Broad Objectives

1. Guided by physical considerations, understand the partial differential equations (PDEs) governing stellar interiors and stellar transients as well as related astrophysical phenomena (such as accretion disks around compact stellar objects).

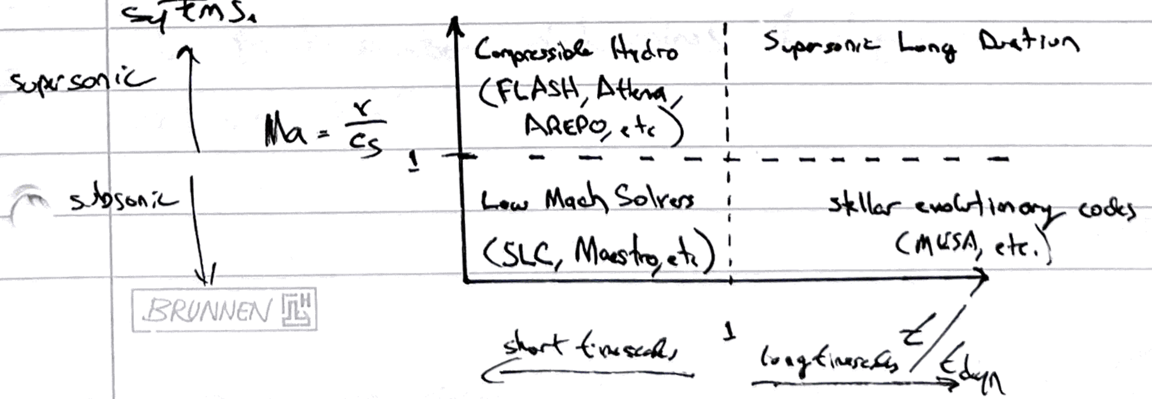
2. Understand the numerical methods employed by modern astrophysical codes to solve these PDEs.

3. Get actual "hands-on" experience in doing multidimensional numerical simulations using an open source hydro code (FLASH-X).

While there is an extensive research literature and a growing number of pedagogically-oriented books on computational astrophysics, still today much of the body of knowledge gets passed on "from master to student." An overarching goal of these lectures is to provide some of this less tangible communal knowledge, and the feeling for the choices which must be made by any practicing computational astrophysicist.

Astrophysical Flow Regimes

No single numerical method is ideally suitable for all astrophysical sytems.



The choice of numerical method depends upon the physical problem and the scales of interest. In astrophysical flows, four numbers largely determine these choices.

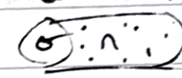
$$
\left.
\begin{array}{lll}
\text{Knudsen number} & Kn \equiv & \dfrac{\lambda_{mf}}{L} \\
\text{Mach number} & Ma \equiv & \dfrac{v}{c_s} \\
\text{Reynolds number} & Re \equiv & \dfrac{Lv}{\nu} \\
\text{Plasma parameter} & \beta \equiv & \dfrac{P_g}{P_m}
\end{array}
\right\}
\begin{array}{l}
\text{Ratio of mean-free path length to typical scale} \\
\text{Ratio of flow to sound speed} \\
\text{Ratio of inertial forces to viscous} \\
\text{Ratio of gas to magnetic pressure}
\end{array}
$$

We examine each of these in turn.

$\lambda_{mf} \sigma n \sim 1$

$\lambda_{mf} \sim vt$

$\lambda_{mf} \sim \dfrac{1}{n\sigma}$



Knudsen Number Kn

The Knudsen number is one of the most basic parameters which qualitatively determines the character of the flow. If $Kn \ll 1$, the mean-free path length is much smaller than length scales of interest, and the fluid flow can therefore be treated as a continuum, with all fluid quantities (mass density, temperature, etc.) smoothly defined (outside of shocks).

Conversely, if $Kn \gtrsim 1$, the fluid flow cannot be represented as a continuous medium, typically requiring the material particles (ions & electrons) to be treated by more fundamental physical descriptions of plasma physics (see next page for estimates).

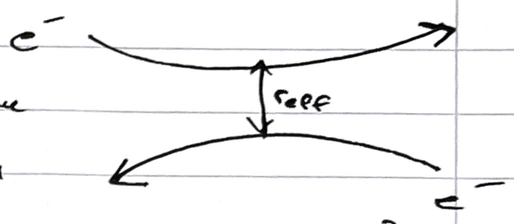
Mach Number Ma

The Mach number determines the degree of compressibility of the fluid.

