Math 135-2, Homework 1

Solutions

Problem 17.1

Find the general solution of each of the following equations:
(p) $16y'' - 8y' + y = 0$
(q) $y'' + 4y' + 5y = 0$
(r) $y'' + 4y' - 5y = 0$

(p)

$$\begin{aligned}
0 &= 16y'' - 8y' + y \\
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2 e^{rx} \\
0 &= 16r^2 e^{rx} - 8re^{rx} + e^{rx} \\
&= (16r^2 - 8r + 1)e^{rx} \\
0 &= 16r^2 - 8r + 1 \\
&= (4r^2 - 1)^2 \\
r &= \frac{1}{4}, \frac{1}{4} \\
y &= (c_0 + c_1 x)e^{\frac{1}{4}x}
\end{aligned}$$

(q)

$$\begin{aligned}
0 &= y'' + 4y' + 5y \\
0 &= r^2 e^{rx} + 4re^{rx} + 5e^{rx} \\
&= (r^2 + 4r + 5)e^{rx} \\
0 &= r^2 + 4r + 5 \\
&= (r + 2)^2 + 1 \\
r &= -2 \pm i \\
y &= c_0 e^{(-2+i)x} + c_1 e^{(-2-i)x} = (c_2 \cos x + c_3 \sin x)e^{-2x}
\end{aligned}$$

(r)

$$
\begin{aligned}
0 &= y'' + 4y' - 5y \\
0 &= r^2 e^{rx} + 4re^{rx} - 5e^{rx} \\
&= (r^2 + 4r - 5)e^{rx} \\
0 &= r^2 + 4r - 5 \\
&= (r+5)(r-1) \\
r &= -1, -5 \\
y &= c_0 e^{-x} + c_1 e^{-5x}
\end{aligned}
$$

Problem 17.5

The equation

$$\boldsymbol{x^2 y'' + pxy' + qy = 0,}$$

where p and q are constants, is called *Euler's equidimensional equation.* Show that the change of independent variable given by $x = e^z$ transforms it into an equation with constant coefficients, and apply this technique to find the general solution of each of the following equations:

(a) $x^2y'' + 3xy' + 10y = 0$

Some degree of care is helpful here. Let $\hat{y}(z) = y(x) = y(e^z)$, where I have avoided calling both functions y since they are in fact different functions.

$$
\begin{aligned}
\hat{y}' &= \frac{d}{dz}(y(e^z)) \\
&= y'(e^z)e^z \\
xy' &= \hat{y}' \\
\hat{y}'' &= \frac{d}{dz}(y'(e^z)e^z) \\
&= y''(e^z)e^z e^z + y'(e^z)e^z \\
&= y''(x)x^2 + y'(x)x \\
x^2y'' &= \hat{y}'' - y'(x)x \\
&= \hat{y}'' - \hat{y}' \\
0 &= x^2y'' + pxy' + qy \\
&= \hat{y}'' - \hat{y}' + p\hat{y}' + q\hat{y} \\
&= \hat{y}'' + (p-1)\hat{y}' + q\hat{y}
\end{aligned}
$$

As expected, this equation has constant coefficients.

$$\begin{aligned}
0 &= x^2y'' + 3xy' + 10y \\
&= \hat{y}'' - \hat{y}' + 3\hat{y}' + 10\hat{y} \\
&= \hat{y}'' + 2\hat{y}' + 10\hat{y} \\
\hat{y} &= e^{rz} \\
0 &= r^2 + 2r + 10 \\
r &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 10}}{2} \\
&= \frac{-2 \pm \sqrt{-36}}{2} \\
&= -1 \pm 3i \\
\hat{y}(z) &= (c_0 \cos(3z) + c_1 \sin(3z))e^{-z} \\
y(x) &= (c_0 \cos(3 \ln x) + c_1 \sin(3 \ln x))e^{-\ln x} \qquad z = \ln x \\
y(x) &= (c_0 \cos(3 \ln x) + c_1 \sin(3 \ln x))x^{-1}
\end{aligned}$$

Note that this process in equivalent to choosing $y = x^r$ as a trial function rather than $\hat{y} = e^{rz}$.

