# Econometrics I
## ECON 7010 Handout

# Matrices: Concepts, Definitions and Some Basic Results

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## Concepts and Definitions

### Vector

A "vector" is a set of *scalar* values, or *elements*, displayed as a *column* or *row* of values. The number of elements in the vector gives us the vector's *dimension*.

The vector $v_1 = \begin{pmatrix} 2 & 6 & 3 & 8 \end{pmatrix}$ is a row vector with 4 elements – it is a $(1 \times 4)$ vector, because it has 1 row with 4 elements. We can also think of these elements as being located in "column" positions, so the vector essentially has one row and 4 columns.

Similarly, the vector

$$v_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 2 \end{pmatrix}$$

is a column vector with 4 elements – it is a $(4 \times 1)$ vector, because it has 1 column with 4 elements. We can think of these elements as being located in "row" positions, so the vector essentially has one column and 4 rows.

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## Matrix

A "matrix" is rectangular array of values, or "elements", obtained by taking several column vectors (of the same dimension) and placing them side-by-side in a specific order. Alternatively, we can think of a matrix as being formed by taking several row vectors (of the same dimension) and placing them one above the other, in a particular order. For example, if we take the vectors

$$v_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 2 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 1 \\ 6 \\ 2 \\ 7 \end{pmatrix}$$

we can form the matrix

$$V_1 = \begin{bmatrix} 2 & 1 \\ 5 & 6 \\ 8 & 2 \\ 2 & 7 \end{bmatrix}$$

If we place the vectors side-by-side in the opposite order, we get a different matrix, of course, namely:

$$V_2 = \begin{bmatrix} 1 & 2 \\ 6 & 5 \\ 2 & 8 \\ 7 & 2 \end{bmatrix}.$$

## Dimension of a Matrix

The "dimension" of a matrix is the number of rows and the number of columns. If there are $m$ rows and $n$ columns, the dimension of the matrix is $(m \times n)$. For example, the matrix

$$A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix}$$

is a $(3 \times 3)$ matrix, while the dimension of the matrix

$$D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$$

is $(3 \times 2)$.

### Matrix Multiplication

The product of two matrices (for example $A$ and $D$) is a new matrix $AD$. The number of rows in $AD$ match the number of rows in $A$, and the number of columns match the number in $D$. Verify that the product of $AD$ is:

$$AD = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 33 & 80 \\ 54 & 95 \\ 125 & 161 \end{bmatrix}.$$

Note that matrix $A$ can not be pre-multiplied by $D$, that is, $DA$ does not exist (the number of columns of the first matrix must match the number of rows of the second matrix).

> $A$ has dimension $(3 \times 3)$, and $D$ has dimension $(3 \times 2)$. To multiply $AD$, the inner dimensions must match, and the outer dimensions determine the dimension of $AD$.

This highlights that the **order of the matrices** matters when performing matrix operations. The order of scalars does not matter: $2 \times 3 = 3 \times 2$, but the **order of matrices matters**. For example, let

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

then

$$PQ = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \text{ and } QP = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}.$$

### Square Matrix

A matrix is "square" if it has the same number of rows as columns. Matrix "$A$" above is square, while matrix "$D$" is not.

### Rectangular Matrix

A rectangular matrix is one whose number of columns is different from its number of rows. Matrix "$D$" from above is rectangular.

### Leading Diagonal

The "leading diagonal" is the string of elements from the top left corner of the matrix to the bottom right corner.

## Diagonal Matrix

A square matrix is said to be “diagonal” if the only non-zero elements in the matrix occur along the leading diagonal. For example, the matrix

$$C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

is a diagonal matrix.

## Scalar Matrix

A square matrix is said to be “scalar” if it is diagonal, and all of the elements of its leading diagonal are the same. The matrix

$$B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

is “scalar”, but the matrix

$$C = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

is not.

## Identity Matrix

An “identity” matrix is one which is scalar, with the value 1 for each element on the leading diagonal. (Because this matrix is scalar, it is also a square and diagonal matrix.) The matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix. (We might also name it $I_3$ to indicate that it is a $(3 \times 3)$ identity matrix.)

An identity matrix serves the same purpose as the number “1” for scalars: if we pre-multiply or post-multiply a matrix by the identity matrix (of the right dimensions), the original matrix is unchanged. For example, if

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix},$$

then $ID = D = DI$.

## Null Matrix

A "null matrix" has the value zero for all of its elements. The matrices

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are both null matrices.

A null matrix serves the same purpose as the number "0" for scalars: if we pre-multiply or post-multiply a matrix by the null matrix (of the right dimensions), the result is a null matrix. For example, if

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix},$$

then $ZD = N$.

$Z$ is $(3 \times 3)$, and $D$ is $(3 \times 2)$, so $ZD$ must be $(3 \times 2)$.

## Trace

The "trace" of a square matrix is the sum of the elements on its leading diagonal. For example, if

$$A = \begin{bmatrix} 7 & 3 & 1 \\ 6 & 4 & 3 \\ 5 & 8 & 9 \end{bmatrix},$$

then $trace(A) = (7 + 4 + 9) = 20$.

