Math 135-2, Homework 2

Solutions

Problem 50.2

Find the inverse Laplace transforms of
(c) $\boldsymbol{\dfrac{p+3}{p^2+2p+5}}$

$$\begin{aligned}
L^{-1}\left[\frac{p+3}{p^2+2p+5}\right] &= L^{-1}\left[\frac{p+3}{(p+1)^2+4}\right] \\
&= L^{-1}\left[\frac{p+1}{(p+1)^2+4}\right] + L^{-1}\left[\frac{2}{(p+1)^2+4}\right] \\
&= e^{-x}L^{-1}\left[\frac{p}{p^2+4}\right] + e^{-x}L^{-1}\left[\frac{2}{p^2+4}\right] \\
&= e^{-x}\cos 2x + e^{-x}\sin 2x
\end{aligned}$$

Problem 50.3

Solve each of the following differential equations by the method of Laplace transforms:
(b) $\boldsymbol{y'' - 4y' + 4y = 0}$**,** $\boldsymbol{y(0) = 0}$ **and** $\boldsymbol{y'(0) = 3}$
(d) $\boldsymbol{y'' + y' = 3x^2}$**,** $\boldsymbol{y(0) = 0}$ **and** $\boldsymbol{y'(0) = 1}$

(b) Let $Y = L[y]$.

$$\begin{aligned}
0 &= y'' - 4y' + 4y \\
0 &= L[y'' - 4y' + 4y] \\
&= L[y''] - 4L[y'] + 4L[y] \\
&= (p^2Y - py(0) - y'(0)) - 4(pY - y(0)) + 4Y \\
&= p^2Y - 3 - 4pY + 4Y \\
(p^2 - 4p + 4)Y &= 3 \\
Y &= \frac{3}{p^2 - 4p + 4} \\
Y &= \frac{3}{(p-2)^2} \\
y &= L^{-1}\left[\frac{3}{(p-2)^2}\right] \\
&= e^{2x}L^{-1}\left[\frac{3}{p^2}\right] \\
&= 3xe^{2x}
\end{aligned}$$

(d) Let $Y = L[y]$.

$$\begin{aligned}
L[y''+y'] &= L[3x^2]\\
L[y''] + L[y'] &= 3L[x^2]\\
(p^2Y - py(0) - y'(0)) + (pY - y(0)) &= \frac{6}{p^3}\\
p^2Y - 1 + pY &= \frac{6}{p^3}\\
p^2Y + pY &= \frac{p^3+6}{p^3}\\
Y &= \frac{p^3+6}{p^4(p+1)}\\
\frac{p^3+6}{p^4(p+1)} &= \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{E}{p+1}\\
p^3 + 6 &= Ap^3(p+1) + Bp^2(p+1) + Cp(p+1) + D(p+1) + Ep^4\\
6 &= D \qquad p = 0\\
5 &= E \qquad p = -1\\
p^3 + 6 &= Ap^3(p+1) + Bp^2(p+1) + Cp(p+1) + 6(p+1) + 5p^4\\
p^3 - 6p - 5p^4 &= Ap^3(p+1) + Bp^2(p+1) + Cp(p+1)\\
p(p+1)(-5p^2 + 6p - 6) &= Ap^3(p+1) + Bp^2(p+1) + Cp(p+1)\\
-5p^2 + 6p - 6 &= Ap^2 + Bp + C\\
A &= -5\\
B &= 6\\
C &= -6\\
\frac{p^3+6}{p^4(p+1)} &= -\frac{5}{p} + \frac{6}{p^2} - \frac{6}{p^3} + \frac{6}{p^4} + \frac{5}{p+1}\\
Y &= \frac{p^3+6}{p^4(p+1)}\\
y &= L^{-1}\left[\frac{p^3+6}{p^4(p+1)}\right]\\
&= L^{-1}\left[-\frac{5}{p} + \frac{6}{p^2} - \frac{6}{p^3} + \frac{6}{p^4} + \frac{5}{p+1}\right]\\
&= -L^{-1}\left[\frac{5}{p}\right] + L^{-1}\left[\frac{6}{p^2}\right] - L^{-1}\left[\frac{6}{p^3}\right] + L^{-1}\left[\frac{6}{p^4}\right] + L^{-1}\left[\frac{5}{p+1}\right]\\
&= -5 + 6x - 3x^2 + x^3 + 5e^{-x}
\end{aligned}$$

