Critical Temperature for Ideal Gas and Radiation Pressure Balance in Pseudo-Enthalpy Formalism

Robert Fisher

June 17, 2025

Overview

In some self-consistent analytic solutions to relativistic stellar structure equations, it is convenient to define a "pseudo-enthalpy" or "log enthalpy" function. A common requirement is that the enthalpy h vanishes at zero pressure:

$$h(P = 0) = 0. \tag{1}$$

For an isothermal gas in local thermodynamic equilibrium (LTE), the gas and radiation are at the same temperature, and total pressure is:

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{\rho k_B T}{\mu m_u} + \frac{1}{3} a T^4, \tag{2}$$

where:

- ρ is the mass density,
- k_B is Boltzmann's constant,
- T is the temperature,
- μ is the mean molecular weight,
- m_u is the atomic mass unit,
- a is the radiation constant.

The specific pseudo-enthalpy (in cgs units of erg/g) is:

$$h = \int_0^P \frac{dP'}{\rho + \epsilon(P') + P'}, \tag{3}$$

which in the isothermal case, with $\epsilon = \epsilon_{\text{gas}} + \epsilon_{\text{rad}}$ and LTE, can be integrated analytically. However, for this to be well-defined and finite, we require the integrand to decay at small P', which implies that the radiation pressure must not dominate the gas pressure as $P \to 0$.

Thus, a **minimum temperature** T_{min} must exist, below which the gas pressure dominates at all pressures and $h(P = 0) = 0$ can be satisfied.

Equating Gas and Radiation Enthalpy Contributions

The specific enthalpy from gas pressure is:

$$h_{\text{gas}} = \frac{\gamma}{\gamma - 1} \cdot \frac{P_{\text{gas}}}{\rho} = \frac{\gamma}{\gamma - 1} \cdot \frac{k_B T}{\mu m_u}, \tag{4}$$

assuming an ideal, fully ionized gas with adiabatic index $\gamma = 5/3$.

The specific enthalpy from radiation pressure is:

$$h_{\text{rad}} = \frac{4}{\rho} a T^4. \tag{5}$$

We define T_{min} as the temperature below which gas pressure dominates:

$$h_{\text{gas}} > h_{\text{rad}}. \tag{6}$$

Setting the two enthalpies equal yields the critical case:

$$\frac{5}{2} \cdot \frac{k_B T_{\text{min}}}{\mu m_u} = \frac{4}{\rho} a T_{\text{min}}^4. \tag{7}$$

Solving for T_{min}:

$$\frac{5}{2} \cdot \frac{k_B}{\mu m_u} = \frac{4}{\rho} a T_{\text{min}}^3, \tag{8}$$

$$T_{\text{min}}^3 = \left(\frac{5}{2} \cdot \frac{k_B \rho}{4 a \mu m_u} \right), \tag{9}$$

$$T_{\text{min}} = \left(\frac{5}{8} \cdot \frac{k_B \rho}{a \mu m_u} \right)^{1/3}. \tag{10}$$

Evaluation for $\rho = 1$ g/cm^3 and 50/50 C/O

Assume full ionization of a 50/50 carbon/oxygen mixture:

$$\frac{1}{\mu} = 0.5 \cdot \frac{Z_C + 1}{A_C} + 0.5 \cdot \frac{Z_O + 1}{A_O} = 0.5 \cdot \frac{7}{12} + 0.5 \cdot \frac{9}{16} \approx \frac{494}{768}, \tag{11}$$

$$\mu \approx 1.555. \tag{12}$$

Constants:

$$\begin{aligned} k_B &= 1.3807 \times 10^{-16}\,\text{erg/K}, \\ a &= 7.5657 \times 10^{-15}\,\text{erg/cm}^3/\text{K}^4, \\ m_u &= 1.6605 \times 10^{-24}\,\text{g}, \\ \rho &= 1\,\text{g/cm}^3. \end{aligned}$$

Plug into the expression:

$$\begin{aligned}
T_{\min} &= \left(\frac{5}{8}\cdot\frac{k_B\rho}{a\mu m_u}\right)^{1/3} && (13)\\
&= \left(\frac{5}{8}\cdot\frac{1.3807\times10^{-16}\cdot 1}{7.5657\times10^{-15}\cdot 1.555\cdot 1.6605\times10^{-24}}\right)^{1/3} && (14)\\
&\approx \left(\frac{8.63\times10^{-17}}{1.95\times10^{-38}}\right)^{1/3} \approx (4.42\times10^{21})^{1/3} \approx \boxed{1.62\times10^{7}\,\mathrm{K}}. && (15)
\end{aligned}$$

Conclusion

For a gas at $\rho = 1\,\mathrm{g/cm^3}$ composed of 50/50 fully ionized C/O, the critical temperature below which analytic solutions for pseudo-enthalpy exist is:

$$\boxed{T_{\min} \approx 1.6\times10^{7}\,\mathrm{K}}.$$

At this temperature, radiation and gas pressure contribute equally to the specific enthalpy, and below this point, the gas pressure dominates, ensuring a well-defined pseudo-enthalpy profile satisfying $h(0) = 0$.