## Transpose

The "transpose" of a matrix is obtained by exchanging all of the rows for all of the columns. That is, the first row becomes the first column; the second row becomes the second column; and so on. For example, if

$$D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}, \text{ then the transpose of } D \text{ is } D' = \begin{bmatrix} 1 & 6 & 8 \\ 8 & 5 & 9 \end{bmatrix}.$$

Sometimes we write $D^T$ rather than $D'$ to denote the transpose of a matrix. Note that if the original matrix is an $(m \times n)$ matrix, then its transpose will be an $(n \times m)$.

Recall that a vector is just a special type of matrix: a matrix with either just one row, or just one column. When we transpose a row vector we get a column vector with the elements in the same order; and when we transpose a column vector we get a row vector, with the order of the elements unaltered.

For example, when we transpose the $(1\times4)$ row vector, $v_1 = \begin{pmatrix} 2 & 6 & 3 & 8 \end{pmatrix}$, we get a column vector which is $(4 \times 1)$:

$$v_1' = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 8 \end{pmatrix}$$

### Symmetric Matrix

A square matrix is "symmetric" if it is equal to its own transpose: transposing the rows and columns of the matrix leaves it unchanged. As we look at elements above and below the leading diagonal, we see the same values in corresponding positions: the element at $(i, j)$ equals the element at $(j, i)$, for all $i \neq j$. For example, let

$$F = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 4 \\ 6 & 4 & 9 \end{bmatrix}.$$

Here the $(1, 3)$ element and the $(3, 1)$ element are both 6, etc. Note that $F' = F$ , so $F$ is symmetric.

### Linear Dependency

Two vectors (and hence two rows, or two columns of a matrix) are "linearly independent" if one vector *cannot* be written as a multiple of the other. For example, the vectors $x_1 = (1, 3, 4, 6)$ and $x_2 = (5, 4, 1, 8)$ are linearly independent, but the vectors $x_3 = (1, 2, 4, 8)$ and $x_4 = (2, 4, 8, 16)$ are "linearly dependent", because $x_4 = 2x_3$.

More generally, a collection of $n$ vectors is linearly independent if no one of the vectors can be written as a linear combination (weighted sum) of the remaining $(n - 1)$ vectors. Consider the vectors $x_1$ and $x_2$ above, together with the vector $x_5 = (4, 1, -3, 2)$. These three vectors are *not* linearly independent, because $x_5 = x_2 - x_1$.

### Rank of a Matrix

The "rank" of a matrix is the (smaller of the) number of linearly independent rows or columns in the matrix. For example, the matrix

$$D = \begin{bmatrix} 1 & 8 \\ 6 & 5 \\ 8 & 9 \end{bmatrix}$$

has a rank of "2". It has 2 columns, and the first column is *not* a multiple of the second column. The columns are linearly independent. It has 3 rows: these three rows make up a group of 3 linearly independent vectors, but by convention we define "rank" in terms of the smaller of the number of rows and columns. So, this matrix has "full rank".

> Full rank: when all of the columns (or rows, if rows $<$ columns) of a matrix are linearly independent.

On the other hand, the matrix

$$G = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 7 \\ 6 & 4 & 10 \end{bmatrix}$$

has a rank of "2", because the third column is the sum of the first two columns. In this case the matrix has "less than full rank", because potentially it could have had a rank of "3", but the one linear dependency reduces the rank below this potential value.

### Determinant of a Matrix

The determinant of a (square) matrix is a particular polynomial in the elements of the matrix, and is a scalar quantity. We usually denote the determinant of a matrix $A$ by $|A|$, or $det.(A)$.

The determinant of a scalar is just the scalar itself. The determinant of a (2 2) matrix is obtained as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22}) - (a_{21}a_{12})$$

If the matrix is $(3 \times 3)$, then

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{11}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right). \end{aligned}$$

## Inverse Matrix

Suppose that we have a square matrix, $A$. If we can find a matrix $B$, with the same dimension as $A$, such that $AB = BA = I$ (an identity matrix), then $B$ is called the "inverse matrix" for $A$, and we denote it as $B = A^{-1}$.

The inverse matrix corresponds to the reciprocal when we are dealing with scalar numbers. Note, however, that many square matrices *do not* have an inverse.

## Singular Matrix

A square matrix that does not have an inverse is said to be a "singular matrix". On the other hand, if the inverse matrix does exist, the matrix is said to be "non-singular".

> A non-singular matrix has an inverse.

For example, every null matrix is singular. Similarly, every identity matrix is non-singular, and equal to its own inverse (just as $^1/_1 = 1$ in the case of scalars).

## Computing an Inverse Matrix

It is worth knowing how to obtain the inverse of a $(2 \times 2)$ non-singular matrix. We first obtain the determinant of the matrix. We then interchange the 2 elements on the leading diagonal of the matrix, and change the signs of the 2 off-diagonal elements. Finally, we divide this transformed matrix by the determinant. This can only be done if the determinant is non-zero! So, a necessary (but not sufficient) condition for a matrix to be non-singular is that its determinant is non-zero.