Problem 18.3

If $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + P(x)y' + Q(x)y = R_1(x)$$

and

$$y'' + P(x)y' + Q(x)y = R_2(x),$$

show that $y(x) = y_1(x) + y_2(x)$ is a solution of

$$y'' + P(x)y' + Q(x)y = R_1(x) + R_2(x).$$

This is called the principle of *superposition*. Use this principle to find the general solution of (b) $y'' + 9y = 2\sin 3x + 4\sin x - 26e^{-2x} + 27x^3$

$$\begin{aligned}
y'' + P(x)y' + Q(x)y &= (y_1 + y_2)'' + P(x)(y_1 + y_2)' + Q(x)(y_1 + y_2) \\
&= (y_1'' + P(x)y_1' + Q(x)y_1) + (y_2'' + P(x)y_2' + Q(x)y_2) \\
&= R_1(x) + R_2(x)
\end{aligned}$$

Next, lets tackle the homogeneous problem

$$\begin{aligned}
0 &= y'' + 9y \\
y &= e^{rt} \\
0 &= (r^2 + 9)e^{rt} \\
r &= \pm 3i \\
y &= c_0 \cos 3x + c_1 \sin 3x
\end{aligned}$$

Finally, lets deal with the right hand sides one by one. First $2\sin 3x$. For this one, note that we will need to introduce a factor of x.

$$\begin{aligned}
y &= ax\sin 3x + bx\cos 3x \\
y' &= a\sin 3x + 3ax\cos 3x + b\cos 3x - 3bx\sin 3x \\
y'' &= 6a\cos 3x - 9ax\sin 3x - 6b\sin 3x - 9bx\cos 3x \\
y'' + 9y &= (6a\cos 3x - 9ax\sin 3x - 6b\sin 3x - 9bx\cos 3x) + 9(ax\sin 3x + bx\cos 3x) \\
&= 6a\cos 3x - 6b\sin 3x \\
&= 2\sin 3x \\
a &= 0 \\
b &= -\frac{1}{3} \\
y &= -\frac{1}{3}x\cos 3x
\end{aligned}$$

Next is $4\sin x$.

$$\begin{aligned}
y &= a\sin x + b\cos x \\
y' &= a\cos x - b\sin x \\
y'' &= -a\sin x - b\cos x \\
y'' + 9y &= (-a\sin x - b\cos x) + 9(a\sin x + b\cos x) \\
&= 8a\sin x + 8b\cos x \\
&= 4\sin x \\
a &= \frac{1}{2} \\
b &= 0 \\
y &= \frac{1}{2}\sin x
\end{aligned}$$

Now for $-26e^{-2x}$.

$$\begin{aligned}
y &= ae^{-2x} \\
y' &= -2ae^{-2x} \\
y'' &= 4ae^{-2x} \\
y'' + 9y &= 13ae^{-2x} \\
&= -26e^{-2x} \\
a &= -2 \\
y &= -2e^{-2x}
\end{aligned}$$

Finally, lets do $27x^3$.

$$\begin{aligned}
y &= ax^3 + bx^2 + cx + d \\
y' &= 3ax^2 + 2bx + c \\
y'' &= 6ax + 2b \\
y'' + 9y &= (6ax + 2b) + 9(ax^3 + bx^2 + cx + d) \\
&= 9ax^3 + 9bx^2 + (9c + 6a)x + (9d + 2b) \\
&= 27x^3 \\
a &= 3 \\
b &= 0 \\
c &= -2 \\
d &= 0 \\
y &= 3x^3 - 2x
\end{aligned}$$

Putting this all together,

$$y = c_0 \cos 3x + c_1 \sin 3x - \frac{1}{3} x \cos 3x + \frac{1}{2} \sin x - 2e^{-2x} + 3x^3 - 2x$$

Problem 48.1

Evaluate the integrals in (8), (9), (11), (12), and (13).

$$\begin{aligned}
L[1] &= \int_0^\infty e^{-px}\,dx \\
&= \left[-\frac{1}{p} e^{-px}\right]_0^\infty \\
&= -\frac{1}{p} \lim_{x\to\infty} e^{-px} + \frac{1}{p} \\
&= \frac{1}{p} \qquad \text{(limit requires } p > 0\text{)}
\end{aligned}$$

$$\begin{aligned}
L[x] &= \int_0^\infty e^{-px} x\,dx \\
&= \left[\left(-\frac{1}{p}\right) e^{-px} x\right]_0^\infty - \int_0^\infty \left(-\frac{1}{p}\right) e^{-px}\,dx \qquad du = e^{-px}\,dx, v = x \\
&= \frac{1}{p} \int_0^\infty e^{-px}\,dx \qquad \text{(limit requires } p > 0\text{)} \\
&= \frac{1}{p^2}
\end{aligned}$$

