MATHEMATICS-II
Complex Analysis

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Note: *Some concepts of Complex Analysis are briefly described here just to help the students. Therefore, the following study material is expected to be useful but not exhaustive for the Mathematics-II course. For detailed study, the students are advised to attend the lecture/tutorial classes regularly, and consult the text book prescribed in the hand out of the course.*

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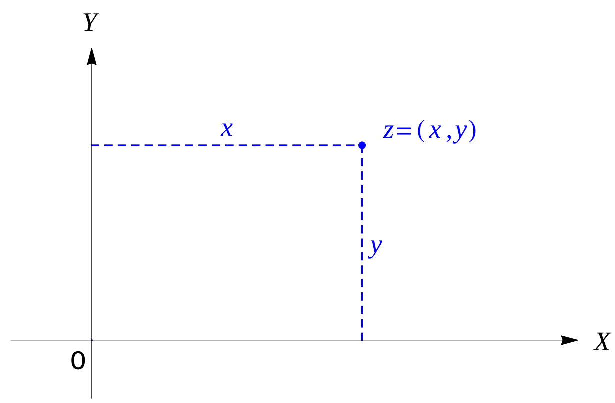
Chapter 1

Complex Numbers

Complex numbers are defined as ordered pairs (x, y) of real numbers which are interpreted as points (x, y) in XY-plane of cartesian system. The complex numbers of the form $(x, 0)$ are points on the X-axis (usually taken as the real number line) and are called pure real numbers. The complex numbers of the form $(0, y)$ are points on the Y-axis and are called pure imaginary numbers. The real numbers x and y are respectively known as the real and imaginary parts of the complex number (x, y). We denote the complex number (x, y) by z, its real part by Rez, and imaginary part by Imz. Thus, we have $z = (x, y)$, Re$z = x$ and Im$z = y$. Further, two complex numbers z_1 and z_2 are said to be equal, that is, $z_1 = z_2$ if and only if Rez_1 =Rez_2 and Imz_1 =Imz_2.



Addition and Multiplication of complex numbers

The addition and multiplication of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Note that the above operations reduce to the usual operations of addition and multiplication when restricted to the real numbers.

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Simplified representation of complex number

Making use of the definitions of addition and multiplication operations of complex numbers, we have

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0)$$

We shall denote the imaginary number $(0, 1)$ by i. For simplified representation, we write x for $(x, 0)$ and y for $(y, 0)$. Thus, the simplified representation of z is

$$z = x + iy.$$

With this representation, the addition and multiplication operations become

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

Further, we have $i^2 = -1$ since $i^2 = i.i = (0, 1)(0, 1) = (-1, 0)$.

Some algebraic properties of complex numbers

It is easy to verify the following properties of complex numbers:
(1) Commutative property: $z_1 + z_2 = z_2 + z_1$, $z_1z_2 = z_2z_1$.
(2) Associative property: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, $z_1(z_2z_3) = (z_1z_2)z_3$.
(3) Identity: $z + 0 = (x, y) + (0, 0) = (x, y) = z$, $z.1 = (x, y)(1, 0) = (x, y) = z$. Therefore, 0 and 1 are respectively the additive and multiplicative identities.
(4) Inverse: For any complex number $z = (x, y)$, there exists a complex number $(-x, -y)$ denoted by $-z$ and called negative of z such that $z + (-z) = 0$. We call $-z$ as additive inverse of z. For any non-zero complex number $z = (x, y)$, there exists a complex number $(x/(x^2 + y^2), -y/(x^2 + y^2))$ denoted by $1/z$ or z^{-1} and called reciprocal of z such that $zz^{-1} = 1$. We call z^{-1} as multiplicative inverse of z. Please also see the footnote[1]
(5) Distributive property: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

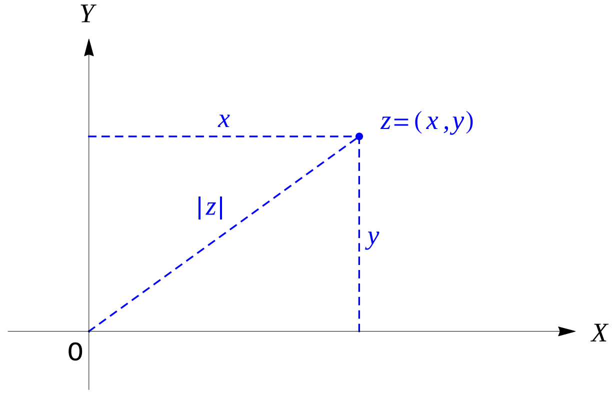
Modulus of a complex number

Recall that the absolute value a real number is its distance from origin on the real number line. In analogy, the modulus or the absolute value of a complex number $z = x + iy$, denoted by $|z|$, is defined as the non-negative real number $\sqrt{x^2 + y^2}$, that is, $|z| = \sqrt{x^2 + y^2}$. Geometrically, it is the distance of the point (x, y) (representing the complex number z) from the origin $(0, 0)$. Some straightforward but important properties of modulus are given below.
(1) For any two complex numbers z_1 and z_2, we have $|z_1| < |z_2|$ or $|z_1| = |z_2|$ or $|z_1| > |z_2|$. Note that the expressions $z_1 < z_2$ or $z_1 > z_2$ are not meaningful unless z_1 and z_2 are pure real numbers.

(2) The equation $|z - z_0| = r$ represents a circle with centre at z_0 and radius r in the complex plane. The inequality $|z - z_0| < r$ represents the interior region of the circle $|z - z_0| = r$ while the inequality $|z - z_0| > r$ represents the entire region in the complex plane exterior to the circle $|z - z_0| = r$.

(3) Let a point $P(x, y)$ in the XY-plane corresponds to a complex number $z = x + iy$. If we assign the position vector $\overrightarrow{OP}$ of P to represent the complex number z, then $|z| = |\overrightarrow{OP}|$. Considering the vector representation of complex numbers, one can easily demonstrate the addition and difference

[1] The expression $z_1 + (-z_2)$ denoted by $z_1 - z_2$ is called the difference of z_1 from z_2. The expression $z_1z_2^{-1}$ denoted by $\frac{z_1}{z_2}$ is called the division of z_1 by z_2.



of two complex numbers in the complex plane as per the rules of vector algebra. However, note that the product of two complex numbers can not be associated with the dot product or scalar product of vectors.

(4) For any two complex numbers z_1 and z_2, we have $|z_1z_2| = |z_1||z_2|$ and $|z_1/z_2| = |z_1|/|z_2|$ provided $z_2 \neq 0$. Further, we have $|z_1+z_2| \leq |z_1|+|z_2|$, known as the triangle inequality. It simply says that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. Also, it can be proved that $|z_1 + z_2| \geq ||z_1| - |z_2||$.

(5) $|z|^2 = (\text{Re}z)^2 + (\text{Im}z)^2$, $\text{Re}z \leq |\text{Re}z| \leq |z|$ and $\text{Im}z \leq |\text{Im}z| \leq |z|$.

Conjugate of a complex number

The conjugate of a complex number $z = x + iy$, denoted by $\overline{z}$, is defined as the complex number $x - iy$, that is, $\overline{z} = \overline{x + iy} = x - iy$. Geometrically, $\overline{z}$ is the reflection of z in the X-axis. Some straightforward properties of complex conjugates are given below.

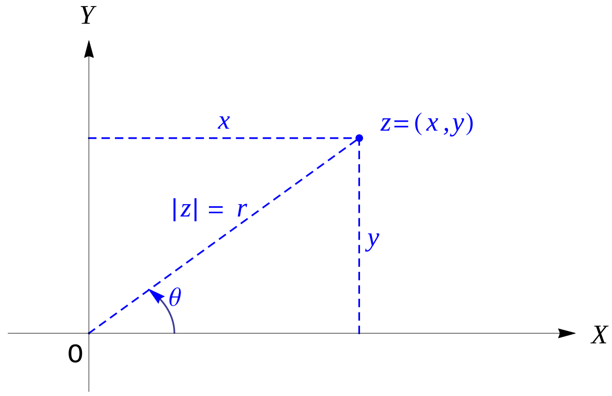
(1) $\overline{\overline{z}} = z$, $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$, $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.

(2) $z\overline{z} = |z|^2$, $\text{Re}z = \frac{z+\overline{z}}{2}$ and $\text{Im}z = \frac{z-\overline{z}}{2i}$.

Polar or exponential form of a complex number

Let (r, θ) be polar coordinates of a point that corresponds to a non-zero complex number $z = x+iy$ in the complex plane. Then $x = r\cos\theta$ and $y = r\sin\theta$. So we have $z = r(\cos\theta + i\sin\theta)$. It is the polar representation of the complex number z. Using the Euler's formula[2] $e^{i\theta} = \cos\theta + i\sin\theta$, we get a more compact form $z = re^{i\theta}$, known as the exponential form of z. Obviously, the modulus of z is $|z| = r$. The angle θ is called the argument of z and is denoted by arg$z$. It is measured in radians and is undefined for $z = 0$. Further, for a non-zero complex number there exists infinitely many values of θ but all differing by an integer multiple of 2π. In the range $-\pi < \theta \leq \pi$, we can find a unique value of θ for any given non-zero complex number z. It is called principal argument of z and is denoted by Arg$z$. So we have $\arg z = \text{Arg}z + 2k\pi$, k is an integer. Some useful points are given below.

[2]It can be proved that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, that is, $(e^{i\theta})^n = e^{in\theta}$. It is known as de Moivre's formula



(1) Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then $z_1 = z_2$ if and if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, k being an integer.

(2) The nth roots of a non-zero complex number z_0 are the roots of the equation $z^n = z_0$. If $z_0 = r_0 e^{i\theta_0}$ and $z = re^{i\theta}$, then $z^n = z_0$ yields $r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$. So $r = \sqrt[n]{r_0}$ and $\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}$, and hence the nth roots of z_0 are given by

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \; k = 0, 1, 2, ..., n-1.$$

Note that the integer values of k less than 0 and greater than $n-1$ do not yield any roots different from the listed ones. Obviously, the nth roots of z_0 lie symmetrically on the circle $|z| = \sqrt[n]{r_0}$.

(3) One can easily prove that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. However, one should be careful that these equalities hold in the sense that given any two arguments in the equality there exists some value of the third argument satisfying the equality.

(4) Note that $z = re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ is parametric representation of the circle $|z| = r$. As the parameter θ increases from 0 to 2π, the point z traverses the circle $|z| = r$ once in the counterclockwise direction starting from the positive real axis.

Regions in complex plane

The following definitions are simple but crucial in the forthcoming analysis. So read and understand these carefully.

Neighbourhoods

Let z_0 be a complex number. Then for any $\epsilon > 0$, the interior of the circle $|z - z_0| = \epsilon$, that is, the region $|z - z_0| < \epsilon$ defines a neighbourhood of z_0. The neighbourhood with z_0 deleted, that is, $0 < |z - z_0| < \epsilon$ is called deleted neighbourhood of z_0. For example, $|z| < 1$ is neighbourhood of 0. In fact, it is neighbourhood of its every point. $0 < |z| < 1$ is deleted neighbourhood of 0.

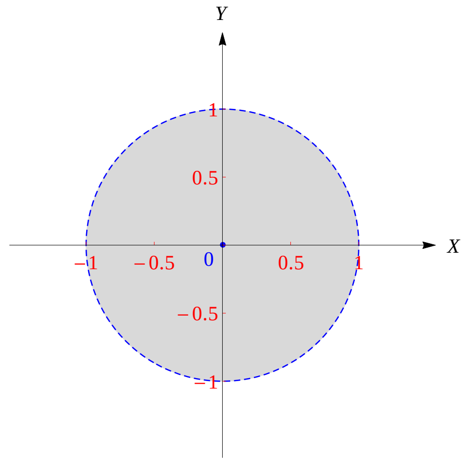


Figure 1: Shaded region stands for $|z| < 1$, the neighbourhood of $z = 0$.

Interior, exterior and boundary points

Let S be a set of complex numbers. A complex number z_0 is said to be an interior point of S if there exists at least one neighbourhood of z_0 that completely lies within S. On the other hand, z_0 is an exterior point of S if there exists at least one neighbourhood of z_0 that completely lies outside S. Next, z_0 is a boundary point of S if it neither an interior nor an exterior point of S. In other words, if each neighbourhood of z_0 contains points from exterior as well as interior of S. For example, $z_0 = 0.5$, $z_1 = 1 + i$ and $z_2 = i$ are respectively the interior, exterior and boundary points of $S = \{z : |z| \leq 1\}$.

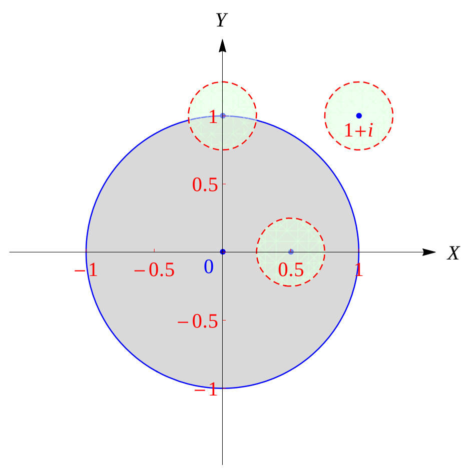


Figure 2: $z_0 = 0.5$, $z_1 = 1+i$ and $z_2 = i$ are respectively the interior, exterior and boundary points of $S = \{z : |z| \leq 1\}$.

Open and closed sets

If all points of a set are its interior points, it is said be an open set. A set containing all its boundary points is a closed set. For example, the $|z| < 1$ is an open set whereas the set $|z| \leq 1$ is a closed set. Note that the punctured unit disc $0 < |z| < 1$ is an open set while the set $0 < |z| \leq 1$ is neither open nor closed. Likewise the annular region $1 < |z| \leq 2$ is neither open nor closed.

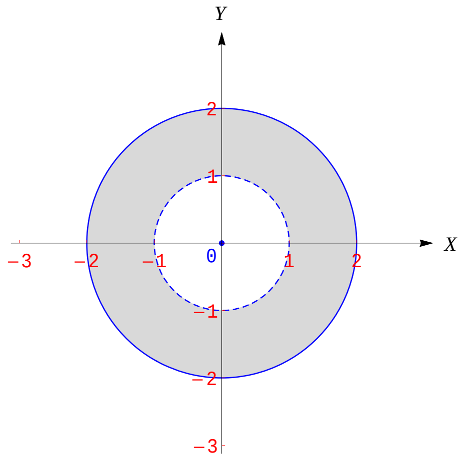


Figure 3: $1 < |z| \leq 2$ is neither open nor closed.

Connected sets

A connected set is the set in which any two points can be joined by a polygonal line consisting of finite number of line segments without leaving the set. For example, $1 < |z| \leq 2$ is a connected set, but the set $S = \{z : \text{Re}z < -1 \quad \text{or} \quad \text{Re}z > 1\}$ is not connected since points of the region $\text{Re}z < -1$ can not be joined with the points of the region $\text{Re}z > 1$ by any polygonal line.