To illustrate these calculations, consider the matrix

$$R = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}.$$

Its determinant is $\Delta = (4 \times -2) - (1 \times -1) = -8 + 1 = -7$. The inverse of the $R$ matrix is:

$$R^{-1} = \left(\frac{1}{\Delta}\right) \begin{bmatrix} -2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ 1/7 & -4/7 \end{bmatrix}.$$

Check that

$$RR^{-1} = R^{-1}R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

## Definiteness of a Matrix

Suppose that $A$ is any square $(n \times n)$ matrix. The matrix $A$ is:

- positive definite if the (scalar) quadratic form, $x'Ax > 0$, for all non-zero $(n \times 1)$ vectors, $x$
- positive *semi*-definite if $x'Ax \geq 0$
- negative definite if $x'Ax < 0$
- negative *semi*-definite if $x'Ax \leq 0$
- indefinite if the sign of $x'Ax$ varies with $x$

For example, let

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} x'Ax &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4x_1 \\ 2x_2 \end{pmatrix} \\ &= 4x_1^2 + 2x_2^2 > 0, \end{aligned}$$

unless *both* $x_1$ and $x_2$ are zero. So, $A$ is positive definite in this case.

## Idempotent Matrix

Suppose that we have a square and symmetric matrix, $Q$, which has the property that $Q^2 = Q$. Because $Q$ is symmetric, this means that $Q'Q = QQ' = QQ = Q^2 = Q$. Any matrix with this property is called an "idempotent matrix".

The identity matrix, and the null matrix, are idempotent. This corresponds with the fact that the only two idempotent scalar numbers are unity and zero. However, other matrices can also be idempotent.

Let $X$ be an $(n \times k)$ matrix, with $n > k$. Let $X$ be such that $(X'X)$ is non-singular and $(X'X)^{-1}$ exists. Let $P = X(X'X)^{-1}X'$.

Note that P is an $(n \times n)$ matrix, so it is square; and also note that

$$
\begin{aligned}
P' &= \left[X\left(X'X\right)^{-1}X'\right]' \\
&= \left(X'\right)'\left[\left(X'X\right)^{-1}\right]'X' \\
&= X\left[\left(X'X\right)'\right]^{-1}X' \\
&= X\left(X'X\right)^{-1}X' = P
\end{aligned}
$$

That is, $P$ is symmetric. Now, observe that

$$
\begin{aligned}
P'P &= \left[X\left(X'X\right)^{-1}X'\right]'X\left(X'X\right)^{-1}X' \\
&= X\left[\left(X'X\right)'\right]^{-1}X'X\left(X'X\right)^{-1}X' \\
&= XI\left(X'X\right)^{-1}X' \\
&= X\left(X'X\right)^{-1}X' \\
&= P
\end{aligned}
$$

and so $P$ is idempotent. You can also check that the matrix $M = (I_n - P)$ is another example of an idempotent matrix.

> An idempotent matrix is invariant to some transformations. It can be pre- or post-multiplied by itself and its transpose, and remain the same.

## Some Matrix Results

Let $A$ be a square $(n \times n)$ matrix. Then:

1. Let $X$ be an $(m \times n)$ matrix with full rank. Then $(XAX')$ is positive definite if $A$ is positive definite.

2. If $A$ is non-singular (that is, it has an inverse) then it is either positive definite, or negative definite, and its determinant is non-zero.

3. If $A$ is positive semi-definite or negative semi-definite, then its determinant is zero, and it is singular (it does not have an inverse).

4. If $A$ is positive definite then the determinant of $A$ is positive.

5. Suppose that $B$ is also $(n \times n)$, and that both $A$ and $B$ are non-singular. Then the definiteness of $(A - B)^{-1}$ is the same as the definiteness of $(B^{-1} - A^{-1})$.

6. If $A$ is either positive definite or negative definite, then $rank(A) = n$.

7. If $A$ is positive semi-definite or negative semi-definite, then $rank(A) = r < n$.
8. If $A$ is idempotent then it is positive semi-definite.
9. If $A$ is idempotent then $rank(A) = trace(A)$, where the trace is the sum of the leading diagonal elements.
10. If $C$ is an $(m \times n)$ matrix, then the rank of $C$ cannot exceed $min.(m, n)$.
11. Suppose that $A$ and $B$ are both $(n \times n)$ matrices. Then $(A + B)' = (A' + B')$.
12. Suppose that $A$ is a non-singular $(n \times n)$ matrix, then $\left(A^{-1}\right)' = (A')^{-1}$.
13. Suppose that $A$ and $B$ have dimensions such that $AB$ is defined. Then $(AB)' = (B'A')$.
14. Suppose that $A$ and $B$ are non-singular $(n \times n)$ matrices such that both $AB$ and $BA$ are defined. Then $(AB)^{-1} = \left(B^{-1}A^{-1}\right)$.