Econometrics I - Applications of FWL

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Applications of FWL

FWL helps understand common transformations of variables in econometrics:

- deviations from the mean (de-meaning or centring the variables)
- de-seasonalizing data
- de-trending data

These are non-singular linear transformations. From the invariance property we know that $\bar{\boldsymbol{y}}$ and $\boldsymbol{e}$ will be unchanged. If the transformed data is orthogonal to certain regressors, then the FWL theorem tells us those certain regressors may be dropped from the model.

These transformations can aid in the visualization of the data, interpretation of the estimated parameters, and in some cases ease computational burden.

Centring data (deviations from the mean, or de-meaning)

Simple model:

$$\boldsymbol{y} = \beta_1 + \beta_2 \boldsymbol{x} + \boldsymbol{\epsilon} \tag{1}$$

Can we drop the constant β_1 from model 1? The LS estimators for β_2 would be different under:

$$\boldsymbol{y} = \beta_2 \boldsymbol{x} + \boldsymbol{\epsilon} \tag{2}$$

```r
un <- read.csv("http://rtgodwin.com/data/centrethis.csv")
lm(y ~ x, data=un)
lm(y ~ x -1, data=un)
```

Figure: A least-squares line fitted through some uncentred data. The estimated intercept of $b_1 = 63.7$ is outside the range of the data, and has little economic meaning in most models.

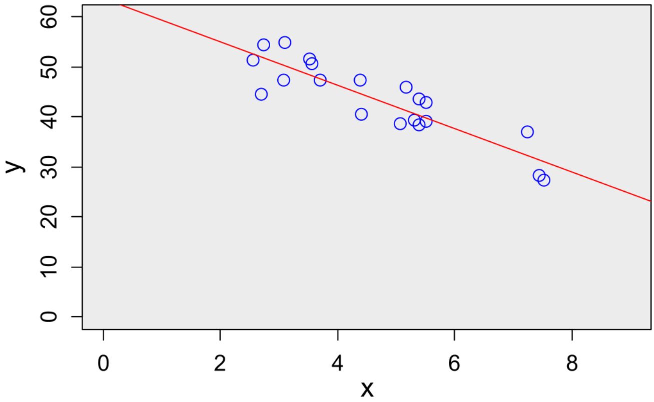
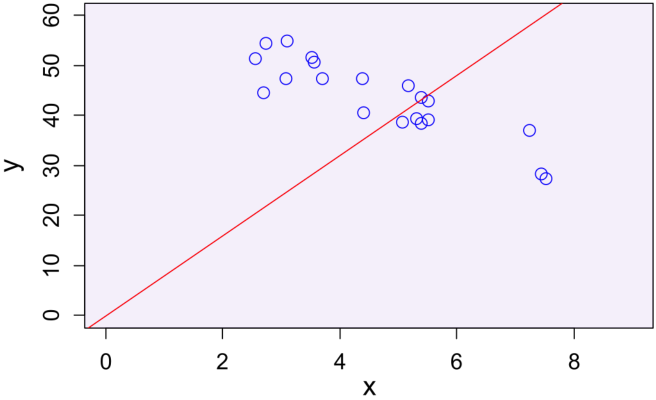


Figure: The least-squares line is forced through the origin if the model does not include an intercept.



Recall that the regression line must pass through the sample means of the data . If the data all has mean zero, then the LS line must pass through the origin anyway, and dropping the intercept has no effect.

To centre data, we *transform* it by subtracting its sample mean:

$$\boldsymbol{y}^\star = \boldsymbol{y} - \boldsymbol{i}\bar{y} \quad ; \quad \boldsymbol{x}^\star = \boldsymbol{x} - \boldsymbol{i}\bar{x}$$

where $\boldsymbol{y}^\star$ and $\boldsymbol{x}^\star$ are the centred variables, and $\boldsymbol{i}$ is a column vector of 1s:

$$\boldsymbol{i} \quad = \quad \underset{(n \times 1)}{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}$$

Estimating the model:

$$\boldsymbol{y}^{\star} = \beta_2 \boldsymbol{x}^{\star} + \boldsymbol{\epsilon} \tag{3}$$

yields identical results to model 1.

The trick is that the variables $\boldsymbol{y}^{\star}$ and $\boldsymbol{x}^{\star}$ are orthogonal to the column vector $\boldsymbol{i}$. The FWL theorem says that exclusion of this regressor ($\boldsymbol{i}$) does not affect the LS estimates.

To prove this, consider the residuals from regressions of $\boldsymbol{x}$ on a constant, and $\boldsymbol{y}$ on a constant. That is, consider the vectors $M_i \boldsymbol{y}$ and $M_i \boldsymbol{x}$.

Let:

$$M_i = I - \boldsymbol{i} \left(\boldsymbol{i}' \boldsymbol{i} \right)^{-1} \boldsymbol{i}' = I - \frac{1}{n} \boldsymbol{i} \boldsymbol{i}' \tag{4}$$

Then,

$$M_i \boldsymbol{y} = \boldsymbol{y} - \boldsymbol{i} \bar{y} = \boldsymbol{y}^{\star}$$

The M_i matrix, when pre-multiplying a vector, creates the deviations-from-means. That is, it centres a variable.

To see how this works, multiply out $M_i\boldsymbol{y}$:

$$M_i\boldsymbol{y} = \left\{ \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} - \begin{bmatrix} 1/n & \dots & 1/n \\ \vdots & \ddots & \vdots \\ 1/n & \dots & 1/n \end{bmatrix} \right\} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{bmatrix} y_1 & - & y_1/n & - & y_2/n & - & \dots & - & y_n/n \\ & & & & \vdots & & & & \\ y_n & - & y_1/n & - & y_2/n & - & \dots & - & y_n/n \end{bmatrix} = \begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

The transformed (centred) variables are now orthogonal to the regressor $\boldsymbol{i}$, that is $(M_i\boldsymbol{y})'\boldsymbol{i} = 0$ and $(M_i\boldsymbol{x})'\boldsymbol{i} = 0$, and so the intercept may be dropped from the model without any substantive effect.