

Recursive kernel density estimators under missing data

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Abstract: In this paper we propose an automatic bandwidth selection of the recursive kernel density estimators with missing data in the context of global and local density estimation. We showed that, using the selected bandwidth and a special stepsize, the proposed recursive estimators outperformed the nonrecursive one in terms of estimation error in the case of global estimation. However, the recursive estimators are much better in terms of computational costs. We corroborated these theoretical results through simulation studies and on the simulated data of the Aquitaine cohort of HIV-1 infected patients and on the coriell cell lines using the chromosome number 11.

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1 Introduction

Kernel density estimator (KDE) were introduced in Parzen (1962) and Rosenblatt (1956) have been widely used in many applications research including clinical trials, epidemiology, genetics as an exploratory tool.

There has been an intensive work in the literature about the bandwidth selection methods of this estimator. See for example Duin (1976), Rudemo (1982), Scott and Terrell (1987) and Marron (1988). However, in many situation the full data are not available and are subjects to missing data. Several methods have been used to overcome the problem and can be found in the literature. For example, the EM algorithm introduced in Dempster et al. (1977), can be used to obtain maximum likelihood estimates when some data are missing, and multiple imputation, which is a Monte Carlo technique in which the missing values are replaced by simulated versions, and the results are combined to produce estimates which incorporate missing data, see the paper of Little and Rubin (2002). Note that all of these methods are concerned with the estimation of unknown parameters. In the framework of nonparametric estimation, Dubnicka (2009) used a modified Horvitz-Thompson-type KDE to estimate the global density under missing data. However, several methods of local density estimation was widely discussed and extended in many directions; see Sheather (1983, 1986), Thombs and Sheather (1992), Hazelton (1999), Chan et al. (2010) and recently Dutta (2014).

In this paper, we propose a Recursive KDE to estimate the global unknown density f of a univariate variable when the data are missing at random, we also give some comparative elements in the case of local density estimation. Then, we present a data-driven bandwidth selection algorithm for the proposed estimators. Data-driven bandwidth selection procedure was proposed by Slaoui (2014a) in the framework of the recursive kernel density estimators, then, Slaoui (2014b) propose a plug-in selection method for recursive kernel distribution estimators, Slaoui (2015) propose a plug-in

selection method for recursive kernel regression estimators with a fixed design setting, and Slaoui (2016) propose a plug-in selection algorithm for the semi-recursive kernel regression estimators, all of this works suppose that the full data are observed. Here, we developed a specific second-generation plug-in bandwidth selection method of the Recursive KDE under missing data.

Interestingly, unlike the EM-algorithm, the proposed Recursive KDE has explicit form, which facilitates its usage and analysis. Moreover, the CPU time using the proposed recursive KDE are approximately two times faster than the CPU time using the nonrecursive Horvitz-Thompson-type KDE.

For further motivation in order to help intuition, let us introduce in some detail two datasets, the first one concerne a simulated data of the Aquitaine cohort of HIV-1 infected patients, see the paper of Thiébault et al. (2000), the second one concerne a real data which correspond to two array Comparative Genomic Hybridization (CGH) studies of fibroblast cell strains. Assuming that all these measurement are corrects, we simulated various proportions of missing at random by removing the corresponding entries before performing our algorithm. The estimated densities are then compared to the shape of the histograms obtained using the full data. Even when 30% of the original measurements are missing, the estimated densities using the proposed estimators remain very accurate thus demonstrating the effectiveness of our approach. Moreover, we compared our approach with the nonrecursive Horvitz-Thompson-type KDE via real dataset as well as simulations. Results showed that our approach outperformed other approaches in terms of estimation accuracy and computing efficiency. The remainder of the paper is organized as follows. In the next section we present our proposed recursive KDE. In Section 3, we state our main results. Section 4 is devoted to our application results, first by simulation (subsection 4.1) and second using two datasets, the first one concerne the simulated data of the Aquitaine cohort of HIV-1 infected patients and the second one concerne the coriell cell lines using the chromosome number 11 (subsection 4.2). We conclude the paper in Section 5, whereas the technical details are deferred to Appendix A and Appendix B gives the results for the Horvitz-Thompson-type KDE.

2 The estimator

Let T, T_1, \dots, T_n be independent, identically distributed random variables, and let f and F denote respectively the probability density and the distribution of T . We suppose that the full data T_1, \dots, T_n are not totally available and are subjects to missing data. The observed random variables are then X_i and δ_i where

$$X_i = \delta_i \times T_i \text{ and } \delta_i = \mathbb{1}_{\{T_i \text{ is observed}\}}, 1 \leq i \leq n.$$

When some T_i are missing, we let : $\pi_i = \mathbb{P}[\delta_i = 1|T_i]$ for $1 \leq i \leq n$.

This probability is often called the propensity score, see Rosenbaum and Rubin (1983). Let us recall that, in order to construct a stochastic algorithm, which approximates the function f at a given point x , we need to define an algorithm of search of the zero of the function $h : y \rightarrow f(x) - y$. Following Robbins-Monro's procedure, this algorithm is defined by setting $f_0(x) \in \mathbb{R}$, and, for all $n \geq 1$,

$$f_n(x) = f_{n-1}(x) + \gamma_n W_n,$$

where $W_n(x)$ is an observation of the function h at the point $f_{n-1}(x)$, and the stepsize (γ_n) is a sequence of positive real numbers that goes to zero. In order to define $W_n(x)$, we follow the

approach of Révész (1973, 1977), Tsybakov (1990) and of Slaoui (2013, 2014a,b), and we introduce a kernel K (that is, a function satisfying $\int_{\mathbb{R}} K(x)dx = 1$), and a bandwidth (h_n) (that is, a sequence of positive real numbers that goes to zero), and sets $W_n(x) = h_n^{-1} \delta_n \pi_n^{-1} K(h_n^{-1}[x - X_n]) - f_{n-1}(x)$. Then, the estimator f_n to recursively estimate the function f at the point x can be written as

$$f_n(x) = (1 - \gamma_n) f_{n-1}(x) + \gamma_n \delta_n \pi_n^{-1} h_n^{-1} K(h_n^{-1}[x - X_n]), \quad (1)$$

where the stepsize (γ_n) is a sequence of positive real numbers that goes to zero. Let us underline that we consider $f_0(x) = 0$ and we let $Q_n = \prod_{j=1}^n (1 - \gamma_j)$, then the equation (1) can be rewritten as follows:

$$f_n(x) = Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k \delta_k \pi_k^{-1} h_k^{-1} K\left(\frac{x - X_k}{h_k}\right),$$

which means that, we can estimate f recursively at the point x by using one of the two previous equations.

Moreover, we show in the next section that the optimal bandwidth which minimize the $\mathbb{E} \int_{\mathbb{R}} [f_n(x) - f(x)]^2 dx$ depend on the choice of the stepsizes (γ_n) ; we show in particular that under some conditions of regularity of f and using the stepsizes $(\gamma_n) = (\gamma_0 n^{-1})$, with $\gamma_0 > 2/5$, the bandwidth (h_n) must equal

$$\left(2^{-1/5} (\gamma_0 - 2/5)^{1/5} \left\{ \frac{\int_{\mathbb{R}} f^2(x) dx}{\int_{\mathbb{R}} (f^{(2)}(x) f(x))^2 dx} \right\}^{1/5} \left\{ \frac{\int_{\mathbb{R}} K^2(z) dz}{(\int_{\mathbb{R}} z^2 K(z) dz)^2} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right).$$

The first aim of the next section is to propose a data-driven bandwidth selection of the proposed recursive KDE in the case of missing data, and the second aim is to give the conditions under which the proposed estimators f_n may behave more efficiently than the nonrecursive Horvitz-Thompson-type KDE see the papers Horvitz and Thompson (1952) and Dubnicka (2009), defined as

$$\tilde{f}_n(x) = \frac{1}{nh_n} \sum_{i=1}^n \delta_i \pi_i^{-1} K\left(\frac{x - X_i}{h_n}\right). \quad (2)$$

In the next section we state our main results: the global density estimation is considered in Section 3.1 and the local density estimation is developed in Section 3.2.

3 Assumptions and main results

We define the following class of regularly varying sequences.

Definition 1. Let $\gamma \in \mathbb{R}$ and $(v_n)_{n \geq 1}$ be a nonrandom positive sequence. We say that $(v_n) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n \rightarrow +\infty} n \left[1 - \frac{v_{n-1}}{v_n} \right] = \gamma. \quad (3)$$

Condition (3) was introduced Galambos and Seneta (1973) to define regularly varying sequences, see also the paper Bojanic and Seneta (1973) and by Mokkadem and Pelletier (2007) in the context of stochastic approximation algorithms. Noting that the acronym \mathcal{GS} stand for (Galambos and Seneta). Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b \in \mathbb{R}$, $n^\gamma (\log n)^b$, $n^\gamma (\log \log n)^b$, and so on.

In this section, we investigate asymptotic properties of the proposed estimators (1). The assumptions to which we shall refer are the following

(A1) $K : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, bounded function satisfying $\int_{\mathbb{R}} K(z) dz = 1$, and, $\int_{\mathbb{R}} zK(z) = 0$ and $\int_{\mathbb{R}} z^2 K(z) < \infty$.

(A2) *i*) $(\gamma_n) \in \mathcal{GS}(-\alpha)$ with $\alpha \in]1/2, 1]$.

ii) $(h_n) \in \mathcal{GS}(-a)$ with $a \in]0, 1[$.

iii) $\lim_{n \rightarrow \infty} (n\gamma_n) \in]\min\{2a, (\alpha - a)/2\}, \infty]$.

(A3) f is bounded, differentiable, and $f^{(2)}$ is bounded.

Assumption (A2) (*iii*) on the limit of $(n\gamma_n)$ as n goes to infinity is usual in the framework of stochastic approximation algorithms. It implies in particular that the limit of $([n\gamma_n]^{-1})$ is finite. For simplicity, we introduce the following notations:

$$\begin{aligned} \xi &= \lim_{n \rightarrow \infty} (n\gamma_n)^{-1}, \\ R(K) &= \int_{\mathbb{R}} K^2(z) dz, \\ \mu_j(K) &= \int_{\mathbb{R}} z^j K(z) dz, \\ \Theta(K) &= R(K)^{4/5} \mu_2(K)^{2/5}, \\ I_1 &= \int_{\mathbb{R}} f^2(x) dx, \\ I_2 &= \int_{\mathbb{R}} \left(f^{(2)}(x)\right)^2 f(x) dx. \end{aligned} \tag{4}$$

In this section, we explicit the choice of (h_n) through a second-generation plug-in method, which minimize the Mean Weighted Integrated Squared Error *MWISE* of the proposed recursive estimators (1) in the case of global estimation and through a second-generation plug-in method, which minimize the Mean Squared Error *MSE* of the proposed recursive estimators (1) in the case of local estimation, in order to provide a comparison with the nonrecursive estimator (2). Our first result is the following proposition, which gives the bias and the variance of f_n .