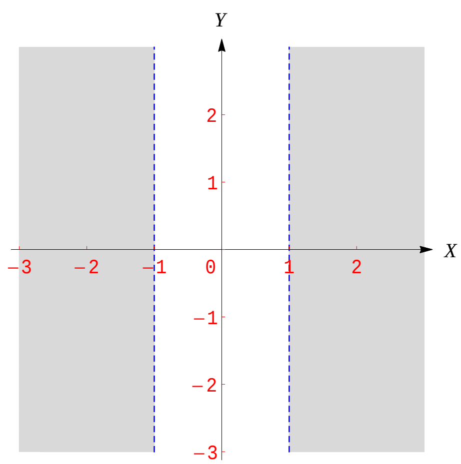


Figure 4: $S = \{z : Rez < -1 \text{ or } Rez > 1\}$ is not connected.

Domains and regions

A non-empty open connected set is called a domain. A region is a domain with some, none or all the boundary points. For example, $1 < |z| < 2$ is domain as well as a region. The set $0 < |z| \leq 1$ is a region but not a domain.

Bounded sets

A set is bounded if it can be enclosed inside the circle $|z| = r$ for some finite value of r. For example, the set $|z| < 1$ is bounded while the set $\text{Re}z > 0$ being the right half of the complex plane is unbounded.

Limit points of a set

A point z_0 is said to be limit point or accumulation point of a set S if every neighbourhood of z_0 carries infinitely many points of S. Obviously, S has no limit points if it is a finite set. Next, all the interior and boundary points of a set are its limit points. A set is closed if and only if it contains all its boundary points. A set may have finite number of limit points. For example, 0 is the only limit point of the set $\{i/n : n = 1, 2, 3....\}$. Note that 0 is not a member of the set. The set $|z| \leq 1$ is the set of all limit points of the set $|z| < 1$.

Chapter 2

Function, Domain and Range

Let B be a subset of the set $\mathbb{C}$ of complex numbers. Then a function f from B to $\mathbb{C}$, also written as $f : B \to \mathbb{C}$, is a rule that assigns to each member z of S a complex number w in $\mathbb{C}$. The complex number w is called the value of f at z (or image of z under f) and is denoted by $f(z)$. So $w = f(z)$. The set B is called the domain of definition of f, and the set $\{f(z) : z \in S\}$ is called the range of f.

The domain, here, should not be confused with the domain (open connected set) defined in the previous chapter. Here, domain appears in analogy with the real functions, and simply stands for discrete or continuous collection of points in the complex plane permissible for the given function. For a given function, the set of all permissible points in the complex plane is called its natural domain. Any proper subset of the natural domain is the restricted domain. We may define functions on restricted domains. Of course, the domain of a function can be an open connected set. For example, $f : \{z : |z| < 1\} \to \mathbb{C}$ given by $f(z) = z^2 + 5$ is a function defined on the unit circular disc $|z| < 1$. Here $|z| < 1$ is the restricted domain of f as its natural domain is $\mathbb{C}$. Also, $|z| < 1$ is an open connected set.

Since $z = x + iy$, we can express the function $f(z) = z^2 + 5$ in terms of x and y as follows:

$$f(x + iy) = (x + iy)^2 + 5 = x^2 - y^2 + 5 + i(2xy) = u(x, y) + iv(x, y),$$

where $u(x, y) = x^2 - y^2 + 5$ and $v(x, y) = 2xy$ are respectively the real and imaginary parts of $f(z)$. If we choose the polar form $z = re^{i\theta}$, then u and v can be determined as functions of r and θ.

Also, note that a function of complex variable need not be complex valued. For example, $f(z) = |z|^2 = x^2 + y^2$ with $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$, is a pure real valued function.

Multiple Valued Functions

Multiple valued functions, which assign more than one value to the domain points, do arise in complex system as these appear in the real system[3]. Such functions are studied by constructing single valued functions in a systematic manner. For instance, the function $w = z^{1/2}$ assigns two values namely $w = \pm\sqrt{r}e^{i\Theta/2}$ corresponding to each non-zero complex number $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$. The functions given by $f_1(z) = \sqrt{r}e^{i\Theta/2}$, $f_1(0) = 0$ and $f_2(z) = -\sqrt{r}e^{i\Theta/2}$, $f_2(0) = 0$ are both single valued functions defined in the entire complex plane.

Limit at an interior point

Let a function $f(z)$ be defined in some deleted neighbourhood of z_0. Then the limit of $f(z)$ as $z \to z_0$ is said to be w_0 denoted as

$$\lim_{z \to z_0} f(z) = w_0$$

[3]The real function given by $x^2 + y^2 = 1$ assigns two values namely $y = \pm\sqrt{1 - x^2}$ to each value of $x \in [-1, 1]$. The functions $y_1 = \sqrt{1 - x^2}$ and $y_2 = -\sqrt{1 - x^2}$ are single valued functions. The graphs of these functions are respectively the upper half and lower half arcs of the circle $x^2 + y^2 = 1$

if given any real number $\epsilon > 0$ (howsoever small), there exists a real number $\delta > 0$ (depending on ϵ) such that

$$|f(z) - w_0| < \epsilon \quad \text{for} \quad 0 < |z - z_0| < \delta.$$

Geometrically speaking, the limit of $f(z)$ as $z \to z_0$ is w_0 if given any ϵ-neighbourhood of w_0, there exists a deleted δ-neighbourhood of z_0 such that the images $f(z)$ of all points z of the deleted δ-neighbourhood of z_0 are contained in the ϵ-neighbourhood of w_0. Also, existence of the limit of $f(z)$ as $z \to z_0$ means that its value w_0 is unique and is independent of the path along which $z \to z_0$.

Ex. Show that $\lim_{z \to 1}(2z + 3) = 5$.
Sol. Let $\epsilon > 0$ be given. Then we have

$$|2z + 3 - 5| = |2z - 2| = 2|z - 1| < \epsilon \quad \text{for} \quad |z - 1| < \epsilon/2.$$

Choosing $\delta = \epsilon/2$, we have

$$|2z + 3 - 5| < \epsilon \quad \text{for} \quad |z - 1| < \delta.$$

$$\therefore \quad \lim_{z \to 1}(2z + 3) = 5.$$

Ex. Show that $\lim_{z \to 0}\left(\frac{z}{\bar{z}}\right)$ does not exist.
Sol. We have

$$\lim_{z \to 0}\left(\frac{z}{\bar{z}}\right) = \lim_{(x,y) \to (0,0)}\left(\frac{x + iy}{x - iy}\right) = \lim_{x \to 0}\left(\frac{x + imx}{x - imx}\right) = \frac{1 + im}{1 - im},$$

where we have chosen the path $y = mx$ along which $z \to 0$. Therefore, the given limit depends on m, that is, on the chosen path and is, therefore, not unique. Hence, it does not exist.

Limit at a boundary point

The definition of the limit at interior point can be extended to the case when z_0 is boundary point of the region where $f(z)$ is defined. In this case, the limit of $f(z)$ as $z \to z_0$ is said to be w_0 if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ for all z common to the δ-neighbourhood of z_0 and the domain region of $f(z)$.

Ex. Suppose $f(z)$ be defined in $|z| < 1$. Show that $\lim_{z \to 1}(iz/2) = i/2$.
Sol. Here $z = 1$ is boundary point of $|z| < 1$. Let $\epsilon > 0$ be given. Then we have

$$|iz/2 - i/2| = (1/2)|z - 1| < \epsilon \quad \text{for} \quad |z - 1| < 2\epsilon \quad \text{and} \quad |z| < 1.$$

Choosing $\delta = 2\epsilon$, we have

$$|iz/2 - i/2| < \epsilon \quad \text{for} \quad |z - 1| < \delta \quad \text{and} \quad |z| < 1.$$

$$\therefore \quad \lim_{z \to 1}(iz/2) = i/2.$$

Algebra of limits

Let $f(z)$ and $g(z)$ be defined in some deleted neighbourhood of z_0. Then following results can be proved.

(1) If $z_0 = x_0 + iy_0$ and $f(z) = u(x, y) + iv(x, y)$, then

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x, y) + i \lim_{(x,y)\to(x_0,y_0)} v(x, y).$$

(2) $\lim_{z\to z_0} [f(z)g(z)] = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)$.

(3) $\lim_{z\to z_0} [f(z) \pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z)$.

(4) $\lim_{z\to z_0} [f(z)/g(z)] = \lim_{z\to z_0} f(z)/ \lim_{z\to z_0} g(z)$, provided $\lim_{z\to z_0} g(z) \neq 0$.

(5) $\lim_{z\to z_0} [kf(z)] = k \lim_{z\to z_0} f(z)$, where k is a constant.

(6) If $f(z) = a_0 + a_1z + ... + a_nz^n$, then $\lim_{z\to z_0} f(z) = f(z_0)$.

Limits involving infinity

It is useful to include the point at infinity (∞) in the complex plane. Such a complex plane is called extended complex plane. To visualize the point at infinity in the complex plane, consider the complex plane passing through the equator of a unit sphere centred at the origin. Now, each point z in the complex plane corresponds to a unique point P on the surface of unit sphere, where P is the point of intersection of the line passing through the north pole N of the sphere and the point z, with the surface of the sphere. Obviously, N does not correspond to any finite point in the complex plane. If we agree N to correspond to the point at infinity, there is a one to one correspondence between the points on the unit sphere and the points in the extended complex plane. Such a correspondence is called stereographic projection, and the unit sphere is called Riemann sphere.

Let $\epsilon > 0$ be a real number. Then the points z in the complex plane such that $|z| > 1/\epsilon$ correspond to a circular region centred at N on the surface of sphere. This in turn suggests that $|z| > 1/\epsilon$ is a neighbourhood of ∞.

Some results on limits involving ∞ are given below.

(1) $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} [1/f(z)] = 0$.

(2) $\lim_{z\to\infty} f(z) = \lim_{z\to 0} f(1/z)$

(3) $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} f(1/z) = 0$.

Ex. $\lim_{z\to 1} \dfrac{z + 2i}{z - 1} = \infty$ since $\lim_{z\to 1} \dfrac{z - 1}{z + 2i} = 0$.

Ex. $\lim_{z\to\infty} \dfrac{iz + 2}{z + i} = \lim_{z\to\infty} \dfrac{i + 2/z}{1 + i/z} = \dfrac{i + 0}{1 + 0} = i$.

Continuity

Let a function $f(z)$ be defined in some neighbourhood of z_0. Then $f(z)$ is said to be continuous at z_0 if given any real number $\epsilon > 0$ (howsoever small), there exists a real number $\delta > 0$ (depending on ϵ) such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{for} \quad |z - z_0| < \delta.$$

So $f(z)$ is continuous at z_0 if and only if $\lim_{z \to z_0} f(z) = f(z_0)$. Further, $f(z)$ is said to be continuous in a region if and only if it is continuous at every point of the region. Some results pertaining to continuous functions are given below.

(1) If $f(z)$ and $g(z)$ are continuous functions at z_0, then so are the functions $f(z) \pm g(z)$, $f(z)g(z)$, $(fog)(z)$ and $f(z)/g(z)$ $(g(z_0) \neq 0)$.

(2) $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z and polynomials all are continuous at every point in the complex plane.

(3) If $f(z)$ is continuous and non-zero at z_0, then there exists some neighbourhood of z_0 such that $f(z)$ is non-zero at each point of the neighbourhood.

For, $f(z)$ is continuous at z_0. Choosing $\epsilon = |f(z_0)|/2$, there exists some $\delta > 0$ such that

$$|f(z) - f(z_0)| < |f(z_0)|/2 \quad \text{for} \quad |z - z_0| < \delta.$$

Now, if there is any point z in the δ-neighbourhood $|z - z_0| < \delta$ of z_0 such that $f(z) = 0$, then we get $|f(z_0)| < |f(z_0)|/2$, which is not true. So $f(z)$ is non-zero at each point of the δ-neighbourhood $|z - z_0| < \delta$ of z_0.

(4) A function $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if the component functions $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

(5) If $f(z)$ is continuous in a closed and bounded region R, then $f(z)$ is bounded in R, that is, there exists some constant $M > 0$ such that $|f(z)| < M$ for all $z \in R$.

For, let $f(z) = u(x, y) + iv(x, y)$. Then continuity of $f(z)$ in R implies the continuity of the component functions $u(x, y)$ and $v(x, y)$ in R. Now, from the theory of real functions, $u(x, y)$ and $v(x, y)$ being continuous in the closed and bounded region R are bounded in R. So there exists constants $M_1 > 0$ and $M_2 > 0$ such that $|u(x, y)| < M_1$ and $|v(x, y)| < M_2$ for all $(x, y) \in R$. Also, $|f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$. It follows that $f(z)$ is bounded in R.

Differentiability

Let a function $w = f(z)$ be defined in some neighbourhood of z_0. Then $w = f(z)$ is said to be differentiable at z_0 if and only if the limit $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists. The derivative of $w = f(z)$ at z_0 is denoted by $f'(z_0)$ or $\left(\frac{dw}{dz}\right)_{z=z_0}$. So we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If we choose $z = z_0 + \delta z$, then we have

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}.$$

Some results pertaining to differentiable functions are given below.

(1) $\dfrac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$.

(2) Product rule: $\dfrac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$.

(3) Quotient rule: $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$ provided $g(z) \neq 0$.

(4) Chain rule: $\dfrac{d}{dz}[f(g(z))] = f'(g(z))g'(z)$.

(5) If $f(z)$ and $g(z)$ are differentiable functions at z_0, then so are the functions $f(z) \pm g(z)$, $f(z)g(z)$, $(fog)(z)$ and $f(z)/g(z)$ $(g(z_0) \neq 0)$.

(6) $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z and polynomials all are differentiable at every point in the complex plane.

Ex. If $f(z) = z^2 + 1$, then

$$f'(2) = \lim_{\delta z \to 0} \frac{f(2 + \delta z) - f(2)}{\delta z} = \lim_{\delta z \to 0} \frac{(2 + \delta z)^2 + 1 - (2^2 + 1)}{\delta z} = \lim_{\delta z \to 0}(4 + \delta z) = 4.$$

Ex. Show that $f(z) = |z|^2$ is not differentiable at any z except $z = 0$.
Sol. We have

$$\begin{aligned}
f'(z) &= \lim_{\delta z \to 0} \frac{|z + \delta z|^2 - |z|^2}{\delta z} \\
&= \lim_{\delta z \to 0} \frac{(z + \delta z)(\overline{z} + \overline{\delta z}) - z\overline{z}}{\delta z} \\
&= \lim_{\delta z \to 0} \frac{(z + \delta z)(\overline{z} + \overline{\delta z}) - z\overline{z}}{\delta z} \\
&= \overline{z} + \lim_{\delta z \to 0} \overline{\delta z} + z \lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z} \\
&= \overline{z} + \lim_{(\delta x, \delta y) \to (0,0)} (\delta x - i\delta y) + z \lim_{(\delta x, \delta y) \to (0,0)} \frac{\delta x - i\delta y}{\delta x + i\delta y} \\
&= \overline{z} + z\frac{1 - im}{1 + im}, \quad (\delta y = m\delta x)
\end{aligned}$$

Clearly, $f'(0) = 0$ and $f'(z)$ depends on m if $z \neq 0$. Thus, $f(z) = |z|^2$ is not differentiable at any z except $z = 0$.

