Math 135-2, Homework 3

Solutions

Problem 53.4

Use the methods of both Examples 1 and 2 to solve each of the following differential equations:

(a) $\boldsymbol{y'' + 5y' + 6y = 5e^{3t}}$, $\boldsymbol{y(0) = y'(0) = 0}$.

(a) First, lets use (13).

$$\begin{aligned}
L[A(t)] &= \frac{1}{p(p^2+5p+6)} \\
&= \frac{1}{p(p+3)(p+2)} \\
&= \frac{B}{p} + \frac{C}{p+2} + \frac{D}{p+3} \\
1 &= B(p+2)(p+3) + Cp(p+3) + Dp(p+2) \\
1 &= 6B \qquad p = 0 \\
1 &= -2C \qquad p = -2 \\
1 &= 3D \qquad p = -3 \\
L[A(t)] &= \frac{1}{6p} - \frac{1}{2(p+2)} + \frac{1}{3(p+3)} \\
A(t) &= \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \\
f(t) &= 5e^{3t} \\
f'(t) &= 15e^{3t} \\
y(t) &= \int_0^t A(t-\tau)f'(\tau)\,d\tau + f(0)A(t) \\
&= \int_0^t \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-\tau)} + \frac{1}{3}e^{-3(t-\tau)}\right)(15e^{3\tau})\,d\tau + 5\left(\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right) \\
&= \int_0^t \frac{5}{2}e^{3\tau} - \frac{15}{2}e^{-2t+5\tau} + 5e^{-3t+6\tau}\,d\tau + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t} \\
&= \left[\frac{5}{6}e^{3\tau} - \frac{3}{2}e^{-2t+5\tau} + \frac{5}{6}e^{-3t+6\tau}\right]_0^t + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t} \\
&= \frac{5}{6}(e^{3t}-1) - \frac{3}{2}(e^{3t} - e^{-2t}) + \frac{5}{6}(e^{3t} - e^{-3t}) + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t} \\
&= \frac{1}{6}e^{3t} - e^{-2t} + \frac{5}{6}e^{-3t}
\end{aligned}$$

Next, lets repeat with (12).

$$
\begin{aligned}
L[h(t)] &= \frac{1}{p^2+5p+6} \\
&= \frac{1}{(p+3)(p+2)} \\
&= \frac{C}{p+2} + \frac{D}{p+3} \\
1 &= C(p+3) + D(p+2) \\
1 &= C \qquad p=-2 \\
1 &= -D \qquad p=-3 \\
L[h(t)] &= \frac{1}{p+2} - \frac{1}{p+3} \\
h(t) &= e^{-2t} - e^{-3t} \\
f(t) &= 5e^{3t} \\
y(t) &= \int_0^t h(t-\tau) f(\tau)\, d\tau \\
&= \int_0^t \left(e^{-2(t-\tau)} - e^{-3(t-\tau)}\right)(5e^{3\tau})\, d\tau \\
&= \int_0^t 5e^{-2t+5\tau} - 5e^{-3t+6\tau}\, d\tau \\
&= \left[e^{-2t+5\tau} - \frac{5}{6}e^{-3t+6\tau}\right]_0^t \\
&= e^{3t} - e^{-2t} - \frac{5}{6}e^{3t} + \frac{5}{6}e^{-3t} \\
&= \frac{1}{6}e^{3t} - e^{-2t} + \frac{5}{6}e^{-3t}
\end{aligned}
$$

Problem 53.8

The current $I(t)$ in an electric circuit with inductance L and resistance R is given by the equation (4) in Section 13:

$$\boldsymbol{L\frac{dI}{dt} + RI = E(t),}$$

where $E(t)$ is the impressed electromotive force. If $I(0) = 0$, use the methods of this section to find $I(t)$ in each of the following cases:
(a) $E(t) = E_0 u(t)$
(b) $E(t) = E_0 \delta(t)$
(c) $E(t) = E_0 \sin \omega t$