Problem 51.1

Show that

$$\boldsymbol{L[x\cos ax] = \frac{p^2 - a^2}{(p^2+a^2)^2}}$$

and use this result to find

$$\boldsymbol{L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right]}$$

The first part can be done with the differentiation rule.

$$\begin{aligned}
L[x\cos ax] &= -\frac{d}{dp}L[\cos ax] \\
&= -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right) \\
&= \frac{p^2-a^2}{(p^2+a^2)^2}
\end{aligned}$$

Next for the second part.

$$\begin{aligned}
\frac{p^2-a^2}{(p^2+a^2)^2} &= \frac{A}{p^2+a^2} + \frac{B}{(p^2+a^2)^2} \\
p^2-a^2 &= A(p^2+a^2)+B \\
-2a^2 &= B \qquad p = ia \\
-a^2 &= Aa^2 + B \\
1 &= A \\
L^{-1}\left[\frac{p^2-a^2}{(p^2+a^2)^2}\right] &= L^{-1}\left[\frac{1}{p^2+a^2}\right] - 2a^2L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] \\
x\cos ax &= \frac{1}{a}\sin ax - 2a^2L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] \\
L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] &= -\frac{1}{2a^2}x\cos ax + \frac{1}{2a^3}\sin ax
\end{aligned}$$

Problem 51.3

Solve each of the following differential equations:
(a) $\boldsymbol{xy'' + (3x-1)y' - (4x+9)y = 0,\ y(0) = 0.}$

Let $Y = L[y]$.

$$
\begin{aligned}
0 &= xy'' + (3x-1)y' - (4x+9)y \\
0 &= L[xy''] + 3L[xy'] - L[y'] - 4L[xy] - 9L[y] \\
&= -(p^2Y - py(0) - y'(0))' - 3(pY - y(0))' - (pY - y(0)) + 4Y' - 9Y \\
&= -2pY - p^2Y' - 3pY' - 3Y - pY + 4Y' - 9Y \\
&= -(p+4)(p-1)Y' - 3(p+4)Y \\
0 &= (p-1)Y' + 3Y \\
\frac{Y'}{Y} &= -\frac{3}{p-1} \\
\ln Y &= -3\ln|p-1| + c_0 \\
Y &= c_1(p-1)^{-3} \\
y &= c_1 L^{-1}\left[(p-1)^{-3}\right] \\
&= c_1 e^x L^{-1}\left[p^{-3}\right] \\
&= \frac{c_1}{2}x^2e^x \\
&= c_2x^2e^x
\end{aligned}
$$

Note that $y'(0)$ is never needed.

Problem 51.7

If $x > 0$, show formally that

(b) $f(x) = \int_0^\infty \frac{\cos xt}{1+t^2}\,dt = \frac{\pi}{2}e^{-x}$

Taking the Laplace transform with respect to x will turn the $\cos xt$ into a rational function. This is useful because we know that we can always integrate rational functions using partial fractions.

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{\cos xt}{1+t^2}\,dt \\
L[f(x)] &= L\left[\int_0^\infty \frac{\cos xt}{1+t^2}\,dt\right] \\
&= \int_0^\infty e^{-px}\int_0^\infty \frac{\cos xt}{1+t^2}\,dt\,dx \\
&= \int_0^\infty \int_0^\infty e^{-px}\frac{\cos xt}{1+t^2}\,dx\,dt \\
&= \int_0^\infty \frac{1}{1+t^2}\int_0^\infty e^{-px}\cos xt\,dx\,dt \\
&= \int_0^\infty \frac{1}{1+t^2}L[\cos xt]\,dt \\
&= \int_0^\infty \frac{1}{1+t^2}\frac{p}{p^2+t^2}\,dt
\end{aligned}
$$

Now we can do partial fractions.