Ex. If $f(z)$ is differentiable at z_0, then show that $f(z)$ is continuous at z_0. However, the converse need not be true.
Sol. If $f'(z_0)$ exists, then we have

$$\lim_{z\to z_0}[f(z)-f(z_0)] = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)\right] = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right].\lim_{z\to z_0}(z-z_0) = f'(z_0).0 = 0.$$

This shows that $f(z)$ is continuous at z_0. However, the converse need not be true. In the previous example, we showed that the function $f(z) = |z|^2 = x^2+y^2$ is not differentiable at any z except $z=0$. However, it is continuous for all z as its real part $u(x,y) = x^2+y^2$ and imaginary part $v(x,y)=0$, both are continuous everywhere in the complex plane.

Necessary condition for differentiability

Let $f(z) = u(x,y)+iv(x,y)$ be differentiable at $z_0 = x_0+iy_0$. Then first order partial derivatives of $u(x,y)$ and $v(x,y)$ exist at (x_0,y_0), and the partial differential equations, known as the Cauchy-Riemann (CR) equations,

$$u_x = v_y, \quad u_y = -v_x$$

are satisfied at (x_0,y_0).
Proof. We have

$$\begin{aligned}
f'(z_0) &= \lim_{\delta z\to 0}\frac{f(z_0+\delta z)-f(z_0)}{\delta z}\\
&= \lim_{(\delta x,\delta y)\to(0,0)}\frac{u(x_0+\delta x,y_0+\delta y)+iv(x_0+\delta x,y_0+\delta y)-u(x_0,y_0)-iv(x_0,y_0)}{\delta x+i\delta y}\\
&= \lim_{(\delta x,\delta y)\to(0,0)}\frac{u(x_0+\delta x,y_0+\delta y)-u(x_0,y_0)}{\delta x+i\delta y}+i\lim_{(\delta x,\delta y)\to(0,0)}\frac{v(x_0+\delta x,y_0+\delta y)-v(x_0,y_0)}{\delta x+i\delta y}
\end{aligned}$$

First solving $f'(z_0)$ along the path $\delta y=0$, we get

$$f'(z_0) = \lim_{\delta x\to 0}\frac{u(x_0+\delta x,y_0)-u(x_0,y_0)}{\delta x}+i\lim_{\delta x\to 0}\frac{v(x_0+\delta x,y_0)-v(x_0,y_0)}{\delta x} = u_x(x_0,y_0)+iv_x(x_0,y_0).$$

Next solving $f'(z_0)$ along the path $\delta x=0$, we get

$$f'(z_0) = -i\lim_{\delta y\to 0}\frac{u(x_0,y_0+\delta x)-u(x_0,y_0)}{\delta x}+i\lim_{\delta y\to 0}\frac{v(x_0,y_0+\delta y)-v(x_0,y_0)}{\delta y} = -iu_y(x_0,y_0)+v_y(x_0,y_0).$$

Since $f'(z_0)$ exists uniquely, so its values along the paths $\delta x=0$ and $\delta y=0$ must be equal. It follows that the CR equations $u_x=v_y$ and $u_y=-v_x$ are satisfied at (x_0,y_0).

Ex. The function $f(z)=z^2$ is differentiable throughout the complex plane as its derivative $f'(z)=2z$ exists uniquely for all z. Now, let us verify the CR equations. We have $f(z)=z^2=x^2-y^2+i(2xy)$. Therefore, $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$. We find that $u_x=2x=v_y$ and $u_y=-2y=-v_x$. So CR equations are satisfied for all (x,y) in the complex plane, as expected.

Ex. We know that the function $f(z)=|z|^2$ is differentiable only at the origin in the complex plane. Now, $f(z)=|z|^2=x^2+y^2$. Therefore, $u(x,y)=x^2+y^2$ and $v(x,y)=0$. So $u_x=2x$

$u_y = 2y$, $v_x = 0$ and $v_y = 0$. We see that the CR equations are satisfied only at $(0, 0)$, as expected.

CR equations in polar form: If $z = re^{i\theta}$, then $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and CR equations in polar form read as $ru_r = v_\theta$ and $u_\theta = -rv_r$.
Proof: We have

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta). \tag{1}$$

Differentiating (1) partially with respect to r and θ, we get

$$f'(re^{i\theta})e^{i\theta} = u_r + iv_r, \tag{2}$$

and

$$f'(re^{i\theta})re^{i\theta}i = u_\theta + iv_\theta, \tag{3}$$

respectively. From (2) and (3), we obtain

$$ir(u_r + iv_r) = u_\theta + iv_\theta$$

$$\text{or} \quad iru_r - rv_r = u_\theta + iv_\theta$$

Equating real and imaginary parts, we get $ru_r = v_\theta$ and $u_\theta = -rv_r$, the CR equations in polar form.
Note. If CR equations are satisfied at a point, then the function need not be differentiable at that point as illustrated in the following example.

Ex. If $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$ for $z = (x, y) \neq 0$ and $f(0) = 0$, then show that CR equations are satisfied at $z = 0$ but $f(z)$ is not differentiable at $z = 0$.
Sol. Let $f(z) = u(x, y) + iv(x, y)$. Then we have

$$u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

Also, $f(0) = 0$ implies $u(0, 0) + iv(0, 0) = 0$. So $u(0, 0) = 0$ and $v(0, 0) = 0$.

$$\therefore\ u_x(0, 0) = \lim_{\delta x \to 0} \frac{u(\delta x, 0) - u(0, 0)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1.$$

$$u_y(0, 0) = \lim_{\delta y \to 0} \frac{u(0, \delta y) - u(0, 0)}{\delta y} = \lim_{\delta y \to 0} \frac{-\delta y}{\delta y} = -1.$$

$$v_x(0, 0) = \lim_{\delta x \to 0} \frac{v(\delta x, 0) - v(0, 0)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1.$$

$$v_y(0, 0) = \lim_{\delta y \to 0} \frac{v(0, \delta y) - u(0, 0)}{\delta y} = \lim_{\delta y \to 0} \frac{\delta y}{\delta y} = 1.$$

Thus, $u_x(0, 0) = 1 = v_y(0, 0)$ and $u_y(0, 0) = -1 = -v_x(0, 0)$. So the CR equations are satisfied at $z = 0$.

Next, we have

$$\begin{aligned}
f'(0) &= \lim_{z\to 0}\frac{f(z)-f(0)}{z-0}\\
&= \lim_{(x,y)\to(0,0)}\frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}\\
&= \lim_{x\to 0}\frac{x^3(1+i)-m^3x^3(1-i)}{(x^2+m^2x^2)(x+imx)} \quad \text{(along the path} \quad y=mx)\\
&= \frac{(1+i)-m^3(1-i)}{(1+m^2)(1+im)}.
\end{aligned}$$

This shows that the value of $f'(0)$ depends on m, and hence $f'(0)$ does not exist.

Sufficient condition for differentiability

Let $f(z)=u(x,y)+iv(x,y)$ be defined in some neighbourhood $z_0=x_0+iy_0$. If the first order partial derivatives of $u(x,y)$ and $v(x,y)$ exist in the neighbourhood of (x_0,y_0), and are continuous at (x_0,y_0) such that the CR equations $u_x=v_y, \quad u_y=-v_x$ are satisfied at (x_0,y_0), then $f(z)$ is differentiable at z_0.

Proof. The continuity of the partial derivatives u_x and u_y at (x_0,y_0) implies the differentiability of $u(x,y)$ at (x_0,y_0), that is,

$$u(x_0+\delta x, y_0+\delta y)-u(x_0,y_0)=u_x(x_0,y_0)\delta x+u_y(x_0,y_0)\delta y+\epsilon_1\delta x+\epsilon_2\delta y,$$

where ϵ_1 and ϵ_2 tend to 0 as $(\delta x,\delta y)\to(0,0)$. Similarly, $v(x,y)$ is differentiable at (x_0,y_0), that is,

$$v(x_0+\delta x, y_0+\delta y)-v(x_0,y_0)=v_x(x_0,y_0)\delta x+v_y(x_0,y_0)\delta y+\epsilon_3\delta x+\epsilon_4\delta y,$$

where ϵ_3 and ϵ_4 tend to 0 as $(\delta x,\delta y)\to(0,0)$.

It follows that

$$\begin{aligned}
f(z_0+\delta z)-f(z_0) &= u(x_0+\delta x,y_0+\delta y)+iv(x_0+\delta x,y_0+\delta y)+i[u(x_0,y_0)+iv(x_0,y_0)]\\
&= u_x(x_0,y_0)\delta x+u_y(x_0,y_0)\delta y+\epsilon_1\delta x+\epsilon_2\delta y\\
&\quad +i[v_x(x_0,y_0)\delta x+v_y(x_0,y_0)\delta y+\epsilon_3\delta x+\epsilon_4\delta y]
\end{aligned}$$

Since CR equations are satisfied at (x_0,y_0), so we replace $u_y(x_0,y_0)$ by $-v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ by $-u_x(x_0,y_0)$ in the above expression, and then divide both sides by $\delta z=\delta x+i\delta y$ to obtain

$$\frac{f(z_0+\delta z)-f(z_0)}{\delta z}=u_x(x_0,y_0)+iv_x(x_0,y_0)+(\epsilon_1+i\epsilon_3)\frac{\delta x}{\delta z}+(\epsilon_2+i\epsilon_4)\frac{\delta y}{\delta z}$$

Both $\frac{\delta x}{\delta z}$ and $\frac{\delta y}{\delta z}$ are bounded since $\left|\frac{\delta x}{\delta z}\right|\leq 1$ and $\left|\frac{\delta y}{\delta z}\right|\leq 1$. So in the limit $\delta z\to 0$, that is, $(\delta x,\delta y)\to(0,0)$, the above expression leads to

$$f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0).$$

This shows that f is differentiable at z_0.

Analyticity

A function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at each point in some neighbourhood of z_0.

Ex. Show that $f(z) = z^2$ is analytic at $z = 0$.
Sol. Consider a neighbourhood of 0, say $|z| < 1$. Then $f'(z) = 2z$, the derivative of $f(z) = z^2$, exists at all points of $|z| < 1$. So $f(z) = z^2$ is analytic at $z = 0$.

Ex. Show that $f(z) = |z|^2$ is not analytic at any point in the complex plane.
Sol. Let z_0 be any point in the complex plane. Since $f(z) = |z|^2$ is nowhere differentiable except $z = 0$, so it can not be differentiable at all points of any neighbourhood of z_0. So $f(z) = |z|^2$ is not analytic at z_0, and hence it is not analytic anywhere in the complex plane.

Remarks:

1. By definition, analyticity at a point implies the differentiability at that point. However, the converse is not true. For example, $f(z) = |z|^2$ is differentiable at $z = 0$ but is not analytic at this point.

2. If a function $f(z)$ is differentiable at each point of an open set S then it is analytic at every point of S since S being an open set is neighbourhood of its every point, and we say that $f(z)$ is analytic in the open set S. For example, $f(z) = z^2$ is analytic in the open set $|z| < 1$.

   In case, S is not open and we say that $f(z)$ is analytic in S, then it should be understood that $f(z)$ is analytic in some open set containing S.

3. The definition of analyticity suggests that the necessary and sufficient conditions for analyticity are same as of differentiability. So, if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the CR equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at all points of the domain D. Conversely, if the CR equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at all points of the domain D, and the partial derivatives u_x, u_y, v_x and v_y are continuous in D, then the function $f(z)$ is analytic in D.

4. If $f(z)$ and $g(z)$ are analytic functions in a domain D, then so are the functions $f(z) \pm g(z)$, $f(z)g(z)$, $(fog)(z)$ and $f(z)/g(z)$ $(g(z_0) \neq 0)$.

5. The terms 'regular' and 'holomorphic' are synonyms of the term 'analytic' in the literature.

Entire function

If a function is analytic at all points in the complex plane, then it is said to be an entire function.

Ex. $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, e^z and polynomials all are entire functions as these are differentiable at every point in the complex plane.

Singularity

If a function $f(z)$ fails to be analytic at z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is said to be singularity or singular point of $f(z)$.

Ex. The function $f(z) = \dfrac{1}{z}$ is analytic everywhere except $z = 0$. So $z = 0$ is the only singular point of $f(z) = \dfrac{1}{z}$.

Ex. The function $f(z) = |z|^2$ has no singularity as it is not analytic anywhere.

Ex. The function $f(z) = \dfrac{1}{(z-1)(z-2)}$ has two singularities $z = 1$ and $z = 2$.

Ex. The singularities of the function $f(z) = \dfrac{1}{\sin z}$ are $z = n\pi$, where n is any integer.

Theorem. If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Given that $f'(z) = 0$ everywhere in a domain D. It implies that $f(z)$ is differentiable everywhere in D, and hence $f(z)$ is analytic everywhere in D. Thus, the CR equations $u_x = v_y$, $u_y = -v_x$ are satisfied at all points in D. Also, $f'(z) = u_x + iv_x = v_y - iu_y$. So $f'(z) = 0$ everywhere in D implies that $u_x = 0$, $u_y = 0$, $v_x = 0$ and $v_y = 0$ everywhere in D.

Now, consider a line segment PQ inside D. Let s denote the arc length from P along PQ, and $\hat{n}$ be a unit vector along PQ in the direction of increasing s. Then directional derivative of $u(x, y)$ along PQ reads as

$$\frac{du}{ds} = \nabla u.\hat{n} = (u_x\hat{i} + u_y\hat{j}).\hat{n} = 0,$$

for all (x, y) on PQ since $u_x = 0$ and $u_y = 0$ everywhere in D. This shows that u is constant for all points on the line segment PQ. Let $u(x, y) = a$ for all (x, y) on PQ. If QR is any line segment inside D, then using the argument as used for the line segment PQ, it follows that u is constant for all points on the line segment QR. But value of $u(x, y)$ is a at Q. It implies that $u(x, y) = a$ for all (x, y) on QR. Now, D being a domain is open and connected. So any two points of D can be joined by a finite number of line segments inside D. It follows that $u(x, y) = a$ for all points in D. Similarly, $v(x, y)$ is constant, say b, for all points in D. Hence, $f(z) = u(x, y) + iv(x, y) = a + ib$, a constant, everywhere in D.

Theorem. If a function $f(z)$ and its conjugate $\overline{f(z)}$, both are analytic everywhere in a domain D, then $f(z)$ is constant throughout D.