Estimating Mean-Free Path in Ionized Plasma

Estimate a large deflection when (*)

$$\frac{e^2}{r_{eff}} \sim m_e w^2 \sim kT_e \quad \Big\}\ w \text{ relative velocity}$$



$$\lambda_{mf} \simeq \frac{1}{n_e \sigma} \sim \frac{1}{n_e \pi r_{eff}^2} \sim \frac{(kT_e)^2}{n_e e^4}$$

For sun, $n_e \simeq \rho / m_u$, assuming $X \simeq 1$. At center of sun:

$$\lambda_{mf} \simeq \frac{(k\overbrace{T_e}^{10^7 K})^2 m_u}{\underbrace{\rho}_{100\,g/cm^3} e^4} \simeq 50 \times 10^{-8}\,cm$$

This length scale is enormously smaller than the radius or pressure scale height of the stellar interior —

$$Kn = \frac{\lambda_{mf}}{R_\odot} \sim \frac{50 \times 10^{-8}\,cm}{7 \times 10^{10}\,cm} \sim 7 \times 10^{-18}$$

The small Kn number also implies collision times between charged species are so short that all species will relax to Maxwellian on times

$$t_c \sim \frac{\lambda_{mf}}{v} \sim \frac{\lambda_{mf}\, m_e^{1/2}}{(kT_e)^{1/2}} \sim 10 \times 10^{-17}\,s$$

Which implies that electrons and ions have a single shared temperature, and that a <u>local</u> thermodynamic approximation applies throughout the stellar interior.

(*) Small angle deflections are typically more important, and bring in a factor of the inverse of the Coulomb logarithm parameter — see Shu, v. II, chapter I.

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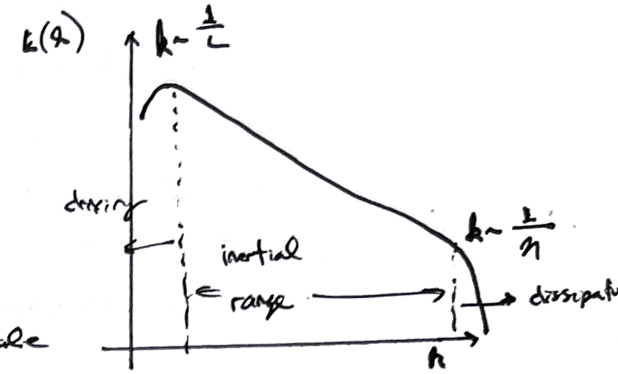
Kolmogorov Turbulence and Reynolds Number

length



Assume self-similar steady cascade from large scale ~L (integral scale), down to the smallest length scale η, which is called the Kolmogorov scale. Further, call $\delta v_L \equiv v_0$.

$$\delta v_r \rightarrow \text{velocity fluctuation on scale } r$$

$$t_{edd}(r) = \frac{r}{\delta v_r}$$

these concepts can be made more rigorous by Fourier transforming the specific kinetic energy v^2 from real to Fourier space

For steady turbulence, ~~$t_{edd}(L) = t_{edd}(r)$~~ flux of spec KE on all scales $\eta < r < L$.

$$\text{rate of spec. KE per time} = \frac{v_0^2}{(L/v_0)} = \frac{\delta v_r^2}{(r/\delta v_r)}$$

$$\frac{v_0^3}{L} = \frac{\delta v_r^3}{r}$$

This carries all the way down to $r \sim \eta$, where turbulence is dissipated.

$$\delta v_r = v_0 \left(\frac{r}{L}\right)^{1/3}$$

Kolmogorov velocity scaling

This rate of turbulent energy dissipation is written as $\varepsilon = \frac{v_0^3}{L}$ and is an invariant constant quantity for steady driven turbulence with fixed v_0 & L.

But this cannot extend to arbitrarily short length scales. When $t_{edd}(\eta) \sim t_{visc}(\eta)$, the eddy cannot turn over before it is dissipated as heat,

Kolmogorov turbulence & Euler Equations

Most astrophysical flows have exceedingly high Reynolds numbers — for example, flows $v_0 \sim 10^3$ km/s, $L \sim 10^2$ km, inside WDs have $Re \sim 10^{16}$. Fundamentally, this is because there exists an enormous dynamic range between the scales $\sim L$ where we examine the flow and the Kolmogorov scale where turbulence is dissipated.

If the physical viscosity were to be employed in a full Navier-Stokes simulation of these flows, its effect would be negligible.