Proposition 1 (Bias and variance of f_n). Let Assumptions (A1) – (A3) hold

1. If $a \in]0, \alpha/5]$, then

$$\mathbb{E}[f_n(x)] - f(x) = \frac{h_n^2}{2(1 - 2a\xi)} f^{(2)}(x) \mu_2(K) + o(h_n^2). \tag{5}$$

If $a \in]\alpha/5, 1[$, then

$$\mathbb{E}[f_n(x)] - f(x) = o\left(\sqrt{\gamma_n h_n^{-1}}\right). \tag{6}$$

2. If $a \in [\alpha/5, 1[$, then

$$\text{Var}[f_n(x)] = \frac{\gamma_n}{h_n} \pi_n^{-1} \frac{1}{(2 - (\alpha - a)\xi)} f(x) R(K) + o\left(\frac{\gamma_n}{h_n}\right). \tag{7}$$

If $a \in]0, \alpha/5[$, then

$$\text{Var}[f_n(x)] = o(h_n^4). \tag{8}$$

3. If $\lim_{n \rightarrow \infty} (n\gamma_n) > \max \{2a, (\alpha - a)/2\}$, then (5) and (7) hold simultaneously.

The bias and the variance of the proposed estimator f_n defined by the stochastic approximation algorithm (1) then heavily depend on the choice of the stepsize (γ_n) . Let us first underline that, it follows from (7) that the stepsize which minimize the variance of f_n is $(\gamma_n) = ([1 - a] n^{-1})$, using this stepsize the variance of f_n is equal to

$$\text{Var} [f_n(x)] = \pi_n^{-1} \frac{1-a}{nh_n} f(x) R(K) + o\left(\frac{1}{nh_n}\right).$$

Now, using the special stepsize $(\gamma_n) = (n^{-1})$, (see Mokkadem et al. (2009) and Slaoui (2013)), the variance of f_n is equal to

$$\text{Var} [f_n(x)] = \frac{\pi_n^{-1}}{1+a} \frac{1}{nh_n} f(x) R(K) + o\left(\frac{1}{nh_n}\right).$$

Let us recall that under the Assumptions (A1), (A2) *ii*) and (A3), we have

$$\text{Var} [\tilde{f}_n(x)] = \frac{\pi_n^{-1}}{nh_n} f(x) R(K) + o\left(\frac{1}{nh_n}\right).$$

Which shows that performing the proposed recursive estimators (1) with one of two proposed stepsizes we get smaller variance than the nonrecursive estimator (2). Similar results was given by Mokkadem et al. (2009) and Slaoui (2013) in the case of complet data.

Let us now state the following theorem, which gives the weak convergence rate of the estimator f_n defined in (1).

Theorem 1 (Weak pointwise convergence rate). Let Assumptions (A1) – (A3) hold

1. If there exists $c \geq 0$ such that $\gamma_n^{-1} h_n^5 \rightarrow c$, then

$$\begin{aligned} & \sqrt{\gamma_n^{-1} \pi_n h_n} (f_n(x) - f(x)) \\ & \xrightarrow{\mathcal{D}} \mathcal{N} \left(\frac{\sqrt{c}}{2(1-2a\xi)} f^{(2)}(x) \mu_2(K), \frac{1}{(2-(\alpha-a)\xi)} f(x) R(K) \right). \end{aligned}$$

2. If $\gamma_n^{-1} h_n^5 \rightarrow \infty$, then

$$\frac{1}{h_n^2} (f_n(x) - f(x)) \xrightarrow{\mathbb{P}} \frac{1}{2(1-2a\xi)} f^{(2)}(x) \mu_2(K),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution, \mathcal{N} the Gaussian-distribution and $\xrightarrow{\mathbb{P}}$ the convergence in probability.

3.1 Global density estimation

In order to measure globally the quality of our recursive estimators (1), we use the following quantity,

$$\begin{aligned} MWISE[f_n] &= \mathbb{E} \int_{\mathbb{R}} [f_n(x) - f(x)]^2 f(x) dx \\ &= \int_{\mathbb{R}} (\mathbb{E}(f_n(x)) - f(x))^2 f(x) dx + \int_{\mathbb{R}} \text{Var}(f_n(x)) f(x) dx. \end{aligned}$$

The following proposition gives the *MWISE* of the recursive estimators defined in (1).

Proposition 2 (MWISE of f_n). Let Assumptions (A1) – (A3) hold.

1. If $a \in]0, \alpha/5[$, then

$$MWISE[f_n] = \frac{h_n^4}{4} \frac{1}{(1 - 2a\xi)^2} I_2 \mu_2^2(K) + o(h_n^4).$$

2. If $a = \alpha/5$, then

$$MWISE[f_n] = \frac{\gamma_n \pi_n^{-1}}{h_n} \frac{1}{(2 - (\alpha - a)\xi)} I_1 R(K) + \frac{h_n^4}{4} \frac{1}{(1 - 2a\xi)^2} I_2 \mu_2^2(K) + o(h_n^4).$$

3. If $a \in]\alpha/5, 1[$, then

$$MWISE[f_n] = \frac{\gamma_n \pi_n^{-1}}{h_n} \frac{1}{(2 - (\alpha - a)\xi)} I_1 R(K) + o\left(\frac{\gamma_n \pi_n^{-1}}{h_n}\right).$$

3.1.1 Estimating propensity scores

In order to estimate the propensity scores π_i , we need to exploit the information contained in the auxiliary variables T_i , which are related to X_i . First, we can estimate π by the empirical proportion based on the observed data (Qi et al. (2005)):

$$\hat{\pi}_i = \hat{\pi}(X_i) = \frac{\sum_{j=1}^n \delta_j \mathbb{1}_{\{X_j = X_i\}}}{\sum_{j=1}^n \mathbb{1}_{\{T_j = T_i\}}}.$$

Moreover, in the case when X_i contains continuous elements, Dubnicka (2009) used the Nadaraya-Watson (local mean) estimator (Nadaraya (1964); Watson (1964)):

$$\hat{\pi}_{NW_i} = \hat{\pi}_{NW}(X_i) = \frac{\sum_{j=1}^n \delta_j K(h_n^{-1}[X_i - X_j])}{\sum_{j=1}^n K(h_n^{-1}[X_i - X_j])}, \quad (9)$$

and in the case of binary nature of the response variable, Tibshirani and Hastie (1987) and Fan et al. (1995) provide an estimation of the propensity score using local likelihood. However, Dubnicka (2009) concluded that, both the Nadaraya-Watson estimates and local likelihood estimates of the propensity scores are more flexible than ordinary binary regression estimates. In this paper we use the recursive version of the estimator of Nadaraya-Watson (called also the semi-recursive estimator) and defined as:

$$\hat{\pi}_{RNW_i} = \hat{\pi}_{RNW}(X_i) = \frac{\sum_{j=1}^n \delta_j h_j^{-1} K(h_j^{-1}[X_i - X_j])}{\sum_{j=1}^n h_j^{-1} K(h_j^{-1}[X_i - X_j])}. \quad (10)$$

For now, we simply use the propensity score estimator (9) in the case of nonrecursive Horvitz-Thompson-type KDE and (10) in the case of the proposed recursive KDE.

3.1.2 Bandwidth selection

In the framework of the nonparametric kernel estimators, the bandwidth selection methods studied in the literature can be divided into three broad classes: the cross-validation techniques, the plug-in ideas and the bootstrap. A detailed comparison of the three practical bandwidth selection can

be found in Delaigle and Gijbels (2004). They concluded that chosen appropriately plug-in and bootstrap selectors both outperform the cross-validation bandwidth, and that none of the two can be claimed to be best in all cases. In this section, we developed a plug-in bandwidth selector that minimizing the *MWISE* of the proposed recursive KDE, using the function $f(x)$ as a weight function.

The following corollary ensures that the bandwidth which minimize the *MWISE* of f_n depend on the stepsize (γ_n) and then the corresponding *MWISE* depend also on the stepsize (γ_n) .

Corollary 1. Let Assumptions (A1) – (A3) hold. To minimize the *MWISE* of f_n , the stepsize (γ_n) must be chosen in $\mathcal{GS}(-1)$, the bandwidth (h_n) must equal

$$\left(\left\{ \frac{(1 - 2a\xi)^2}{(2 - (\alpha - a)\xi)} \frac{I_1}{I_2} \right\}^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} \gamma_n^{1/5} \right).$$

Then, we have

$$MWISE[f_n] = \frac{5}{4} (1 - 2a\xi)^{-2/5} (2 - (\alpha - a)\xi)^{-4/5} I_1^{4/5} I_2^{1/5} \Theta(K) \pi_n^{-4/5} \gamma_n^{4/5} + o\left(\pi_n^{-4/5} \gamma_n^{4/5}\right).$$

The following corollary shows that, for a special choice of the stepsize $(\gamma_n) = (\gamma_0 n^{-1})$, which fulfilled that $\lim_{n \rightarrow \infty} n\gamma_n = \gamma_0$ and that $(\gamma_n) \in \mathcal{GS}(-1)$, the optimal value for h_n depend on γ_0 and then the corresponding *MWISE* depend on γ_0 .

Corollary 2. Let Assumptions (A1) – (A3) hold, and suppose that $(\gamma_n) = (\gamma_0 n^{-1})$. To minimize the *MWISE* of f_n , the stepsize (γ_n) must be chosen in $\mathcal{GS}(-1)$, $\lim_{n \rightarrow \infty} n\gamma_n = \gamma_0$, the bandwidth (h_n) must equal

$$\left(2^{-1/5} (\gamma_0 - 2/5)^{1/5} \left(\frac{I_1}{I_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right),$$

and we then have

$$MWISE[f_n] = \frac{5}{4} \frac{1}{2^{4/5}} \frac{\gamma_0^2}{(\gamma_0 - 2/5)^{6/5}} I_1^{4/5} I_2^{1/5} \Theta(K) \pi_n^{-4/5} n^{-4/5} + o\left(\pi_n^{-4/5} n^{-4/5}\right).$$

Moreover, the minimum of $\gamma_0^2 (\gamma_0 - 2/5)^{-6/5}$ is reached at $\gamma_0 = 1$, then the bandwidth (h_n) must equal

$$\left(\left(\frac{3}{10} \right)^{1/5} \left(\frac{I_1}{I_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right), \quad (11)$$

and then the asymptotic *MWISE*

$$AMWISE[f_n] = \frac{5}{4} \frac{1}{2^{4/5}} \left(\frac{5}{3} \right)^{6/5} I_1^{4/5} I_2^{1/5} \Theta(K) \pi_n^{-4/5} n^{-4/5}.$$

In order to estimate the optimal bandwidth (11), we must estimate the unknown quantites I_1 and I_2 . For this purpose, we used the following modified version of the kernel estimators introduced in Slaoui (2014a):

$$\widehat{I}_1 = \frac{\Psi_n}{n} \sum_{i,k=1}^n \Psi_k^{-1} \beta_k b_k^{-1} \delta_k \widehat{\pi}_{RNWk}^{-1} K_b \left(\frac{X_i - X_k}{b_k} \right) \quad (12)$$

$$\widehat{I}_2 = \frac{\Phi_n^2}{n} \sum_{\substack{i,j,k=1 \\ j \neq k}}^n \Phi_j^{-1} \Phi_k^{-1} \beta'_j \beta'_k \delta_j \delta_k \widehat{\pi}_{RNWj}^{-1} \widehat{\pi}_{RNWk}^{-1} b_j'^{-3} b_k'^{-3} K_{b'}^{(2)} \left(\frac{X_i - X_j}{b'_j} \right) K_{b'}^{(2)} \left(\frac{X_i - X_k}{b'_k} \right) \quad (13)$$

where K_b and $K_{b'}$ are a kernels, b_n and b'_n are respectively the associated bandwidth (called pilot bandwidth) and β_n and β'_n are the two pilot stepsizes for the estimation of I_1 and I_2 respectively, and $\Psi_n = \prod_{i=1}^n (1 - \beta_i)$ and $\Phi_n = \prod_{i=1}^n (1 - \beta'_i)$.