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Then $\overline{f(z)} = u(x, y) - iv(x, y)$. Given that $f(z)$ and $\overline{f(z)}$, both are analytic everywhere in the domain D. So CR equations for $f(z) = u(x, y) + iv(x, y)$ given by

$$u_x = v_y, \quad u_y = -v_x,$$

and the CR equations for $\overline{f(z)} = u(x, y) - iv(x, y)$ given by

$$u_x = -v_y, \quad u_y = v_x,$$

are satisfied everywhere in D. Adding and subtracting the CR equations in pairs for $f(z)$ and $\overline{f(z)}$, we get $u_x = 0$, $u_y = 0$ and $v_x = 0$, $v_y = 0$, respectively, everywhere in D. So by previous theorem, $f(z)$ is constant throughout D.

Theorem. If a function $f(z)$ is analytic and $|f(z)|$ is constant everywhere in a domain D, then $f(z)$ is constant throughout D.

Proof. Let $|f(z)| = c$, a constant for all $z \in D$. If $c = 0$, then $f(z) = 0$ for all $z \in D$ and we are through. Assume that $c \neq 0$. Then we have $c^2 = |f(z)|^2 = f(z)\overline{f(z)}$ or $\overline{f(z)} = c^2/f(z)$. Given that $f(z)$ is analytic and $|f(z)| = c \neq 0$ throughout D. It implies that $\overline{f(z)} = c^2/f(z)$ is also analytic throughout D. So by previous theorem, $f(z)$ is constant throughout D.

Harmonic function

Let $h(x, y)$ be a real valued function defined in a domain D in the xy-plane. Then $h(x, y)$ is said to be harmonic in D if there exist continuous partial derivatives of the first and second order of $h(x, y)$ throughout D, and satisfy the Laplace equation $h_{xx} + h_{yy} = 0$.

Ex. Consider the function $h(x, y) = e^{-y} \sin x$. Then we have

$$h_x = e^{-y} \cos x, \quad h_y = -e^{-y} \sin x, \quad h_{xx} = -e^{-y} \sin x, \quad h_{yy} = e^{-y} \sin x.$$

We see that $h(x, y)$ has continuous partial derivatives of the first and second order throughout the xy-plane. Also, $h_{xx} + h_{yy} = -e^{-y} \sin x + e^{-y} \sin x = 0$. Therefore, $h(x, y) = e^{-y} \sin x$ is harmonic in the entire xy-plane.

Ex. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then show that $u(x, y)$ and $v(x, y)$ are harmonic in D.
Sol. Given that $f(z) = u(x, y) + iv(x, y)$ is analytic in D. So CR equations are satisfied in D, that is,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

It will be proved later that if $f(z) = u(x, y) + iv(x, y)$ is analytic, then $u(x, y)$ and $v(x, y)$ possess continuous partial derivatives of all orders. In particular, u_{xy}, u_{yx}, v_{xy} and v_{yx} are continuous. Then from the theory of partial differentiation, it follows that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

Now, differentiating the first CR equation partially with respect to x while second with respect to y, and adding the resulting equations, we get

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = v_{xy} - v_{xy} = 0.$$

This shows that $u(x, y)$ is harmonic in D.

Similarly, differentiating the first CR equation partially with respect to y while second with respect to x, and adding the resulting equations, we get

$$v_{xx} + v_{yy} = u_{xy} - u_{yx} = u_{xy} - u_{xy} = 0.$$

This shows that $v(x, y)$ is harmonic in D.

Harmonic conjugate

If two real valued functions $u(x, y)$ and $v(x, y)$ are harmonic in a domain D and satisfy the CR equations throughout D, then $v(x, y)$ is known as harmonic conjugate of $u(x, y)$ in D.

It is easy to deduce that a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if $v(x, y)$ is harmonic conjugate of $u(x, y)$ in D.

Note that if $v(x, y)$ is harmonic conjugate of $u(x, y)$, then $u(x, y)$ need not be harmonic conjugate of $v(x, y)$. For instance, consider the function $f(z) = z^2 = x^2 - y^2 + i(2xy)$. Here, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. So $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. Thus, CR equations are satisfied. Also, $u_{xx} + u_{yy} = 2 - 2 = 0$ and $v_{xx} + v_{yy} = 0 + 0 = 0$. Hence, $v(x, y)$ is harmonic conjugate of $u(x, y)$. Now, let $u(x, y) = 2xy$ and $v(x, y) = x^2 - y^2$. Then, $u_x = 2y \neq -2y = v_y$. This shows that the CR equations are not satisfied in the opposite case. Thus, $u(x, y)$ is not harmonic conjugate of $v(x, y)$.

Ex. Find harmonic conjugate of $u(x, y) = y^3 - 3x^2y$.
Sol. Let $v(x, y)$ be harmonic conjugate of $u(x, y) = y^3 - 3x^2y$. So by CR equations, we have

$$v_x = -u_y = -3y^2 + 3x^2, \quad \text{and} \quad v_y = u_x = -6xy.$$

Now integrating $v_x = -3y^2 + 3x^2$ with respect to x, we get

$$v(x, y) = -3xy^2 + x^3 + \phi(y)$$

Differentiating partially with respect to y, we have

$$v_y = -6xy + \phi'(y).$$

But from the second CR equation, $v_y = -6xy$. It follows that $\phi'(y) = 0$. So $\phi(y) = C$, a constant. Thus, $v(x, y) = -3xy^2 + x^3 + C$ is the required harmonic conjugate.

Chapter 3

Exponential function

Complex exponential function, denoted by e^z or $\exp z$, is defined as

$$e^z = e^x e^{iy},$$

where $z = x + iy$. Obviously, $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$, where k is any integer. Also, the complex exponential function e^z reduces to the real exponential function e^x for $y = 0$. Some properties of e^z different from e^x are given below.

- e^x is positive for all real values of x. However, e^z can attain negative values. For instance, $e^{i\pi} = \cos\pi + i\sin\pi = -1$.
- Since $e^{2\pi i} = 1$, so $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$. This shows that e^z is a periodic function with imaginary period $2\pi i$. On the other hand, e^x is not a periodic function.

Logarithmic function

The logarithm of a non-zero complex number z, denoted by $\log z$ is defined as the complex number w such that $e^w = z$. To determine w explicitly, let $w = u + iv$ and $z = re^{i\Theta}$. Then we have

$$e^u e^{iv} = re^{i\Theta}.$$

It follows that

$$e^u = r \quad \text{and} \quad v = \Theta + 2k\pi, \qquad \text{where k is any integer.}$$

Thus, the logarithm of z is given by

$$w = \log z = \ln r + i(\Theta + 2k\pi), \quad k = 0, \pm1, \pm2, ...$$

Obviously, $\log z$ is a multiple valued function. The value of $\log z$ corresponding to $k = 0$ is called its principal value, and is denoted by $\text{Log}z$, that is,

$$\text{Log } z = \ln r + i\Theta.$$

It is, of course, a well defined single valued function of z provided $z \neq 0$. It reduces to the usual logarithm of real numbers when z is a positive real number. Now, the multiple valued function $\log z$ can be rewritten as

$$\log z = \text{Log } z + 2k\pi i, \quad k = 0, \pm1, \pm2, ...$$

Next, we have

$$e^{\log z} = e^{\text{Log } z} e^{2k\pi i} = e^{\ln r + i\Theta}.1 = e^{\ln r} e^{i\Theta} = re^{i\Theta} = z.$$

On the other hand

$$\log e^z = \log(e^x e^{iy}) = \ln e^x + i(y + 2k\pi) = x + iy + 2k\pi i = z + 2k\pi i.$$

Complex exponents

Let c be any complex number. Then the function z^c, where $z \neq 0$, is defined as

$$z^c = e^{c \log z}.$$

Since $\log z$ is a multiple valued function, so is z^c. For instance,

$$i^i = e^{i \log i} = e^{i[\ln 1 + i(2k\pi + \pi/2)]} = e^{-(4k+1)\pi/2}, \quad k = 0, \pm 1, \pm 2, ...$$

Following the definition of z^c, we have

$$c^z = e^{z \log c},$$

the exponential function with base $c (\neq 0)$. It is indeed a multiple valued function. The usual interpretation of e^z occurs when the principal value of logarithm is taken into account. For the principal value, $\log e = 1$.

Trigonometric functions

By Euler's formula, we know that $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$. So the sine and cosine functions of the real variable x can be written as

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

In analogy, the sine and cosine functions of a complex variable z are defined as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Clearly, these functions reduce to $\sin x$ and $\cos x$ in case z is restricted to the real number x. This fact serves as a motivation for the above definitions of $\sin z$ and $\cos z$. Next, we define the other elementary trigonometric functions given by $\tan z = \sin z / \cos z$, $\cot z = \cos z / \sin z$, $\sec z = 1/\cos z$ and $\csc z = 1/\sin z$, allowing the values of z for which the denominator of the function under consideration is non-zero.

Let z, z_1 and z_2 be any complex numbers. Then the following identities can easily be established:

(1) $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

(2) $\sin(z_1 + z_2) + \sin(z_1 - z_2) = 2 \sin z_1 \cos z_2$

(3) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(4) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(5) $\sin^2 z + \cos^2 z = 1$, $1 + \tan^2 z = \sec^2 z$, $1 + \cot^2 z = \csc^2 z$

(6) $\sin 2z = 2 \sin z \cos z$, $\cos 2z = \cos^2 z - \sin^2 z$

(7) $\sin(z + \pi/2) = \cos z$, $\cos(z + \pi/2) = -\sin z$

(8) $\sin(iy) = i \sinh y$, $\cos(iy) = \cosh y$, where $\sinh y = (e^y - e^{-y})/2$ and $\cosh y = (e^y + e^{-y})/2$

(9) $\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$

(10) $\cos z = \cos(x+iy) = \cos x\cos(iy) + \sin x\sin(iy) = \cos x\cosh y + i\sin x\sinh y$

(11) $\sin(z+2\pi) = \sin z$, $\cos(z+2\pi) = \cos z$

(12) $|\sin z|^2 = \sin^2 x + \sinh^2 y$
For, $|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x\cosh^2 y + (1-\sin^2 x)\sinh^2 y = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y = \sin^2 x + \sinh^2 y$

(13) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Note that the identities in (11) indicate that $\sin z$ and $\cos z$ are periodic functions with period 2π each. The identities (12) and (13) tell us that $\sin z$ and $\cos z$ are unbounded functions in the complex plane since $\sinh^2 y \to \infty$ as $y \to \infty$ and therefore $|\sin z| \to \infty$ as well as $|\cos z| \to \infty$ in the limit $y \to \infty$. In contrast, $\sin x$ and $\cos x$ are bounded functions in their domain of real numbers since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all $x \in \mathbb{R}$.

Inverse Trigonometric functions

If $z = \sin w$, then w defines inverse sine of z and is denoted by $\sin^{-1} z$. So $w = \sin^{-z}$ is the inverse sine function. Now, the relation

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

yields the following quadratic equation in e^{iw}.

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0.$$

Solving for e^{iw}, we get

$$e^{iw} = iz + (1-z^2)^{1/2} \quad \text{or} \quad w = -i\log[iz + (1-z^2)^{1/2}].$$

Thus, we have

$$\sin^{-1} z = -i\log[iz + (1-z^2)^{1/2}].$$

Note that $(1+z^2)^{1/2}$ is a double valued function. Also, the logarithmic function is multiple valued function. Consequently, $\sin^{-1} z$ is a multiple valued function.

In analogy to $\sin^{-1} z$, we define inverse cosine and tangent functions denoted by $\cos^{-1} z$ and $\tan^{-1} z$ from the relations $w = \cos z$ and $w = \tan z$, respectively. Further, we can prove that the multiple valued functions $\cos^{-1} z$ and $\tan^{-1} z$ are given by

$$\cos^{-1} z = -i\log[z + i(1-z^2)^{1/2}].$$

$$\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}.$$

Hyperbolic functions

The hyperbolic sine and cosine functions, denoted by $\sinh z$ and $\cosh z$ respectively, are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

These definitions are motivated by the fact that these functions reduce to their counterparts $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$ in real system when z is restricted to the real variable x. The other elementary hyperbolic functions are defined as $\tanh z = \sinh z/\cosh z$, $\coth z = \cosh z/\sinh z$, $\operatorname{sech} z = 1/\cosh z$ and $\operatorname{csch} z = 1/\sinh z$, allowing the values of z for which the denominator of the function under consideration is non-zero. Some useful identities related to the hyperbolic functions are given below.

(1) $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$

(2) $\cosh^2 z - \sinh^2 z = 1$

(3) $-i\sinh(iz) = \sin z$, $\cosh(iz) = \cos z$, $-i\sin(iz) = \sinh z$, $\cos(iz) = \cosh z$

(4) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

(5) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(6) $\sinh z = \sinh(x + iy) = \sinh x \cosh(iy) + \cosh x \sinh(iy) = \sinh x \cos y + i \cosh x \sin y$

(7) $\cosh z = \cosh(x + iy) = \cosh x \cosh(iy) + \sinh x \sinh(iy) = \cosh x \cos y + i \sinh x \sin y$

(8) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

(9) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

Inverse hyperbolic functions

If $z = \sinh w$, then w defines inverse hyperbolic sine of z and is denoted by $\sinh^{-1} z$. So $w = \sinh^{-1} z$ is the inverse hyperbolic sine function. Now, the relation

$$z = \sinh w = \frac{e^w - e^{-w}}{2}.$$

yields

$$w = \log[z + (z^2 + 1)^{1/2}].$$

Thus, we have

$$\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}].$$

Note that $(1 + z^2)^{1/2}$ is a double valued function. Also, the logarithmic function is multiple valued function. Consequently, $\sinh^{-1} z$ is a multiple valued function.

In analogy to $\sinh^{-1} z$, we define inverse hyperbolic cosine and tangent functions denoted by $\cosh^{-1} z$ and $\tanh^{-1} z$ from the relations $w = \cosh z$ and $w = \tanh z$, respectively. Further, we can prove that the multiple valued functions $\cosh^{-1} z$ and $\tanh^{-1} z$ are given by

$$\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}].$$

$$\tan^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}.$$

Chapter 4

Definition . **(Differentiation of a complex-valued function of real variable)** If $w(t) = u(t) + iv(t)$ is a complex-valued function of a real variable t, then its derivative with respect to t is defined as

$$w'(t) = u'(t) + iv'(t),$$

where prime denotes the derivative with respect to t.

Obviously, the existence of the derivatives $u'(t)$ and $v'(t)$ is essential for the existence of $w'(t)$.

Remark . Note that complex-valued functions of real variable do not respect the mean value theorem of differential calculus[4]. For, consider the function $w(t) = e^{it}$, $t \in [0, 2\pi]$. Then $w(t)$ is continuous in $[0, 2\pi]$, and is differentiable in $(0, 2\pi)$ since $w'(t) = ie^{it}$ exists for all $t \in (0, 2\pi)$. Also $w(0) = 1 = w(2\pi)$. But $|w'(t)| = 1$ for all $t \in (0, 2\pi)$. Thus, $w'(t)$ does not vanish at any point in $(0, 2\pi)$.

