Birla Institute of Technology and Science, Pilani
Second Semester 2015–2016, MATH F112 (Mathematics-II)
Assignment-II

1. Sketch the region $S = \{z : |z - 4i| > 2\}$ and answer the following:
   (i) Is S open? Justify.
   (ii) Is S connected? Justify.
   (iii) Is S a domain? Justify.
   (iv) Is S bounded? Justify.
   (v) Sketch the closure of S.

2. Prove that $f(z) = \frac{z}{z^4+1}$ is continuous at all points inside and on the unit circle $|z| = 1$ except at four points, and determine these points.

3. Determine all the points where $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$ is differentiable and analytic.

4. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic, and find it's harmonic conjugate.

5. Find all roots of the following equations:

   (a) $\cos z = 2$ (b) $z^{1-i} = 4$

6. Show that if e^z is real, then $\operatorname{Im} z = n\pi$ (n = 0, ±1, ±2,…).

7. Let f be a function defined by

$$f(z) = \begin{cases} \frac{3}{2}, & x > 0 \\ 3x, & x < 0 \end{cases}$$

   and C be the arc from $z = -1 + 2i$ to $z = 1 + 2i$ along the curve $y = x^2 + 1$. Then evaluate $\int_C f(z)\,dz$.

8. (i) Find an upper bound for the absolute value of $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z| = 4$.

(ii) Evaluate $\int_C |z| \bar{z} \, dz$, where C is the counter-clockwise oriented semi-circular part of the circle $|z| = 2$ lying in the second and third quadrants.

9. Find the coefficient of $\frac{1}{z^3}$ in the Laurent series expansion of $f(z) = \frac{1}{z^2-3z+2}$ for $|z| > 2$.

10. Find all zeros and their respective orders of the function $f(z) = z^2(1-\cos z)$.

11. Using Cauchy integral formula, evaluate the following integrals:

(i) $\oint_C \frac{z^3(z-3)^7}{(z-1)(z-2)^4} dz$, where C: $|z| = \frac{3}{2}$, in counter-clockwise direction.

(ii) $\oint_C \frac{z+1}{z(z-2)^3} dz$, where C: $|z-1| = \frac{3}{2}$, in counter-clockwise direction

12. Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2+9} dx$.

13. Use residues to evaluate $\int_{-\pi}^{\pi} \frac{a \cos\theta}{a+\cos\theta} d\theta$ where $a > 1$.