In practice, we take

$$b_n = n^{-\beta} \min \left\{ \widehat{s}, \frac{Q_3 - Q_1}{1.349} \right\}, \quad \beta \in]0, 1[\quad (14)$$

(see Silverman (1986)) with \widehat{s} the sample standard deviation, and Q_1, Q_3 denoting the first and third quartiles, respectively.

We followed the same steps as in Slaoui (2014a) and we showed that in order to minimize the *AMISE* of \widehat{I}_1 the pilot bandwidth (b_n) must belong to $\mathcal{GS}(-2/5)$ and the pilot stepsize (β_n) should be equal to $(1.36n^{-1})$, and in order to minimize the *AMISE* of \widehat{I}_2 the pilot bandwidth (b'_n) must belong to $\mathcal{GS}(-3/14)$ and the pilot stepsize (β'_n) should be equal to $(1.48n^{-1})$.

Finally, the plug-in estimator of the bandwidth (h_n) using the recursive estimators defined in (1) with the stepsizes $(\gamma_n) = (n^{-1})$ is equal to

$$\left(\left(\frac{3}{10} \right)^{1/5} \left(\frac{\widehat{I}_1}{\widehat{I}_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \widehat{\pi}_{RNWn}^{-1/5} n^{-1/5} \right), \quad (15)$$

and the associated plug-in *AMWISE* is equal to

$$\widehat{AMWISE}[f_n] = \frac{5}{4} \frac{1}{2^{4/5}} \left(\frac{5}{3} \right)^{6/5} \widehat{I}_1^{4/5} \widehat{I}_2^{1/5} \Theta(K) \widehat{\pi}_{RNWn}^{-4/5} n^{-4/5}.$$

Now, using the stepsize $(\gamma_n) = ([4/5] n^{-1})$, the stepsize which minimize the variance of the proposed estimators defined in (1), it follows from Corollary 2 that, the plug-in estimator of the bandwidth (h_n) in this particular case is equal to

$$\left(5^{-1/5} \left(\frac{\widehat{I}_1}{\widehat{I}_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \widehat{\pi}_{RNWn}^{-1/5} n^{-1/5} \right), \quad (16)$$

and the associated plug-in *AMWISE* is equal to

$$\widehat{AMWISE}[f_n] = 5^{1/5} \widehat{I}_1^{4/5} \widehat{I}_2^{1/5} \Theta(K) \widehat{\pi}_{RNWn}^{-4/5} n^{-4/5}.$$

Moreover, following similar steps as in Slaoui (2014a), we prove the following corollary

Corollary 3. Let the assumptions (A1) – (A3) hold, and the bandwidth (h_n) equal to (15) and the stepsize $(\gamma_n) = (n^{-1})$. We have

$$\frac{\mathbb{E} \left[\widehat{AMWISE}(f_n) \right]}{\mathbb{E} \left[\widetilde{AMWISE}(\tilde{f}_n) \right]} < 1.$$

Then, the expectation of the estimated $AMWISE$ of the proposed recursive estimators defined by (1) using the special stepsize $(\gamma_n) = (n^{-1})$ is smaller than the expectation of the estimated $AMWISE$ of the nonrecursive Horvitz-Thompson-type KDE defined by (2) (see Appendix B for some results on the nonrecursive Horvitz-Thompson-type KDE estimator).

3.2 Local density estimation

In order to measure locally the quality of our recursive estimators (1), we use the following quantity,

$$\begin{aligned} MSE[f_n] &= \mathbb{E}[f_n(x) - f(x)]^2 \\ &= (\mathbb{E}(f_n(x)) - f(x))^2 + Var(f_n(x)). \end{aligned}$$

Following similar step as in the case of the global density estimation, we show that the value of h_n which minimizes the asymptotic mean square error of $f_n(x)$ is equal to

$$\left(\left(\frac{3}{10} \right)^{1/5} \left(\frac{f(x)}{(f^{(2)}(x))^2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right). \quad (17)$$

In practice the kernel K is known but $f(x)$ and $f^{(2)}(x)$ are not. Thus, we estimate $f(x)$ by $f_n(x)$ and $f^{(2)}(x)$ by $f_n^{(2)}(x)$, and so the plug-in bandwidth selection in the case of locally estimation is equal to

$$\left(\left(\frac{3}{10} \right)^{1/5} \left(\frac{\hat{f}_n(x)}{(\hat{f}_n^{(2)}(x))^2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \hat{\pi}_{RNW}^{-1/5} n^{-1/5} \right), \quad (18)$$

where

$$\begin{aligned} \hat{f}_n(x) &= \Psi_n \sum_{k=1}^n \Psi_k^{-1} \beta_k b_k^{-1} \delta_k \hat{\pi}_{RNWk}^{-1} K_b \left(\frac{x - X_k}{b_k} \right) \\ \hat{f}_n^{(2)}(x) &= \Phi_n \sum_{k=1}^n \Phi_k^{-1} \beta'_k \delta_k \hat{\pi}_{RNWk}^{-1} b_k'^{-3} K_{b'}^{(2)} \left(\frac{x - X_k}{b'_k} \right) \end{aligned}$$

where K_b and $K_{b'}$ are a kernels, b_n and b'_n are respectively the associated bandwidth (called pilot bandwidth) and β_n and β'_n are the two pilot stepsizes for the estimation of $f(x)$ and $f^{(2)}(x)$ respectively, and $\Psi_n = \prod_{i=1}^n (1 - \beta_i)$ and $\Phi_n = \prod_{i=1}^n (1 - \beta'_i)$.

4 Applications

The aim of our applications is to compare the performance of the recursive kernel density estimators under missing data defined in (1) with that of the nonrecursive Horvitz-Thompson-type KDE defined in (2).

When applying f_n one need to choose four quantities:

1. The function K , we choose the Normal kernel.
2. The stepsizes (γ_n) equal to (cn^{-1}) , with $c \in [4/5, 1]$.
3. The propensity score (π_n) is chosen to be equal to (10).
4. The bandwidth (h_n) is chosen to be equal to (15) in the case when $(\gamma_n) = (n^{-1})$ and to (16) in the case when $(\gamma_n) = ([4/5]n^{-1})$.
 - (a) To estimate I_1 , we use (12); The pilot bandwidth is chosen to be equal to (14) with the choice of $\beta = 2/5$ and the pilot stepsize equal to $(1.36n^{-1})$.
 - (b) To estimate I_2 , we use (13); The pilot bandwidth is chosen to be equal to (14) with the choice of $\beta = 3/14$ and the pilot stepsize equal to $(1.48n^{-1})$.

When applying \tilde{f}_n one need to choose three quantities:

1. The function K , as in the recursive framework, we use the Normal kernel.
2. The propensity score (π_n) is chosen to be equal to (9).
3. The bandwidth (h_n) is chosen to be equal to (29).
 - (a) To estimate I_1 , we use (27); The pilot bandwidth is chosen to be equal to (14) with the choice of $\beta = 2/5$.
 - (b) To estimate I_2 , we use (28); The pilot bandwidth is chosen to be equal to (14) with the choice of $\beta = 3/14$.

4.1 Simulations

4.1.1 Global density estimation

In order to investigate the comparison between the proposed estimators, we consider three sample sizes: $n = 100, 200$, and 500 . In each case, we consider 0%, 30%, 50% and 70% of missing data, the number of the design points was fixed to be equal to 500, which variate from the lowest value to highest value over ($N = 500$ number of simulations) with equally spaced setting. We then use the bandwidth selection method proposed in the applications Section. Moreover, we consider three densities functions f : 1- the standard normal: $X \sim \mathcal{N}(0, 1)$ (see Table 1), 2- the normal mixture distribution: $X \sim 1/2\mathcal{N}(2, 1) + 1/2\mathcal{N}(-3, 1)$ (see Table 2), 3- the weibull distribution with shape parameter 2 and scale parameter 1: $X \sim \text{Weibul}(2, 1)$ (see Table 3). We considered also the case of one auxiliary variable and generated the pairs $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$ by first generating Y distributed as F_Y and X^* distributed as F_{X^*} , and we let $X = \rho Y + \sqrt{1 - \rho^2}X^*$, $-1 < \rho < 1$, then, Y and X were correlated. Moreover, we generated X^*, X_1^*, \dots, X_n^* from the standard normal distribution and we set $\rho = 0.3, 0.5$ and 0.8 and we generated the responses Y, Y_1, \dots, Y_n from one of the three previous densities functions f (see Table 4), in this case we present results for the situations in which 70% of the responses are missing. For each situations, we compute the Mean Weighted Integrated Squared Error $MWISE$ (over 500 samples).