Definition . **(Integration of a complex-valued function of real variable)** Let $w(t) = u(t) + iv(t)$ be a piecewise continuous function in $[a, b]$. Then the integral of $w(t) = u(t) + iv(t)$ over $[a, b]$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Remark . Mean value theorem of integral calculus[5] is also not respected by the complex-valued functions of real variable. For, consider the function $w(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$\int_0^{2\pi} w(t)dt = \int_a^b e^{it}dt = \frac{e^{it}}{i}\bigg]_0^{2\pi} = 0.$$

But $|w(c)(2\pi - 0)| = 2\pi|e^{ic}| = 2\pi$ for all $c \in (0, 2\pi)$. So

$$\int_0^{2\pi} w(t)dt = 0 \neq w(c)(2\pi - 0)$$

for any $c \in (0, 2\pi)$.

Contours

Definition . **(Simple Arc)** The function $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous on $[a, b]$, defines an arc in the complex plane from $z(a)$ to $z(b)$. The curve is traversed with increasing values of t in $[a, b]$, that is, from $z(a)$ to $z(b)$. If the curve does not cross itself, that is, $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$, it is said to be a simple arc or Jordan arc.

[4] If a function f is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then $f'(x)$ vanishes at least once in (a, b).

[5] If a function f is continuous in $[a, b]$, then $\displaystyle\int_a^b f(x)dx = c(b - a)$ for some $c \in (a, b)$.

Definition . (Simple Closed Curve) If the arc $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ does not cross itself except for $z(a) = z(b)$, it is said to be simple closed curve or Jordan curve. If the simple closed curve traverses from $z(a)$ to $z(b)$ in counterclockwise direction, then it is said to be positively oriented.

Definition . (Smooth Arc) If $z'(t) = x'(t) + iy'(t)$ is continuous on $[a, b]$ and non-zero on (a, b), then the arc or curve $z(t) = x(t) + iy(t)$, $a \leq t \leq b$ is said to be a smooth arc.

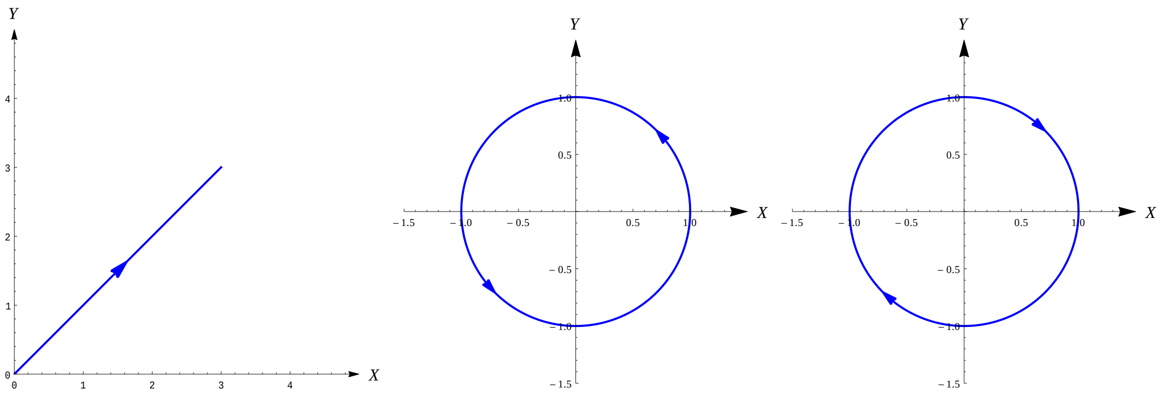


Figure 5: **Left:** $z(t) = t + it$, $0 \leq t \leq 4$; **Middle:** $z(t) = e^{it} = \cos t + i \sin t$, $0 \leq t \leq 2\pi$; **Right:** $z(t) = e^{-it} = \cos t - i \sin t$, $0 \leq t \leq 2\pi$.

Example . $z(t) = t + it$, $0 \leq t \leq 4$ is a smooth arc. It is the line segment $y = x$, $0 \leq x \leq 4$ (See left plot in Figure 5).

Example . $z(t) = e^{it} = \cos t + i \sin t$, $0 \leq t \leq 2\pi$ is a positively oriented simple closed smooth curve. It is the unit circle $x^2 + y^2 = 1$ traversed in counterclockwise direction (See middle plot in Figure 5).

Example . $z(t) = e^{-it} = \cos t - i \sin t$, $0 \leq t \leq 2\pi$ is a negatively oriented simple closed smooth curve. It is the unit circle $x^2 + y^2 = 1$ traversed in clockwise direction (See right plot in Figure 5).

Definition . (Contour) A contour is a piecewise smooth arc, that is, an arc consisting of finite number of smooth arcs joined end to end. A simple closed contour is a contour that does not intersect itself except for the end points.

Example . $z(t) = t + it$, $0 \leq t \leq 4$ is a simple contour but not closed.

Example . $z(t) = e^{it} = \cos t + i \sin t$, $0 \leq t \leq 2\pi$ is a simple closed contour.

Example . Any triangle is a simple closed contour consisting of three smooth arcs (the three sides) joined end to end. Similarly, any rectangle is a simple closed contour.

Contour Integrals

Definition . (Contour Integral) Let f be a piecewise continuous function on a contour C : $z(t) = x(t) + iy(t)$, $a \leq t \leq b$. Then the integral of f along the contour C, denoted by $\int_C f(z)dz$ is defined as

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt.$$

Note that the piecewise continuity of f on C and the piecewise continuity of $z'(t)$ (as $C: \ z(t) = x(t) + iy(t)$ is a contour) in $[a, b]$ imply the piecewise continuity of the integrand $f[z(t)]z'(t)$ in $[a, b]$, and hence ensure the existence of the contour integral.

Next, we have

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = -\int_b^a f[z(t)]z'(t)dt = -\int_{-C} f(z)dz.$$

where $-C$ is the notation for the contour $C: z(t) = x(t) + iy(t)$, $a \le t \le b$, when it is traversed from $z(b)$ to $z(a)$.

If the contour $C: z(t) = x(t) + iy(t)$, $a \le t \le b$ consists of two smooth arcs $C_1: z_1(t) = x_1(t) + iy_1(t)$ $(a \le t \le c)$ and $C_2: z_2(t) = x_2(t) + iy_2(t)$ $(c \le t \le b)$, then we have

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = \int_a^c f[z_1(t)]z_1'(t)dt + \int_c^b f[z_2(t)]z_2'(t)dt = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

In general, if the contour C consists of n piecewise smooth arcs C_1, C_2, ..., C_n joined end to end in succession, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz.$$

Example . Evaluate $\displaystyle\int_C \overline{z}dz$, where $C: \ |z| = 1$.

Solution . Given contour C can be written as $z = e^{it}$, $0 \le t \le 2\pi$.

$$\therefore \quad \int_C \overline{z}dz = \int_0^{2\pi} e^{-it}e^{it}.idt = \int_0^{2\pi} idt = 2\pi i.$$

Example . Evaluate $\displaystyle\int_C f(z)dz$, where $f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$, $(|z| > 0,\ 0 < \arg z < 2\pi)$ and C is the semicircular path $z = 2e^{i\theta}$, $0 \le \theta \le \pi$ (Figure 6).

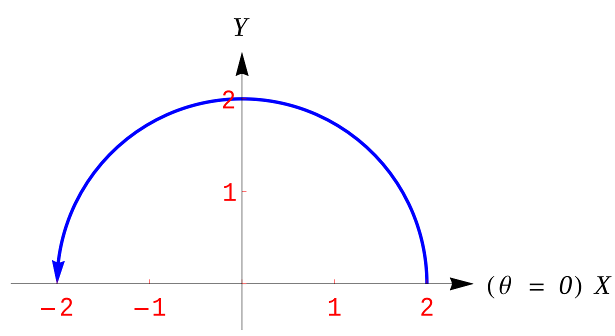


Figure 6: $z = 2e^{i\theta}$, $0 \le \theta \le \pi$.

Solution . We notice that the point $z = 2$ of the semicircular path $C: z(\theta) = 2e^{i\theta}$, $0 \le \theta \le \pi$, lies on the ray $\theta = 0$, where $f(z)$ is not defined. So first we test the piecewise continuity of the integrand $f[z(\theta)]z'(\theta)$ in the interval $[0, \pi]$. We have

$$f[z(\theta)] = \exp\left[\frac{1}{2}(\ln 2 + i\theta)\right] = \sqrt{2}e^{i\theta/2},$$

$$f[z(\theta)]z'(\theta) = \sqrt{2}e^{i\theta/2}2ie^{i\theta} = 2\sqrt{2}ie^{i3\theta/2} = -2\sqrt{2}\sin(3\theta/2) + i2\sqrt{2}\cos(3\theta/2), \ (0 < \theta \leq \pi).$$

So the right hand limit of $f[z(\theta)]z'(\theta)$ as $\theta \to 0$ is $i2\sqrt{2}$, which is finite. This shows that $f[z(\theta)]z'(\theta)$ is piecewise continuous on $[0, \pi]$. Hence the given integral exists, and is given by

$$\int_C f(z)dz = \int_0^\pi f[z(\theta)]z'(\theta)d\theta = 2\sqrt{2}i \int_0^\pi e^{i3\theta/2}d\theta = 2\sqrt{2}i \left.\frac{2e^{i3\theta/2}}{3i}\right]_0^\pi = -\frac{4\sqrt{2}}{3}(1+i).$$

Boundedness of Contour Integrals

Theorem . If $w(t)$ is a complex-valued piecewise continuous function in $[a, b]$, then $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$.

Proof. Let $r_0 = \left|\int_a^b w(t)dt\right|$. If $r_0 = 0$, then the required inequality holds clearly and we have nothing to prove. So let us assume $r_0 \neq 0$. Then we have

$$r_0 e^{i\theta_0} = \int_a^b w(t)dt \quad \text{or} \quad r_0 = \int_a^b e^{-i\theta_0} w(t)dt.$$

Since r_0 is real, so right hand side must also be real.

$$\therefore \quad r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)dt = \int_a^b \text{Re}[e^{-i\theta_0} w(t)]dt.$$

Now, $\text{Re}[e^{-i\theta_0} w(t)] \leq |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}||w(t)| = |w(t)|$.

$$\therefore \quad r_0 \leq \int_a^b |w(t)|dt \quad \text{or} \quad \left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt.$$ □

Theorem . If a function f is piecewise continuous on a contour $C: \ z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then $\int_C f(z)dz$ is bounded.

Proof. Given that f is piecewise continuous on $C: \ z(t) = x(t) + iy(t)$, $a \leq t \leq b$. It implies that $f[z(t)]$ is piecewise continuous on $[a, b]$. Therefore, $f[z(t)]$ must be bounded on $[a, b]$, that is, there exists a constant M such that $|f[z(t)]| \leq M$ for all $t \in [a, b]$. The length of the contour C reads as

$$\int_a^b |z'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt = L \text{ (say)}.$$

Then, in view of the previous theorem, we have

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f[z(t)]z'(t)dt\right| \leq \int_a^b |f[z(t)]z'(t)|dt = \int_a^b |f[z(t)]||z'(t)|dt \leq M \int_a^b |z'(t)|dt = ML.$$

Thus, $\int_C f(z)dz$ is bounded. □

Use of antiderivatives in contour integrals

Definition . (Antiderivative) Let f be continuous in a domain D (open connected set). Then antiderivative of f is a function F such that $F'(z) = f(z)$ for all $z \in D$.

Theorem . If f is a continuous function in a domain D, then the following statements are equivalent:

(i) Antiderivative of f exists throughout D.

(ii) If C is any contour inside D from a point z_1 to z_2, then $\displaystyle\int_C f(z)dz = F(z_2) - F(z_1)$, where $F'(z) = f(z)$.

(iii) Integral of f along any closed contour lying completely inside D is 0.

Remark . It should be noted that the three statements in the above theorem imply each other, and the independent existence/validity of the statements may not be true. Further, the concept of antiderivative and path independence is very useful in evaluating contour integrals as illustrated in the examples given below.

Remark . If antiderivative of a function f exists throughout D, and C is any contour inside D from a point z_1 to z_2, then we use the notation $\displaystyle\int_{z_1}^{z_2} f(z)dz$ for the contour integral of f along C. In other words, when the contour integral $\displaystyle\int_C f(z)dz$ is independent of the path joining the points z_1 and z_2, we write

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz.$$

Example . Evaluate $\int_C zdz$, where C is any contour extending from 0 and $1-i$.

Solution . The integrand function z is continuous everywhere in the complex plane. Also, its antiderivative is $z^2/2$, which exists everywhere in the complex plane. It implies that $\displaystyle\int_C zdz$ is independent of the contour C joining the points 0 and $1-i$. So we have

$$\int_C zdz = \int_0^{1-i} zdz = \frac{z^2}{2}\bigg]_0^{1-i} = \frac{1}{2}(1-i)^2 = i.$$

Example . Evaluate $\displaystyle\int_C \frac{1}{z^3}dz$, where C is any contour extending from 2 and i but not passing through origin.

Solution . The integrand function $1/z^3$ is continuous everywhere in the complex plane except at $z = 0$. Also, its antiderivative is $-3/z^2$, which exists everywhere in the complex plane except at $z = 0$. The given contour C extending from 2 to i does not pass through $z = 0$. It implies that $\displaystyle\int_C zdz$ is independent of the contour C. So we have

$$\int_C zdz = \int_2^i \frac{1}{z^3}dz = -\frac{3}{z^2}\bigg]_2^i = 3 + \frac{3}{4} = \frac{15}{4}.$$

Example . Use the approach of antiderivatives to evaluate $\int_C \frac{1}{z} dz$, where $C : |z| = 2$.