$$
\begin{aligned}
L[e^{ax}] &= \int_0^\infty e^{-px} e^{ax}\,dx \\
&= \int_0^\infty e^{-(p-a)x}\,dx \\
&= \left[-\frac{1}{p-a} e^{-(p-a)x}\right]_0^\infty \\
&= -\frac{1}{p-a} \lim_{x\to\infty} e^{-(p-a)x} + \frac{1}{p-a} \\
&= \frac{1}{p-a} \qquad \text{(limit requires } p > a\text{)}
\end{aligned}
$$

Let $A = L[\sin ax]$ and $B = L[\cos ax]$.

$$
\begin{aligned}
A &= L[\sin ax] \\
&= \int_0^\infty e^{-px} \sin ax\,dx \\
&= \left[\left(-\frac{1}{p}\right) e^{-px} \sin ax\right]_0^\infty - \int_0^\infty \left(-\frac{1}{p}\right) e^{-px} a \cos ax\,dx \qquad du = e^{-px}\,dx, v = x \\
&= \frac{a}{p} \int_0^\infty e^{-px} \cos ax\,dx \qquad \text{(limit requires } p > a\text{)} \\
&= \frac{a}{p} B \\
B &= L[\cos ax] \\
&= \int_0^\infty e^{-px} \cos ax\,dx \\
&= \left[\left(-\frac{1}{p}\right) e^{-px} \cos ax\right]_0^\infty - \int_0^\infty \left(-\frac{1}{p}\right) e^{-px} (-a) \sin ax\,dx \qquad du = e^{-px}\,dx, v = x \\
&= \frac{1}{p} - \frac{a}{p} \int_0^\infty e^{-px} \sin ax\,dx \qquad \text{(limit requires } p > a\text{)} \\
&= \frac{1}{p} - \frac{a}{p} A \\
&= \frac{1}{p} - \frac{a^2}{p^2} B \\
(p^2 + a^2) B &= p \\
B &= \frac{p}{p^2 + a^2} \\
A &= \frac{a}{p} B \\
&= \frac{a}{p^2 + a^2}
\end{aligned}
$$

Note that the last two could also be obtained by using $e^{iax} = \cos ax + i \sin ax$, which produces a simpler integral. In that case,

$$\begin{aligned}
L[\cos ax] + iL[\sin ax] &= L[\cos ax + i\sin ax] \\
&= L[e^{iax}] \\
&= \int_0^\infty e^{-px}e^{iax}\,dx \\
&= \int_0^\infty e^{-(p-ia)x}\,dx \\
&= \frac{1}{p-ia} \qquad \text{(limit requires } p>0) \\
&= \frac{p}{p^2+a^2} + \frac{a}{p^2+a^2}i
\end{aligned}$$

Problem 48.2

Without integrating, show that
(a) $\boldsymbol{L[\sinh ax] = \dfrac{a}{p^2-a^2},\ p>|a|}$

$$\begin{aligned}
L[\sinh ax] &= L\left[\frac{e^{ax}-e^{-ax}}{2}\right] \\
&= \frac{L[e^{ax}]-L[e^{-ax}]}{2} \\
&= \frac{\frac{1}{p-a}-\frac{1}{p+a}}{2} \\
&= \frac{a}{p^2-a^2}
\end{aligned}$$

Note that the Laplace transforms used require $p>a$ and $p>-a$, leading to $p>|a|$.

Problem 48.4

Use the formulas given in the text to find the transform of each of the following functions:
(d) $\boldsymbol{4\sin x\cos x + 2e^{-x}}$

$$\begin{aligned}
L[4\sin x\cos x + 2e^{-x}] &= L[2\sin 2x + 2e^{-x}] \\
&= 2L[\sin 2x] + 2L[e^{-x}] \\
&= \frac{4}{p^2+4} + \frac{2}{p+1}
\end{aligned}$$

Problem 48.5

Find a function $\boldsymbol{f(x)}$ **whose transform is**
(e) $\boldsymbol{\dfrac{1}{p^4+p^2}}$

$$\begin{aligned}
\frac{1}{p^4+p^2} &= \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^2+1} \\
1 &= Ap(p^2+1) + B(p^2+1) + Cp^2 \\
1 &= B \qquad p = 0 \\
1 &= -C \qquad p = i \\
1 &= 2A + 2B + C \qquad p = 1 \\
0 &= 2A \qquad p = 1 \\
\frac{1}{p^4+p^2} &= \frac{1}{p^2} - \frac{1}{p^2+1} \\
f(x) &= x - \sin x
\end{aligned}$$