$$
\begin{aligned}
L[h(t)] &= \frac{1}{Lp+R} \\
&= \frac{1}{L}\frac{1}{p+R/L} \\
h(t) &= \frac{1}{L}e^{-Rt/L} \\
I(t) &= \int_0^t h(t-\tau)E(\tau)\,d\tau \\
&= \int_0^t \frac{1}{L}e^{-R(t-\tau)/L}E(\tau)\,d\tau \\
&= \frac{1}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}E(\tau)\,d\tau \\
I(t) &= \frac{1}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}E_0 u(\tau)\,d\tau \qquad \text{part (a)} \\
&= \frac{E_0}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}\,d\tau \\
&= \frac{E_0}{L}e^{-Rt/L}\left[\frac{L}{R}e^{R\tau/L}\right]_0^t \\
&= \frac{E_0}{L}e^{-Rt/L}\left(\frac{L}{R}e^{Rt/L}-\frac{L}{R}\right) \\
&= \frac{E_0}{R}\Big(1-e^{-Rt/L}\Big) \\
I(t) &= \frac{1}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}E_0\delta(\tau)\,d\tau \qquad \text{part (b)} \\
&= \frac{E_0}{L}e^{-Rt/L} \\
I(t) &= \frac{1}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}E_0\sin\omega\tau\,d\tau \qquad \text{part (c)} \\
&= \frac{E_0}{L}e^{-Rt/L}\int_0^t e^{R\tau/L}\sin\omega\tau\,d\tau \\
&= \frac{E_0}{L}e^{-Rt/L}\mathrm{Im}\bigg(\int_0^t e^{(R/L+i\omega)\tau}\,d\tau\bigg) \\
&= \frac{E_0}{L}e^{-Rt/L}\mathrm{Im}\Bigg(\bigg[\frac{1}{R/L+i\omega}e^{(R/L+i\omega)\tau}\bigg]_0^t\Bigg) \\
&= \frac{E_0}{L}e^{-Rt/L}\mathrm{Im}\bigg(\frac{R/L-i\omega}{(R/L)^2+\omega^2}\Big(e^{(R/L+i\omega)t}-1\Big)\bigg) \\
&= \frac{E_0}{L((R/L)^2+\omega^2)}e^{-Rt/L}\mathrm{Im}\Big((R/L-i\omega)\Big(e^{Rt/L}\cos\omega t+ie^{Rt/L}\sin\omega t-1\Big)\Big) \\
&= \frac{E_0}{L((R/L)^2+\omega^2)}e^{-Rt/L}\bigg(\frac{R}{L}e^{Rt/L}\sin\omega t-\omega\Big(e^{Rt/L}\cos\omega t-1\Big)\bigg) \\
&= \frac{E_0}{R^2+L^2\omega^2}\Big(R\sin\omega t-L\omega\cos\omega t+L\omega e^{-Rt/L}\Big)
\end{aligned}
$$

Problem 69.2

Show that $f(x, y) = y^{1/2}$
(a) does not satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $0 \leq y \leq 1$.
(b) does satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $c \leq y \leq d$ where $0 < c < d$.

(a) Let $y_1 = 0$ and $y_2 = \varepsilon$.

$$\begin{aligned}
\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} &= \frac{\sqrt{y_1} - \sqrt{y_2}}{y_1 - y_2} \\
&= \frac{1}{\sqrt{y_1} + \sqrt{y_2}} \\
&= \frac{1}{\sqrt{\epsilon}}
\end{aligned}$$

which is unbounded.
(b) Noting $y_1, y_2 \geq c > 0$,

$$\begin{aligned}
\left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| &= \left|\frac{\sqrt{y_1} - \sqrt{y_2}}{y_1 - y_2}\right| \\
&= \frac{1}{\sqrt{y_1} + \sqrt{y_2}} \\
&\leq \frac{1}{2\sqrt{c}} \\
&= R
\end{aligned}$$

provides a bound.

Problem 69.4

Show that $f(x, y) = xy^2$
(a) satisfies a Lipschitz condition on the rectangle $a \leq x \leq b$ and $c \leq y \leq d$.
(b) does not satisfy a Lipschitz condition on any strip $a \leq x \leq b$ and $-\infty \leq y \leq \infty$.

(a) Note that $|x| \leq \max(|a|, |b|) = A$ and $|y| \leq \max(|c|, |d|) = C$.

$$\begin{aligned}
\left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| &= \left|\frac{xy_1^2 - xy_2^2}{y_1 - y_2}\right| \\
&= |x(y_1 + y_2)| \\
&\leq |x|(|y_1| + |y_2|) \\
&\leq 2AC
\end{aligned}$$

is a bound.
(b) Choose any $x \neq 0$ (possible unless $a = b$), $y_1 = 0$, and $y_2 \to \infty$.

$$\begin{aligned}
\left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| &= \left|\frac{xy_1^2 - xy_2^2}{y_1 - y_2}\right| \\
&= |x(y_1 + y_2)| \\
&= |x|y_2 \\
&\to \infty
\end{aligned}$$

is unbounded.

Problem A

The problem $yy' = 1$, $y(0) = 0$ seems like it should have no solution. Show that it actually has two solutions. How is this possible? This demonstrates that plugging the initial conditions into an ODE and producing a contradiction does not suffice to show that there is no solution.

$$\begin{aligned} yy' &= 1 \\ \frac{1}{2}y^2 &= x + c \\ y &= \pm\sqrt{2(x+c)} \\ 0 = y(0) &= \pm\sqrt{2c} \\ y &= \pm\sqrt{2x} \end{aligned}$$

The two solutions are $y = \sqrt{2x}$ and $y = -\sqrt{2x}$. Plugging $y = 0$ into $yy' = 1$ and deriving a contradiction implicitly assumes that y' is finite, which it is not. In actuality, plugging in $y = 0$ produces $0 \cdot \infty$, which is indeterminate.