$X \sim \mathcal{N}(0, 1)$						
	$n = 100$		$n = 200$		$n = 500$	
0%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$3.86e^{-03}$	364	$2.33e^{-03}$	2448	$1.20e^{-03}$	14432
Recursive 1	$3.70e^{-03}$	194	$2.30e^{-03}$	1225	$1.20e^{-03}$	7118
Recursive 2	$3.85e^{-03}$	184	$2.39e^{-03}$	1169	$1.25e^{-03}$	7122
30%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$1.47e^{-02}$	365	$5.93e^{-03}$	2448	$2.59e^{-03}$	14134
Recursive 1	$1.27e^{-02}$	164	$5.73e^{-03}$	1206	$2.56e^{-03}$	7112
Recursive 2	$1.32e^{-02}$	167	$6.17e^{-03}$	1216	$2.70e^{-03}$	7043
50%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$2.23e^{-02}$	356	$1.87e^{-02}$	2567	$3.32e^{-03}$	14123
Recursive 1	$2.11e^{-02}$	169	$1.69e^{-02}$	1146	$3.24e^{-03}$	7089
Recursive 2	$2.17e^{-02}$	176	$1.85e^{-02}$	1154	$3.46e^{-03}$	7134
70%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$2.89e^{-02}$	346	$2.43e^{-02}$	2267	$4.82e^{-03}$	14134
Recursive 1	$2.63e^{-02}$	168	$2.32e^{-02}$	1166	$4.63e^{-03}$	7156
Recursive 2	$2.77e^{-02}$	177	$2.39e^{-02}$	1175	$4.86e^{-03}$	7143

Table 1: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the normal distribution $X \sim \mathcal{N}(0, 1)$ with the proportion of missing at random equal respectively to 0% (in the first block), equal to 10% (in the second block), equal to 20% (in the third block) and equal to 30% (in the last block), we consider three sample sizes $n = 100$, $n = 200$ and $n = 500$, the number of simulations is 500, and we compute the Mean Weighted Integrated Squared Error (*MWISE*) and the CPU time in seconds.

Computational cost The advantage of recursive estimators on their nonrecursive version is that their update, from a sample of size n to one of size $n + 1$, require less computations. This property can be generalized, one can check that it follows from (1) that for all $n_1 \in [0, n - 1]$,

$$f_n(x) = \prod_{j=n_1+1}^n (1 - \gamma_j) f_{n_1}(x) + \sum_{k=n_1}^{n-1} \left[\prod_{j=k+1}^n (1 - \gamma_j) \right] \gamma_k h_k^{-1} \delta_k \pi_k^{-1} K(h_k^{-1}(x - X_k)).$$

In order to give some comparative elements with nonrecursive Horvitz-Thompson-type KDE (2), including computational costs. We consider a 500 samples of size $n_1 = \lfloor n/2 \rfloor$ (the lower integer part of $n/2$) generated from respectively the five considered distributions, moreover, we suppose that we receive an additional 500 samples of size $n - n_1$ generated also from the same five considered distributions. Performing the two methods, we report the total CPU time values for each considered distribution and in all cases given in Tables 1, 2, 3 and 4, the CPU time is given in seconds.

$X \sim \frac{1}{2}\mathcal{N}(2, 1) + \frac{1}{2}\mathcal{N}(-3, 1)$						
	$n = 100$		$n = 200$		$n = 500$	
0%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$9.96e^{-04}$	372	$6.09e^{-04}$	2448	$3.38e^{-04}$	14104
Recursive 1	$8.64e^{-04}$	174	$5.50e^{-04}$	1168	$3.09e^{-04}$	7016
Recursive 2	$9.00e^{-04}$	177	$5.72e^{-04}$	1189	$3.23e^{-04}$	7003
30%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$1.30e^{-03}$	366	$8.31e^{-04}$	2449	$3.90e^{-04}$	14142
Recursive 1	$1.25e^{-03}$	175	$7.69e^{-04}$	1246	$3.82e^{-04}$	7215
Recursive 2	$1.30e^{-03}$	168	$8.01e^{-04}$	1238	$3.97e^{-04}$	7225
50%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$1.41e^{-03}$	354	$1.55e^{-03}$	2246	$8.07e^{-04}$	13994
Recursive 1	$1.29e^{-03}$	166	$1.31e^{-03}$	1185	$6.52e^{-04}$	6946
Recursive 2	$1.35e^{-03}$	173	$1.36e^{-03}$	1174	$6.79e^{-04}$	6987
70%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$1.95e^{-03}$	356	$1.87e^{-03}$	2347	$1.13e^{-03}$	13965
Recursive 1	$1.80e^{-03}$	168	$1.69e^{-03}$	1164	$1.09e^{-03}$	7064
Recursive 2	$2.02e^{-03}$	179	$1.72e^{-03}$	1162	$1.11e^{-03}$	7086

Table 2: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the normal mixture distribution $X \sim \frac{1}{2}\mathcal{N}(2, 1) + \frac{1}{2}\mathcal{N}(-3, 1)$, with the proportion of missing at random equal respectively to 0% (in the first block), equal to 30% (in the second block), equal to 50% (in the third block) and equal to 70% (in the last block), we consider three sample sizes $n = 100$, $n = 200$ and $n = 500$, the number of simulations is 500, and we compute the Mean Weighted Integrated Squared Error (*MWISE*) and the CPU time in seconds.

$X \sim Weibull(2, 1)$						
	$n = 100$		$n = 200$		$n = 500$	
0%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$1.82e^{-02}$	374	$1.11e^{-02}$	2357	$4.75e^{-03}$	14256
Recursive 1	$1.70e^{-02}$	164	1.09e⁻⁰²	1184	4.74e⁻⁰⁴	7122
Recursive 2	$1.77e^{-04}$	178	$1.15e^{-02}$	1194	$4.94e^{-04}$	7089
30%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$2.09e^{-02}$	366	$1.22e^{-02}$	2452	$5.19e^{-03}$	14327
Recursive 1	1.93e⁻⁰³	177	1.16e⁻⁰⁴	1154	5.12e⁻⁰³	7012
Recursive 2	$2.01e^{-03}$	167	$1.23e^{-04}$	1124	$5.27e^{-03}$	6984
50%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$2.25e^{-02}$	356	$1.65e^{-02}$	2354	$1.14e^{-02}$	14232
Recursive 1	2.14e⁻⁰³	166	1.59e⁻⁰²	1163	1.07e⁻⁰²	7048
Recursive 2	$2.19e^{-02}$	168	$1.62e^{-02}$	1165	$1.12e^{-02}$	7044
70%	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	$2.45e^{-02}$	364	$2.12e^{-02}$	2346	$1.82e^{-02}$	13982
Recursive 1	2.36e⁻⁰²	174	2.03e⁻⁰²	1187	1.72e⁻⁰²	7086
Recursive 2	$2.41e^{-02}$	166	$2.08e^{-02}$	1184	$1.83e^{-02}$	7094

Table 3: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the weibull distribution with shape parameter 2 and scale parameter 1, $X \sim Weibull(2, 1)$, with the proportion of missing at random equal respectively to 0% (in the first block), equal to 30% (in the second block), equal to 50% (in the third block) and equal to 70% (in the last block), we consider three sample sizes $n = 100$, $n = 200$ and $n = 500$, the number of simulations is 500, and we compute the Mean Weighted Integrated Squared Error *MWISE* and the CPU time in seconds.

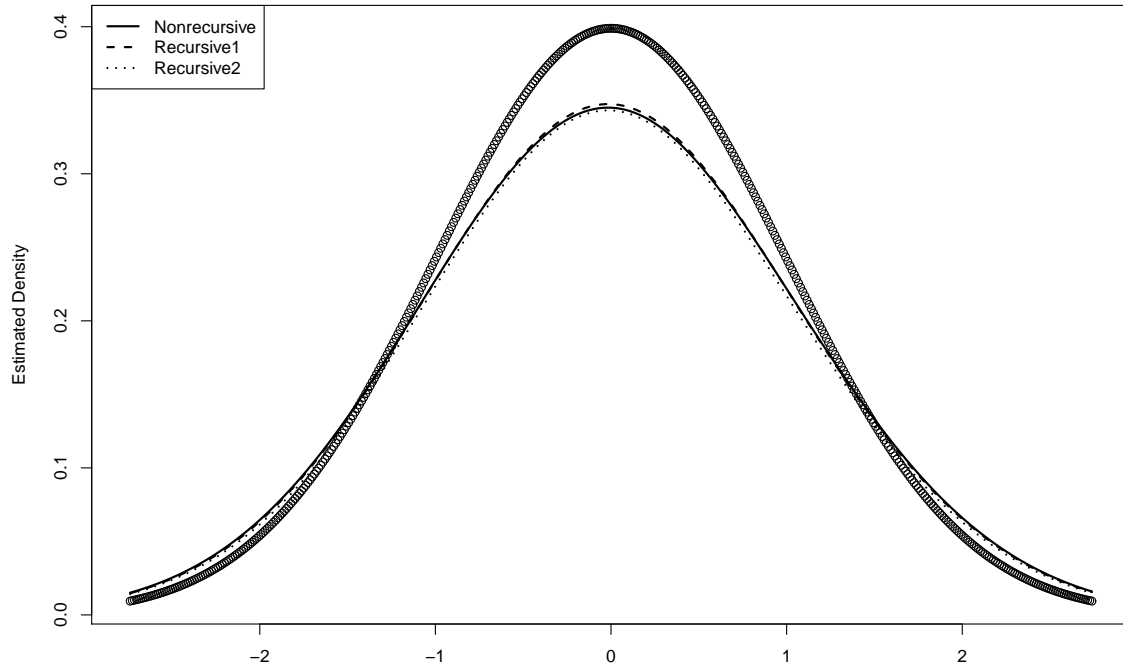


Figure 1: Qualitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the normal distribution $\mathcal{N}(0, 1)$ under 70% of the missing data and we consider 500 samples of size $n = 200$.

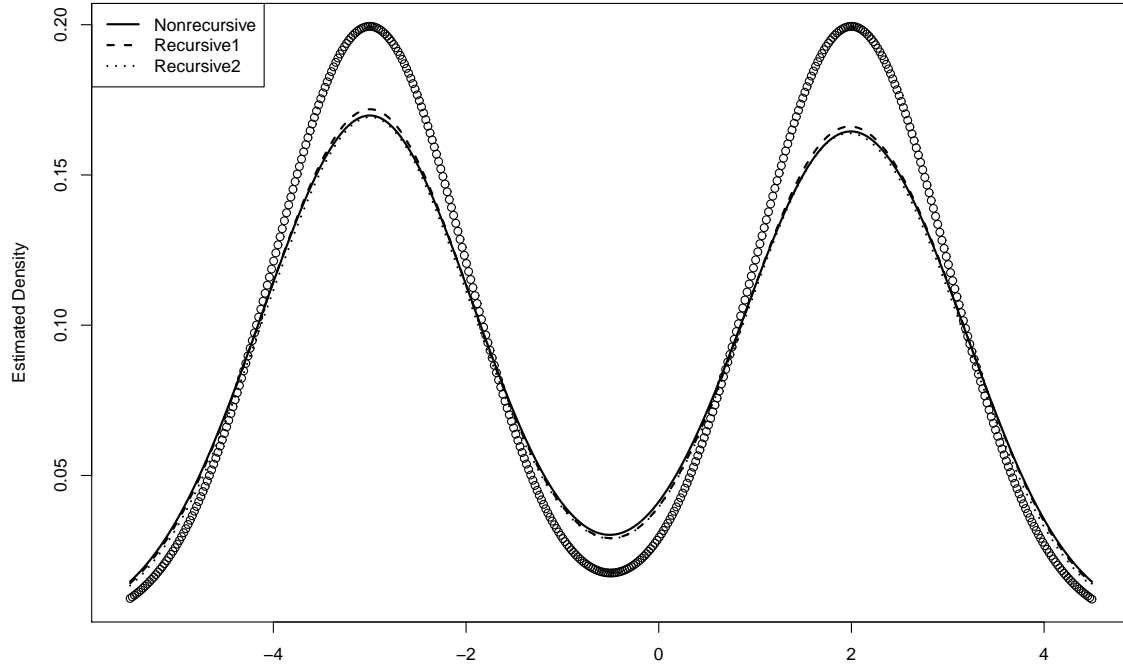


Figure 2: Qualitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the normal mixture distribution $X \sim \frac{1}{2}\mathcal{N}(2, 1) + \frac{1}{2}\mathcal{N}(-3, 1)$ under 70% of the missing data and we consider 500 samples of size $n = 200$.

