Binomial Theorem

- The **binomial theorem** is a theorem from algebra which expands $(a+b)^k$.

  Example 1: $(a+b)^2 = a^2 + 2ab + b^2$
  Example 2: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
  Example 3: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

- The statement of the **binomial theorem** is

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

  Where $\binom{k}{n}$ are the **binomial coefficients**

- We are interested in a special case of the **binomial theorem**.

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

- We calculate the **binomial coefficients** as follows.

$$\binom{k}{n} = \frac{\overbrace{k(k-1)(k-2)\cdots(k-n+1)}^{n-\text{terms}}}{n!}$$

  Example 1: $\binom{5}{2} = \dfrac{(5)(4)}{2!}$

  Example 2: $\binom{11}{5} = \dfrac{(11)(10)(9)(8)(7)}{5!}$

  Example 3: $\binom{9}{7} = \dfrac{(9)(8)(7)(6)(5)(4)(3)}{7!}$

  Example 4: $\binom{k}{0} = 1$ (Special case)

- The **binomial coefficients** can also be calculated using Pascal's Triangle.

- A common formula for the **binomial coefficients** is

$$\binom{k}{n} = \frac{k!}{n!(k-n)!}$$

- **Warning!** In this course we will be interested in finding the **binomial coefficients** for fractional and negative values of k. In these cases the above formula will require factorials of fractional (or negative) numbers and it is easier to follow the above examples.

Binomial Series

- The **binomial series** extends the binomial theorem to work with fractional and negative powers.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

- The **binomial series** converges for all $|x| < 1$.
- The **binomial series** is the Taylor series about $x = 0$ for functions of the form $(1+x)^k$.
- The **binomial coefficients** are calculated as before

$$\binom{k}{n} = \frac{\overbrace{k(k-1)(k-2)\cdots(k-n+1)}^{n-\text{terms}}}{n!}$$

But we can use fractional or negative values for k.

Example 1: $\displaystyle \binom{\frac{1}{2}}{4} = \frac{\overbrace{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}^{4-\text{terms}}}{4!}$

Example 2: $\displaystyle \binom{-3}{7} = \frac{\overbrace{(-3)(-4)(-5)(-6)(-7)(-8)(-9)}^{7-\text{terms}}}{7!}$

Example 3: $\displaystyle \binom{5}{7} = \frac{\overbrace{(5)(4)(3)(2)(1)(0)(-1)}^{7-\text{terms}}}{7!} = 0$

- Example 3 shows that if k is a positive integer then $\binom{k}{n} = 0$ for all $n > k$.
- In the case k is a positive integer the **binomial series** is the **binomial theorem** and there are only a finite number of non-zero terms.
- Many useful series are **binomial series** such as the **geometric series**, $(1-x)^{-1}$.
- Some elementary functions are integrals of **binomial series** such as

Example 1: $\displaystyle \ln(1+x) = \int_0^x \frac{dt}{1+t}$

Example 2: $\displaystyle \sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$

Examples

In these **binomial series** examples the formula for a_n is not always obvious. In the assignment you will only be asked for the first few terms and not required to provide a formula in terms of n.

Example 1:
$$\begin{aligned}(1+x)^{-1} &= \sum_{n=0}^{\infty} \binom{-1}{n} x^n \\ &= 1 + \frac{(-1)}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n\end{aligned}$$

Example 2:
$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \\ &= 1 + \frac{(\frac{1}{2})}{1!}x + \frac{(\frac{1}{2})(\frac{-1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{348}x^4 + \cdots \\ &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}n!(n-2)!}x^n\end{aligned}$$

Example 3:
$$\begin{aligned}(1-x^2)^{-1/2} &= \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} x^n \\ &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}(-x^2)^3 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2}x^{2n}\end{aligned}$$

Example 3*:
$$\begin{aligned}\sin^{-1}(x) &= \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2}t^{2n}dt \\ &= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \cdots\right) dt \\ &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1}\end{aligned}$$