Solution . The integrand function $f(z) = 1/z$ is continuous everywhere in the complex plane except at $z = 0$. Also, its antiderivative is $F(z) = \log z = \ln r + i\theta$ $(\alpha < \theta < \alpha + 2\pi)$, which exists everywhere in the complex plane except at $z = 0$ and the branch cut $\theta = \alpha$. Obviously, for any value of α, the antiderivative $F(z)$ does not exist at the point of intersection of the branch cut $\theta = \alpha$ and the given contour $C : z = 2e^{i\theta}$, $0 \le \theta \le 2\pi$. To resolve this issue, we evaluate the integral along C in two parts: (i) along upper half $C_1 : z = 2e^{i\theta}$, $0 \le \theta \le \pi$ and, (ii) along lower half $C_2 : z = 2e^{i\theta}$, $\pi \le \theta \le 2\pi$. Notice that the branch cut $\theta = -\pi/2$ of $F(z) = \log z = \ln r + i\theta$ $(-\pi/2 < \theta < 3\pi/2)$ does not intersect C_1, which extends from $z = 2$ $(\theta = 0\ )$ to $z = -2$ $(\theta = \pi)$. So we have

$$\int_{C_1} \frac{1}{z} dz = \log z]_2^{-2} = \log(-2) - \log(2) = (\ln 2 + i\pi) - \ln 2 = \pi i.$$

Similarly, the branch cut $\theta = \pi/2$ of $F(z) = \log z = \ln r + i\theta$ $(\pi/2 < \theta < 5\pi/2)$ does not intersect C_2, which traverses from $z = -2$ $(\theta = \pi\ )$ to $z = 2$ $(\theta = 2\pi)$. So we have

$$\int_{C_2} \frac{1}{z} dz = \log z]_{-2}^{2} = \log(2) - \log(-2) = (\ln 2 + i.2\pi) - (\ln 2 + i\pi) = \pi i.$$

Finally, we have

$$\int_C \frac{1}{z} dz = \int_{C_1} \frac{1}{z} dz + \int_{C_2} \frac{1}{z} dz = \pi i + \pi i = 2\pi i.$$

Cauchy's Theorem

Theorem . (Cauchy) If f is analytic and f' is continuous within and on a simple closed contour C, then $\int_C f(z)dz = 0$.

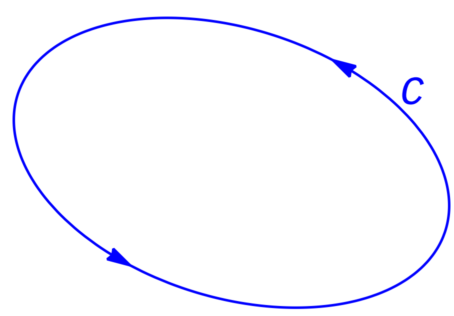


Figure 7: A simple closed contour C.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ and $C : z(t) = x(t) + iy(t),\ a \leq t \leq b$. Then we have

$$\begin{aligned} \int_C f(z)dz &= \int_a^b f[z(t)]z'(t)dt \qquad (4) \\ &= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]dt \\ &= \int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt + i\int_a^b [v(x(t), y(t))x'(t) - u(x(t), y(t))y'(t)]dt \\ &= \int_C [u(x, y)dx - v(x, y)dy] + i\int_C [v(x, y)dx + u(x, y)dy] \end{aligned}$$

It is given that f is analytic. So $f'(z) = u_x + iv_x = v_y - iu_x$, $u_x = v_y$ and $u_y = -v_x$. Further, f' is continuous within and on C. It follows that u, v, u_x, u_y, v_x and v_y all are continuous within and on C. So by Green's theorem[6], (4) becomes

$$\int_C f(z)dz = \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy = \iint_R (-v_x + v_x)dxdy + i\iint_R (v_y - v_y)dxdy = 0,$$

where R is the closed region carrying all points within and on C. □

Cauchy-Goursat Theorem

Goursat proved that the condition of continuity of f', as mentioned in Cauchy's theorem, can be dropped.

Theorem . (Cauchy-Goursat) If f is analytic within and on a simple closed contour C, then $\int_C f(z)dz = 0$.

It is a very useful result regarding the contour integrals. For example, suppose we wish to solve the contour integral $\int_C e^{z^2}dz$, where C is some simple closed contour. Solving this integral directly without the aid of Cauchy-Goursat theorem would be a tricky business (you can try!). We know that the function e^{z^2} is analytic everywhere in the complex plane. So by Cauchy-Goursat theorem it follows that $\int_C e^{z^2}dz = 0$

Simply and multiply connected domains

Definition . (Simply and multiply connected domains) A domain D is said to be simply connected if any closed contour inside D contains points of D only. A domain which is not simply connected is called multiply connected.

Example . $|z| < 1$ is a simply connected domain since any simple closed contour inside $|z| < 1$ would carry points of $|z| < 1$ only.

[6]If ϕ, ψ, ϕ_y and ψ_x are continuous within and on a simple closed curve C in the XY-plane, then $\int_C [\phi dx + \psi dy] = \iint_R (\psi_x - \phi_y)dxdy$, where R is the closed region carrying all points within and on C.

Example . The ring shaped domain $1 < |z| < 2$ is multiply connected. For, the simple closed contour $|z| = 3/2$ lies inside $1 < |z| < 2$. The region $|z| \leq 1$ is a part of interior of $|z| = 3/2$. However, $|z| \leq 1$ is not lying in $1 < |z| < 2$.

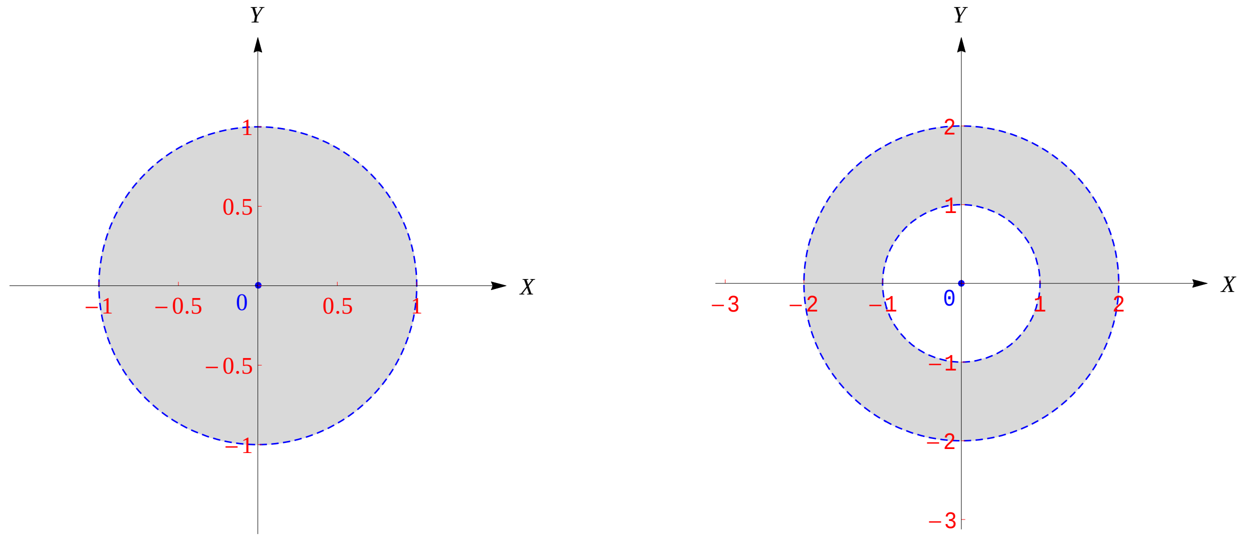


Figure 8: **Left:** $|z| < 1$ is simply connected. **Right:** The ring shaped domain $1 < |z| < 2$ is multiply connected

Theorem . If a function f is analytic throughout a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour lying in D. Furthermore, f must have antiderivative everywhere in D.

Proof. If C is any simple closed contour inside D, then f is analytic within and on C. So by Cauchy-Goursat theorem, $\int_C f(z)dz = 0$. If C is closed but not simple, then C consists of a finite number of simple closed contours, say C_1, C_2, ..., C_n such that the integral of f along C is equal to the sum of integrals along the simple closed contours C_1, C_2, ..., C_n. But integral of f along any simple closed contour is 0. It follows that $\int_C f(z)dz = 0$.

We have proved that $\int_C f(z)dz = 0$ for every closed contour C inside the domain D. So from theorem , it follows that antiderivative of f exists throughout D. □

Extension of Cauchy-Goursat theorem

In the following, we discuss some theorems which pertain to the extension of Cauchy-Goursat theorem to multiply connected regions.

Theorem . (Principle of deformation of path) Suppose a positively oriented simple closed contour C_1 completely lies in the interior of a positively oriented simple closed contour C. If a function f is analytic within and on the closed region constituted by the interior and boundary of C, and exterior of C_1, then $\int_C f(z)dz = \int_{C_1} f(z)dz$.

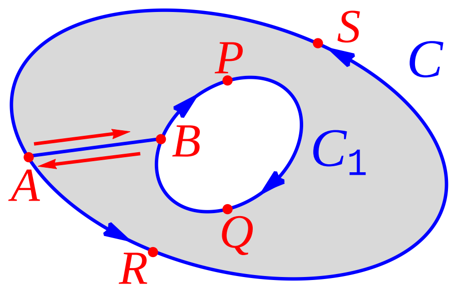


Figure 9: The curve $ABPQBARSA$ constitutes a simple closed contour as it is traversed along the path shown by arrowheads starting from A and ending at A.

Proof. Let AB be a line segment such that A lies on C and B lies on C_1. Also, choose distinct points P, Q on C_1 while R, S on C as shown in Figure 9. Then we see that the path $ABPQBARSA$ constitutes a simple closed contour within and on which the function f is analytic. So by Cauchy-Goursat theorem, we have

$$\int_{ABPQBARSA} f(z)dz = 0.$$

$$\implies \int_{AB} f(z)dz + \int_{BPQB} f(z)dz + \int_{BA} f(z)dz + \int_{ARSA} f(z)dz = 0.$$

$$\implies \int_{AB} f(z)dz - \int_{BQPB} f(z)dz - \int_{AB} f(z)dz + \int_{ARSA} f(z)dz = 0.$$

$$\implies -\int_{C_1} f(z)dz + \int_{C} f(z)dz = 0.$$

$$\implies \int_{C} f(z)dz = \int_{C_1} f(z)dz.$$

This result is called **principle of deformation of path** in the sense that the contour C is deformed to the contour C_1 for evaluating the integral. It proves to be very useful for solving the integrals along contours of arbitrary shape. □

Theorem . Suppose two positively oriented and non-intersecting simple closed contour C_1 and C_2 completely lie in the interior of a positively oriented simple closed contour C. If a function f is analytic within and on the closed region constituted by the interior and boundary of C and exteriors of C_1, C_2, then $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$.

Proof. Let AB, DE and FG be line segments or polygonal paths connecting C and C_1, C_1 and C_2, and C_2 and C, respective as shown in the Figure 10. Choose points P, Q on Also, choose distinct points P, Q on C_1 while R, S on C_2 and T, U on C as shown in Figure 10. Then we see that the path $ABPDERFGTAUGSEDQBA$ constitutes a simple closed contour within and on which the function f is analytic. So by Cauchy-Goursat theorem, we have

$$\int_{ABPDERFGTAUGSEDQBA} f(z)dz = 0.$$

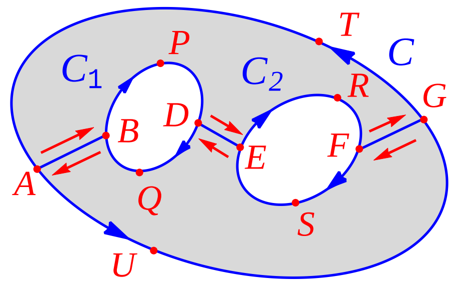


Figure 10: The curve $ABPDERFGTAUGSEDQBA$ constitutes a simple closed contour as it is traversed along the path shown by arrowheads starting from A and ending at A.

$$\implies \int_{AB} f(z)dz + \int_{BPD} f(z)dz + \int_{DE} f(z)dz + \int_{ERF} f(z)dz + \int_{FG} f(z)dz + \int_{GTA} f(z)dz + \int_{AUG} f(z)dz + \int_{GF} f(z)dz + \int_{FSE} f(z)dz + \int_{ED} f(z)dz + \int_{DQB} f(z)dz + \int_{BA} f(z)dz = 0.$$

$$\implies \int_{AB} f(z)dz + \int_{BPD} f(z)dz + \int_{DE} f(z)dz + \int_{ERF} f(z)dz + \int_{FG} f(z)dz - \int_{ATG} f(z)dz - \int_{GUA} f(z)dz - \int_{FG} f(z)dz + \int_{FSE} f(z)dz - \int_{DE} f(z)dz + \int_{DQB} f(z)dz - \int_{AB} f(z)dz = 0.$$

$$\implies \int_{BPD} f(z)dz + \int_{DQB} f(z)dz + \int_{ERF} f(z)dz + \int_{FSE} f(z)dz - \int_{ATG} f(z)dz - \int_{GUA} f(z)dz = 0.$$

$$\implies \int_{BPDQB} f(z)dz + \int_{ERFSE} f(z)dz - \int_{ATGUA} f(z)dz = 0.$$

$$\implies \int_{C_1} f(z)dz + \int_{C_2} f(z)dz - \int_{C} f(z)dz = 0.$$

$$\implies \int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

□

Remark . Let C be a positively oriented simple closed contour, and C_1, C_2, ..., C_n be positively oriented non-intersecting simple closed contours lying inside C. If a function f is analytic within and on the closed region constituted by the interior and boundary of C, and exteriors of C_1, C_2,......., C_n, then following the analogy of theorem , we get

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz.$$

This result is also known as **Cauchy-Goursat theorem for multiply connected domains**.

Example . Solve $\displaystyle\int_C \frac{dz}{z}$, given that C is any contour such that a circle of radius less than 2 and centred at origin lies inside C.