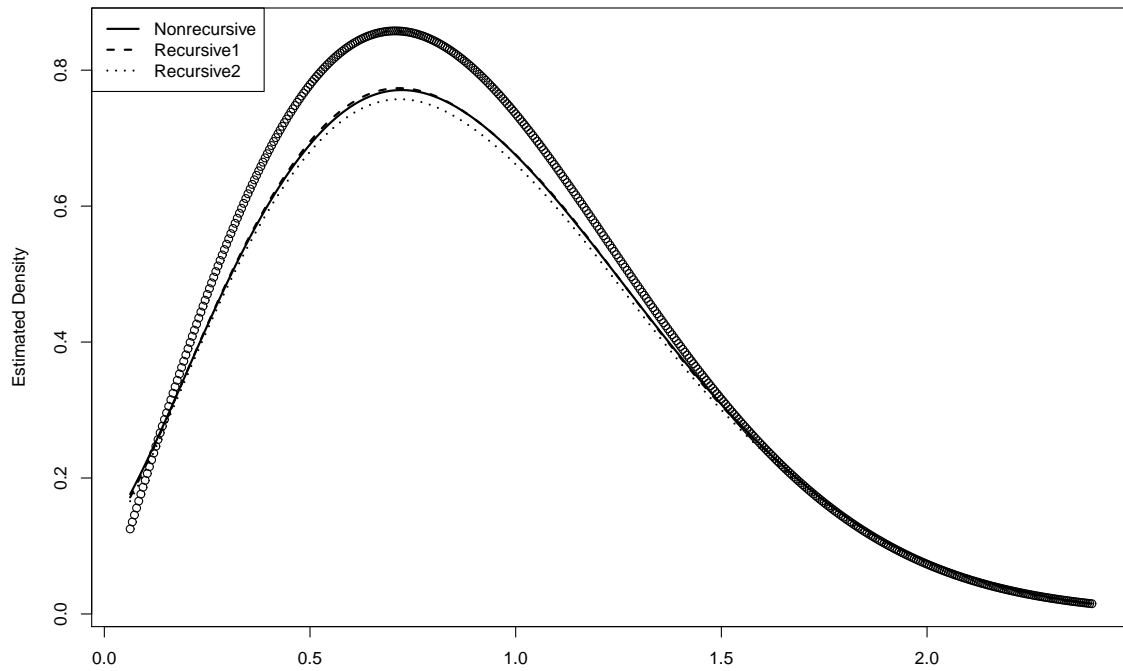


Figure 3: Qualitative comparison between the nonrecursive Horvitz-Thompson-type KDE estimator (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the weibull distribution with shape parameter 2 and scale parameter 1, $X \sim \text{Weibull}(2, 1)$ under 70% of the missing data and we consider 500 samples of size $n = 200$.

$n = 100$			$n = 200$		$n = 500$	
$Y \sim \mathcal{N}(0, 1)$						
$\rho = 30\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02882	357	0.02584	2335	0.05042	14342
Recursive 1	0.02622	187	0.0244	1244	0.0484	7332
Recursive 2	0.02746	194	0.02542	1255	0.05044	7412
$\rho = 50\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02866	366	0.02564	2534	0.05054	14542
Recursive 1	0.02634	191	0.02454	1254	0.04864	7353
Recursive 2	0.02724	187	0.02544	1243	0.05124	7366
$\rho = 80\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02894	362	0.02586	2447	0.05084	14121
Recursive 1	0.02664	188	0.02486	1214	0.00493	7215
Recursive 2	0.02784	184	0.02564	1209	0.05164	7226
$Y \sim 1/2\mathcal{N}(2, 1) + 1/2\mathcal{N}(-3, 1)$						
$\rho = 30\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.00328	354	0.00314	2126	0.00186	14054
Recursive 1	0.03014	184	0.00306	1207	0.00172	7105
Recursive 2	0.00329	177	0.00316	1198	0.00187	7115
$\rho = 50\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.00332	357	0.00221	2345	0.00192	14153
Recursive 1	0.00318	192	0.00215	1209	0.00180	7121
Recursive 2	0.00334	184	0.00224	1199	0.00194	7144
$\rho = 80\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.00334	366	0.00244	2543	0.00194	14532
Recursive 1	0.00324	194	0.00234	1321	0.00184	7342
Recursive 2	0.00338	188	0.00246	1306	0.00194	7224
$Y \sim Weibull(2, 1)$						
$\rho = 30\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02342	345	0.02082	2442	0.01742	14121
Recursive 1	0.02265	173	0.001944	1235	0.01638	7214
Recursive 2	0.02348	163	0.02105	1226	0.01744	7116
$\rho = 50\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02354	356	0.02142	2448	0.01754	14098
Recursive 1	0.02289	174	0.02008	1212	0.01644	7032
Recursive 2	0.02358	186	0.02144	1176	0.01760	7034
$\rho = 80\%$	MWISE	CPU	MWISE	CPU	MWISE	CPU
Nonrecursive	0.02372	352	0.02152	2356	0.001758	14024
Recursive 1	0.02314	176	0.02089	1209	0.01654	7014
Recursive 2	0.02369	184	0.02154	1165	0.01760	7006

Table 4: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5]n^{-1})$. We consider the proportion of missing at random equal to 70% and the auxiliary variable ρ , equal to 30% (in the first block), to 50% (in the second block) and equal to 70% (in the third block) of each considered distribution, we consider three sample sizes $n = 100$, $n = 200$ and $n = 500$, the number of simulations is 500, and we compute the Mean Weighted Integrated Squared Error (*MWISE*) and the CPU time in seconds.

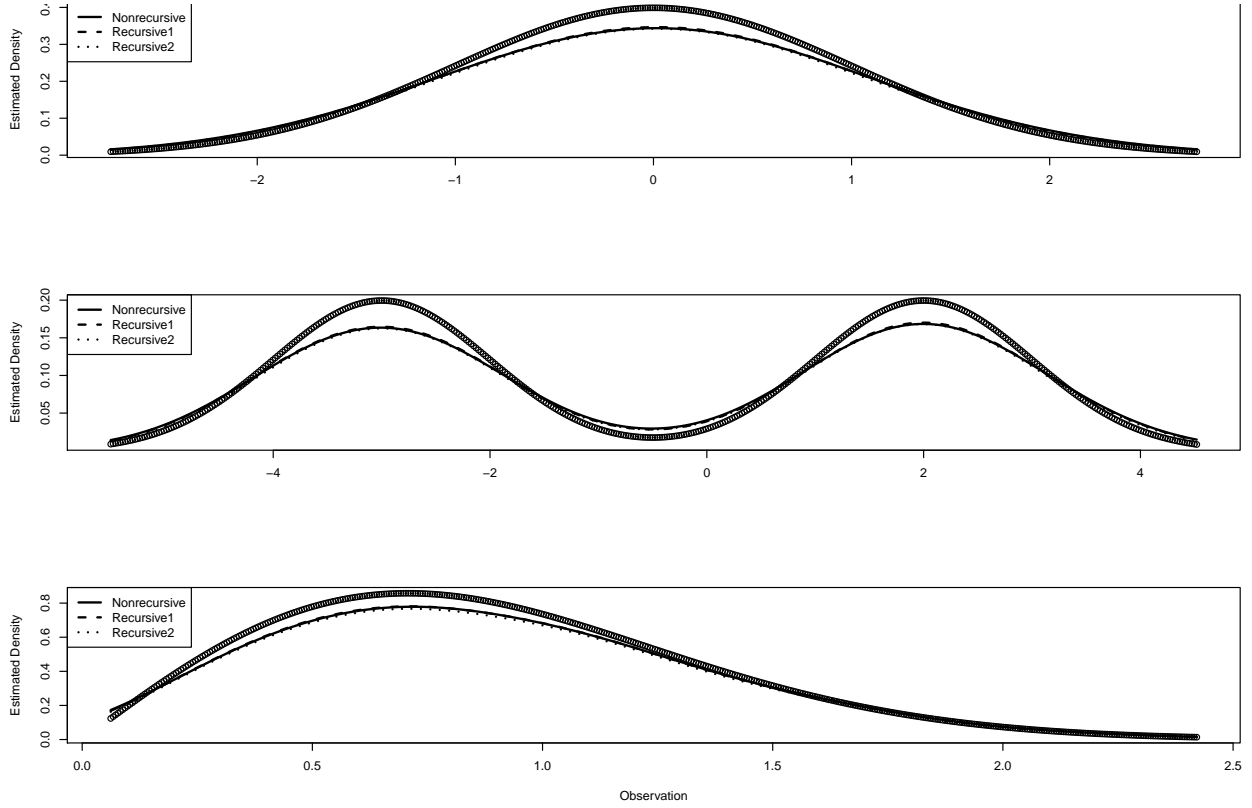


Figure 4: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$ for (a) normal, (b) normal mixture and (c) weibull distributions with 70% missing data and correlation between the response and auxiliary variable of $\rho = 0.8$. Based on 500 simulated datasets of size 200.

From Figures 1 2, 3, 4, Tables 1, 2, 3 and 4, we conclude that

- The proposed recursive kernel density estimator (1), with the stepsize $(\gamma_n) = (n^{-1})$ is closer to the true density function as compared with the nonrecursive Horvitz-Thompson-type KDE (2).
- In all the considered densities, the proposed recursive KDE with the stepsize $(\gamma_n) = (n^{-1})$ outperformed the nonrecursive Horvitz-Thompson-type KDE (2) in terms of estimation error and computational costs.
- The estimators get closer to the true density function as sample size increase and the proportion of the missing at random decrease.
- The CPU time using the proposed recursive estimators are approximately two times faster than the CPU time using the nonrecursive Horvitz-Thompson-type KDE.