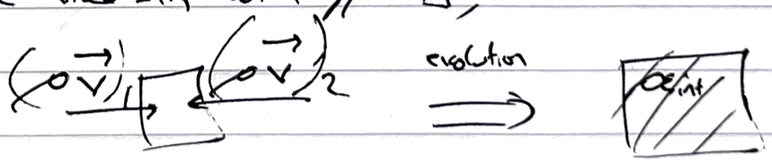
Physical viscosity (almost!) never matters in astrophysics.

The reason why can again be understood from the

$$Re^{3/4} = \frac{L}{\eta}$$

scaling. To imply we are simulating a $Re = 10^{16}$ flow means we would need $\eta = 10^{-12} L$! Clearly, no simulation completed to date even approaches this value. Instead, the record-holding simulations have of order $(10^4)^3$ cells, implying $Re \sim 10^5$, effectively.

What is happening inside the simulation is that the grid effectively dissipates kinetic energy on the grid scale Δ, and acts as an effective viscosity with $\eta \sim \Delta$,



momentum conservation of flows into cell produce heating

Because the effective Reynolds number is always much smaller than the physical Reynolds number, we employ the Euler equations of inviscid hydrodynamics.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

body force

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_j \frac{\partial \Pi_{ij}}{\partial x_j} = F_{v,i}$$

momentum flux tensor $\Pi_{ij} = p\delta_{ij} + \rho v_i v_j$

body work, specific total energy

$$\frac{\partial(\rho E)}{\partial t} + \sum_i \frac{\partial(\rho E + p)v_i}{\partial x_i} = \sum_i F_{v,i} v_i$$

$E = \hat{\epsilon}_{int} + \frac{1}{2}v^2$

There is no physical viscosity in these ~~simulations~~ equations, but the numerical simulations posess numerical viscosity as described above, ~~or~~ artificial viscosity (usually very small in modern numerical methods) introduced "by hand." This is an extremely important point about simulating hydrodynamical flows, even though it is extremely challenging to accurately characterize the precise level of viscosity introduced (see Bazi, Biferale, Fisher +, PRL, 100, 23, 234503).

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad \Big\} \text{ Kolmogorov length scale}$$

The ratio of the large-scale integral driving length scale, down to the smallest Kolmogorov length scale is

$$\frac{L}{\eta} = \frac{L}{\nu^{3/4}}\,\varepsilon^{1/4} = \frac{L\, v_0^{3/4}}{\nu^{3/4}\, L^{1/4}}$$

$$= \left(\frac{L v_0}{\nu}\right)^{3/4} = Re^{3/4}$$

The Reynolds number is normally introduced as the ratio of the importance of inertial to viscous forces

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_j \left(\frac{\partial \Pi_{ij}}{\partial x_j} + \overbrace{\frac{\partial \sigma_{ij}}{\partial x_j}}^{\text{viscous stress tensor}}\right) = 0 \quad \Big\} \text{ zero net force}$$

$$\sigma_{ij} = \rho\nu\left(\underbrace{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}}_{\text{deformation}} - \underbrace{\frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}}_{\text{dilation}}\right)$$

$$\text{Ratio} = \frac{\Pi_{ij}}{\sigma_{ij}} \sim \frac{\cancel{\rho} v_0^{\cancel{2}}}{\cancel{\rho\nu v_0}\,/\,L} \sim \frac{L v_0}{\nu}$$

But turbulence provides us with another way to think about the Reynolds number — as simply the dynamic range (to the 4/3 power) between the largest and smallest scales of turbulence.

$$Re = \left(\frac{L}{\eta}\right)^{4/3}$$

Numerical Solutions of Conservation Law Systems

Consider the model system of advection of a scalar field φ in the presence of a ~~uniform~~ fixed background velocity $\vec{V}$,

$$\frac{\partial \varphi}{\partial t} + \vec{\nabla} \cdot \vec{f} = 0 \quad \Big\} \; \vec{f} \text{ flux vector} = \varphi \vec{V}$$

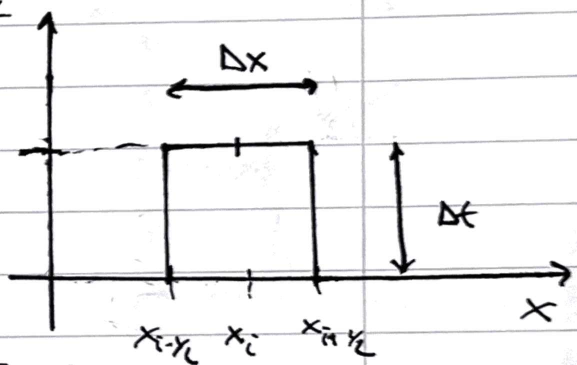
This equation would for example apply to the mass density ρ in hydrodynamics, with a fixed velocity field $\vec{V}$, and many other examples besides (eg, compositions in non-reacting flows).

