; \| U^T U \; \Sigma \; V^T - U^T B \|_F^2 \quad \text{by } ①$$

$$= \operatorname{argmin} \; \| \Sigma \; V^T - U^T B \|_F^2 \quad \text{by } U^T U = I$$

$$= \operatorname{argmin} \; \| V \Sigma^T - B \; U \|_F^2 \quad \text{by } ②$$

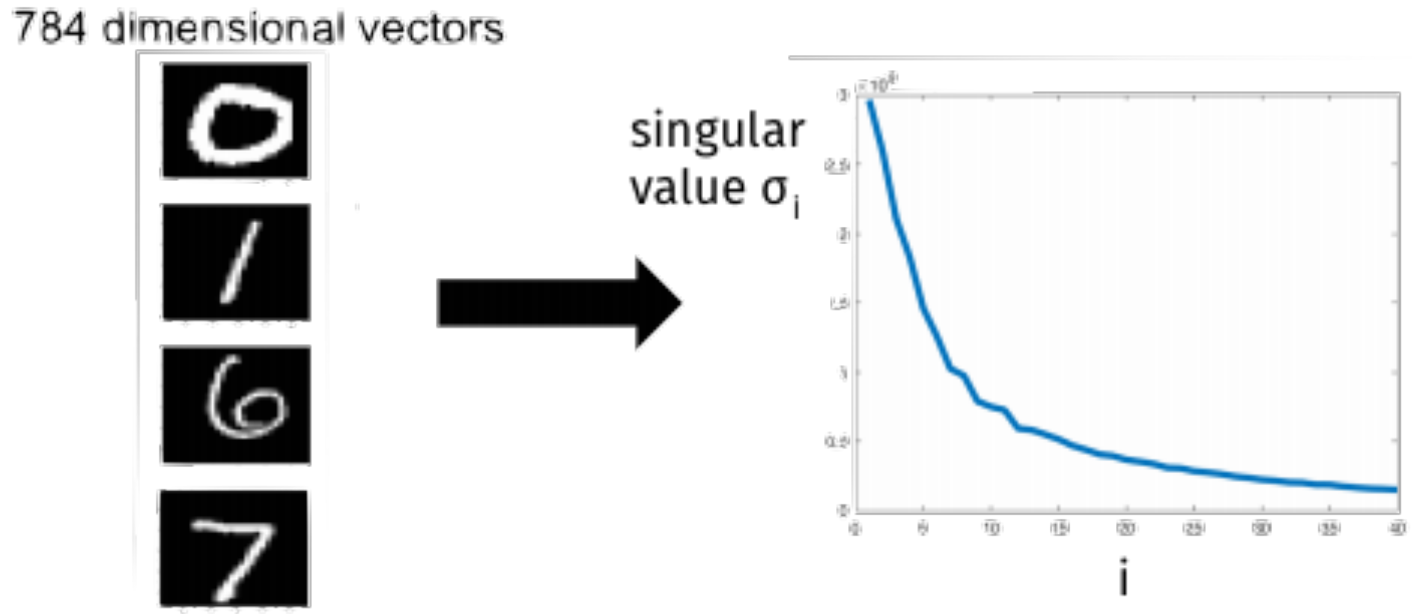
$$= \operatorname{argmin} \; \| \Sigma - V^T B U \|_F^2 \quad \text{by } ①$$

Intuition: $B = \Sigma_k = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}$ best approximation

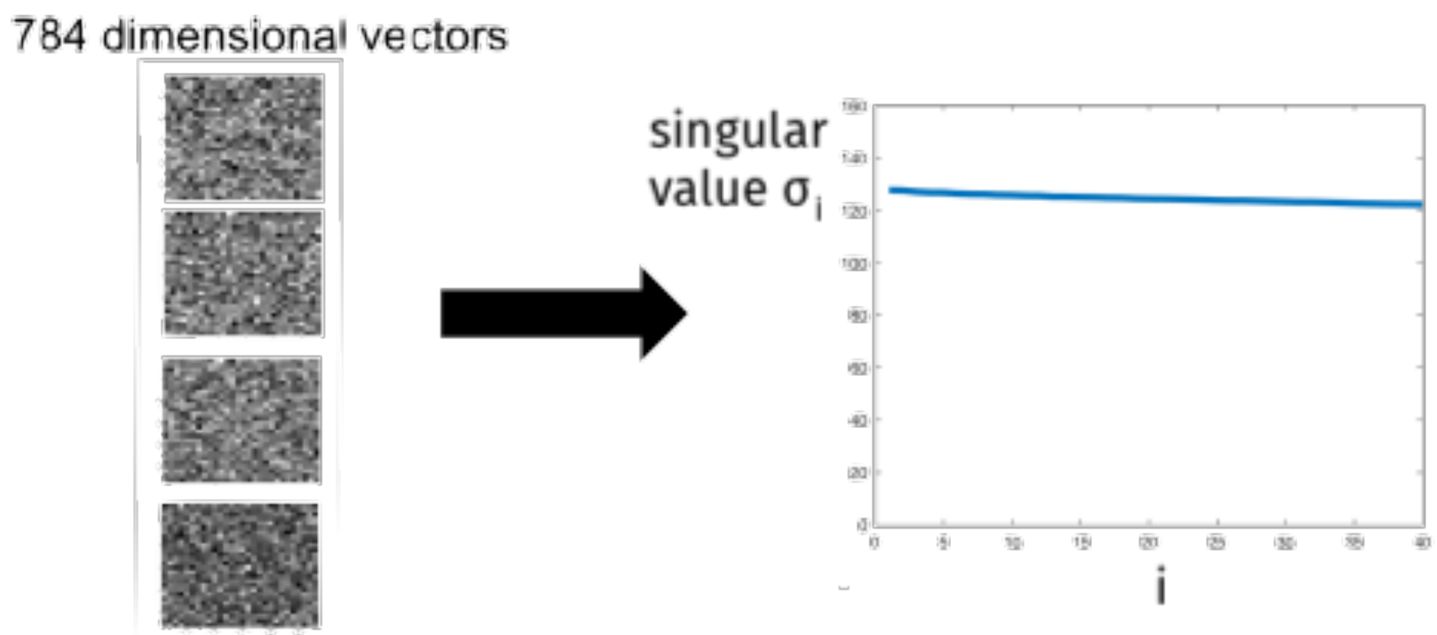
(Proved formally in typed notes.)

$$
\begin{aligned}
\|X - X_k\|_F^2 &= \|U \Sigma V^T - U \Sigma_k V^T\|_F^2 \\
&= \|U (\Sigma - \Sigma_k) V^T\|_F^2 \\
&= \|(\Sigma - \Sigma_k) V^T\|_F^2 \quad \text{by } 1 \\
&= \|\Sigma - \Sigma_k\|_F^2 \quad \text{by } 2, 1 \\
&= \sum_{i=k+1}^{n} \sigma_i^2
\end{aligned}
$$

Quality of approximation depends on $\underbrace{\text{sum smaller singular values}}_{\text{How well do top singular values fit matrix?}}$



Structured



Unstructured

Finding SVD $\quad X \in \mathbb{R}^{n \times d}$

↳ Compute $X^T X$ in time

↳ Decompose $X^T X = V \Lambda V^T$ in time

↳ Compute $L = XV$, in time

$\sigma_i = \|L_i\|_2$

$u_i = L_i / \sigma_i$

Total: $O(nd^2)$

but then only use rank k approximation?

We'll see how to compute faster in $O(ndk)$ time