Density	(x_0, n)	h_{CLP}	CLP	h_{NRPI}	NRPI	h_{RPI}	RPI
$\frac{1}{2}\mathcal{N}\left(-1, \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\mathcal{N}\left(1, \frac{1}{\sqrt{2}}\right)$	(0, 100)	0.192	0.227	0.606	0.191	0.716	0.189
	(0, 1000)	0.151	0.233	0.274	0.086	0.579	0.158
$\frac{1}{2}\mathcal{N}(-1, 1) + \frac{1}{2}\mathcal{N}(1, 1)$	(0, 100)	0.099	0.118	0.806	0.083	0.865	0.085
	(0, 1000)	0.039	0.048	0.474	0.037	0.751	0.046
$\mathcal{N}(0, 1)$	(-1.282, 100)	0.149	0.175	0.558	0.161	0.757	0.135
	(-1.282, 1000)	0.070	0.131	0.321	0.071	0.445	0.067
	(0, 100)	0.131	0.107	0.408	0.170	0.646	0.133
	(0, 1000)	0.056	0.045	0.334	0.069	0.560	0.074
$\mathcal{Exp}(1)$	(1.282, 100)	0.168	0.178	0.566	0.148	0.701	0.134
	(1.282, 1000)	0.070	0.129	0.353	0.069	0.523	0.069
	(0.1054, 100)	0.435	0.180	0.177	0.338	0.234	0.403
	(0.1054, 1000)	0.260	0.088	0.083	0.130	0.196	0.357
	(0.693, 100)	0.145	0.135	0.387	0.071	0.527	0.118
	(0.693, 1000)	0.070	0.071	0.301	0.038	0.411	0.029
	(2.303, 100)	0.213	0.232	0.406	0.267	0.300	0.296
	(2.303, 1000)	0.077	0.101	0.277	0.101	0.309	0.100
$Cauchy(0, 1)$	(-3.078, 100)	0.390	1.573	0.350	0.533	0.471	0.450
	(-3.078, 1000)	0.147	1.514	0.205	0.203	0.358	0.150
	(0, 100)	0.218	0.135	0.332	0.152	0.447	0.163
	(0, 1000)	0.076	0.046	0.190	0.070	0.305	0.088
	(3.078, 100)	0.435	1.662	0.336	0.456	0.398	0.415
	(3.078, 1000)	0.151	1.513	0.213	0.211	0.471	0.149

Table 5: Monte Carlo estimates of $\frac{\sqrt{MSE(f_n(x_0))}}{f(x_0)}$ and the bandwidth obtained respectively by CLP method, the nonrecursive Horvitz-Thompson-type KDE (2) using the proposed plug-in method and the proposed recursive estimator using the proposed plug-in method.

4.1.2 Local density estimation

In order to investigate the comparison between the proposed estimators in the case of local density estimation, we followed the paper of Dutta (2014) and we consider two sample sizes: $n = 100$ and 1000, the number of simulations is $N = 200$. Moreover, we consider five densities functions f : 1- the normal mixture distribution : $X \sim 1/2\mathcal{N}(-1, 1/\sqrt{2}) + 1/2\mathcal{N}(1, 1/\sqrt{2})$, 2- the normal mixture distribution : $X \sim 1/2\mathcal{N}(-1, 1) + 1/2\mathcal{N}(1, 1)$, 3- the standard normal : $X \sim \mathcal{N}(0, 1)$, 4- the exponential distribution with rate 1: $X \sim \mathcal{Exp}(1)$, 5- the Cauchy distribution with location 0 and scale 1. We report in Table 5, in the first two columns the bandwidth and the Monte Carlo estimates of the ratio of the square root of the MSE of the estimator proposed by Chan et al. (2010) (see Table 1 of Dutta (2014)), then, we give the bandwidth and the Monte Carlo estimates of the ratio of the square root of the MSE of the nonrecursive Horvitz-Thompson-type KDE using the proposed plug-in bandwidth selection (31) and in the last two columns we give the Monte Carlo estimates of the square ratio of the root of the MSE of the proposed recursive estimator using the proposed plug-in bandwidth selection (18). In Table 5, we consider the estimation of (a) the $1/2\mathcal{N}(-1, 1/\sqrt{2}) + 1/2\mathcal{N}(1, 1/\sqrt{2})$ density at $x_0 = 0$, (b) the $1/2\mathcal{N}(-1, 1) + 1/2\mathcal{N}(1, 1)$ density at $x_0 = 0$, and the estimation of (c) $\mathcal{N}(0, 1)$, (d) $\mathcal{Exp}(1)$, and (e) $Cauchy(0, 1)$ densities at x_0 equal to the 10th, 50th, and 90th percentiles.

From Table 5, we have the following observations.

- (i) The proposed recursive estimator and the Horvitz-Thompson-type KDE, performed better than the CLP method in many situations: using the mixed normal densities ((a) and (b)), using the standard normal density (c) and standard Cauchy (e) in the tail regions (10th and 90th percentiles), using the exponential density (d) with x_0 equal to the median.
- (ii) The proposed recursive estimator and the Horvitz-Thompson-type KDE, seems to be consistent in all the cases (its MISE seems to decrease as n increased), however the CLP estimator does not seem to be consistent (see for example the mixed normal density (a)).
- (ii) No one between the proposed recursive estimator and the Horvitz-Thompson-type KDE can be claimed to be the best in all the considered cases.

4.2 Examples

In this section we report our analysis of the two datasets mentioned in the Introduction, the first one concerne the simulated data of the Aquitaine cohort of HIV-1 infected patients and the second one concerne the coriell cell lines using the chromosome number 11.

Simulated data : Aquitaine cohort of HIV-1 infected patients The aim of this part was to compare the performance of the proposed recursive KDE to the nonrecursive Horvitz-Thompson-type KDE using the Aquitaine cohort of HIV-1 infected patients, see the paper Thiébault et al. (2000) and receiving highly active anti-retroviral therapy.

We generated 200 samples of 100 subjects. The number of repeated measures for each subject was randomly distributed between 2 and 7 (mean 4) and the times of measurements were uniformly distributed between 0 and 6.

For comparisons, we simulated data according to the following linear mixed effects model:

$$Y_{i,k} = A1 + A2 * X_{i,k} + b_i + \varepsilon_{i,k}$$

with the assumption that $b_i \sim \mathcal{N}(0, \sigma_1^2)$ and $\varepsilon_{i,k} \sim \mathcal{N}(0, \sigma^2)$. We assumed that random coefficient $(b_i, \varepsilon_{i,k})$ were independent of each other. The values of parameters were chosen to be similar to those obtained from the Aquitaine cohort of HIV-1 infected patients. The parameters were $A1 = 4$, $A2 = -0.5$, $\sigma_1^2 = 0.25$ and $\sigma^2 = 1$.

Then, we fixed the proportion of missing at random to be equal to 30%.

In Figure 5, we qualitatively compared the proposed recursive KDE using the two discussed step-sizes to the nonrecursive Horvitz-Thompson-type KDE. We observed that even when 30% of the original measurements are missing, the proposed recursive estimators remain very accurate thus demonstrating the effectiveness of our approach.

Moreover, our proposed recursive KDE was over two times faster than the nonrecursive Horvitz-Thompson-type KDE algorithm (2).

Our second application concerned real dataset, for which we compared the proposed recursive KDE approach to the nonrecursive Horvitz-Thompson-type KDE algorithm using a special data using essentially in the context of change point problem.

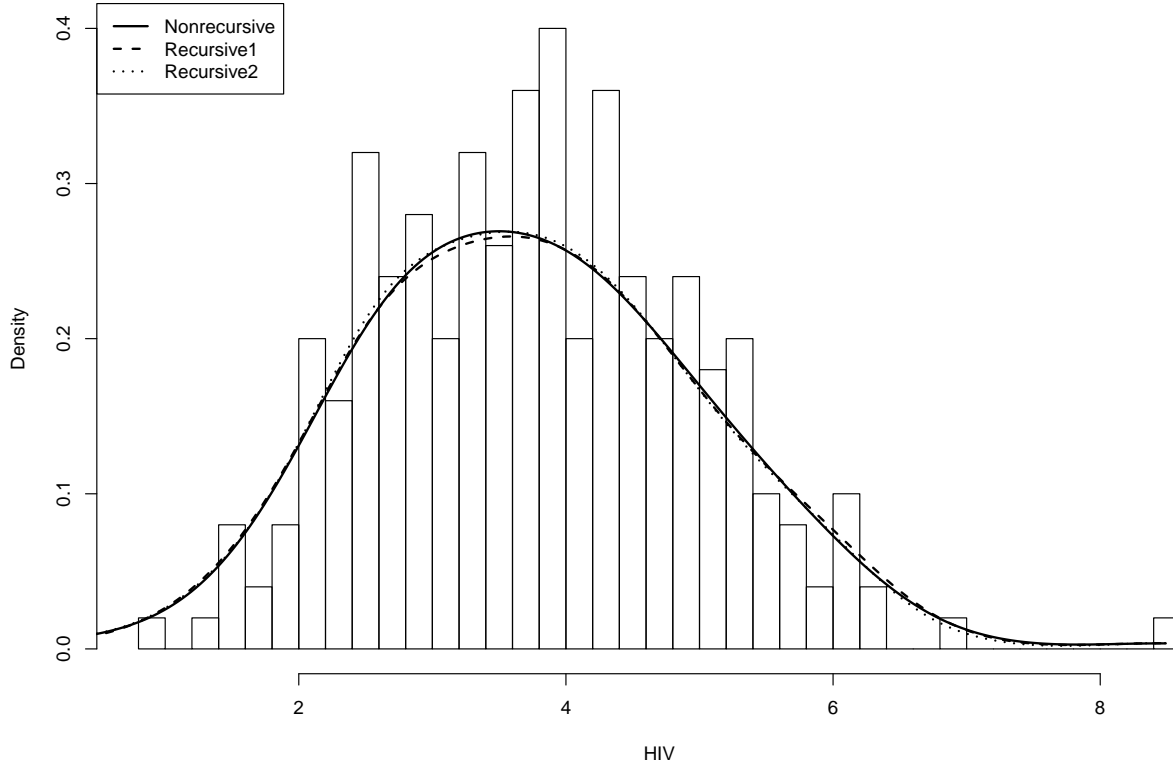


Figure 5: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the fibroblast cell strains using the chromosome number 11 with the proportion of missing at random equal to 30%. Running time was roughly 483s using the Horvitz-Thompson-type KDE, 248s using recursive 1 and 253s using recursive 2

Real dataset: Coriell cell lines using the chromosome number 11 The data correspond to two array Comparative Genomic Hybridization (CGH) studies of Coriell cell lines, which appears in the `DNACopy` package and `bcp` package (see the paper Erdman and Emerson (2007)), and for more details about the data we advise the reader to see the paper of Snijders et al. (2001).

The data correspond to two array CGH studies of fibroblast cell strains. In particular in Venkatraman and Olshe (2007), they chose the studies **GM05296** and **GM13330**. After selecting only the mapped data from chromosomes 1-22 and X , there are 2271 data points. Here we perform an analysis on the **GM13330** array CGH study described above. We simulated three different proportions of missing at random.