Problem 49.2

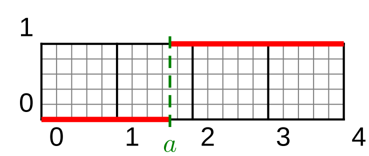
In each of the following cases, graph the function and find its Laplace transform:
(a) $f(x) = u(x-a)$ where a is a positive number and $u(x)$ is the unit step function defined by

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

(b) $f(x) = [x]$ where $[x]$ denotes the greatest integer $\leq x$
(c) $f(x) = x - [x]$
(d) $f(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x > \pi \end{cases}$

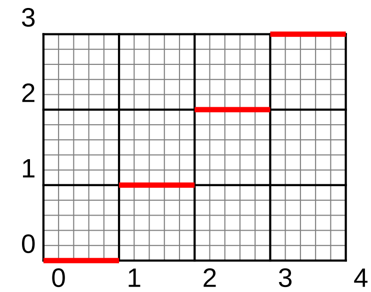
(a)

$$\begin{aligned}
L[u(x-a)] &= \int_0^\infty u(x-a)e^{-px}\,dx \\
&= \int_0^a 0e^{-px}\,dx + \int_a^\infty 1e^{-px}\,dx \\
&= \left[-\frac{1}{p}e^{-px}\right]_a^\infty \\
&= \frac{1}{p}e^{-pa}
\end{aligned}$$



(b)

$$
\begin{aligned}
L[([x])] &= \int_0^\infty [x]e^{-px}\,dx \\
&= \sum_{k=0}^\infty \int_k^{k+1} ke^{-px}\,dx \\
&= \sum_{k=0}^\infty \left[-\frac{k}{p}e^{-px}\right]_k^{k+1} \\
&= \sum_{k=0}^\infty \left(-\frac{k}{p}e^{-p(k+1)} + \frac{k}{p}e^{-pk}\right) \\
&= \frac{1-e^{-p}}{p}\sum_{k=0}^\infty ke^{-pk}
\end{aligned}
$$

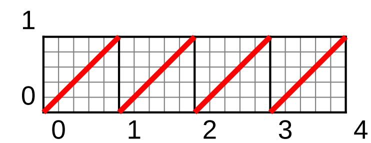


To evaluate this summation, note that

$$
\begin{aligned}
\sum_{k=0}^\infty z^k &= \frac{1}{1-z} \\
\sum_{k=1}^\infty kz^{k-1} &= \frac{1}{(1-z)^2} \\
\sum_{k=1}^\infty kz^k &= \frac{z}{(1-z)^2} \\
\sum_{k=0}^\infty ke^{-pk} &= \frac{e^{-p}}{(1-e^{-p})^2} \\
L[([x])] &= \frac{1-e^{-p}}{p}\sum_{k=0}^\infty ke^{-pk} \\
&= \frac{1-e^{-p}}{p}\left(\frac{e^{-p}}{(1-e^{-p})^2}\right) \\
&= \frac{1}{p}\left(\frac{e^{-p}}{1-e^{-p}}\right) \\
&= \frac{1}{p(e^p-1)}
\end{aligned}
$$

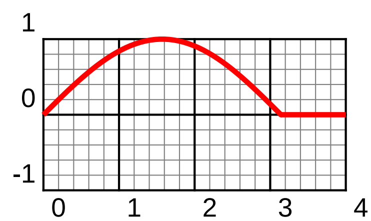
(c)

$$
\begin{aligned}
L[(x-[x])] &= L[x] - L[([x])] \\
&= \frac{1}{p^2} - \frac{1}{p(e^p-1)}
\end{aligned}
$$



(d)

$$
\begin{aligned}
L[f(x)] &= \int_0^\infty f(x)e^{-px}\,dx \\
&= \int_0^\pi \sin xe^{-px}\,dx + \int_\pi^\infty 0e^{-px}\,dx \\
&= \int_0^\pi \sin xe^{-px}\,dx
\end{aligned}
$$



This integral is similar to one above. Let $A = \int_0^\pi \sin x e^{-px}\, dx$ and $B = \int_0^\pi \sin x e^{-px}\, dx$.