Solution . Consider the unit circle $C_1 : |z| = 1$. Then the function $1/z$ is analytic in the interior of C but exterior of C_1. So by principle of deformation of path, we have

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = 2\pi i.$$

Cauchy Integral Formula and its extension

Theorem . (Cauchy Integral Formula) If f is analytic within and on a positively oriented simple closed contour C and z_0 is a point in the interior of C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0}. \tag{5}$$

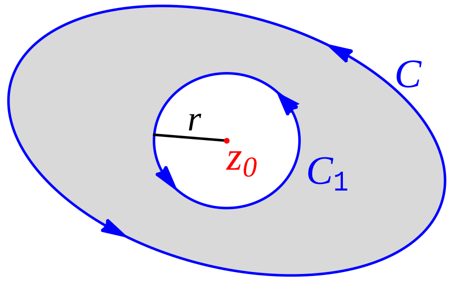


Figure 11:

Proof. Given that f is analytic within and on the simple closed contour C. So the function $\frac{f(z)}{z-z_0}$ has the only singularity $z = z_0$ that lies inside C. Let $C_1 : \ |z - z_0| = r$ be a circle with radius r so small that it completely lies in the interior of C. Then the function $\frac{f(z)}{z-z_0}$ is analytic in the interior of C but exterior of C_1. So by principle of deformation of path, we have

$$\int_C \frac{f(z)dz}{z - z_0} = \int_{C_1} \frac{f(z)dz}{z - z_0} = \int_0^{2\pi} \frac{f(z_0 + re^{it})ire^{it}dt}{re^{it}} = i\int_0^{2\pi} f(z_0 + re^{it})dt.$$

Here, r is at our discretion. In the limit $r \to 0$, the continuity of f implies that

$$\int_C \frac{f(z)dz}{z - z_0} = i\int_0^{2\pi} f(z_0)dt = 2\pi i f(z_0),$$

which leads to the Cauchy Integral Formula. □

Note that the Cauchy Integral Formula can be generalized to obtain

$$\int_C \frac{f(z)dz}{(z - z_0)^{n+1}} = 2\pi i f^{(n)}(z_0)\ n!.$$

For, let z be a point inside C. Then by Cauchy Integral Formula, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{s - z}. \tag{6}$$

Without any argument (in analogy of differentiation of real functions under integral sign[7]), let us differentiate both sides of (6) with respect to z to obtain

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s - z)^2}. \tag{7}$$

[7] Leibniz Rule for differentiation under integral sign: $\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)dx + \frac{db}{d\alpha}f(b,\alpha) - \frac{da}{d\alpha}f(a,\alpha)$

Now, we prove the formula obtained in (7). Let d be the smallest distance of z from points s on C and $0 < |\delta z| < d$. Then, in view of (6), we have

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\delta z}ds = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\delta z)(s-z)}. \quad (8)$$

Subtracting right hand side of (7) from both sides of (8), we get

$$\frac{f(z+\delta z)-f(z)}{\delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\delta z f(s)ds}{(s-z-\delta z)(s-z)^2}. \quad (9)$$

Now, let $|f(s)| \leq M$ on C and L be the length of C. We notice that $|s-z| \geq d$ and $|\delta| < d$, and therefore

$$|s-z-\delta z| = |(s-z)-\delta z| \geq ||s-z|-|\delta z|| \geq d - |\delta z| > 0.$$

Then, from (9), it follows that

$$\left|\frac{f(z+\delta z)-f(z)}{\delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}\right| = \frac{1}{2\pi}\left|\int_C \frac{\delta z f(s)ds}{(s-z-\delta z)(s-z)^2}\right| \leq \frac{|\delta z|ML}{(d-|\delta z|)d^2}. \quad (10)$$

In the limit $\delta z \to 0$, right side of the inequality (10) tends to 0 and hence we get

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2} = 0.$$

$$\therefore \quad f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}.$$

Differentiating again with respect to z and following the procedure adopted for proving (7), it can be shown that

$$\therefore \quad f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^3}.$$

Repeating the same procedure $n-2$ more times, we get the general formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}, \quad n = 0, 1, 2, ...$$

If we denote z by z_0 and replace s by z, the above formula reads as

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0, 1, 2, ...$$

We shall refer this formula to as the generalized Cauchy Integral Formula since it yields the Cauchy Integral Formula (5) for $n = 0$. It is very useful for evaluating contour integrals as illustrated in the following examples.

Example . Evaluate $\displaystyle\int_C \frac{dz}{z}$, $C : |z| = 1$.

Solution . Comparing the given integral with $\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$, we have $f(z) = 1$, $z_0 = 0$ and $n = 0$. We see that $f(z) = 1$ is analytic within and on C. Also, $z_0 = 0$ lies inside $C : |z| = 1$. So by Cauchy integral formula, we have

$$\int_C \frac{dz}{z} = 2\pi i f(0) = 2\pi i.$$

Example . Evaluate $\int_C \frac{z^3 + 2z^2 - 1}{(z-1)^3} dz$, $C : |z| = 2$.

Solution . Comparing the given integral with $\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$, we have $f(z) = z^3 + 2z^2 - 1$, $z_0 = 1$ and $n = 2$. So $f''(z) = 6z + 4$. We see that $f(z) = z^3 + 2z^2 - 1$ is analytic within and on C. Also, $z_0 = 1$ lies inside $C : |z| = 2$. So by Cauchy integral formula, we have

$$\int_C \frac{z^3 + 2z^2 - 1}{(z-1)^3} dz = 2\pi i f''(1) = 2\pi i(10) = 20\pi i.$$

Example . Evaluate $\int_C \frac{z^3 + 2z^2 - 1}{(z-1)^3} dz$, $C : |z| = 1/2$.

Solution . The integrand function $\frac{z^3+2z^2-1}{(z-1)^3}$ has only one singularity $z = 1$ that lies outside $C : |z| = 1/2$. So $\frac{z^3+2z^2-1}{(z-1)^3}$ is analytic within and on $C : |z| = 1/2$. So by Cauchy-Goursat theorem, the given integral vanishes.

Example . Evaluate $\int_C \frac{z^2 + 2}{(z-1)(z-4)} dz$, $C : |z| = 3$.

Solution . The given integrand function $\frac{z^2 + 2}{(z-1)(z-4)}$ has two singularities namely $z = 1$ and $z = 4$. But $z = 4$ lies outside $C : |z| = 3$. So rewriting the given integral as $\int_C \frac{(z^2 + 2)/(z-4)}{z-1} dz$ and comparing with $\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$, we have $f(z) = (z^2 + 2)/(z-4)$, $z_0 = 1$ and $n = 0$. We see that $f(z) = (z^2 + 2)/(z-4)$ is analytic within and on C. Also, $z_0 = 1$ lies inside $C : |z| = 3$. So by Cauchy integral formula, we have

$$\int_C \frac{z^2 + 2}{(z-1)(z-4)} dz = 2\pi i f(1) = 2\pi i(-1) = -2\pi i.$$

Implications of Cauchy Integral Formula

In this section, we discuss some remarkable implications of the Cauchy Integral Formula.

Theorem . **(Derivatives of an analytic function)** If a function f is analytic at z_0, then derivatives of all orders of f are also analytic at z_0.

Proof. Given that f is analytic at z_0. Therefore, f is analytic in some neighbourhood of z_0, say, $|z - z_0| < \epsilon$. Then f is analytic within and on the circle $C : |z - z_0| = \epsilon/2$. So by Cauchy Integral Formula, f'' exists at all points inside C. It implies that f' is analytic at z_0. Applying the same argument on f', it is easy to show that f'' is analytic at z_0. In general, we reach to the conclusion that derivatives of all orders of f are analytic at z_0. This completes the proof. □

Remark . Note that the above result is not true in case of real functions. For, consider $f(x) = x^{3/2}$. Then the first derivative of f, that is, $f'(x) = (3/2)x^{1/2}$ exists at $x = 0$, which is $f'(0) = 0$. But the second derivative $f''(x) = (3/4)x^{-1/2}$ does not exists at $x = 0$.

Theorem . If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then $u(x, y)$ and $v(x, y)$ possess continuous partial derivatives of all orders in D.

Proof. Given that $f(z) = u(x, y)+iv(x, y)$ is analytic D. So by theorem , derivatives of all orders of f exist in D. Also, by theorem (necessary condition for differentiability), $f'(z) = u_x+iv_x = v_y-iu_y$, $f''(z) = u_{xx}+iv_{xx} = v_{xy}-iu_{xy}$ and so forth. Now by theorem(?), differentiability of f in D implies continuity of u and v in D, differentiability of f' in D implies continuity of u_x, u_y, v_x, v_y in D and so forth. This completes the proof. It □

Theorem . **(Morera's Theorem)** If a function f is continuous in a domain D, and $\int_C f(z)dz = 0$ for every closed contour C inside D, then f is analytic throughout the domain D.

Proof. Given that f is continuous in the domain D, and $\int_C f(z)dz = 0$ for every closed contour C inside D. So by the theorem of antiderivatives, there exists a function F such that $F'(z) = f(z)$ for all $z \in D$. It implies that $F(z)$ is analytic in D. We know that derivative of analytic function is also analytic. Therefore, f being derivative of F is analytic in D. This completes the proof. □

Theorem . **(Liouville's Theorem)** Every entire and bounded function is constant.

Proof. Let f be an entire and bounded function. Let z_0 be any point in the complex plane. Then to prove the Liouville's theorem, it is sufficient to show that $f'(z_0) = 0$. Now consider a circle $C : |z - z_0| = R$ in the complex plane. Then f being entire function is analytic within and on C. So by Cauchy integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2}dz$$

Now f being a bounded function there exists a constant M such that $|f(z)| \leq M$ for all z on $C : |z - z_0| = R$. So we have

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2}dz\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})Re^{it}idt}{R^2e^{i2t}}\right| = \frac{1}{2\pi R}\int_0^{2\pi} |f(z_0+Re^{it})|dt \leq \frac{1}{2\pi R}M.2\pi = \frac{M}{R}$$

Since R is at our discretion, in the limit $R \to \infty$, the inequality $|f'(z_0)| \leq \frac{M}{R}$ implies that $f'(z_0) = 0$. This completes the proof. □

Ex. The function $\sin z$ is unbounded in the complex plane. For, if $\sin z$ is bounded, then $\sin z$ being an entire function should be constant by Liouville's theorem.

Remark . f If a function f is analytic within and on a positively oriented circle $C : |z-z_0| = R$ and $|f(z)| \leq M$ for all z on C, then using the generalized Cauchy Integral Formula and the procedure adopted in last step of the proof of Liouville's theorem, it is easy to show that $|f^{(n)}(z_0)| \leq n!M/R^n$.

Theorem . (Fundamental theorem of algebra) Any polynomial of degree $n \geq 1$ has at least one zero in the complex plane.

Proof. Let $P_n(z)$ be a polynomial of degree $n \geq 1$. Then there exists constants a_0, a_1, ..., a_{n-1}, $a_n \neq 0$ such that $P_n(z) = a_0 + a_1 z + ... + a_{n-1}z^{n-1} + a_n z^n$. Assume, if possible, that $P_n(z)$ has no zero in the complex plane. Then the function $1/P_n(z) = f(z)$ (say) being a rational function with denominator non-zero for all z, is analytic throughout the complex plane. Now, we shall prove that $f(z)$ is bounded. We have

$$\begin{aligned} |P_n(z)| &= \left|\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + ... + \frac{a_{n-1}}{z} + a_n\right) z^n\right| && (11) \\ &\geq \left(|a_n| - \left|\frac{a_0}{z^n}\right| - \left|\frac{a_1}{z^{n-1}}\right| - .... - \left|\frac{a_{n-1}}{z}\right|\right) |z|^n && (12) \end{aligned}$$

We can choose a positive real number R sufficiently large such that $\left|\frac{a_i}{z^{n-i}}\right| < \frac{|a_n|}{2n}$, $(i = 0, 1, .., n-1)$, for all z with $|z| > R$. So we have

$$|P_n(z)| > \left(|a_n| - \frac{|a_n|}{2}\right) R^n = \frac{|a_n|}{2} R^n.$$

It leads to $|f(z)| = |1/P_n(z)| < \frac{2}{|a_n|R^n}$ for all z with $|z| > R$, which is turn implies that $f(z)$ is bounded in the exterior of the disk $|z| = R$. Also $f(z)$ being an entire function is bounded within and on the disk $|z| = R$. Thus $f(z)$ is bounded function. By Liouville's theorem, therefore, $f(z)$ must be constant, which is not true. So our assumption that $P_n(z)$ has no zero in the complex plane is wrong. This completes the proof. □

Remark . By the above theorem, $P_n(z)$ has at least one zero, say z_n. Then $z - z_n$ is a factor of $P_n(z)$ and consequently $P_n(z) = (z - z_n)Q_{n-1}(z)$, where $Q_{n-1}(z)$ is a polynomial of degree $n - 1$. Again, by the above theorem, the polynomial $Q_{n-1}(z)$ has at least one zero, say z_{n-1}. So $z - z_{n-2}$ is a factor of $Q_{n-1}(z)$ and this in turn implies that $P_n(z) = (z - z_n)(z - z_{n-1})R_{n-2}(z)$, where $R_{n-2}(z)$ is a polynomial of degree $n - 2$. Continuing the same process for $n - 2$ more times, we find

$$P_n(z) = (z - z_n)(z - z_{n-1}).......(z - z_1)a_n$$

This shows that the polynomial $P_n(z)$ has exactly n zeros given by z_1,...., z_n.

Chapter 5

Theorem . (Laurent series) If a function f is analytic throughout an open annular or ring shaped domain $R_1 < |z-z_0| < R_2$, and C is any positively oriented simple closed contour C inside the ring shaped domain encircling z_0, then for any z in $R_1 < |z-z_0| < R_2$, $f(z)$ has the series representation given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0, 1, 2, ...$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}, \quad n = 1, 2, ...$$

Remark . If f is analytic throughout the domain $|z-z_0| < R_2$, then $f(z)(z-z_0)^{n-1}$ $(n = 1, 2, ...)$ is analytic within and on closed the contour C. So by Cauchy Goursat theorem, it follows that

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} = 0, \quad n = 1, 2, ...$$

and consequently the above Laurent series of f reduces to the Taylor series of f in the domain $|z-z_0| < R_2$.

Example . The function $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ has two singularities $z = 1$ and $z = 2$. So this function is analytic in three distinct domains $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$. Let us find find series of f valid (convergent) in the three domains.

Solution . (i) For $|z| < 1$, we have $|z/2| < 1$. It follows that

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\frac{1}{1-z/2} + \frac{1}{1-z} = -\frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}(1-2^{-n-1})z^n$$

It is Taylor series of the given function in the domain $|z| < 1$. Equivalently, it is Laurent series with all $b_n = 0$.

(ii) For $1 < |z| < 2$, we have $|z/2| < 1$ and $|1/z| < 1$. It follows that

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z} = -\frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = -\frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n - \sum_{n=1}^{\infty} z^{-n}$$

It is Laurent series of the given function in the domain $1 < |z| < 2$.

(iii) For $|z| > 2$, we have $|2/z| < 1$ and $|1/z| < 1$. It follows that

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z}\frac{1}{1-2/z} - \frac{1}{z}\frac{1}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty}(2/z)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=1}^{\infty}(2^n - 1)z^{-n}$$

It is Laurent series of the given function in the domain $1 < |z| < 2$.

Chapter 6

Isolated Singularity

We know that if a function fails to be analytic at z_0 but is analytic at some point in each neighbourhood of z_0, then z_0 is a singularity of f. Further, if f is analytic in some deleted neighbourhood of z_0, then z_0 is said to be isolated singularity of f. In other words, z_0 is isolated singularity of f provided there exists some neighbourhood of z_0 that contains no singularity of f other than z_0.

Ex. The function $\frac{1}{(z-1)(z-2)}$ has two singularities namely $z = 1$ and $z = 2$. Both are isolated since singularity free deleted neighbourhoods can easily be constructed for both the singularities. For instance, $0 < |z - 1| < 1/2$ and $0 < |z - 2| < 1/2$ are deleted neighbourhoods of $z = 1$ and $z = 2$ respectively, not containing any other singularity of f.

Ex. Consider the function $\frac{1}{\sin(\pi/z)}$. We see that $z = 0$ is a singularity of this function as it is not defined at $z = 0$. The other singularities are given by $\sin(\pi/z) = 0$ or $\pi/z = n\pi$ or $z = 1/n$ where n is any non-zero integer. Thus, the given function has infinitely many singularities. The singularities $z = 1/n$ are all isolated. But $z = 0$ is not isolated as each neighbourhood of 0 carries infinitely many singularities of the type $1/n$.