It is interesting to note from Figure 6 that the three estimators are quite similar, and closely follow the shape of the histogram. Moreover, we can claim that the proposed recursive estimators with the choice of the stepsize $(\gamma_n) = (n^{-1})$ is closer to the shape of the histogram of the true data as compared with the nonrecursive Horvitz-Thompson-type KDE (2).

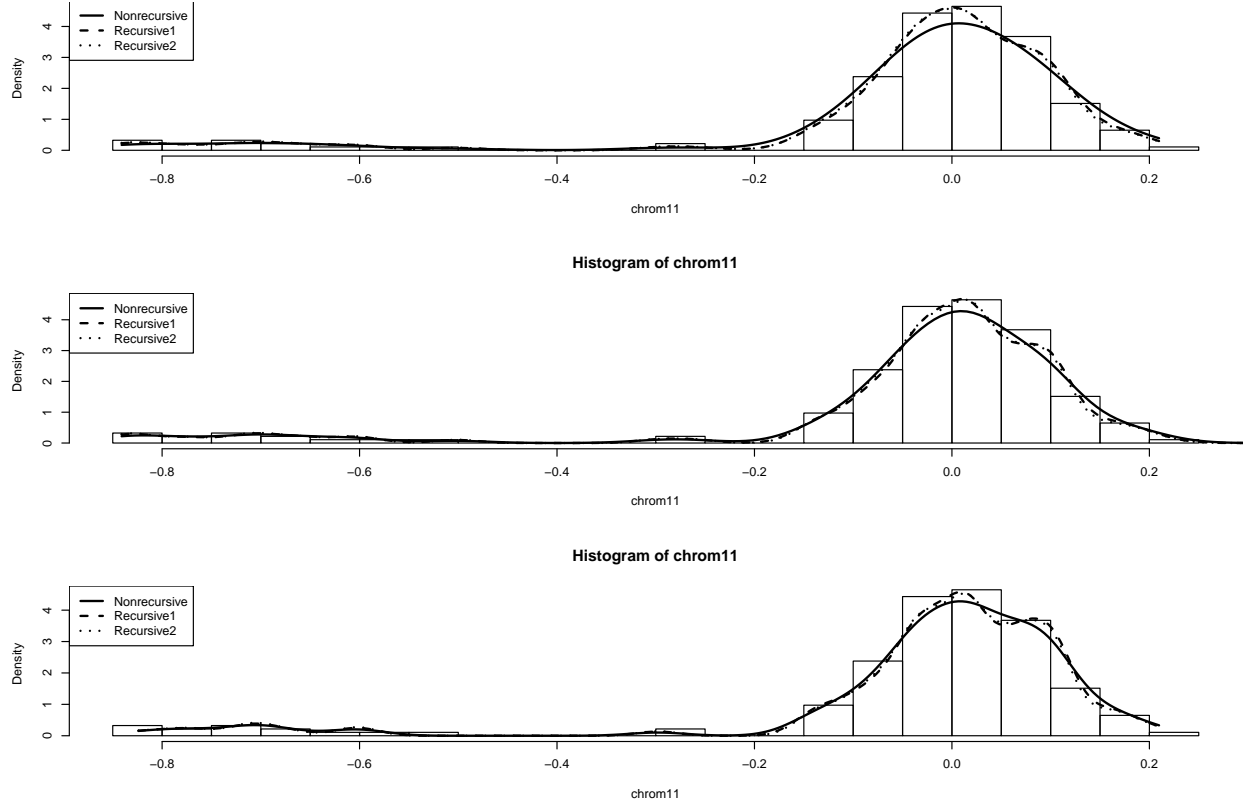


Figure 6: Quantitative comparison between the nonrecursive Horvitz-Thompson-type KDE (2) and two recursive estimators; recursive 1 correspond to the proposed estimator (1) with the choice of $(\gamma_n) = (n^{-1})$, recursive 2 correspond to the estimator (1) with the choice of $(\gamma_n) = ([4/5] n^{-1})$. Here we consider the fibroblast cell strains using the chromosome number 11 with the proportion of missing at random equal respectively to 0% (in the first block), equal to 15% (in the second block), and equal to 30% (in the last block). Running time was roughly 211s using the Horvitz-Thompson-type KDE, 107s using recursive 1 and 113s using recursive 2, when the proportion of missing at random equal to 30%

5 Conclusion

This paper propose an automatic bandwidth selection of the recursive KDE under missing data (1). The proposed estimators are asymptotically follows normal distribution. The proposed estimators are compared to the nonrecursive Horvitz-Thompson-type KDE algorithm under missing data (2). We showed that, using a recursive version of the Nadaraya-Watson estimator to estimate the propensity scores and using a specific data-driven bandwidth and some particularly stepsizes, the proposed recursive estimators can give better results compared to the nonrecursive Horvitz-Thompson-type estimator in terms of estimation error in the case of global estimation and quite similar results in the case of local estimation. Noting that the two estimators ((1) and (2)) can deal effectively with both missing and complete data. The simulation study confirms the nice features of our proposed recursive estimators and satisfactory improvement in the CPU time in comparison to the nonrecursive Horvitz-Thompson-type KDE estimator.

In conclusion, the proposed recursive estimators allowed us to obtain quite better results compared to the nonrecursive Horvitz-Thompson-type KDE under missing data in terms of estimation error and much better in terms of computational costs.

A Technical proofs for the recursive KDE

Throughout this section we use the following notation:

$$\begin{aligned} Q_n &= \prod_{j=1}^n (1 - \gamma_j), \\ Z_n(x) &= h_n^{-1} \delta_n \pi_n^{-1} K\left(\frac{x - X_n}{h_n}\right). \end{aligned} \tag{19}$$

Let us first state the following technical lemma.

Lemma 1. Let $(v_n) \in \mathcal{GS}(v^*)$, $(\eta_n) \in \mathcal{GS}(-\eta)$, and $m > 0$ such that $m - v^*\xi > 0$ where ξ is defined in (4). We have

$$\lim_{n \rightarrow +\infty} v_n Q_n^m \sum_{k=1}^n Q_k^{-m} \gamma_k v_k^{-1} = (m - v^*\xi)^{-1}.$$

Moreover, for all positive sequence (α_n) such that $\lim_{n \rightarrow +\infty} \alpha_n = 0$, and all $C \in \mathbb{R}$,

$$\lim_{n \rightarrow +\infty} v_n Q_n^m \left[\sum_{k=1}^n Q_k^{-m} \eta_k v_k^{-1} \alpha_k + C \right] = 0.$$

Lemma 1 is widely applied throughout the proofs. Let us underline that it is its application, which requires Assumption (A2)(iii) on the limit of $(n\gamma_n)$ as n goes to infinity.

Our proofs are organized as follows. Propositions 1 and 2 in Sections A.1 and A.2 respectively, Theorem 1 in Section A.3.

A.1 Proof of Proposition 1

In view of (1) and (19), we have

$$\begin{aligned}
f_n(x) - f(x) &= (1 - \gamma_n)(f_{n-1}(x) - f(x)) + \gamma_n(Z_n(x) - f(x)) \\
&= \sum_{k=1}^{n-1} \left[\prod_{j=k+1}^n (1 - \gamma_j) \right] \gamma_k(Z_k(x) - f(x)) + \gamma_n(Z_n(x) - f(x)) \\
&\quad + \left[\prod_{j=1}^n (1 - \gamma_j) \right] (f_0(x) - f(x)) \\
&= Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k(Z_k(x) - f(x)) + Q_n(f_0(x) - f(x)).
\end{aligned}$$

It follows that

$$\mathbb{E}(f_n(x)) - f(x) = Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k(\mathbb{E}(Z_k(x)) - f(x)) + Q_n(f_0(x) - f(x)).$$

Moreover, we have

$$\begin{aligned}
\mathbb{E}[Z_k^p(x)] &= h_k^{-p} \pi_k^{-p} \mathbb{E} \left[\delta_k K^p \left(\frac{x - X_k}{h_k} \right) \right] \\
&= h_k^{-p} \pi_k^{-p} \mathbb{E} \left[\mathbb{1}_{\{T_k = X_k\}} K^p \left(\frac{x - X_k}{h_k} \right) \right] \\
&= h_k^{-p} \pi_k^{-p} \mathbb{P}[T_k = X_k] \mathbb{E} \left[K^p \left(\frac{x - X_k}{h_k} \right) \mid \{T_k = X_k\} \right] \\
&= h_k^{-p} \pi_k^{-p} \mathbb{P}[\delta_k = 1 \mid T_k] \mathbb{E} \left[K^p \left(\frac{x - T_k}{h_k} \right) \right] \\
&= h_k^{-p} \pi_k^{-p+1} \mathbb{E} \left[K^p \left(\frac{x - T_k}{h_k} \right) \right] \\
&= h_k^{-p} \pi_k^{-p+1} \int_{\mathbb{R}} K^p \left(\frac{x - t}{h_k} \right) f(t) dt \\
&= h_k^{-p+1} \pi_k^{-p+1} \int_{\mathbb{R}} K^p(z) f(x - zh_k) dz
\end{aligned} \tag{20}$$

Then, it follows from (20), for $p = 1$, that

$$\begin{aligned}
\mathbb{E}[Z_k(x)] - f(x) &= \int_{\mathbb{R}} K(z) [f(x - zh_k) - f(x)] dz \\
&= \frac{h_k^2}{2} f^{(2)}(x) \mu_2(K) + \eta_k(x),
\end{aligned} \tag{21}$$

with

$$\eta_k(x) = \int_{\mathbb{R}} K(z) \left[f(x - zh_k) - f(x) - \frac{1}{2} z^2 h_k^2 f^{(2)}(x) \right] dz,$$

and, since f is bounded and continuous at x , we have $\lim_{k \rightarrow \infty} \eta_k(x) = 0$. In the case $a \leq \alpha/5$, we have $\lim_{n \rightarrow \infty} (n\gamma_n) > 2a$; the application of Lemma 1 then gives

$$\begin{aligned}\mathbb{E}[f_n(x)] - f(x) &= \frac{1}{2} f^{(2)}(x) \int_{\mathbb{R}} z^2 K(z) dz Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k h_k^2 [1 + o(1)] \\ &\quad + Q_n (f_0(x) - f(x)) \\ &= \frac{1}{2(1-2a\xi)} f^{(2)}(x) \mu_2(K) [h_n^2 + o(1)],\end{aligned}$$

and (5) follows. In the case $a > \alpha/5$, we have $h_n^2 = o(\sqrt{\gamma_n h_n^{-1}})$, and $\lim_{n \rightarrow \infty} (n\gamma_n) > (\alpha - a)/2$, then Lemma 1 ensures that

$$\begin{aligned}\mathbb{E}[f_n(x)] - f(x) &= Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k o\left(\sqrt{\gamma_k h_k^{-1}}\right) + O(Q_n) \\ &= o\left(\sqrt{\gamma_n h_n^{-1}}\right).\end{aligned}$$

which gives (6). Further, we have

$$\begin{aligned}\text{Var}[f_n(x)] &= Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 \text{Var}[Z_k(x)] \\ &= Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 \left(\mathbb{E}(Z_k^2(x)) - (\mathbb{E}(Z_k(x)))^2 \right).\end{aligned}\tag{22}$$