To solve this equation, let's take the simplified case of a 1D flow with constant background velocity, $\vec{V} = (v_x, 0, 0)$, Let's also assume the spatial and time discretization is uniform (all of those assumptions can, of course, be relaxed in more general schemes).

Integrate the scalar field conservation equation first in space — from $x_{i-1/2}$ to $x_{i+1/2}$

$(n+1)\Delta t \quad t_{n+1}$

$n\Delta t = t_n$



$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi}{\partial t} dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial f}{\partial x} dx = 0$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi}{\partial t} dx + \left[f(x_{i+1/2}) - f(x_{i-1/2}) \right] = 0$$

Next, integrate this result in time from t_n to t_{n+1}:

$$\int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi}{\partial t} dx\,dt + \int_{t_n}^{t_{n+1}} \left[f(x_{i+1/2}) - f(x_{i-1/2}) \right] dt = 0$$

$(\partial\varphi/\partial t)$

Assuming the integrand is continuous, the order of integration in the first integral can be exchanged:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t_n}^{t_{n+1}} \frac{\partial\varphi}{\partial t} \, dt \, dx + \int_{t_n}^{t_{n+1}} \left[f(x_{i+1/2}) - f(x_{i-1/2}) \right] dt = 0$$

$$(*) \quad \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(x, t_{n+1}) - \varphi(x, t_{n}) \, dx + \int_{t_n}^{t_{n+1}} \left(f(x_{i+1/2}) - f(x_{i-1/2}) \right) dt = 0$$

The first term can be cast in a more familiar form by expressing it in terms of the spatial average over the cell width Δx

$$\bar{\varphi}(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(x, t) \, dx$$

So $(*)$ becomes

$$\Delta x \left[\bar{\varphi}(t_{n+1}) - \bar{\varphi}(t_n) \right] + \int_{t_n}^{t_{n+1}} \left[f(x_{i+1/2}) - f(x_{i-1/2}) \right] dt = 0$$

This equation is so far exact, and has a very simple interpretation — the update to the conserved scalar field is simply determined by the time-integrated fluxes around the boundaries of cells.

$$\bar{\varphi}(t_{n+1}) = \bar{\varphi}(t_n) + \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \left[f(x_{i+1/2}) - f(x_{i-1/2}) \right] dt$$

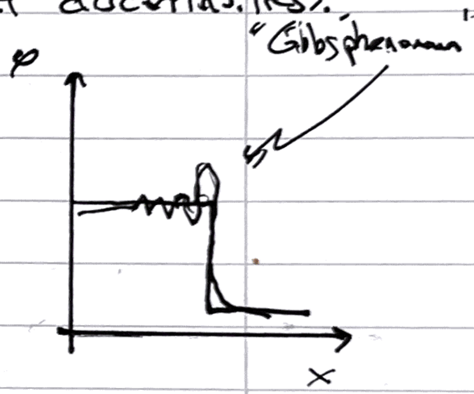
The values of φ are naturally "cell-centered" in this approach — meaning the values stored at $\varphi(t_n)$ are naturally the spatial averaged quantities in each cell. One more step represents the fluxes as time-averages —

$$\bar{f}(x_{i-1/2}) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(x_{i-1/2})\, dt$$

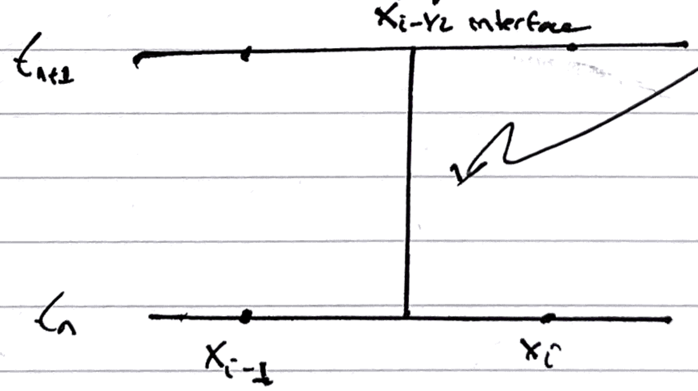
So that the entire problem boils down to obtaining the time-averaged fluxes $\bar{f}(x_{i-1/2})$, etc. around cell interfaces and applying these to the conservative update through

$$\bar{\varphi}(t_{n+1}) = \bar{\varphi}(t_n) + \frac{\Delta t}{\Delta x}\left[\bar{f}(x_{i+1/2}) - \bar{f}(x_{i-1/2})\right]$$

Finite difference methods apply discretized stencils typically directly ~~the~~ beginning PDEs. In supersonic flows, strong shocks lead to discontinuities where the discretization leads to problematic oscillations around the shock fronts (also contact discontinuities).