Problem B

Consider the ODE $x^3y' = 2y$.
(a) Find all solutions if $y(0) = 0$.
(b) Find all solutions if $y(0) = 1$.

(a) First, lets find the general solution. The equation is separable.

$$\begin{aligned} x^3y' &= 2y \\ y^{-1}y' &= 2x^{-3} \\ \ln|y| &= -x^{-2} + c_0 \\ y &= c_1e^{-x^{-2}} \end{aligned}$$

This satisfies is a solution for any c_1. We have not lost any solutions by dividing by zero, since $y = 0$ is captured by $c_1 = 0$.
(b) We also know that $y(0) = 1$ is not possible since we have already worked out the general solution. Note that it is not sufficient to plug $x = 0$ and $y = 1$ into the ODE to derive a contradiction, as demonstrated by Problem A.

Problem C

Find the Lipschitz constant (or show that it does not have one) for each of the following functions on the indicated interval. (The Lipschitz constant is a *tight* bound for the Lipschitz condition.)
(a) $\cos x \sin x$, $(-\infty, \infty)$
(b) $|\sin x|$, $(-\infty, \infty)$

Note that if $f(x)$ is differentiable on some interval $[a, b]$, then the Lipschitz constant L for that interval is obtained by looking at its derivative.

$$\frac{f(a)-f(b)}{a-b} = f'(c) \qquad \text{for some } c \in [a,b]$$

Thus, any value of this fraction that can be obtained is also obtained by the derivative somewhere in the interval. What is more, the derivative is obtained in the limit $a \to c$ and $b \to c$, so

$$\begin{aligned} L &= \sup_{a\le x<y\le b} \left|\frac{f(x)-f(y)}{x-y}\right| \\ &= \sup_{a\le z\le b} |f'(z)| \end{aligned}$$

(a) This one is differentiable. $L = \max_x |f'(x)| = \max_x |\cos 2x| = 1$.
(b) From $|x| = |y+(x-y)| \le |y|+|x-y|$ and $|y| = |x+(y-x)| \le |x|+|y-x|$ we deduce $||x|-|y|| \le |x-y|$. Then,

$$\begin{aligned} \left|\frac{f(x)-f(y)}{x-y}\right| &= \left|\frac{|\sin x|-|\sin y|}{x-y}\right| \\ &\le \left|\frac{\sin x-\sin y}{x-y}\right| \\ &\le \sup_z |\cos z| \\ &= 1 \end{aligned}$$

Now, I need to show that $L=1$ is tight. This follows from

$$\begin{aligned} \lim_{\epsilon\to 0^+} \left|\frac{f(\epsilon)-f(0)}{\epsilon-0}\right| &= \lim_{\epsilon\to 0^+} \left|\frac{|\sin \epsilon|-|\sin 0|}{\epsilon-0}\right| \\ &= \lim_{\epsilon\to 0^+} \frac{\sin\epsilon}{\epsilon} \\ &= 1 \end{aligned}$$

Problem D

Derive the time delay rule

$$\boldsymbol{L[u(x-a)f(x-a)] = e^{-ap}F(p).}$$

For which choices a is this rule valid?

$$
\begin{aligned}
L[u(x-a)f(x-a)] &= \int_0^\infty e^{-px}u(x-a)f(x-a)\,dx \\
&= \int_{-a}^\infty e^{-p(z+a)}u(z)f(z)\,dz \qquad x = z+a \\
&= \int_0^\infty e^{-p(z+a)}u(z)f(z)\,dz + \int_{-a}^0 e^{-p(z+a)}u(z)f(z)\,dz \\
&= \int_0^\infty e^{-p(z+a)}f(z)\,dz + \int_{-a}^0 e^{-p(z+a)}u(z)f(z)\,dz \\
&= e^{-ap}\int_0^\infty e^{-pz}f(z)\,dz + \int_{-a}^0 e^{-p(z+a)}u(z)f(z)\,dz \\
&= e^{-ap}F(p) + \int_{-a}^0 e^{-p(z+a)}u(z)f(z)\,dz
\end{aligned}
$$

If $a \geq 0$, then the remaining integral is over negative values of z, for which $u(z) = 0$. Thus, we will have the desired identity. If $a < 0$, then the remaining integral will in general be nonzero, since all three factors will generally be nonzero. Thus, the identity is true only for $a \geq 0$.