Moreover, in view of (20), for $p = 2$, that

$$\begin{aligned}\mathbb{E}(Z_k^2(x)) &= h_k^{-1} \pi_k^{-1} \int_{\mathbb{R}} f(x - zh_k) K^2(z) dz \\ &= h_k^{-1} \pi_k^{-1} f(x) \int_{\mathbb{R}} K^2(z) dz + \nu_k(x),\end{aligned}\tag{23}$$

with

$$\nu_k(x) = h_k^{-1} \pi_k^{-1} \int_{\mathbb{R}} K^2(z) [f(x - zh_k) - f(x)] dz.$$

Moreover, it follows from (21), that

$$\mathbb{E}[Z_k(x)] = f(x) + \tilde{\nu}_k(x),\tag{24}$$

with

$$\tilde{\nu}_k(x) = \int_{\mathbb{R}} K(z) [f(x - zh_k) - f(x)] dz.$$

Then, it follows from (22), (23) and (24), that

$$\begin{aligned}\text{Var}[f_n(x)] &= f(x) R(K) Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 h_k^{-1} \pi_k^{-1} + Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 \nu_k(x) \\ &\quad - f^2(x) Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 - 2f(x) Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 \tilde{\nu}_k(x) \\ &\quad - Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 \tilde{\nu}_k^2(x).\end{aligned}$$

Since f is bounded continuous, we have $\lim_{k \rightarrow \infty} \nu_k(x) = 0$ and $\lim_{k \rightarrow \infty} \tilde{\nu}_k(x) = 0$. In the case $a \geq \alpha/5$, we have $\lim_{n \rightarrow \infty} (n\gamma_n) > (\alpha - a)/2$, and the application of Lemma 1 gives

$$\text{Var}[f_n(x)] = \gamma_n h_n^{-1} \pi_n^{-1} (2 - (\alpha - a)\xi)^{-1} f(x) R(K) + o(\gamma_n h_n^{-1}),$$

which proves (7). Now, in the case $a < \alpha/5$, we have $\gamma_n h_n^{-1} = o(h_n^4)$, and $\lim_{n \rightarrow \infty} (n\gamma_n) > 2a$, then the application of Lemma 1 gives

$$\begin{aligned} \text{Var}[f_n(x)] &= Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k o(h_k^4) \\ &= o(h_n^4), \end{aligned}$$

which proves (2).

A.2 Proof of Proposition 2

Following similar steps as the proof of the Proposition 2 of Mokkadem et al. (2009), we proof the Proposition 2.

A.3 Proof of Theorem 1

Let us at first assume that, if $a \geq \alpha/5$, then

$$\sqrt{\gamma_n^{-1} h_n \pi_n} (f_n(x) - \mathbb{E}[f_n(x)]) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, (2 - (\alpha - a)\xi)^{-1} f(x) R(K)\right). \quad (25)$$

In the case when $a > \alpha/5$, Part 1 of Theorem 1 follows from the combination of (6) and (25). In the case when $a = \alpha/5$, Parts 1 and 2 of Theorem 1 follow from the combination of (5) and (25). In the case $a < \alpha/5$, (2) implies that

$$h_n^{-2} (f_n(x) - \mathbb{E}(f_n(x))) \xrightarrow{\mathbb{P}} 0,$$

and the application of (5) gives Part 2 of Theorem 1.

We now prove (25). In view of (1), we have

$$f_n(x) - \mathbb{E}[f_n(x)] = Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k (Z_k(x) - \mathbb{E}[Z_k(x)]).$$

Set

$$Y_k(x) = \Pi_k^{-1} \gamma_k (Z_k(x) - \mathbb{E}[Z_k(x)]).$$

The application of Lemma 1 ensures that

$$\begin{aligned} v_n^2 &= \sum_{k=1}^n \text{Var}(Y_k(x)) \\ &= \sum_{k=1}^n \Pi_k^{-2} \gamma_k^2 \text{Var}(Z_k(x)) \\ &= \sum_{k=1}^n \Pi_k^{-2} \gamma_k^2 h_k^{-1} \pi_k^{-1} [f(x) R(K) + o(1)] \\ &= Q_n^{-2} \gamma_n h_n^{-1} \pi_n^{-1} \left[(2 - (\alpha - a)\xi)^{-1} f(x) R(K) + o(1) \right]. \end{aligned}$$

On the other hand, we have, for all $p > 0$,

$$\mathbb{E} \left[|Z_k(x)|^{2+p} \right] = O \left(h_k^{-1-p} \right),$$

and, since $\lim_{n \rightarrow \infty} (n\gamma_n) > (\alpha - a)/2$, there exists $p > 0$ such that $\lim_{n \rightarrow \infty} (n\gamma_n) > \frac{1+p}{2+p}(\alpha - a)$. Applying Lemma 1, we get

$$\begin{aligned} \sum_{k=1}^n \mathbb{E} \left[|Y_k(x)|^{2+p} \right] &= O \left(\sum_{k=1}^n Q_k^{-2-p} \gamma_k^{2+p} \mathbb{E} \left[|Y_k(x)|^{2+p} \right] \right) \\ &= O \left(\sum_{k=1}^n \frac{\Pi_k^{-2-p} \gamma_k^{2+p}}{h_k^{1+p}} \right) \\ &= O \left(\frac{\gamma_n^{1+p}}{Q_n^{2+p} h_n^{1+p}} \right), \end{aligned}$$

and we thus obtain

$$\frac{1}{v_n^{2+p}} \sum_{k=1}^n \mathbb{E} \left[|Y_k(x)|^{2+p} \right] = O \left([\gamma_n h_n^{-1}]^{p/2} \right) = o(1).$$

The convergence in (25) then follows from the application of Lyapounov's Theorem.

B Results for Horvitz-Thompson-type KDE

B.1 Global density estimation using HT KDE

Now, let us recall that the bias and variance of the nonrecursive Horvitz-Thompson-type KDE defined by (2) are given by

$$\mathbb{E} \left[\tilde{f}_n(x) \right] - f(x) = \frac{h_n^2}{2} f^{(2)}(x) \mu_2(K) + o(h_n^2),$$

and

$$\text{Var} \left[\tilde{f}_n(x) \right] = \frac{\pi_n^{-1}}{n h_n} f(x) R(K) + o \left(\frac{1}{n h_n} \right).$$

It follows that,

$$AMWISE \left[\tilde{f}_n \right] = \frac{\pi_n^{-1}}{n h_n} I_1 R(K) + \frac{h_n^4}{4} I_2 h_n^4 \mu_2^2(K).$$

Then, to minimize the *AMWISE* of \tilde{f}_n , the bandwidth (h_n) must equal to

$$\left(\left(\frac{I_1}{I_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right), \quad (26)$$

and we have

$$AMWISE \left[\tilde{f}_n \right] = \frac{5}{4} I_1^{4/5} I_2^{1/5} \Theta(K) \pi_n^{-4/5} n^{-4/5}.$$

To estimate the optimal bandwidth (30), we must estimate I_1 and I_2 . For this purpose, we use the following two kernel estimators :

$$\tilde{I}_1 = \frac{1}{n(n-1)b_n} \sum_{\substack{i,j=1 \\ i \neq j}}^n \delta_j \hat{\pi}_{NWj}^{-1} K_b \left(\frac{X_i - X_j}{b_n} \right), \quad (27)$$

$$\tilde{I}_2 = \frac{1}{n^3 b_n'^6} \sum_{\substack{i,j,k=1 \\ j \neq k}}^n \delta_j \delta_k \hat{\pi}_{NWj}^{-1} \hat{\pi}_{NWk}^{-1} K_{b'}^{(2)} \left(\frac{X_i - X_j}{b_n'} \right) K_{b'}^{(2)} \left(\frac{X_i - X_k}{b_n'} \right). \quad (28)$$

where K_b and $K_{b'}$ are a kernels, b_n and b_n' are respectively the associated bandwidth given in (14). We showed that in order to minimize the $AMISE$ of \tilde{I}_1 respectively of \tilde{I}_2 , the pilot bandwidth (b_n) respectively (b_n') must belong to $\mathcal{GS}(-2/5)$, respectively to $\mathcal{GS}(-3/14)$.

Then the plug-in estimator of the bandwidth (h_n) using the nonrecursive estimator (2), is given by

$$\left(\left(\frac{\tilde{I}_1}{\tilde{I}_2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \hat{\pi}_{NWn}^{-1/5} n^{-1/5} \right), \quad (29)$$

and the plug-in of the $AMWISE$ of the nonrecursive estimator (2), is given by

$$\widetilde{AMWISE}[\tilde{f}_n] = \frac{5}{4} \tilde{I}_1^{4/5} \tilde{I}_2^{1/5} \Theta(K) \hat{\pi}_{NWn}^{-4/5} n^{-4/5}.$$

B.2 Local density estimation using the HT KDE

Now, using the nonrecursive Horvitz-Thompson-type KDE and in order to minimize the $AMSE$ of \tilde{f}_n , the bandwidth (h_n) must equal to

$$\left(\left(\frac{f(x)}{(f^{(2)}(x))^2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \pi_n^{-1/5} n^{-1/5} \right). \quad (30)$$

Moreover, since the $f(x)$ and $f^{(2)}(x)$ are not known, we estimate $f(x)$ and $f^{(2)}(x)$ by

$$\begin{aligned} \tilde{f}_n(x) &= \frac{1}{nb_n} \sum_{i=1}^n \delta_i \pi_{NW_i}^{-1} K_b \left(\frac{x - X_i}{b_n} \right), \\ \tilde{f}_n^{(2)}(x) &= \frac{1}{nb_n'^3} \sum_{i=1}^n \delta_i \pi_{NW_i}^{-1} K_{b'}^{(2)} \left(\frac{x - X_i}{b_n'} \right). \end{aligned}$$

where K_b and $K_{b'}$ are a kernels, b_n and b_n' are respectively the associated bandwidth given in (14). Then the plug-in estimator of the bandwidth (h_n) using the nonrecursive estimator (2), to estimate locally the density f at a point x is given by

$$\left(\left(\frac{\tilde{f}_n(x)}{(\tilde{f}_n^{(2)}(x))^2} \right)^{1/5} \left\{ \frac{R(K)}{\mu_2^2(K)} \right\}^{1/5} \hat{\pi}_{NWn}^{-1/5} n^{-1/5} \right). \quad (31)$$

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