Types of Isolated Singularities

Let z_0 be an isolated singularity of a function f. Then there exists some deleted neighbourhood of z_0, say $0 < |z - z_0| < R$, in which f is analytic. So f is analytic in the ring shaped annular region $0 < |z - z_0| < R$ (We may treat $z = 0$ as the point circle). So Laurent series of f in $0 < |z - z_0| < R$ reads as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, ...$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}, \quad n = 1, 2, ...$$

and C is any positively oriented simple closed contour inside $0 < |z - z_0| < R$ around z_0.

The first part of Laurent series namely $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called regular part while the second part $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called principal part. Depending on the number terms in the principal part of the Laurent series, we classify the isolated singularity in the following three ways.

Removable singularity

If the principal part of Laurent series contains no term, that is, $b_n = 0$ for all n, then z_0 is called removable singularity of f.

Ex. Consider the function $\frac{\sin z}{z}$. Obviously, 0 is an isolated singularity of $\frac{\sin z}{z}$. Next, we have

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + ...\right) = 1 - \frac{z^2}{3!} + ...$$

the Laurent series of $\frac{\sin z}{z}$ valid in the domain $0 < |z| < \infty$. Note that we cancelled z in the series expansion since $z \neq 0$ in the domain $0 < |z| < \infty$. We see that principal part of the Laurent series contains no term, and therefore 0 is a removable singularity of $\frac{\sin z}{z}$. If we define the value of $\frac{\sin z}{z}$ equal to 1 at $z = 0$, then 0 is no more a singularity of $\frac{\sin z}{z}$. In other words the singularity is removed by redefining the function. That is why the name removable singularity is there.

Pole

If the principal part of Laurent series contains finite number of terms, that is, there exists some positive integer m such that $b_m = 0$ and $b_n = 0$ for all $n > m$, then z_0 is called pole order m of f. A pole of order 1 is called simple pole.

Ex. Consider the function $f(z) = \frac{1}{(z-1)^2}$. We see that $z = 1$ is the only singularity of f. Also, $f(z) = (z-1)^{-2}$ is already in the Laurent series form about $z = 1$ with $b_1 = 0$, $b_2 = 1$ and $b_n = 0$ for all $n > 2$. So $z = 1$ is a pole of order 2.

Note: Let $f(z) = g(z)/(z-z_0)^m$, where $g(z)$ is non-zero and analytic at z_0. Then z_0 is a pole of order m of $f(z)$. For, $g(z)$ being analytic at z_0, its Taylor series about z_0 reads as

$$g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + ... + a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + ...$$

$$\therefore \quad f(z) = a_0(z-z_0)^{-m} + a_1(z-z_0)^{-m+1} + ... + a_m + a_{m+1}(z-z_0) + ...$$

Now $g(z_0) = a_0 \neq 0$ since $g(z)$ is non-zero at z_0. So from Laurent series of $f(z)$, we observe that $z = z_0$ is a pole of order m.

Ex. The function $f(z) = \frac{1}{(z-1)(z-3)^2}$ has two isolated singularities namely 1 and 3. To see the behavior of the singularity 1, we rewrite $f(z)$ as

$$f(z) = \frac{1/(z-3)^2}{z-1},$$

and notice that $1/(z-3)^2$ is analytic and non-zero at $z = 1$. So $z = 1$ is a simple pole.

Again, rewriting $f(z)$ as

$$f(z) = \frac{1/(z-1)}{(z-3)^2},$$

we see that $1/(z-1)$ is analytic and non-zero at $z = 3$. So $z = 3$ is a pole of order 2.

Essential singularity

If the principal part of Laurent series contains infinitely number of terms, then z_0 is called an essential singularity of f.

Ex. The function e^{1/z^2} has an isolated singularity at $z = 0$. The Laurent series of e^{1/z^2} about $z = 0$ is given by

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \dots\dots\dots\dots$$

We see that there are infinite number of terms with negative powers of z. Hence, 0 is an essential singularity of e^{1/z^2}.

Residue at Isolated Singularity

In Laurent series of $f(z)$ about the isolated singularity $z = z_0$, the coefficient of $(z - z_0)^{-1}$ is given by

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz.$$

It is called residue of f at z_0 and we shall denote it by $\text{Res}[f(z), z_0]$. Therefore, we have

$$\text{Res}[f(z), z_0] = b_1 = \frac{1}{2\pi i}\int_C f(z)dz.$$

It can be rewritten as

$$\int_C f(z)dz = 2\pi i\ \text{Res}[f(z), z_0] = 2\pi i b_1.$$

This shows that the value of b_1 or $\text{Res}[f(z), z_0]$ can readily provide us the value of the contour integral $\int_C f(z)dz$.

Ex. Evaluate the contour integral $\int_C e^{1/z^2}dz$, $C : |z| = 1$.

Sol. The function e^{1/z^2} has only one singularity namely $z = 0$ inside C. The Laurent series of e^{1/z^2} is given by

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \dots\dots\dots\dots$$

We see that the coefficient of z^{-1} is 0, that is, $b_1 = 0$. Hence, the value of the given integral is 0.

Remark. The above example shows that if integral of a function along a simple closed contour is 0, the function need not be analytic. On the other hand, if a function is analytic within and on a simple closed contour C, then the integral of the function along C is 0, by Cauchy-Goursat theorem.

Residue at pole

If z_0 is a pole of order m, then Laurent series of $f(z)$ reads as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + ... + b_m(z-z_0)^{-m}.$$

Multiplying both sides by $(z-z_0)^m$, we have

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + ... + b_m.$$

Differentiating both sides $m-1$ times with respect z, and evaluating the resulting two sides at $z = z_0$, we get

$$\phi^{m-1}(z) = b_1(m-1)!,$$

where $\phi(z) = (z-z_0)^m f(z)$.

$$\therefore \quad \text{Res}[f(z), z_0] = b_1 = \frac{\phi^{m-1}(z)}{(m-1)!}$$

Ex. Find the residues at the poles of $f(z) = \frac{1}{(z-1)(z-3)^2}$.
Sol. The given function has two poles namely 1 and 3 of orders 1 and 2, respectively.
Residue at the simple pole $z = 1$
In this case, we have $\phi(z) = (z-1)f(z) = 1/(z-3)^2$. So $\phi(1) = 1/4$ and

$$\text{Res}[f(z), 1] = \frac{\phi(1)}{0!} = 1/4.$$

Residue at $z = 3$
In this case, we have $\phi(z) = (z-3)^2 f(z) = 1/(z-1)$. So $\phi'(z) = -1/(z-1)^2$, $\phi'(3) = -1/4$ and

$$\text{Res}[f(z), 3] = \frac{\phi'(3)}{1!} = -1/4.$$

Cauchy Residue Theorem

If a function is analytic within and on a positively oriented simple closed contour C except some finite number of singular points, say z_1, z_2, ..., z_n inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k].$$

Proof. The singularities z_1, z_2, ..., z_n being finite in number are isolated. Let us enclose these singularities inside non-overlapping n circles $C_k : |z-z_k|$, $(k = 1, 2, ..., n)$ completely lying inside C. Then f is analytic inside the contour C and outside the circles C_k. So by Cauchy-Goursat theorem, we have

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]$$

since

$$\int_{C_k} f(z)dz = 2\pi i \,\text{Res}[f(z), z_k], \qquad (k = 1, 2, ..., n).$$

Remark. The Cauchy Residue Theorem is very useful for calculating contour integral of a function having finite number of singularities inside the contour.

Ex. Use Cauchy residue theorem to evaluate $\displaystyle\int_C \frac{1}{(z-1)(z-3)^2}dz$, $C : |z| = 4$.

Sol. We see that the integrand $f(z) = \frac{1}{(z-1)(z-3)^2}$ has two poles namely 1 and 3, both lying in the interior of the circle $C : |z| = 4$. So by Cauchy residue theorem

$$\int_C \frac{1}{(z-1)(z-3)^2}dz = 2\pi i[\text{Res}[f(z), 1] + \text{Res}[f(z), 3]] = 2\pi i(1/4 - 1/4) = 0$$

Ex. Use Cauchy residue theorem to evaluate $\displaystyle\int_C \frac{1}{(z-1)(z-3)^2}dz$, $C : |z| = 2$.

Sol. In this case, the pole 1 lies inside $C : |z| = 2$ while the pole 3 lies outside it. So by Cauchy residue theorem

$$\int_C \frac{1}{(z-1)(z-3)^2}dz = 2\pi i\text{Res}[f(z), 1] = 2\pi i(1/4) = \pi i/2.$$

Caution. Observe carefully the positions of singularities with respect to the given contour C. In the formula of Cauchy residue theorem, use residues at only those singularities which lie inside C.

Ex. Use Cauchy residue theorem to evaluate $\displaystyle\int_C \frac{5z-2}{z(z-1)}dz$, $C : |z| = 2$.

Sol. We see that the integrand $f(z) = \frac{5z-2}{z(z-1)}$ has two simple poles, namely 0 and 1 both lying inside the circle $C : |z - 1| = 1/2$. So by Cauchy residue theorem, we have

$$\int_C f(z)dz = 2\pi i\ [\text{Res}[f(z), 0] + \text{Res}[f(z), 1]]$$

Residue at $z = 0$

In this case, we have $\phi(z) = zf(z) = \frac{5z-2}{z-1}$. So $\phi(0) = 2$ and

$$\text{Res}[f(z), 1] = \frac{\phi(0)}{0!} = 2.$$

Residue at $z = 3$

In this case, we have $\phi(z) = (z-1)f(z) = \frac{5z-2}{z}$. So $\phi(1) = 3$ and

$$\text{Res}[f(z), 1] = \frac{\phi(1)}{0!} = 3.$$

$$\therefore \quad \int_C f(z)dz = 2\pi i(2 + 3) = 10\pi i.$$

Residue at Infinity

Sometimes it is useful to include the point at ∞ in the complex plane, which we call the extended complex plane. If a function f is analytic in the domain $R_1 < |z| < \infty$, then ∞ is said to be an isolated singularity of f. Further, residue of f at ∞ is defined as

$$\text{Res}[f(z),\infty] = \frac{1}{2\pi i}\int_{C_0} f(z)dz, \tag{13}$$

where $C_0 : |z| = R_0$ $(R_0 > R_1)$ is a negatively oriented circle.

Next assume that f is analytic in the entire complex plane except some finite number of singular points inside a positively oriented simple closed contour C that lies in the interior of $C_1 : |z| = R_1$. Then f is analytic in the domain between C and C_0. Therefore by principle of the deformation of path

$$\int_C f(z)dz = -\int_{C_0} f(z)dz = -2\pi i\text{Res}[f(z),\infty]. \tag{14}$$

Here minus sign appears since C_0 is negatively oriented.

The function f is analytic in $R_1 \leq |z| < \infty$ with isolated singularity at ∞. So the function $f(1/w)$ is analytic in $0 < |w| \leq 1/R_1$ with isolated singularity at $w = 0$. Now let us use the transformation $z = \frac{1}{w}$ in $\int_{C_0} f(z)dz$ so that

$$\int_{C_0} f(z)dz = -\int_{C_*} \frac{1}{w^2} f\left(\frac{1}{w}\right) dw. \tag{15}$$

where $C_* : |w| = 1/R_1$ is a circle with positive orientation since the transformation $z = 1/w$ changes the orientation. Now $\frac{1}{w^2} f\left(\frac{1}{w}\right)$ is analytic within and on the positively oriented circle C_* except 0. So by Cauchy residue theorem

$$\int_{C_*} \frac{1}{w^2} f\left(\frac{1}{w}\right) dw = 2\pi i\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right]. \tag{16}$$

From (14), (15) and (16), we get

$$\text{Res}[f(z),\infty] = -\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right]. \tag{17}$$

and,

$$\int_C f(z)dz = 2\pi i\text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

This formula is more economical than the Cauchy residue theorem for evaluating the integral of f along the contour C as it involves the calculation of single residue. Remember the hypothesis for this formula that the function f is analytic in the entire complex plane except a finite number of points inside the positively oriented simple closed contour C.

Ex. Evaluate $\int_C \frac{5z-2}{z(z-1)}dz$, $C : |z| = 2$.

Sol. We see that the integrand $f(z) = \frac{5z-2}{z(z-1)}$ is analytic in the entire complex plane except the two singularities, namely 0 and 1, which lie inside the circle $C : |z| = 2$. So we have

$$\int_C f(z)dz = 2\pi i \text{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right].$$

Now,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)}.$$

We see that 0 is a simple pole of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$. Here $\phi(z) = z.\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{1-z}$ and so $\phi(0) = 5$. It follows that

$$\int_C f(z)dz = 2\pi i(5) = 10\pi i.$$

Zeros and Poles

Let a function f be analytic at z_0. Then z_0 is said to be a zero of order m of f if $f(z_0) = 0, f^{(n)}(z_0) = 0$ $(n = 1, 2, ..., m-1)$ and $f^{(m)}(z_0) = 0$. For example, $z = 1$ is a zero of order 2 of the function $f(z) = (z-1)^2$ since $f(1) = 0$, $f'(1) = 0$ and $f''(1) = 2 \neq 0$.

Further, it can be shown that f has a zero z_0 of order m if and only if $f(z) = (z-z_0)^m g(z)$, where $g(z)$ is analytic and non-zero at z_0.

We know that a function f having a pole z_0 of order m can be written in the form $f(z) = g(z)/(z-z_0)^m$, where $g(z)$ is analytic and non-zero at z_0. Then the function $h(z) = 1/g(z)$ is also analytic and non-zero at z_0. It then implies that the function $1/f(z) = (z-z_0)^m h(z)$ has a zero of order m at z_0.

More generally, we can conclude that if two functions p and q are analytic at z_0 such that $p(z_0) \neq 0$ and q has a zero of order m at z_0, then the function p/q has a pole of order m at z_0. In case, $m = 1$, that is, z_0 is a simple pole of p/q, then it can easily be shown that

$$\text{Res}\left[p(z)/q(z), z_0\right] = p(z_0)/q'(z_0)$$

Ex. Find residues at the poles of $\cot z$.

Sol. We have $\cot z = \cos z/\sin z$. The singularities of $\cot z$ are given by $\sin z = 0$ or $z = n\pi$, where n is any integer. For each integer value of n, the singularity $z = n\pi$ is a simple pole of $\cot z$. Also, $\cos z$ is analytic and non-zero at $z = n\pi$ as $\cos n\pi = (-1)^n$. So residue at $z = n\pi$ is given by

$$\text{Res}\left[\cot z, n\pi\right] = \text{Res}\left[\cos z/\sin z, n\pi\right] = \cos n\pi/\cos n\pi = 1.$$

Ex. We know $z = \pi$ is a zero of $\sin z$. So $\sin z = (z-\pi)g(z)$, where $g(z)$ is analytic and non-zero at $z = \pi$. Determine $g(z)$.

Sol. It is given by

$$g(z) = \sin z/(z-\pi) \text{ for } z \neq \pi \text{ and } g(\pi) = -1.$$

Note that the value $g(\pi) = -1$ is defined so that $g(z)$ is analytic at $z = \pi$.