$$\begin{aligned}
A &= \int_0^\pi e^{-px} \sin x \, dx \\
&= \left[\left(-\frac{1}{p}\right) e^{-px} \sin x\right]_0^\pi - \int_0^\pi \left(-\frac{1}{p}\right) e^{-px} \cos x \, dx \qquad du = e^{-px}\, dx, v = x \\
&= \frac{1}{p} \int_0^\pi e^{-px} \cos x \, dx \\
&= \frac{1}{p} B \\
B &= \int_0^\pi e^{-px} \cos x \, dx \\
&= \left[\left(-\frac{1}{p}\right) e^{-px} \cos x\right]_0^\pi - \int_0^\pi \left(-\frac{1}{p}\right) e^{-px} \sin x \, dx \qquad du = e^{-px}\, dx, v = x \\
&= \frac{1}{p} e^{-p\pi} + \frac{1}{p} - \frac{1}{p} \int_0^\pi e^{-px} \sin x \, dx \\
&= \frac{1}{p} e^{-p\pi} + \frac{1}{p} - \frac{1}{p} A \\
&= \frac{1}{p} e^{-p\pi} + \frac{1}{p} - \frac{1}{p^2} B \\
(p^2+1)B &= p(e^{-p\pi}+1) \\
B &= \frac{p(e^{-p\pi}+1)}{p^2+1} \\
A &= \frac{1}{p} B \\
&= \frac{e^{-p\pi}+1}{p^2+1}
\end{aligned}$$

Problem 49.4

Show explicitly that $L[x^{-1}]$ does not exist.

$$\begin{aligned}
L[x^{-1}] &= \int_0^\infty x^{-1} e^{-px} \, dx \\
&= \int_0^1 x^{-1} e^{-px} \, dx + \int_1^\infty x^{-1} e^{-px} \, dx
\end{aligned}$$

Using $x^{-1} \leq 1$ shows that the second integral exists. The first integral is the problem. Indeed,

$$\begin{aligned}
\int_0^1 x^{-1} e^{-px} \, dx &\geq \int_0^1 x^{-1} e^{-p} \, dx \\
&= e^{-p} \int_0^1 x^{-1} \, dx
\end{aligned}$$

This integral diverges by the p-test.

Problem 50.1

Find the Laplace transforms of
(b) $(1 - x^2)e^{-x}$

$$
\begin{aligned}
L[1 - x^2] &= \frac{1}{p} - \frac{2}{p^3} \\
L[(1 - x^2)e^{-x}] &= \frac{1}{p+1} - \frac{2}{(p+1)^3}
\end{aligned}
$$

Note that this technique can also be used to deal with factors like $\sin ax$ or $\cos ax$. For example, Problem 49.2(d) was very tedious to do directly. Using the shift rule, this can be simplified somewhat.

$$
\begin{aligned}
f(x) &= (1 - u(x - \pi)) \sin x \\
&= \sin x - \operatorname{Im}(u(x - \pi)e^{ix}) \\
L[f(x)] &= L[\sin x - \operatorname{Im}(u(x - \pi)e^{ix})] \\
L[f(x)] &= L[\sin x] - \operatorname{Im}\big(L[u(x - \pi)e^{ix}]\big) \\
L[f(x)] &= \frac{1}{p^2 + 1} - \operatorname{Im}\Big(L[u(x - \pi)]\big|_{p \to p-i}\Big) \\
L[f(x)] &= \frac{1}{p^2 + 1} - \operatorname{Im}\left(\left.\frac{1}{p}e^{-p\pi}\right|_{p \to p-i}\right) \\
L[f(x)] &= \frac{1}{p^2 + 1} - \operatorname{Im}\left(\frac{1}{p - i}e^{-(p-i)\pi}\right) \\
L[f(x)] &= \frac{1}{p^2 + 1} - \operatorname{Im}\left(\frac{p + i}{p^2 + 1}e^{-p\pi}(\cos \pi + i \sin \pi)\right) \\
L[f(x)] &= \frac{1}{p^2 + 1} - \operatorname{Im}\left(\frac{p + i}{p^2 + 1}e^{-p\pi}(-1)\right) \\
L[f(x)] &= \frac{1}{p^2 + 1} + \frac{1}{p^2 + 1}e^{-p\pi} \\
L[f(x)] &= \frac{e^{-p\pi} + 1}{p^2 + 1}
\end{aligned}
$$