Alternatively, Godunov methods, which underly most modern astrophysical grid and mesh codes (and even smoothed particle hydrodynamics codes) employs a clever conception of the cell data which does not presume the smoothness of the underlying flow.

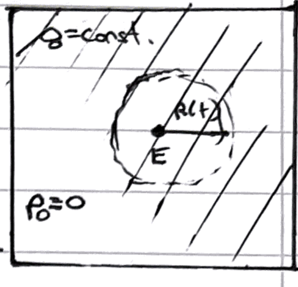


need to find time-averaged flux here

The Riemann problem addresses this problem, treating the initial data as discontinuous, and asking what the solution along the interface is.

The scheme described above turns out to only be first-order accurate in both space and time — that is, the errors diminish with $\mathcal{O}(\Delta x)$. Godunov's theorem proves that } in 1959 linear schemes which preserve monotonicity (eg. if $\varphi_n(x)$ is monotonically increasing or decreasing in x, so too is $\varphi_{n+1}(x)$) are necessarily only first-order accurate. This may seem to be a roadblock to developing higher-order accuracy schemes, but the solution (which took 20+ years to fully develop) is to utilize nonlinear reconstruction slopes and slope limiters on the initial data at t_n. This is the basis of the PPM and many other methods.

Taylor-Sedov Blastwave



Consider a delta function injection of energy, E, into an inviscid fluid medium with uniform mass density, ρ_0, and zero pressure.

If the medium was self-gravitating, one would have the characteristic gravitational Jeans length $\lambda_J = c_s\left(\frac{\pi}{G\rho_0}\right)^{1/2}$. However, if gravity is neglected, $G=0$, $\lambda_J \to \infty$, and there exists no such length scale in this problem. Indeed, no characteristic length scale can be found for the pointwise injection of energy into the inviscid flow, implying that the problem is self-similar.

Because $p_0=0$, the blast wave also never transitions to subsonic.

Some simple estimates can shed light on the evolution of the blast wave. Assume the problem is in 3D, and that flow is adiabatic - no thermal energy is transported across fluid elements.

E conserved

$$\left\{ E = \underbrace{\frac{1}{2}\int d\Omega \int_0^{R(t)} r^2\,dr\, \rho(r,t)\, v^2(r,t)}_{KE(t)} + \underbrace{\int d\Omega \int_0^{R(t)} r^2\,dr\, \rho(r,t)\, \varepsilon_{int}(r,t)}_{E_{therm}(t)} \right.$$

thermal energy/mass

specific internal energy

$$E = KE(t) + E_{therm}(t)$$

But, because $p_0=0$, these two energies must always scale proportionate to one another. Otherwise, one could find a final a time (say t_*), such that

$$\frac{KE(t_*)}{E_{therm}(t_*)} = 1$$

But such a time would break self-similarity [one could have $r_* = v(R(t), t_*)$]. So $KE(t) \propto E_{therm}(t)$ proportionately.

We can therefore write

$$\alpha \, KE(t) = E_{therm}(t)$$

$$E = (1+\alpha) \, KE(t)$$

Approximately,

$$E \sim Mv^2 \sim R^3(t) \rho_0 \left(\frac{R(t)}{t}\right)^2$$

$$R(t) \sim \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} = \beta \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$$

Drop $R(t)$ notation

The value of β depends upon the geometry (here assumed spherical, $\beta \sim 1.033$).

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Exercise: Derive the scaling for a 2D cylindrical blast wave.

$$E \sim Mv^2 \sim R^2(t) \rho_0 \left(\frac{R(t)}{t}\right)^2$$

$$R(t) \sim \left(\frac{E}{\rho_0}\right)^{1/4} t^{1/2} = \beta \left(\frac{E}{\rho_0}\right) t^{1/2}$$

One can construct similarity variables

$$\xi = \frac{r}{R(t)} \Rightarrow G(\xi) = \frac{\rho}{\rho_0}$$

Th

Shock speed

$$D = \frac{dR}{dt} = \frac{2}{5} \frac{R}{t}$$

$$V(\xi) = \frac{v}{\dot{R}(t)} = \frac{5}{2} \frac{t v}{r}$$

$$Z(\xi) = \frac{25 t^2 c^2}{4 r^2}$$

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