$$
\begin{aligned}
\frac{1}{1+t^2}\frac{p}{p^2+t^2} &= \frac{A}{1+t^2} + \frac{B}{p^2+t^2} \\
p &= A(p^2+t^2) + B(t^2+1) \\
p &= A(p^2-1) \qquad t = i \\
A &= \frac{p}{p^2-1} \\
p &= B(-p^2+1) \qquad t = ip \\
B &= -\frac{p}{p^2-1} \\
\frac{1}{1+t^2}\frac{p}{p^2+t^2} &= \frac{p}{p^2-1}\frac{1}{1+t^2} - \frac{p}{p^2-1}\frac{1}{p^2+t^2} \\
L[f(x)] &= \int_0^\infty \frac{1}{1+t^2}\frac{p}{p^2+t^2}\,dt \\
&= \int_0^\infty \left(\frac{p}{p^2-1}\frac{1}{1+t^2} - \frac{p}{p^2-1}\frac{1}{p^2+t^2}\right) dt \\
&= \frac{p}{p^2-1}\left[\tan^{-1} t\right]_0^\infty - \frac{p}{p^2-1}\left[\frac{1}{p}\tan^{-1}\left(\frac{t}{p}\right)\right]_0^\infty \\
&= \frac{\pi}{2}\frac{p}{p^2-1} - \frac{\pi}{2}\frac{1}{p^2-1} \\
&= \frac{\pi}{2}\frac{p-1}{p^2-1} \\
&= \frac{\pi}{2}\frac{1}{p+1} \\
f(x) &= L^{-1}\left[\frac{\pi}{2}\frac{1}{p+1}\right] \\
&= \frac{\pi}{2}e^{-x}
\end{aligned}
$$

Problem 52.2

Solve each of the following integral equations:

(b) $\boldsymbol{y(x) = e^x\left[1 + \int_0^x e^{-t}y(t)\,dt\right]}$

$$\begin{aligned}
y(x) &= e^x\left[1+\int_0^x e^{-t}y(t)\,dt\right] \\
L[y(x)] &= L\left[e^x+\int_0^x e^{x-t}y(t)\,dt\right] \\
&= L[e^x]+L[e^x]L[y(x)] \\
(1-L[e^x])L[y(x)] &= L[e^x] \\
L[y(x)] &= \frac{L[e^x]}{1-L[e^x]} \\
&= \frac{\frac{1}{p-1}}{1-\frac{1}{p-1}} \\
&= \frac{1}{p-2} \\
y(x) &= e^{2x}
\end{aligned}$$

Problem 52.5

Show that the differential equation

$$\boldsymbol{y'' + a^2 y = f(x), y(0) = y'(0) = 0}$$

has

$$\boldsymbol{y(x) = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt}$$

as its solution.

$$\begin{aligned}
L[y''+a^2y] &= L[f(x)] \\
p^2L[y]-py(0)-y'(0)+a^2L[y] &= L[f(x)] \\
p^2L[y]+a^2L[y] &= L[f(x)] \\
L[y] &= \frac{1}{p^2+a^2}L[f(x)] \\
&= \frac{1}{a}L[\sin ax]L[f(x)] \\
y &= \frac{1}{a}\int_0^x \sin a(x-t)f(t)\,dt
\end{aligned}$$

Problem 53.2

Find the convolution of each of the following pairs of functions:
(a) 1, $\boldsymbol{\sin at}$
(c) $\boldsymbol{t}$, $\boldsymbol{e^{at}}$

(a)

$$\begin{aligned}
1 * \sin at &= \int_0^t 1 \sin a\tau \, d\tau \\
&= \left[-\frac{1}{a} \cos a\tau \right]_0^t \\
&= \frac{1}{a}(1 - \cos at)
\end{aligned}$$

(c)

$$\begin{aligned}
t * e^{at} &= \int_0^t (t - \tau) e^{a\tau} \, d\tau \\
&= \left[(t - \tau) \frac{1}{a} e^{a\tau} \right]_0^t - \int_0^t (-1) \frac{1}{a} e^{a\tau} \, d\tau \\
&= -\frac{t}{a} + \frac{1}{a} \int_0^t e^{a\tau} \, d\tau \\
&= -\frac{t}{a} + \frac{1}{a} \left[\frac{1}{a} e^{a\tau} \right]_0^t \\
&= -\frac{t}{a} + \frac{1}{a^2}(e^{at} - 1)
\end{aligned}$$